Electronic Research Archive

http://www.aimspress.com/journal/era

ERA, 33(5): 3065–3073.
DOI: 10.3934/era.2025134
Received: 11 March 2025
Revised: 22 April 2025
Accepted: 06 May 2025
Published: 19 May 2025

Research article

Quasi-tilted property of generalized lower triangular matrix algebras

Xiu-Jian Wang and Jia-Bao Liu*

School of Finance and Mathematics, West Anhui University, Luan 237012, China

* **Correspondence:** Email:liujiabaoad@163.com.

Abstract: In this paper, we investigated the generalized lower triangular matrix algebra, and gave the sufficient and necessary condition for the generalized lower triangular matrix algebra to be quasi-tilted.

Keywords: generalized lower triangular matrix algebra; quasi-tilted algebra; one-point extension algebra; semisimple algebra; hereditary algebra

1. Introduction

Throughout this paper, we denote a field by k, an algebra R means a finite dimensional algebra over k, $\mathrm{mod}\,R$ denotes the category of all finitely generated right R-modules, while $\mathrm{ind}\,R$ denotes a full subcategory of $\mathrm{mod}\,R$ containing one representative of each isomorphism class of indecomposable R-modules. We will use freely Auslander-Reiten translation $\tau = D\mathrm{Tr}$, irreducible maps and properties of the Auslander-Reiten sequences. For an R-module X, the injective hull of X will be denoted by $\mathrm{E}(X)$, the projective (resp. injective) dimension of X will be denoted by $\mathrm{pd}_R X$ (resp. $\mathrm{id}_R X$), and the global dimension of R will be denoted by $\mathrm{gl.dim}R$.

As a proper generalization of tilted algebras, Happel has introduced quasi-tilted algebras in [1]. An algebra R is called quasi-tilted if it satisfies the two conditions: (i) If U is an indecomposable R-module, then $\mathrm{pd}_R U \leq 1$ or $\mathrm{id}_R U \leq 1$, and (ii) $\mathrm{gl.dim}R \leq 2$.

Let R and S be algebras and N be a finite dimensional left-S, right-R bimodule over k, and set

$$T = \begin{pmatrix} R & 0 \\ N & S \end{pmatrix} = \left\{ \begin{pmatrix} r & 0 \\ n & s \end{pmatrix} \mid r \in R,\, n \in N,\, s \in S \right\}.$$

Meanwhile, the addition and multiplication are defined as follows:

$$\begin{pmatrix} r_1 & 0 \\ n_1 & s_1 \end{pmatrix} + \begin{pmatrix} r_2 & 0 \\ n_2 & s_2 \end{pmatrix} = \begin{pmatrix} r_1 + r_2 & 0 \\ n_1 + n_2 & s_1 + s_2 \end{pmatrix},$$

$$\begin{pmatrix} r_1 & 0 \\ n_1 & s_1 \end{pmatrix}\begin{pmatrix} r_2 & 0 \\ n_2 & s_2 \end{pmatrix} = \begin{pmatrix} r_1 r_2 & 0 \\ n_1 r_2 + s_1 n_2 & s_1 s_2 \end{pmatrix}.$$

Then, T is a finite-dimensional k-algebra, and T is called a generalized lower triangular matrix algebra.

It follows from a result of [1] that every quasi-tilted artin algebra is isomorphic to a generalized lower triangular matrix algebra $\begin{pmatrix} R & 0 \\ N & S \end{pmatrix}$, where R and S are hereditary. So to find all quasi-tilted algebras, it is enough to investigate when such generalized lower triangular matrix algebras are quasi-tilted.

The conditions for the one-point extension algebra of a quasi-tilted algebra to be again quasi-tilted were given by Huard [2] and Coelho [3, 4]. Recently, [5] has obtained a new proof concerning quasi-tilted algebras. In [6], Assem characterized cluster-tilted and quasi-tilted algebras. For more results, one can see [7–9].

