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***Research article***

# The fourth power mean of the three–term exponential sums

**Xiaoge Liu**[1,*] **and Guohui Chen**[2]

[1] Research Center for Number Theory and its Applications, School of Mathematics, Northwest University, Xi'an, China

[2] College of Mathematics and Statistics, Hainan Normal University, Hainan, China

* **Correspondence:** Email: xiaogeliu@stumail.nwu.edu.cn.

**Abstract:** By employing trigonometric sums and solving congruence equations, this paper explores the power mean of a specific class of high–order three–term exponential sums, and derives intriguing identities in the process. The findings presented in this paper offer insights into establishing the lower bound for these sums.

**Keywords:** three–term exponential sums; elementary methods; fourth power mean; identity

## 1. Introduction

Let $p$ be an odd prime. Then rational exponential sum modulo $p$ is defined by

$$S\left(f(x);p\right) = \sum_{x=0}^{p-1} e\left(\frac{f(x)}{p}\right), \tag{1.1}$$

where $f(x) = a_1x^{k_1} + a_2x^{k_2} + \cdots + a_rx^{k_r}$, where the $a_i$ are integers satisfy $(a_i, p) = 1$, $k_i \not\equiv k_j \bmod p-1$, $i,\ j \in 1, 2, \ldots, r$ and $e(x) = e^{2\pi ix}$. This type of sum plays a crucial role in the Waring type problem [1–3]. Obviously, due to $|e(x)| = 1$ for every real number $x$. Thus, the trivial bound of $S\left(f(x);p\right)$ is $|S\left(f(x);p\right)| \leq p$. Over time, many outstanding mathematicians such as Weyl [4], Hua [5], and Weil [6] strove to obtain a stronger upper bound estimate with order less than $p$ for this sum. Perhaps the most beautiful result about (1.1) is Weil's [6] result, i.e.,

$$S\left(f(x);p\right) \leq (t-1)\cdot p^{\frac{1}{2}},$$

where $p$ is a prime, $\deg f(x) = t$ and $f(x)$ is not a constant polynomial.

The exponential sum $S\left(f(x);p\right)$ is classified as a $k$–term exponential sum when the polynomial $f(x)$ contains exactly $k$ monomial terms. Furthermore, when the exponential function in Eq (1.1) is weighted by a Dirichlet character, the resulting sum is referred to as a mixed rational exponential sum, which is

$$S(f(x),\chi;p)=\sum_{x=1}^{p-1}\chi(x)e\left(\frac{f(x)}{p}\right), \tag{1.2}$$

where $\chi$ is a Dirichlet character modulo $p$. Weil [6] demonstrated that

$$|S(f(x),\chi;p)|\le d(f)\cdot p^{\frac{1}{2}},$$

where $p$ is a prime and (assuming that $k_1<k_2<\cdots<k_r$)

$$d(f)=\begin{cases} k_r & \text{if all } k_i \text{ are positive;}\\ |k_1| & \text{if all } k_i \text{ are negative;}\\ k_r-k_1 & \text{otherwise.}\end{cases}$$

The above is nontrivial only if $d(f)<\sqrt{p}$. In addition, some insights into Eq (1.2) are provided in [7–9].

In this paper, we focus on two types of three–term exponential sums which are $E(m,n,r,u,v,w;p)=\sum_{x=0}^{p-1}e\left(\frac{mx^u+nx^v+rx^w}{p}\right)$ and $E(m,n,r,u,v,w,\chi;p)=\sum_{x=1}^{p-1}\chi(x)e\left(\frac{mx^u+nx^v+rx^w}{p}\right)$. Moreover, we will study the bound of $E(m,n,r,u,v,w;p)$ and $E(m,n,r,u,v,w,\chi;p)$ from the perspective of power mean.

Let $p\ge 3$ be an odd prime with primitive root $g$. For arbitrary integers $a,m_0\in\mathbb{Z}$ , the Dirichlet character $\chi$ modulo $p$ is defined as:

$$\chi(a)=\begin{cases}0 & \text{if } (a,p)>1;\\ e\left(\frac{m_0\mathrm{ind}_g(a)}{p-1}\right) & \text{if } (a,p)=1,\end{cases}$$

where $\mathrm{ind}_g(a)$ is the index satisfying $g^{ind_g(a)}\equiv a \bmod p$. When $w=1$, Liu and Li [10] obtained that for any integers $m$, $n$ with $(mn,p)=1$,

$$\sum_{r=0}^{p-1}\sum_{\chi \bmod p}|E(m,n,r,u,v,1,\chi;p)|^4=2p^4-7p^3+8p^2-3p.$$

