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Research article

Spatial decay estimates for a coupled system of wave-plate type

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Abstract: In this article, we studied the spatial property for a coupled system of wave-plate type in a two-dimensional cylindrical domain. Using an integral differential inequality, we obtained the spatial decay estimates result that the solution can decay exponentially as the distance from the entry section tended to infinity. The result can be viewed as a version of Saint-Venant principle.

Keywords: wave-plate type; spatial decay estimates; Saint-Venant principle; biharmonic equation

1. Introduction

Saint-Venant's principle was formulated and conjectured by Saint-Venant in 1856 in [1]. An extensive investigation on this principle was carried in the framework of applied mathematics. Now, Saint-Venant's principle is a very famous mathematical and mechanical principle. The main purpose of Saint-Venant's principle is to obtain an exponential decay estimate of energy with axial distance from the near end of a semi-infinite strip or cylinder. In order to obtain this result, an a priori decay assumption on solution at infinity must be added. The study of the spatial decay estimates belongs to the study of the Saint-Venant's principle. The spatial decay estimates show that the solution can decay exponentially as the distance from the entry section tends to infinity.

Many investigations have expanded the applications of the Saint-Venant principle. Horgan [2, 3] and Horgan and Knowles [4] in their review papers have summarized the results of these studies. Edelstein [5] first studied the spatial behavior study for the transient heat conduction. Then, many authors began to study the spatial property for parabolic equations (see [6], for example). Knops and Payne [7] may be the first to study the Saint-Venant's principle for the hyperbolic equation. In order to understand the progresses of the problems regarding the studies for hyperbolic or quasi-hyperbolic equations in the Saint-Venant principle, one could refer to [8].

In recent years, the bi-harmonic equation is used to describe the behaviors of the two-dimensional physical field within a plane. It can represent many different physical phenomena, including sound waves, electric fields, and magnetic fields. Many important applications are studied in applied mathematics and mechanics. In order to obtain the Saint-Venant type result for the bi-harmonic equations, many studies and various methods have been proposed for researching the spatial behaviors for the solutions of the bi-harmonic equations in a semi-infinite strip in R^2. We mention the studies by Knowles [9, 10], Flavin [11], Flavin and Knops [12], and Horgan [13]. We note that some time-dependent problems concerning the bi-harmonic operator were considered in the literature. We mention the papers by Knops and Lupoli [14], and Song [15, 16] in connection with the spatial behaviors of solutions for a fourth-order transformed problem associated with the slow flow of an incompressible viscous fluid along a semi-infinite strip. Other results for Phragmén-lindelöf alternative may be found in [17–19].

Our problem is considered on the domain Ω_0, which is an unbounded region defined by

$$\Omega_0 := \{(x_1, x_2) \mid x_1 > 0, 0 < x_2 < h\}, \tag{1.1}$$

with h constant. We use the notation

$$L_z = \{(x_1, x_2) \mid x_1 = z \geq 0, 0 \leq x_2 \leq h\}. \tag{1.2}$$

The problem is considered in the time interval $[0, T]$, where T is a fixed positive constant.

In [20], the coupled system of wave-plate type with thermal effect was studied, precisely,

$$\rho_1 u_{,tt} - \triangle u - \mu \triangle u_{,t} + a \triangle v = 0, \tag{1.3}$$

$$\rho_2 v_{,tt} + \gamma \triangle^2 v + a \triangle u + m \triangle \theta = 0, \tag{1.4}$$

$$\tau \theta_{,t} - k \triangle \theta - m \triangle v_{,t} = 0. \tag{1.5}$$

The generation of the thermal effect can be attributed to various types of heat conduction, such as the Fourier Law which postulates a direct proportionality between the heat flux and the temperature gradient, and the Cattaneo Law which represents a hyperbolic version of heat conduction, suggesting a finite velocity for the propagation of thermal signals (see [20, 21]). In [20], the authors studied the existence, analyticity, and the exponential decay of the solutions of (1.3)–(1.5).