The construction of the generalized lower triangular matrix ring or algebra is a major issue in ring and algebraic representation theory, and it has been often used to construct counterexamples in rings and modules. Following [10], Gorenstein conditions over formal triangular matrix rings are studied by Enochs, and Gorenstein-projective modules over triangular matrix Artin algebras are also studied by Xiong and Zhang in [8]. As for more examples of generalized lower triangular matrix algebras, we refer to [7, 11, 12] and so on.

Based on the above discussion, it is meaningful to discuss quasi-tilted property of generalized lower triangular matrix algebra. In the present paper, we consider the problem of characterizing when a generalized lower triangular matrix algebra is quasi-tilted.

It is widely known that $\operatorname{mod} T$ is equivalent to a category Ω of triples (U, W, φ), where $U \in \operatorname{mod} R$, $W \in \operatorname{mod} S$ and $\varphi : W \otimes_S N \to U$ is a map in $\operatorname{mod} R$. The right T-module which corresponds to (U, W, φ) is the additive group $U \oplus W$ with right T-action given by:

$$(u, w)\begin{pmatrix} r & 0 \\ n & s \end{pmatrix} = (ur + \varphi(w \otimes n), ws).$$

We consider the right T-modules as triples (U, W, φ). Given (U_1, W_1, φ_1) and (U_2, W_2, φ_2), a morphism $f : (U_1, W_1, \varphi_1) \longrightarrow (U_2, W_2, \varphi_2)$ is a pair (f_1, f_2), where $f_1 : U_1 \to U_2$ is homomorphism in $\operatorname{mod} R$ and $f_2 : W_1 \to W_2$ is homomorphism in $\operatorname{mod} S$; furthermore, we can get that

$$\begin{array}{ccc} W_1 \otimes_S N & \xrightarrow{\varphi_1} & U_1 \\ f_2 \otimes 1 \downarrow & & \downarrow f_1 \\ W_2 \otimes_S N & \xrightarrow{\varphi_2} & U_2 \end{array}$$

is a commutative diagram. If $f = (f_1, f_2) : (U_1, W_1, \varphi_1) \longrightarrow (U_2, W_2, \varphi_2)$ is a homomorphism map, then we have f is surjective (resp. injective) if and only if $f_1 : U_1 \to U_2$ and $f_2 : W_1 \to W_2$ are surjective (resp., injective).

It is well-known that the right T-module (U, W, φ) is a projective module if, and only if, $(U/\varphi(W \otimes N))_R$ and W_S are projective modules and $\varphi : W \otimes N \longrightarrow U$ is one-one. So, we can get the following.

The T-module (U, W, φ) is a projective module if, and only if, W_S is a projective module, $\varphi : W \otimes N \longrightarrow U$ is monic, and $U = P \oplus \varphi(W \otimes N)$, where P_R is projective.

For a right T-module (U, W, φ), define

$$\widetilde{\varphi} : W \longrightarrow \mathrm{Hom}_R(N, U)$$

given by $\widetilde{\varphi}(w)(n) = \varphi(w \otimes n)$ for $n \in N$, $w \in W$. Then, we can get $\widetilde{\varphi}$ is an S-homomorphism.

The right T-module (U, W, φ) is an injective module if, and only if, U_R is an injective module and the map $\Phi : W \longrightarrow \mathrm{Hom}_R(N, U) \oplus \mathrm{E}(\ker\widetilde{\varphi})$ given by $\Phi(w) = (\widetilde{\varphi}(w), \tau(w))$ is an isomorphism of right S-modules, where $\tau : W \to \mathrm{E}(\ker\widetilde{\varphi})$ is an extension of the inclusion $\ker\widetilde{\varphi} \hookrightarrow \mathrm{E}(\ker\widetilde{\varphi})$.

We know that a right T-module $(U_R, 0, 0)$ is a projective module if, and only if, U_R is projective, hence, $\mathrm{pd}_T(U_R, 0, 0) = \mathrm{pd}_R U_R$. A right T-module $(U_R, 0, 0)$ is an injective module if, and only if, U_R is an injective module and $\mathrm{Hom}_R(N, U) = 0$. A right T-module $(0, W_S, 0)$ is a projective module if, and only if, W_S is projective and $W \otimes N = 0$. A right T-module $(0, W_S, 0)$ is injective if, and only if, W_S is injective, hence, $\mathrm{id}_T(0, W_S, 0) = \mathrm{id}_S W_S$.