Besides, if $r$ is an integer with $(r,p)=1$. They [10] got

$$\sum_{m=0}^{p-1}\sum_{n=0}^{p-1}|E(m,n,r,4,2,1,\chi;p)|^4=\begin{cases}2p^4-11p^3+16p^2 & \text{if } \chi=\chi_0;\\ 2p^4-7p^3+12p^2 & \text{if } \chi\ne\chi_0 \text{ and } 2\mid m_0;\\ 2p^4-3p^3 & \text{if } \chi\ne\chi_0 \text{ and } 2\nmid m_0,\end{cases}$$

and

$$\sum_{m=0}^{p-1}\sum_{n=0}^{p-1}|E(m,n,r,6,2,1,\chi;p)|^4$$

$$= \begin{cases} 2p^4 - 15p^3 + 36p^2 & \text{if } \chi = \chi_0 \text{ and } p \equiv 1 \bmod 4; \\ 2p^4 - 11p^3 + 16p^2 & \text{if } \chi = \chi_0 \text{ and } p \equiv 3 \bmod 4; \\ 2p^4 - 7p^3 + 28p^2 & \text{if } \chi \neq \chi_0, p \equiv 1 \bmod 4, \text{ and } 4 \mid m_0; \\ 2p^4 - 7p^3 + 12p^2 & \text{if } \chi \neq \chi_0, p \equiv 1 \bmod 4, 2 \mid m_0, \text{ and } 4 \nmid m_0; \\ 2p^4 - 7p^3 + 12p^2 & \text{if } \chi \neq \chi_0, p \equiv 3 \bmod 4 \text{ and } 2 \mid m_0; \\ 2p^4 - 3p^3 & \text{if } \chi \neq \chi_0 \text{ and } 2 \nmid m_0. \end{cases}$$

When $w = 2$, the power mean of this type of sum becomes very complex, and it's difficult to obtain some identities about it. Zhang [11] primary considered the case of $w = 2$. Let $p$ be an odd prime. Then he got

$$\sum_{m=0}^{p-1} \sum_{n=0}^{p-1} |E(m, n, 1, 6, 3, 2; p)|^4$$
$$= \begin{cases} 2p^4 - 5p^3 + 34p^2 + p^{\frac{5}{2}} \cdot \left(8p - 36 - 2\left(\frac{2}{p}\right)\right) & \text{if } p \equiv 1 \bmod 12 \text{ and } 3 \nmid \alpha; \\ 2p^4 - 5p^3 + 34p^2 - p^{\frac{5}{2}} \cdot \left(8p - 28 - 2\left(\frac{2}{p}\right)\right) & \text{if } p \equiv 1 \bmod 12 \text{ and } 3 \mid \alpha; \\ 2p^4 - p^3 & \text{if } p \equiv 5 \bmod 12; \\ 2p^4 - 13p^3 + 34p^2 & \text{if } p \equiv 7 \bmod 12; \\ 2p^4 - p^3 & \text{if } p \equiv 11 \bmod 12, \end{cases}$$

where $\alpha = \sum_{x=1}^{\frac{p-1}{2}} \left(\frac{x + \bar{x}}{p}\right)$ and $\left(\frac{*}{p}\right)$ denotes the Legendre's symbol modulo $p$.

In addition, relevant articles on rational exponential sum can be found in [12–17]. In fact, when the degree of $f(x)$ increases and $w = 2$, these problems become more complicated. Can we still obtain some identities about it? Thus, we consider three–term exponential sums $E(m, n, r, 8, 4, 2; p)$ and $E(m, n, r, 8, 4, 2, \chi; p)$ in what follows. The principal challenge in analyzing mean value problems fundamentally resides in solving systems of congruence equations. Our investigation focuses on exponents of specific orders (particularly 8–th, 4–th, and 2–nd) because lower–order congruence equations (4–th and 2–nd) provide a systematic mechanism for resolving higher–order (8–th) congruence equations through reduction techniques.

**Theorem 1.1.** Let $p$ be an odd prime. Then

$$\sum_{m=0}^{p-1} \sum_{n=0}^{p-1} |E(m, n, 1, 8, 4, 2; p)|^4$$
$$= \begin{cases} 8p^4 + 16p^{\frac{7}{2}} - 40p^3 - 64p^{\frac{5}{2}} + 81p^2 & \text{if } p \equiv 1 \bmod 8; \\ 8p^4 - 16p^{\frac{7}{2}} - 40p^3 + 48p^{\frac{5}{2}} + 81p^2 & \text{if } p \equiv 5 \bmod 8; \\ 8p^4 - 16p^3 + 9p^2 & \text{if } p \equiv 3 \bmod 4. \end{cases}$$

**Theorem 1.2.** Let $p$ be an odd prime and $m$, $n$ be arbitrary integers. Then

$$\sum_{r=0}^{p-1} \sum_{\chi \bmod p} |E(m, n, r, 8, 4, 2, \chi; p)|^4 = 4p(p-1)^2(p-2).$$

**Theorem 1.3.** Let $p$ be an odd prime and $m$, $r$ be arbitrary integers with $(r, p) = 1$. Then

$$\sum_{n=0}^{p-1} \sum_{\chi \bmod p} |E(m, n, r, 8, 4, 2, \chi; p)|^4$$
$$= \begin{cases} p(p-1)(4p^2 - 16p - 12\left(\frac{r}{p}\right) \cdot p^{\frac{1}{2}} + 24) & \text{if } p \equiv 1 \bmod 8; \\ p(p-1)(4p^2 - 16p + 4\left(\frac{r}{p}\right) \cdot p^{\frac{1}{2}} + 24) & \text{if } p \equiv 5 \bmod 8; \\ 4p(p-1)^2(p-2) & \text{if } p \equiv 3 \bmod 4. \end{cases}$$

**Corollary 1.1.** From our theorems, we can obtain that if $p$ is an odd prime, then there exist Dirichlet characters $\chi$ modulo $p$ and $m_0, n_0, m_1, n_1, r_1 \in 0, 1, \ldots, p-1$ satisfying

$$E(m_0, n_0, 1, 8, 4, 2; p) \gg p^{\frac{1}{2}},$$
$$E(m_1, n_1, r_1, 8, 4, 2, \chi; p) \gg p^{\frac{1}{2}},$$

where $f(x) \gg g(x)$ denotes that there exists constant $c$ satisfies $|f(x)| \geq c \cdot g(x)$.