2. Quasi-tilted algebra

In this section, we consider the problem of characterizing when a generalized lower triangular matrix algebra is quasi-tilted.

Suppose $T = \begin{pmatrix} R & 0 \\ N & S \end{pmatrix}$ is a generalized lower triangular matrix algebra, while Goodearl [2] gave a condition for T to be hereditary.

Theorem 1 [13]. T is a hereditary ring if, and only if,

(1) R and S are both hereditary.

(2) ${}_S N$ is a projective module.

(3) $(N/IN)_R$ is projective for all $I \leq S_S$.

We know $\mathrm{Hom}_R(W \otimes_S N, U) \overset{\sigma}{\cong} \mathrm{Hom}_S(W, \mathrm{Hom}_R(N, U))$; for $\varphi : W \otimes N \longrightarrow U$, let $\varphi' : W \longrightarrow \mathrm{Hom}_R(N, U)$ be the map corresponding to φ, that is, $\sigma(\varphi) = \varphi'$. So, we can get the following result.

Theorem 2. Suppose $T = \begin{pmatrix} R & 0 \\ N & S \end{pmatrix}$, (U, W, φ) is a right T-module such that φ' is a monomorphism. Then, the following conditions are equivalent.

(1) $\mathrm{id}_T(U, W, \varphi) \leq 1$.

(2) There exists an exact sequence $0 \longrightarrow W \longrightarrow \mathrm{Hom}_R(N, I_0) \longrightarrow \mathrm{Hom}_R(N, I_1) \oplus I_2 \longrightarrow 0$ with I_0, I_1, I_2 injective R-modules and $\mathrm{id}_R U \leq 1$.

Proof: Since $\mathrm{Hom}_R(W \otimes_S N, U) \overset{\sigma}{\cong} \mathrm{Hom}_S(W, \mathrm{Hom}_R(N, U))$, we can view (U, W, φ) as the triple (U, W, φ').

Suppose

$$0 \longrightarrow U \overset{i}{\longrightarrow} I_0(U) \longrightarrow I_1(U)$$

is a minimal injective resolution of U; thus, we can get

$$\mathrm{Hom}_R(N, U) \overset{\mathrm{Hom}_R(N,i)}{\longrightarrow} \mathrm{Hom}_R(N, I_0(U)).$$

Set $\varphi'' = \mathrm{Hom}_R(N, i) \circ \varphi'$, then φ'' is a monomorphism because φ' is a monomorphism. We have that

$$(U, W, \varphi') \overset{(i,\varphi'')}{\longrightarrow} (I_0(U), \mathrm{Hom}_R(N, I_0(U)), 1)$$

is an injective envelope of T-module (U, W, φ').

Then, we can get the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow W & \overset{\varphi''}{\longrightarrow} & \mathrm{Hom}_R(N, I_0(U)) & \rightarrow & \mathrm{Hom}_R(N, I_0(U))/\varphi''(W) & \rightarrow 0 \\ & \varphi'\downarrow & & \| & & \downarrow g & \\ 0 & \rightarrow \mathrm{Hom}_R(N, U) & \rightarrow & \mathrm{Hom}_R(N, I_0(U)) & \rightarrow & \mathrm{Hom}_R(N, I_0(U)/i(U)). & \end{array}$$

So, $\mathrm{Coker}(i, \varphi'') = (I_0(U)/i(U), \mathrm{Hom}_R(N, I_0(U))/\varphi''(W), g)$. The injective envelope of $\mathrm{Coker}(i, \varphi'')$ is

$$(I_1(U), \mathrm{Hom}_R(N, I_1(U)), 1) \oplus (0, E(I), 0),$$

where $I = \mathrm{ker} g \cong \mathrm{Coker}\varphi'$. Set $F = \mathrm{Hom}_R(N, -)$, hence, $\mathrm{id}_T(U, W, \varphi) \leq 1$ if, and only if,

$$0 \rightarrow (U, W, \varphi') \overset{(i,\varphi'')}{\longrightarrow} (I_0(U), F(I_0(U)), 1) \rightarrow (I_1(U), F(I_1(U)), 1) \oplus (0, E(I), 0) \rightarrow 0$$

is exact if, and only if,

$$0 \longrightarrow U \longrightarrow I_0(U) \longrightarrow I_1(U) \longrightarrow 0$$

and

$$0 \longrightarrow W \longrightarrow F(I_0(W)) \longrightarrow F(I_1(U)) \oplus E(I) \longrightarrow 0$$

are exact.

Given an algebra C, suppose $\mathcal{L}_C$ denotes the subset of $\mathrm{ind}C$ defined by

$$\mathcal{L}_C = \{U \in \mathrm{ind}C \mid \mathrm{pd}_C W \leq 1 \text{ for each predecessor } W \text{of } U\}.$$

Also, we denote the full subcategory of $\mathcal{L}_C$ of direct summands of finite direct sums of copies of $\mathcal{L}_C$ by $\mathrm{add}\mathcal{L}_C$.

We now give the following necessary conditions for generalized lower triangular matrix algebra R to be quasi-tilted.

Theorem 3. Let $T = \begin{pmatrix} R & 0 \\ N & S \end{pmatrix}$. If T is quasi-tilted , then

(1) Both R and S are quasi-tilted.

(2) $N_R \in \mathrm{add}\mathcal{L}_R$.

Proof: Since T is quasi-tilted, we have $\mathrm{gl.dim}T \leq 2$, and it follows from [3] that $\mathrm{gl.dim}R \leq 2$, $\mathrm{gl.dim}S \leq 2$.

(1) If S is not quasi-tilted, then there is an indecomposable R-module W_S with $\mathrm{pd}_S W = 2 = \mathrm{id}_S W$. Since $\mathrm{id}_T(0, W_S, 0) = \mathrm{id}_S W = 2$ and T is quasi-tilted, we know that $\mathrm{pd}_T(0, W_S, 0) \leq 1$. Assume that $\pi : P \rightarrow W_S$ is a projective cover of W_S, let Z_S be the kernel of π, then $(0, \pi) : (P \otimes N, P, 1) \rightarrow (0, W_S, 0)$ is a projective cover of $(0, W_S, 0)$ with kernel $(P \otimes N, Z_S, g)$. Since $\mathrm{pd}_T(0, W_S, 0) \leq 1$, we know that $(P \otimes N, Z_W, g)$ is a projective T-module, and then Z_W is a projective S-module, thus, $\mathrm{pd}_S W \leq 1$. This contradicts that $\mathrm{pd}_S W = 2$, so S is quasi-tilted.

If R is not quasi-tilted, then there exists an indecomposable R-module U_R with $\mathrm{pd}_R U = 2 = \mathrm{id}_R U$. Since $\mathrm{pd}_T(U_R, 0, 0) = \mathrm{pd}_R U = 2$ and T is quasi-tilted, we know $\mathrm{id}_T(U_R, 0, 0) \leq 1$, and by Theorem 2, $\mathrm{id}_R U \leq 1$. This contradicts that $\mathrm{id}_R U = 2$, so R is quasi-tilted.

(2) By (1), R is quasi-tilted. If some indecomposable summand N_1 of N is not in $\mathcal{L}_R$, then, by definition of $\mathcal{L}_R$, there exists some indecomposable predecessor U_R of N_1 such that $\mathrm{pd}_R U = 2$. Then,

$\mathrm{pd}_T(U,0,0) = 2$, and by [4], there exists an indecomposable injective T-module L such that $\mathrm{Hom}_T(L, \tau(U,0,0)) \neq 0$. Thus, we have a sequence

$$L \to \tau(U,0,0) \to * \to (U,0,0) \to \cdots \to (N_1,0,0) \to (N_1,S,1)$$

of nonzero maps between indecomposable T-modules starting with an injective module and ending with a projective module. Because no refinement of the path can be sectional, we have a contradiction to T being quasi-tilted. So, $M_R \in \mathrm{add}\mathcal{L}_R$.