## 2. Formulas that satisfy the congruence equation

In this section, we will use some concepts from elementary number theory and analytic number theory, specifically focusing on the properties of trigonometric sums and residue systems modulo $p$. For a more comprehensive understanding, please refer to the books [18–20].

**Lemma 2.1.** Let $p$ be an odd prime. Then

$$\sum_{\substack{a=0 \\ a^4+b^4 \equiv c^4+1 \bmod p \\ a^4 \cdot b^4 \equiv c^4 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} 1 = \begin{cases} 8(4p-11) & \text{if } p \equiv 1 \bmod 4; \\ 8p - 12 & \text{if } p \equiv 3 \bmod 4, \end{cases}$$

and

$$\sum_{\substack{a=0 \\ a^4+b^4 \equiv c^4+1 \bmod p \\ a^4 \cdot b^4 \equiv c^4 \bmod p \\ a^2+b^2 \equiv c^2+1 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} 1 = 8p - 12.$$

**Proof.** Firstly, let's prove the first identities.

$$\sum_{\substack{a=0 \\ a^4+b^4 \equiv c^4+1 \bmod p \\ a^4 \cdot b^4 \equiv c^4 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} 1 = \sum_{\substack{a=0 \\ a^4+b^4 \equiv a^4 \cdot b^4+1 \bmod p \\ a^4 \cdot b^4 \equiv c^4 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} 1 = \sum_{\substack{a=0 \\ (a^4-1)(b^4-1) \equiv 0 \bmod p \\ a^4 \cdot b^4 \equiv c^4 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} 1$$
$$= \sum_{\substack{a=0 \\ a^4 \equiv b^4 \equiv c^4 \equiv 1 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} 1 + 2 \sum_{\substack{a=0 \\ a^4 \equiv 1 \bmod p,\, b^4 \not\equiv 1 \bmod p \\ b^4 \equiv c^4 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} 1.$$

If $p \equiv 1 \bmod 4$, we know that if $a$ runs through a complete residue system of modulo $p$, then the congruence equation $a^4 \equiv 1 \bmod p$ has four solutions, which are $\pm 1$ and $\pm w$ satisfy $(\pm 1)^2 \equiv 1 \bmod p$ and $(\pm w)^2 \equiv -1 \bmod p$. And if $p \equiv 3 \bmod 4$, then congruence equation $a^4 \equiv 1 \bmod p$ has two solutions, which are $\pm 1$. Thus, if $p \equiv 1 \bmod 4$, then

$$\begin{aligned}
\sum_{\substack{a=0 \\ a^4+b^4\equiv c^4+1 \bmod p \\ a^4\cdot b^4\equiv c^4 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} 1 &= 4\cdot 4\cdot 4 + 2\cdot 4\cdot \sum_{\substack{b=0 \\ b^4\not\equiv 1 \bmod p \\ b^4\equiv c^4 \bmod p}}^{p-1}\sum_{c=0}^{p-1} 1 \\
&= 64 + 8\left(\sum_{\substack{b=1 \\ b^4\not\equiv 1 \bmod p \\ b^4\equiv c^4 \bmod p}}^{p-1}\sum_{c=0}^{p-1} 1 + 1\right) = 64 + 8\left(\sum_{\substack{b=1 \\ b^4\not\equiv 1 \bmod p}}^{p-1} 1\cdot \sum_{\substack{c=0 \\ c^4\equiv 1 \bmod p}}^{p-1} 1 + 1\right) \\
&= 8(4p-11).
\end{aligned}$$

If $p \equiv 3 \bmod 4$, then

$$\begin{aligned}
\sum_{\substack{a=0 \\ a^4+b^4\equiv c^4+1 \bmod p \\ a^4\cdot b^4\equiv c^4 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} 1 &= 2\cdot 2\cdot 2 + 2\cdot 2\cdot \sum_{\substack{b=0 \\ b^4\not\equiv 1 \bmod p \\ b^4\equiv c^4 \bmod p}}^{p-1}\sum_{c=0}^{p-1} 1 \\
&= 8 + 4\left(\sum_{\substack{b=1 \\ b^4\not\equiv 1 \bmod p \\ b^4\equiv c^4 \bmod p}}^{p-1}\sum_{c=0}^{p-1} 1 + 1\right) = 8 + 4\left(\sum_{\substack{b=1 \\ b^4\not\equiv 1 \bmod p}}^{p-1} 1\cdot \sum_{\substack{c=0 \\ c^4\equiv 1 \bmod p}}^{p-1} 1 + 1\right) \\
&= 8p-12.
\end{aligned}$$

Having proved the first identity, we now prove the second.

$$\begin{aligned}
\sum_{\substack{a=0 \\ a^4+b^4\equiv c^4+1 \bmod p \\ a^4\cdot b^4\equiv c^4 \bmod p \\ a^2+b^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} 1 &= \sum_{\substack{a=0 \\ a^2\cdot b^2\equiv c^2 \bmod p \\ a^4\cdot b^4\equiv c^4 \bmod p \\ a^2+b^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} 1 = \sum_{\substack{a=0 \\ a^2\cdot b^2\equiv c^2 \bmod p \\ a^2+b^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} 1 \\
&= \sum_{\substack{a=0 \\ (a^2-1)\cdot(b^2-1)\equiv 0 \bmod p \\ a^2+b^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} 1 = \sum_{\substack{a=0 \\ a^2\equiv b^2\equiv c^2\equiv 1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} 1 + 4\sum_{\substack{b=0 \\ b^2\not\equiv 1 \bmod p \\ b^2\equiv c^2 \bmod p}}^{p-1}\sum_{c=0}^{p-1} 1 \\
&= 8 + 4\cdot(2p-5) = 8p-12.
\end{aligned}$$

The proof of lemma 2.1 is completed.