Corollary 4 [1]. Suppose k is a field, and $T = R[N] = \begin{pmatrix} R & 0 \\ N & k \end{pmatrix}$ is the one-point extension of R by N. If T is quasi-tilted, then

(1) R is quasi-tilted.

(2) $N \in \mathrm{add}\mathcal{L}_R$.

Next, we will investigate what conditions need to be added to ensure that $T = \begin{pmatrix} R & 0 \\ N & S \end{pmatrix}$ is quasi-tilted.

To investigate when $T = \begin{pmatrix} R & 0 \\ N & S \end{pmatrix}$ is quasi-tilted, we first investigate when $\mathrm{gl.dim}T \leq 2$. By [1], we have:

Lemma 5. Let R and S be finite dimensional algebras over a field k and N a S-R bimodule finite dimensional over k. Let $T = \begin{pmatrix} R & 0 \\ N & S \end{pmatrix}$. Then, we can get

$$\mathrm{gl.dim}T = \max\{\mathrm{gl.dim}R,\ \mathrm{gl.dim}S,\ \mathrm{pd}_{S\otimes_k R^{op}}N + 1\}.$$

From this, we get that $\mathrm{gl.dim}T \leq 2$ if, and only if, $\mathrm{gl.dim}R \leq 2$ and $\mathrm{gl.dim}S \leq 2$ and $\mathrm{pd}_{S\otimes_k R^{op}}N \leq 1$. From Lemma 5, we have

Theorem 6. Let $T = \begin{pmatrix} R & 0 \\ N & S \end{pmatrix}$, (U, W, φ) be a right T-module. Then,

$$\mathrm{pd}_T(U,W,\varphi) \leq \sup\{\mathrm{pd}_R U,\ \mathrm{pd}_T(0,W,0)\}.$$

If W_S is a projective module, then

$$\mathrm{pd}_T(U,W,\varphi) \leq \sup\{\mathrm{pd}_R U,\ \mathrm{pd}_R N + 1\}.$$

Proof: Consider the following exact sequence:

$$0 \longrightarrow (U,0,0) \overset{(1_U,0)}{\longrightarrow} (U,W,\varphi) \overset{(0,1_W)}{\longrightarrow} (0,W,0) \longrightarrow 0$$

in mod T. Then, it follows from a well-known result that

$$\mathrm{pd}_T(U,W,\varphi) \leq \sup\{\mathrm{pd}_T(U,0,0),\ \mathrm{pd}_T(0,W,0)\}.$$

Since $\mathrm{pd}_T(U,0,0) = \mathrm{pd}_R U$, then

$$\mathrm{pd}_T(U,W,\varphi) \leq \sup\{\mathrm{pd}_R U,\ \mathrm{pd}_T(0,W,0)\}.$$

If W_S is a projective module, then we can get the following exact sequence:

$$0 \longrightarrow (W \otimes N, 0, 0) \xrightarrow{(1,0)} (W \otimes N, W, 1) \xrightarrow{(0,1)} (0, W, 0) \longrightarrow 0$$

in mod T. Because $(W \otimes N, W, 1)$ is projective, we can get that

$$\mathrm{pd}_T(0, W, 0) = \mathrm{pd}_T(W \otimes N, 0, 0) + 1 = \mathrm{pd}_R W \otimes N + 1 \leq \mathrm{pd}_R N + 1.$$

In the equation above, suppose W_S is not projective, and

$$\pi : Q_S \longrightarrow W_S \longrightarrow 0$$

is a projective cover of W_S with the first syzygy Z_S, then we have the projective cover

$$(0, \pi) : (Q \otimes N, Q_S, 1) \longrightarrow (0, W_S, 0) \longrightarrow 0$$

with the first syzygy $(Q \otimes N, Z_S, \eta \otimes 1)$, where $\eta : Z_S \hookrightarrow Q_S$ is the inclusion map. Hence,

$$\mathrm{pd}_T(0, W, 0) = \mathrm{pd}_T(Q \otimes N, Z_S, \eta \otimes 1) + 1.$$

Corollary 7 [2]. Suppose k is a field and $T = R[N] = \begin{pmatrix} R & 0 \\ N & k \end{pmatrix}$ is the one-point extension of R by N. If (U, k^s, φ) is a right T-module, then

$$\mathrm{pd}_T(U, k^s, \varphi) \leq \sup\{\mathrm{pd}_R U,\ \mathrm{pd}_R N + 1\}.$$

Corollary 8. Let $T = \begin{pmatrix} R & 0 \\ N & S \end{pmatrix}$. If S is semi-simple and N_R is projective, then $\mathrm{pd}_T(U, W, \varphi) \leq \sup\{\mathrm{pd}_R U,\ 1\}$ for any T-module (U, W, φ).

Corollary 9 [13]. Let $T = \begin{pmatrix} R & 0 \\ N & S \end{pmatrix}$. If R is hereditary, N_R is projective, and S is semi-simple, then T is hereditary.

From Theorem 6, we also have

Corollary 10. Let $T = \begin{pmatrix} R & 0 \\ N & S \end{pmatrix}$. If gl.dim$R \leq 2$ and S is semi-simple and $\mathrm{pd}_R N \leq 1$, then gl.dim$T \leq 2$.

Proof: By Theorem 6, for any T-module (U, W, φ), we can get

$$\mathrm{pd}_T(U, W, \varphi) \leq \sup\{\mathrm{pd}_R U,\ \mathrm{pd}_R N + 1\} \leq 2.$$

Theorem 11. Let $T = \begin{pmatrix} R & 0 \\ N & S \end{pmatrix}$ with gl.dim$R \leq 2$. For any T-module (U, W, φ), if kerφ is not projective, then $\mathrm{pd}_T(U, W, \varphi) \geq 2$.

Proof: Let $\varepsilon_2 : H \to W$ be a projective cover, and $\delta_1 : P \to U/\phi(W \otimes N)$ is a projective cover. Set $\theta = \varphi \circ (\varepsilon_2 \otimes 1_N) : H \otimes N \to U$. Let $\eta : U \to U/\varphi(W \otimes N)$ be a natural homomorphism, $\varepsilon_1 : (H \otimes N) \oplus P \to U$ such that $\varepsilon_1|_{H \otimes N} = \theta$, $\varepsilon_1|_P = \gamma_1$, where $\gamma_1 : P \to U$ such that $\delta_1 = \eta \circ \gamma_1$. We have a projective cover

$$((H \otimes N) \oplus P, H, j) \xrightarrow{\varepsilon = (\varepsilon_1, \varepsilon_2)} (U, W, \varphi).$$

According to the above discussion, we can get the following commutative diagram:

$$\begin{array}{ccccccccc} & & \ker\varepsilon_2\otimes N & \xrightarrow{\sigma\otimes 1_N} & H\otimes N & \xrightarrow{\varepsilon_2\otimes 1_N} & W\otimes N & \longrightarrow 0 \\ & & \psi\downarrow & & (1,0)\downarrow & & \varphi\downarrow & \\ 0 & \longrightarrow & K & \longrightarrow & (H\otimes N)\oplus P & \xrightarrow{\varepsilon_1} & U & \longrightarrow 0. \end{array}$$

According to the snake lemma, it is easy to get the two exact sequences

$$0\longrightarrow \ker f\longrightarrow \mathrm{coker}\psi\longrightarrow P\longrightarrow \mathrm{coker}\varphi\longrightarrow 0$$

and

$$0\longrightarrow \ker\varphi\longrightarrow \mathrm{coker}\psi\longrightarrow \Omega^1(\mathrm{coker}\varphi)\longrightarrow 0.$$

We know that $\mathrm{pd}_T(U,W,\varphi)\leq 1$ if, and only if, $(K,\ker\varepsilon_2,\varphi)$ is projective. Since $\mathrm{gl.dim}R\leq 2$, we have $\mathrm{pd}_R\mathrm{coker}\varphi\leq 2$. Hence, $\ker\varphi$ is projective. Therefore, if $\ker\varphi$ is not projective, then $\mathrm{pd}_T(U,W,\varphi)\geq 2$.