**Lemma 2.2.** Let $p$ be an odd prime. Then

$$\sum_{\substack{a=0 \\ a^4+b^4\equiv c^4 \bmod p \\ ab\equiv 0 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} e\left(\frac{a^2+b^2-c^2}{p}\right) = \begin{cases} 4p+4\left(\frac{2}{p}\right)\sqrt{p}-7 & \text{if } p\equiv 1 \bmod 4; \\ 4p-3 & \text{if } p\equiv 3 \bmod 4. \end{cases}$$

**Proof.** By utilizing the properties of the reduced residue system of modulo $p$, we can obtain

$$\sum_{\substack{a=0 \\ a^4+b^4\equiv c^4 \bmod p \\ ab\equiv 0 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} e\left(\frac{a^2+b^2-c^2}{p}\right) = 1+2\sum_{\substack{b=1 \\ b^4\equiv c^4 \bmod p}}^{p-1}\sum_{c=0}^{p-1} e\left(\frac{b^2-c^2}{p}\right)$$
$$= 1+2\sum_{\substack{b=1 \\ c^4\equiv 1 \bmod p}}^{p-1}\sum_{c=0}^{p-1} e\left(\frac{b^2(1-c^2)}{p}\right).$$

If $p\equiv 1 \bmod 4$, then

$$\sum_{\substack{a=0 \\ a^4+b^4\equiv c^4 \bmod p \\ ab\equiv 0 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} e\left(\frac{a^2+b^2-c^2}{p}\right) = 1+2\sum_{\substack{b=0 \\ c^4\equiv 1 \bmod p}}^{p-1}\sum_{c=0}^{p-1} e\left(\frac{b^2(1-c^2)}{p}\right) - 2\sum_{\substack{c=0 \\ c^4\equiv 1 \bmod p}}^{p-1} 1$$
$$= 1+2\sum_{b=0}^{p-1}\left(2e\left(\frac{0\cdot b^2}{p}\right)+2e\left(\frac{2\cdot b^2}{p}\right)\right)-8 = 4p+4\left(\frac{2}{p}\right)\tau(\chi_2)-7,$$

where $\chi_2(a)=\left(\frac{a}{p}\right)$ and $\tau(\chi_2)=\sum_{a=1}^{p-1}\chi_2(a)e\left(\frac{a}{p}\right)$. We know that if $p\equiv 1 \bmod 4$, then $\tau(\chi_2)=\sqrt{p}$. Thus,

$$\sum_{\substack{a=0 \\ a^4+b^4\equiv c^4 \bmod p \\ ab\equiv 0 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} e\left(\frac{a^2+b^2-c^2}{p}\right) = 4p+4\left(\frac{2}{p}\right)\sqrt{p}-7.$$

If $p\equiv 3 \bmod 4$, then

$$\sum_{\substack{a=0 \\ a^4+b^4\equiv c^4 \bmod p \\ ab\equiv 0 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} e\left(\frac{a^2+b^2-c^2}{p}\right) = 1+2\sum_{\substack{b=0 \\ c^4\equiv 1 \bmod p}}^{p-1}\sum_{c=0}^{p-1} e\left(\frac{b^2(1-c^2)}{p}\right) - 2\sum_{\substack{c=0 \\ c^4\equiv 1 \bmod p}}^{p-1} 1$$
$$= 1+2\sum_{b=0}^{p-1}2e\left(\frac{0\cdot b^2}{p}\right)-4 = 4p-3.$$

The proof of lemma 2.2 is completed.

**Lemma 2.3.** Let $p$ be an odd prime. Then

$$\sum_{\substack{a=0 \\ a^4+b^4\equiv c^4+1 \bmod p \\ a^4\cdot b^4\equiv c^4 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\left(\frac{a^2+b^2-c^2-1}{p}\right)=\begin{cases}(16p-60)\cdot\left(\frac{2}{p}\right)-8 & \text{if } p\equiv 1 \bmod 4;\\ 0 & \text{if } p\equiv 3 \bmod 4.\end{cases}$$

**Proof.**

$$\begin{aligned}&\sum_{\substack{a=0 \\ a^4+b^4\equiv c^4+1 \bmod p \\ a^4\cdot b^4\equiv c^4 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\left(\frac{a^2+b^2-c^2-1}{p}\right)=\sum_{\substack{a=0 \\ (a^4-1)(b^4-1)\equiv 0 \bmod p \\ a^4\cdot b^4\equiv c^4 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\left(\frac{a^2+b^2-c^2-1}{p}\right)\\ &=\sum_{\substack{a=0 \\ a^4\equiv b^4\equiv c^4\equiv 1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\left(\frac{a^2+b^2-c^2-1}{p}\right)+2\sum_{\substack{a=0 \\ a^4\equiv 1 \bmod p,\, b^4\not\equiv 1 \bmod p \\ b^4\equiv c^4 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\left(\frac{a^2+b^2-c^2-1}{p}\right).\end{aligned}\tag{2.1}$$

If $p\equiv 1 \bmod 4$, then

$$\begin{aligned}&\sum_{\substack{a=0 \\ a^4\equiv b^4\equiv c^4\equiv 1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\left(\frac{a^2+b^2-c^2-1}{p}\right)\\ &=2\sum_{\substack{a=0 \\ a^4\equiv b^4\equiv 1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\left(\frac{a^2+b^2-1-1}{p}\right)+2\sum_{\substack{a=0 \\ a^4\equiv b^4\equiv 1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\left(\frac{a^2+b^2+1-1}{p}\right)\\ &=8\left(\frac{1+1-1-1}{p}\right)+8\left(\frac{1-1-1-1}{p}\right)+8\left(\frac{-1+1-1-1}{p}\right)\\ &\quad+8\left(\frac{-1-1-1-1}{p}\right)+8\left(\frac{1+1+1-1}{p}\right)+8\left(\frac{1-1+1-1}{p}\right)\\ &\quad+8\left(\frac{-1+1+1-1}{p}\right)+8\left(\frac{-1-1+1-1}{p}\right)\\ &=32\left(\frac{2}{p}\right)+8.\end{aligned}\tag{2.2}$$

$$\begin{aligned}&\sum_{\substack{a=0 \\ a^4\equiv 1 \bmod p,\, b^4\not\equiv 1 \bmod p \\ b^4\equiv c^4 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\left(\frac{a^2+b^2-c^2-1}{p}\right)\\ &=2\sum_{\substack{b=0 \\ b^4\not\equiv 1 \bmod p \\ b^4\equiv c^4 \bmod p}}^{p-1}\sum_{c=0}^{p-1}\left(\frac{b^2-c^2}{p}\right)+2\sum_{\substack{b=0 \\ b^4\not\equiv 1 \bmod p \\ b^4\equiv c^4 \bmod p}}^{p-1}\sum_{c=0}^{p-1}\left(\frac{b^2-c^2-2}{p}\right)\end{aligned}$$

$$= 2 \sum_{\substack{b=1 \\ b^4 \not\equiv 1 \bmod p \\ b^4 \equiv c^4 \bmod p}}^{p-1} \sum_{c=0}^{p-1} \left(\frac{b^2 - c^2}{p}\right) + 2 \sum_{\substack{b=0 \\ b^4 \not\equiv 1 \bmod p \\ (b^2+c^2)(b^2-c^2) \equiv 0 \bmod p}}^{p-1} \sum_{c=0}^{p-1} \left(\frac{b^2 - c^2 - 2}{p}\right)$$

$$= 2 \sum_{\substack{b=1 \\ b^4 \not\equiv 1 \bmod p}}^{p-1} \left(\frac{b^2}{p}\right) \sum_{\substack{c=0 \\ c^4 \equiv 1 \bmod p}}^{p-1} \left(\frac{1 - c^2}{p}\right) + 2 \sum_{\substack{b=0 \\ b^4 \not\equiv 1 \bmod p \\ b^2+c^2 \equiv b^2-c^2 \equiv 0 \bmod p}}^{p-1} \sum_{c=0}^{p-1} \left(\frac{0 - 2}{p}\right)$$

$$+2 \sum_{\substack{b=0 \\ b^4 \not\equiv 1 \bmod p \\ b^2-c^2 \equiv 0 \bmod p,\, b^2+c^2 \not\equiv 0 \bmod p}}^{p-1} \sum_{c=0}^{p-1} \left(\frac{0 - 2}{p}\right) + 2 \sum_{\substack{b=0 \\ b^4 \not\equiv 1 \bmod p \\ b^2-c^2 \not\equiv 0 \bmod p,\, b^2+c^2 \equiv 0 \bmod p}}^{p-1} \sum_{c=0}^{p-1} \left(\frac{b^2 - c^2 - 2}{p}\right)$$

$$= 2(p-5) \cdot 2\left(\frac{2}{p}\right) + 2\left(\frac{-2}{p}\right) + 2 \cdot 2(p-5)\left(\frac{-2}{p}\right) + 2 \cdot 2 \sum_{\substack{b=0 \\ b \neq 0, \pm 1, \pm w}}^{p-1} \left(\frac{2b^2 - 2}{p}\right)$$

$$= (8p - 38)\left(\frac{2}{p}\right) + 4\left(\frac{2}{p}\right)\left(\sum_{b=0}^{p-1} \left(\frac{b^2 - 1}{p}\right) - \left(\frac{-1}{p}\right) - 2\left(\frac{-2}{p}\right)\right)$$

$$= (8p - 46) \cdot \left(\frac{2}{p}\right) - 8, \tag{2.3}$$

where we use the identity

$$\sum_{a=0}^{p-1} \left(\frac{a^2 + n}{p}\right) = \begin{cases} p - 1 & \text{if } (p, n) = p; \\ -1 & \text{if } (p, n) = 1. \end{cases}$$