From the above proof, if R is a hereditary ring, then we can get that $\mathrm{pd}_T(U,W,\varphi)\leq 1$ if, and only if, $\ker\varphi$ is projective.

A sufficient condition for T to be quasi-tilted is given by the following theorem.

Theorem 12. Let $T=\begin{pmatrix} R & 0\\ N & S\end{pmatrix}$. If R is quasi-tilted, S is semi-simple and N_R is projective, then T is quasi-tilted.

Proof: According to Corollary 10, $\mathrm{gl.dim}T\leq 2$. Let (U,W,φ) be an indecomposable T-module, then W is a semi-simple S-module and U is an indecomposable R-module. Because R is quasi-tilted, then $\mathrm{pd}_RU\leq 1$ or $\mathrm{id}_RU\leq 1$.

If $\mathrm{pd}_RU\leq 1$, then, by Theorem 6, $\mathrm{pd}_T(U,W,\varphi)\leq 1$.

Assume that $\mathrm{id}_RU\leq 1$ and $0\to U\xrightarrow{i} I_0\xrightarrow{j} I_1\to 0$ is the minimal injective resolution of U, then

$$0\to \mathrm{Hom}_R(N,U)\xrightarrow{i_*}\mathrm{Hom}_R(N,I_0)\xrightarrow{j_*}\mathrm{Hom}_R(N,I_1)\to 0$$

is exact since N_R is projective.

Since $\mathrm{Hom}_R(Y\otimes_S N,U)\overset{\sigma}{\cong}\mathrm{Hom}_S(W,\mathrm{Hom}_R(N,U))$, for $\varphi: W\otimes N\longrightarrow U$, let $\varphi': W\longrightarrow \mathrm{Hom}_R(N,U)$ be the map corresponding to φ, that is, $\sigma(\varphi)=\varphi'$. We can then view (U,W,φ) as the triple (U,W,φ') where $\varphi': W\longrightarrow\mathrm{Hom}_R(N,U)$ is a S-module homomorphism. Hence, (U,W,φ') is an indecomposable T-module. If φ' is not a monomorphism, then $\varphi'=0$ and $(U,W,\varphi')=(0,W,0)$, hence, $\mathrm{id}_T(U,W,\varphi')<1$. Thus, we can assume that φ' is a monomorphism. Set $\varphi''=i_*\circ\varphi'$, then φ'' is a monomorphism. We know that

$$(U,W,\varphi')\xrightarrow{(i,\varphi'')}(I_0,\mathrm{Hom}_R(N,I_0),1)$$

is an injective envelope of T-module (U,W,φ').

Furthermore, we have the following commutative diagram:

$$\begin{array}{ccccccc} 0\to & W & \xrightarrow{\varphi''} & \mathrm{Hom}_R(N,I_0) & \to & \mathrm{Hom}_R(N,I_0)/\varphi''(W) & \to 0\\ & \varphi'\downarrow & & \| & & g\downarrow & \\ 0\to & \mathrm{Hom}_R(N,U) & \to & \mathrm{Hom}_R(N,I_0) & \to & \mathrm{Hom}_R(N,I_1) & \to 0. \end{array}$$

By the Five Lemma in homological algebra, g is epic.

We know that $\mathrm{Coker}(i, \varphi'') = (I_1, \mathrm{Hom}_R(N, I_0)/\varphi''(W), g)$, and the injective envelope of $\mathrm{Coker}(i, \varphi'')$ is $(I_1, \mathrm{Hom}_R(N, I_1), 1) \oplus (0, \ker g, 0)$.

We now show that $\mathrm{Hom}_R(N, I_0)/\varphi''(W) \cong \mathrm{Hom}_R(N, I_1) \oplus \ker g$.