So, if $p \equiv 1 \bmod 4$, combining (2.1)–(2.3) we get

$$\sum_{\substack{a=0 \\ a^4+b^4 \equiv c^4+1 \bmod p \\ a^4 \cdot b^4 \equiv c^4 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \left(\frac{a^2 + b^2 - c^2 - 1}{p}\right)$$
$$= 32\left(\frac{2}{p}\right) + 8 + 2\left((8p - 46) \cdot \left(\frac{2}{p}\right) - 8\right) = (16p - 60)\left(\frac{2}{p}\right) - 8.$$

If $p \equiv 3 \bmod 4$, then congruence equation $a^4 \equiv 1 \bmod p$ has two solutions, which are $\pm 1$. Besides, we have $(\pm 1)^2 \equiv 1 \bmod p$. Thus,

$$\sum_{\substack{a=0 \\ a^4 \equiv b^4 \equiv c^4 \equiv 1 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \left(\frac{a^2 + b^2 - c^2 - 1}{p}\right) = 0. \tag{2.4}$$

$$\sum_{\substack{a=0 \\ a^4\equiv 1 \bmod p,\, b^4\not\equiv 1 \bmod p \\ b^4\equiv c^4 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\left(\frac{a^2+b^2-c^2-1}{p}\right)$$
$$= 2\sum_{\substack{b=0 \\ b^4\not\equiv 1 \bmod p \\ b^4\equiv c^4 \bmod p}}^{p-1}\sum_{c=0}^{p-1}\left(\frac{b^2-c^2}{p}\right) = 2\sum_{\substack{b=1 \\ b^4\not\equiv 1 \bmod p \\ b^4\equiv c^4 \bmod p}}^{p-1}\sum_{c=0}^{p-1}\left(\frac{b^2-c^2}{p}\right)$$
$$= 2\sum_{\substack{b=1 \\ b^4\not\equiv 1 \bmod p}}^{p-1}\left(\frac{b^2}{p}\right)\sum_{\substack{c=0 \\ c^4\equiv 1 \bmod p}}^{p-1}\left(\frac{1-c^2}{p}\right) = 0. \tag{2.5}$$

So, if $p \equiv 3 \bmod 4$, combining (2.1), (2.4) and (2.5) we get

$$\sum_{\substack{a=0 \\ a^4+b^4\equiv c^4+1 \bmod p \\ a^4\cdot b^4\equiv c^4 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\left(\frac{a^2+b^2-c^2-1}{p}\right) = 0.$$

The proof of lemma 2.3 is completed.

## 3. Proofs of the Theorems

In this section, we will complete the proof of our theorems. Firstly, we will prove theorem 1.1. Combining lemma 2.1–2.3, the properties of residue systems modulo $p$ and $|z|^2 = z \cdot \bar{z}$, where $z$ is a complex number, we can get

$$\frac{1}{p^2}\sum_{m=0}^{p-1}\sum_{n=0}^{p-1}|E(m,n,1,8,4,2;p)|^4$$
$$= \frac{1}{p^2}\sum_{m=0}^{p-1}\sum_{n=0}^{p-1}\left|\sum_{x=0}^{p-1} e\left(\frac{mx^8+nx^4+x^2}{p}\right)\right|^4$$
$$= \frac{1}{p^2}\sum_{m=0}^{p-1}\sum_{n=0}^{p-1}\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\sum_{d=0}^{p-1} e\left(\frac{m(a^8+b^8-c^8-d^8)+n(a^4+b^4-c^4-d^4)}{p}\right)$$
$$\cdot e\left(\frac{a^2+b^2-c^2-d^2}{p}\right)$$
$$= \sum_{\substack{a=0 \\ a^8+b^8\equiv c^8+d^8 \bmod p \\ a^4+b^4\equiv c^4+d^4 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\sum_{d=0}^{p-1} e\left(\frac{a^2+b^2-c^2-d^2}{p}\right)$$

$$= \sum_{\substack{a=0 \\ a^4+b^4\equiv c^4+d^4 \bmod p \\ a^4\cdot b^4\equiv c^4\cdot d^4 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \sum_{d=0}^{p-1} e\left(\frac{a^2+b^2-c^2-d^2}{p}\right)$$

$$= \sum_{\substack{a=0 \\ a^4+b^4\equiv c^4+d^4 \bmod p \\ a^4\cdot b^4\equiv c^4\cdot d^4 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \sum_{d=1}^{p-1} e\left(\frac{a^2+b^2-c^2-d^2}{p}\right) + \sum_{\substack{a=0 \\ a^4+b^4\equiv c^4 \bmod p \\ a^4\cdot b^4\equiv 0 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} e\left(\frac{a^2+b^2-c^2}{p}\right)$$

$$= \sum_{\substack{a=0 \\ a^4+b^4\equiv c^4+1 \bmod p \\ a^4\cdot b^4\equiv c^4 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \sum_{d=0}^{p-1} e\left(\frac{d^2(a^2+b^2-c^2-1)}{p}\right) - \sum_{\substack{a=0 \\ a^4+b^4\equiv c^4+1 \bmod p \\ a^4\cdot b^4\equiv c^4 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} 1$$

$$+ \sum_{\substack{a=0 \\ a^4+b^4\equiv c^4 \bmod p \\ ab\equiv 0 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} e\left(\frac{a^2+b^2-c^2}{p}\right)$$

$$= p \cdot \sum_{\substack{a=0 \\ a^4+b^4\equiv c^4+1 \bmod p \\ a^4\cdot b^4\equiv c^4 \bmod p \\ a^2+b^2\equiv c^2+1 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} 1 + \tau(\chi_2) \cdot \sum_{\substack{a=0 \\ a^4+b^4\equiv c^4+1 \bmod p \\ a^4\cdot b^4\equiv c^4 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \left(\frac{a^2+b^2-c^2-1}{p}\right)$$

$$- \sum_{\substack{a=0 \\ a^4+b^4\equiv c^4+1 \bmod p \\ a^4\cdot b^4\equiv c^4 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} 1 + \sum_{\substack{a=0 \\ a^4+b^4\equiv c^4 \bmod p \\ ab\equiv 0 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} e\left(\frac{a^2+b^2-c^2}{p}\right).$$

Combining lemma 2.1–2.3, we can obtain that if $p \equiv 3 \bmod 4$,

$$\begin{aligned}
&\frac{1}{p^2} \sum_{m=0}^{p-1} \sum_{n=0}^{p-1} |E(m,n,1,8,4,2;p)|^4 \\
&= p \cdot (8p-12) + \tau(\chi_2) \cdot 0 - (8p-12) + (4p-3) \\
&= 8p^2 - 16p + 9.
\end{aligned}$$

Similarly, if $p \equiv 1 \bmod 4$,

$$\begin{aligned}
&\frac{1}{p^2} \sum_{m=0}^{p-1} \sum_{n=0}^{p-1} |E(m,n,1,8,4,2;p)|^4 \\
&= p(8p-12) + \sqrt{p} \cdot \left((16p-60)\left(\frac{2}{p}\right) - 8\right) - 8(4p-11) \\
&+4p + 4\left(\frac{2}{p}\right)\sqrt{p} - 7 \\
&= 8p^2 + 16\left(\frac{2}{p}\right)p^{\frac{3}{2}} - 40p - \left(56\left(\frac{2}{p}\right) + 8\right)\sqrt{p} + 81.
\end{aligned}$$

Combining $\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}$, we can derive theorem 1.1.

Secondly, we will prove theorem 1.2. Using the same methods, we can obtain

$$\begin{aligned}
&\frac{1}{p(p-1)}\sum_{r=0}^{p-1}\sum_{\chi \bmod p}|E(m,n,r,8,4,2,\chi;p)|^4\\
&=\frac{1}{p(p-1)}\sum_{r=0}^{p-1}\sum_{\chi \bmod p}\left|\sum_{x=1}^{p-1}\chi(x)e\left(\frac{mx^8+nx^4+rx^2}{p}\right)\right|^4\\
&=\frac{1}{p(p-1)}\sum_{r=0}^{p-1}\sum_{\chi \bmod p}\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a)e\left(\frac{mb^8(a^8-1)+nb^4(a^4-1)+rb^2(a^2-1)}{p}\right)\right|^2\\
&=\frac{1}{p(p-1)}\sum_{r=0}^{p-1}\sum_{\chi \bmod p}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\chi(a\bar{c})e\left(\frac{m(b^8(a^8-1)-d^8(c^8-1))}{p}\right)\\
&\cdot e\left(\frac{n(b^4(a^4-1)-d^4(c^4-1))+r(b^2(a^2-1)-d^2(c^2-1))}{p}\right)\\
&=\frac{1}{p(p-1)}\sum_{r=0}^{p-1}\sum_{\chi \bmod p}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\chi(a\bar{c})e\left(\frac{md^8(b^8(a^8-1)-(c^8-1))}{p}\right)\\
&\cdot e\left(\frac{nd^4(b^4(a^4-1)-(c^4-1))+rd^2(b^2(a^2-1)-(c^2-1))}{p}\right)\\
&=\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{\substack{c=1\\ a\equiv c \bmod p\\ b^2(a^2-1)\equiv c^2-1 \bmod p}}^{p-1}\sum_{d=1}^{p-1} e\left(\frac{md^8(b^8(a^8-1)-(c^8-1))+nd^4(b^4(a^4-1)-(c^4-1))}{p}\right)\\
&=\sum_{a=1}^{p-1}\sum_{\substack{b=1\\ (b^2-1)(a^2-1)\equiv 0 \bmod p}}^{p-1}\sum_{d=1}^{p-1} e\left(\frac{md^8(b^8-1)(a^8-1)+nd^4(b^4-1)(a^4-1)}{p}\right)\\
&=2\cdot 2\cdot\sum_{b=1}^{p-1}\sum_{d=1}^{p-1}1-\sum_{a=1}^{p-1}\sum_{\substack{b=1\\ a^2\equiv b^2\equiv 1 \bmod p}}^{p-1}\sum_{d=1}^{p-1}1\\
&=4(p-1)(p-2).
\end{aligned}$$

Finally, using similar arguments as in proving theorem 1.2, we will now complete the proof of theorem 1.3.

$$\begin{aligned}
&\frac{1}{p(p-1)}\sum_{n=0}^{p-1}\sum_{\chi \bmod p}|E(m,n,r,8,4,2,\chi;p)|^4\\
&=\frac{1}{p(p-1)}\sum_{n=0}^{p-1}\sum_{\chi \bmod p}\left|\sum_{x=1}^{p-1}\chi(x)e\left(\frac{mx^8+nx^4+rx^2}{p}\right)\right|^4
\end{aligned}$$

$$= \sum_{\substack{a=1 \\ (b^4-1)(a^4-1)\equiv 0 \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{d=1}^{p-1} e\left(\frac{md^8(b^8-1)(a^8-1)+rd^2(b^2-1)(a^2-1)}{p}\right)$$

$$= 2\sum_{\substack{a=1 \\ a^4\equiv 1 \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{d=1}^{p-1} e\left(\frac{rd^2(b^2-1)(a^2-1)}{p}\right) - \sum_{\substack{a=1 \\ a^4\equiv b^4\equiv 1 \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{d=1}^{p-1} e\left(\frac{rd^2(b^2-1)(a^2-1)}{p}\right). \tag{3.1}$$