Since j_* and g are epics, we have

$$\mathrm{Hom}_R(N, I_1) \cong \mathrm{Hom}_R(N, I_0)/\ker j_*,$$

$$\mathrm{Hom}_R(N, I_1) \cong \mathrm{Hom}_R(N, I_0)/\varphi''(W)/\ker g.$$

Therefore,

$$\mathrm{Hom}_R(N, I_0)/\varphi''(Y) \cong \mathrm{Hom}_R(N, I_0)/\ker j_* + \ker g \cong \mathrm{Hom}_R(N, I_1) \oplus \ker g.$$

So $\mathrm{id}(U, W, \varphi') \leq 1$ and T is quasi-tilted.

From Theorems 3 and 12, we have:

Corollary 13 [2]. Let R be an algebra with $\mathrm{gl.dim}R \leq 2$, N a projective R-module, and let $T = R[N]$ be the one-point extension of R by N. Then, T is quasi-tilted if, and only if, R is quasi-tilted.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

We would like to thank the referees for the very helpful comments and suggestions which improved the original version of this paper. This work is supported by the Natural Science Foundation for Yang Scholars of Anhui Province (No.1508085QA04). The authors are grateful to professor Xianneng Du for careful guidance and assistance in this research.

Conflict of interest

The authors declare there is no conflicts of interest.

References

1. D. Happel, I. Reiten, S. O. Smalϕ, Tilting in abelian categories and quasi-tilted algebras, *Am. Math. Soc.*, **120** (1996), 575.
2. F. Huard, One-point extensions of quasi-tilted algebras by projectives, *Commun. Algebra*, **29** (2001), 3055–3060. https://doi.org/10.1081/AGB-5005
3. F. U. Coelho, M. I. R. Martins, J. A. de la Pena, Quasitilted extensions of algebras I, *Proc. Am. Math. Soc.*, **129** (2000), 1289–1297. https://doi.org/10.1090/S0002-9939-00-05667-7
4. F. U. Coelho, M. I. R. Martins, J. A. de la Pena, Quasitilted extensions of algebras II, *J. Algebra*, **227** (2000), 582–594. https://doi.org/10.1006/jabr.1999.8199

5. S. Zito, A new proof concerning quasi-tilted algebras, *Proc. Am. Math. Soc.*, **147** (2019), 2757–2760.
6. I. Assem, R. Schiffler, K. Serhiyenko, Cluster-tilted and quasi-tilted algebras, *J. Pure Appl. Algebra*, **221** (2016), 2266–2288. https://doi.org/10.1016/j.jpaa.2016.12.008
7. H. Gao, Z. Huang, Silting modules over triangular matrix rings, *Taiwanese J. Math.*, **24** (2020), 1417–1437. https://doi.org/10.11650/tjm/200204
8. B. L. Xiong, P. Zhang, Gorenstein-projective modules over triangular matrix Artin algebras, *J. Algebra Appl.*, **11** (2012), 125006. https://doi.org/10.1142/S0219498812500661
9. P. Zhang, Monomorphism categories, cotilting theory and Gorenstein projective modules, *J. Algebra*, **339** (2011), 181–202. https://doi.org/10.1016/j.jalgebra.2011.05.018
10. E. E. Enochs, M. Cortes-Izurdiaga, B. Torrecillas, Gorenstein conditions over triangular matrix rings, *J. Pure Appl. Algebra*, **218** (2014), 1544–1554. https://doi.org/10.1016/j.jpaa.2013.12.006
11. H. Eshraghi, R. Hafezi, S. Salarian, Z. W. Li, Gorenstein projective modules over triangular matrix rings, *Algebra Colloq.*, **23** (2016), 97–104.
12. L. X. Mao, Duality pairs and FP-injective modules over formal triangular matrix rings, *Commun. Algebra*, **48** (2020), 5296–5310. https://doi.org/10.1080/00927872.2020.1786837
13. K. R. Goodearl, *Ring Theory, Non Singular Rings and Modules*, Marcel Dekker, New York and Basel, 1976.