If $p \equiv 3 \bmod 4$, then

$$\begin{aligned} &\frac{1}{p(p-1)} \sum_{n=0}^{p-1} \sum_{\chi \bmod p} |E(m,n,r,8,4,2,\chi;p)|^4 \\ &= 2\cdot 2\cdot \sum_{b=1}^{p-1}\sum_{d=1}^{p-1} 1 - 4\sum_{d=1}^{p-1} 1 \\ &= 4(p-1)(p-2). \end{aligned} \tag{3.2}$$

If $p \equiv 1 \bmod 4$, combining $\tau(\chi_2) = \sqrt{p}$ and $\left(\frac{-1}{p}\right) = 1$. When $(r,p)=1$, we can get

$$\begin{aligned} &\sum_{\substack{a=1 \\ a^4\equiv 1 \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{d=1}^{p-1} e\left(\frac{rd^2(b^2-1)(a^2-1)}{p}\right) \\ &= 2\cdot \sum_{b=1}^{p-1}\sum_{d=1}^{p-1} 1 + 2\cdot \sum_{b=1}^{p-1}\sum_{d=1}^{p-1} e\left(\frac{-2rd^2(b^2-1)}{p}\right) \\ &= 2(p-1)^2 + 2\cdot \sum_{b=1}^{p-1}\sum_{d=1}^{p-1} e\left(\frac{-2rd(b^2-1)}{p}\right) + 2\cdot \sum_{b=1}^{p-1}\sum_{d=1}^{p-1} \left(\frac{d}{p}\right) e\left(\frac{-2rd(b^2-1)}{p}\right) \\ &= 2(p-1)^2 + 4(p-1) + 2\sum_{\substack{b=1 \\ b^2\not\equiv 1 \bmod p}}^{p-1} \sum_{d=1}^{p-1} e\left(\frac{-2rd(b^2-1)}{p}\right) + 2\tau(\chi_2)\sum_{b=1}^{p-1}\left(\frac{-2r(b^2-1)}{p}\right) \\ &= 2(p-1)^2 + 4(p-1) - 2(p-3) + 2\tau(\chi_2)\left(\frac{-2r}{p}\right)\left(\sum_{b=0}^{p-1}\left(\frac{b^2-1}{p}\right) - \left(\frac{-1}{p}\right)\right) \\ &= 2(p^2-p+2) - 4\sqrt{p}\left(\frac{2r}{p}\right). \end{aligned} \tag{3.3}$$

$$\begin{aligned} &\sum_{\substack{a=1 \\ a^4\equiv b^4\equiv 1 \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{d=1}^{p-1} e\left(\frac{rd^2(a^2-1)(b^2-1)}{p}\right) \\ &= \sum_{\substack{a=1 \\ a^2\equiv b^2\equiv 1 \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{d=1}^{p-1} e\left(\frac{rd^2(a^2-1)(b^2-1)}{p}\right) + 2\sum_{\substack{a=1 \\ a^2\equiv 1 \bmod p,\, b^2\equiv -1 \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{d=1}^{p-1} e\left(\frac{rd^2(a^2-1)(b^2-1)}{p}\right) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{\substack{d=1\\ a^2\equiv b^2\equiv -1 \bmod p}}^{p-1} e\left(\frac{rd^2(a^2-1)(b^2-1)}{p}\right)\\
&= 4\sum_{d=1}^{p-1}1 + 2\cdot 4\cdot \sum_{d=1}^{p-1}1 + 4\cdot\sum_{d=1}^{p-1} e\left(\frac{4rd^2}{p}\right)\\
&= 12(p-1) + 4\left(\sum_{d=1}^{p-1} e\left(\frac{4rd}{p}\right) + \sum_{d=1}^{p-1}\left(\frac{d}{p}\right) e\left(\frac{4rd}{p}\right)\right)\\
&= 12(p-1) + 4\left(-1 + \left(\frac{4r}{p}\right)\tau(\chi_2)\right) = 12(p-1) + 4\left(-1+\left(\frac{r}{p}\right)\sqrt{p}\right)\\
&= 4\left(3p-4+\left(\frac{r}{p}\right)\sqrt{p}\right). \qquad (3.4)
\end{aligned}
$$

Combining (3.1), (3.3) and (3.4) we can get if $p \equiv 1 \bmod 4$,

$$
\begin{aligned}
&\frac{1}{p(p-1)}\sum_{n=0}^{p-1}\sum_{\chi \bmod p} |E(m,n,r,8,4,2,\chi;p)|^4\\
&= 4p^2 - 16p - 4\sqrt{p}\left(\left(\frac{r}{p}\right) + 2\left(\frac{2r}{p}\right)\right) + 24. \qquad (3.5)
\end{aligned}
$$

The proof of theorem 1.3 can be completed by combining (3.2), (3.5), and $\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}$.

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare there is no conflicts of interest.

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