The above model can be used to describe the evolution of a system consisting of an elastic membrane and an elastic plate, subject to an elastic force that attracts the membrane to the plate with coefficient a, subject to a thermal effect. Here u and v represent the vertical deflections of the membrane and of the plate, respectively. θ denotes the difference of temperature. The coefficient ρ_1 is the density of the elastic membrane, ρ_2 is the density of the elastic plate, μ is the damping coefficient for the membrane, a is the elastic coupling coefficient, γ the bi-harmonic coefficient for the plate, m is the thermal coupling coefficient, τ is the thermal relaxation time, and k is the thermal conductivity coefficient. They are all nonnegative constants.

In this paper, we consider the special case of the system (1.3)–(1.5). We choose $\tau = 0$. The physical significance of setting the coefficient τ to 0 in the wave-plate type equations lies in simplifying the thermal effect component of the system. Specifically, the wave-plate type equations model the evolution of a system comprising an elastic membrane and an elastic plate, subject to an elastic force attracting the membrane to the plate, as well as a thermal effect. When τ is set to 0, it implies that the thermal effect is simplified or modified, potentially removing terms related to the rate of change of temperature or altering the nature of heat conduction within the system.

The Eqs (1.3)–(1.5) turn to

$$\rho_1 u_{,tt} - \triangle u - \mu \triangle u_{,t} + a \triangle v = 0, \tag{1.6}$$

$$\rho_2 v_{,tt} + \gamma \triangle^2 v + a \triangle u - \frac{m^2}{k} \triangle v_{,t} = 0. \tag{1.7}$$

The initial boundary conditions are

$$u(x_1, 0, t) = v(x_1, 0, t) = v_{,2}(x_1, 0, t) = 0, \quad x_1 > 0, t > 0, \tag{1.8}$$

$$u(x_1, h, t) = v(x_1, h, t) = v_{,2}(x_1, h, t) = 0, \quad x_1 > 0, t > 0, \tag{1.9}$$

$$u(0, x_2, t) = g_1(x_2, t), \quad 0 \leq x_2 \leq h, t > 0, \tag{1.10}$$

$$v(0, x_2, t) = g_2(x_2, t), \quad 0 \leq x_2 \leq h, t > 0, \tag{1.11}$$

$$v_{,1}(0, x_2, t) = g_3(x_2, t), \quad 0 \leq x_2 \leq h, t > 0, \tag{1.12}$$

and

$$u(x_1, x_2, 0) = v(x_1, x_2, 0) = u_{,t}(x_1, x_2, 0) = v_{,t}(x_1, x_2, 0) = 0, \quad 0 \leq x_2 \leq h, x_1 > 0. \tag{1.13}$$

In this paper, we add some a priori asymptotical decay assumptions for solution at the infinity.

$$\begin{aligned} &u_{,t}(x_1, x_2, t), u_{,\alpha}(x_1, x_2, t), u_{,\alpha t}(x_1, x_2, t), v_{,t}(x_1, x_2, t), v_{,\alpha}(x_1, x_2, t), v_{,\alpha t}(x_1, x_2, t), \\ &v_{,\alpha\beta}(x_1, x_2, t) \to 0 \quad (uniformly \quad in \quad x_2) \quad as \quad x_1 \to \infty. \end{aligned} \tag{1.14}$$

In this paper, $g_i(x_2, t)$ $i = 1, 2, 3$ are prescribed functions satisfying the compatibility:

$$g_1(0, t) = g_1(h, t) = g_{1,2}(0, t) = g_{1,2}(h, t) = 0,$$

$$g_2(0, t) = g_2(h, t) = g_{2,2}(0, t) = g_{2,2}(h, t) = 0,$$

$$g_3(0, t) = g_3(h, t) = g_{3,2}(0, t) = g_{3,2}(h, t) = 0,$$

$$g_1(x_2, 0) = g_2(x_2, 0) = g_3(x_2, 0) = 0.$$

Here, $\triangle$ is the harmonic operator, and $\triangle^2$ is the bi-harmonic operator. The comma is used to indicate partial differentiation, and the differentiation with respect to the direction x_k is denoted as $,k$, thus, $u_{,\alpha}$ denotes $\frac{\partial u}{\partial x_\alpha}$, and $u_{,t}$ denotes $\frac{\partial u}{\partial t}$. The usual summation convection is employed with repeated Greek subscripts α summed from 1 to 2. Hence, $u_{,\alpha\alpha} = \sum_{\alpha=1}^{2} \frac{\partial^2 u}{\partial x_\alpha^2}$. Physically, the interactions between u and v are intricate. The membrane's deflection u influences the plate's deflection v, and vice versa, through the elastic force denoted by coefficient a. This mutual interaction is captured in Eqs (1.6) and (1.7), where the Laplace operator $\triangle$ and bi-harmonic operator $\triangle^2$ terms involving u and v are coupled. In proving the existence of the solutions in [20], the authors added some restrictions on the prescribed functions $g_i(x_2, t)$ and the coefficients. However, in the present paper, we want to use the energy method to obtain the result of the Saint-Venant type. We don't add any restrictions on them. If we follow the restrictions added in [20], all the derivations of this paper are also valid. We can get the same result with no change.

In [20], the authors concentrated on the analytic properties of the system, including behavioral characteristics under specific conditions. They employed the attractors within the framework of the C_0-*semigroups* to explore the analytic properties of the system. In the present paper, we focus on the spatial decay estimates for the solutions of the system in a semi-infinite channel. We use the integral differential inequality and energy expressions to derive the spatial decay estimates. The two researches are different in research focuses and mathematical methods. Eqs (1.6)–(1.14) were studied by [22] , and the spatial decay estimates results were obtained by using both the first order differential inequality and the second order differential inequality. In [23], the authors obtained some structural stability results for the same equations by using a second order differential inequality. In [24], hyperbolic-parabolic equations were studied, and the Saint-Venant type result was obtained for the weighted energy by using a second order differential inequality. In the present paper, we will use a new method to obtain the result for the unweighted energy. Recently, in papers [25, 26], the authors studied the stability for some fluid equations. [27] studied both the spatial property and the stability for the Darcy plane flow.

Prior works, primarily dealt with elliptic or parabolic equations. The current paper demonstrates the validity of Saint-Venant's principle for hyperbolic equations, which presents unique challenges in constructing and controlling energy functions. The methodology used to obtain spatial decay estimates involves formulating energy expressions and deriving an integral differential inequality. This approach is novel in the context of Saint-Venant's principle for coupled hyperbolic systems. Unlike previous methods that relied on controlling the energy function by its own derivative, this work introduces the integral of energy for control, a method rarely used in previous Saint-Venant principle research. What's more, the vertical deflections of u and v interact with each other, and how to overcome the interactions between u and v will be another difficulty in this article. We have never seen such a result for the coupled system. Since the main difficulty in studying the wave-plate type equations is how to tackle the bi-harmonic operator, the method proposed in this paper is valid in overcoming it. We think this method is applicable to the study of other biharwave-plate type equations. From this point, our paper is new and interesting. The result obtained in this paper shows that the Saint-Venant principle is also valid for the hyperbolic-hyperbolic systems.

In this paper, we are concerned with the spatial decay estimates for a coupled system of wave-plate type in a semi-infinite channel. We formulate the energy expressions and derive an integral differential inequality, which is useful in deriving our main result in Section 2. In Section 3, we obtain the spatial decay estimates for the solution. A is an area element on the $x_1 - x_2$ plane, $dA = dx_2dx_1$. η is a time variable.

2. The definitions of the energy functions

Before stating our main result (i.e., Theorem 3.1), let us state some preliminaries for the definition of the energy expressions.

Proposition 2.1: Let (u, v) be the classical solution (the solution is smooth and differentiable) of the initial boundary value problems (1.6)–(1.14), and we define a function

$$
\begin{aligned}
F_1(z,t) = {} & \frac{\rho_1\omega}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\eta}^2\mathrm{d}A\mathrm{d}\eta + \frac{\rho_1}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)u_{,t}^2\mathrm{d}A \\
& + \frac{\omega}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\alpha}u_{,\alpha}\mathrm{d}A\mathrm{d}\eta + \frac{1}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)u_{,\alpha}u_{,\alpha}\mathrm{d}A \\
& + \mu\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\alpha\eta}u_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta + a\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\eta}v_{,\alpha\alpha}\mathrm{d}A\mathrm{d}\eta.
\end{aligned}
\tag{2.1}
$$

$F_1(z,t)$ can also be expressed as

$$
F_1(z,t) = -\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\eta}u_{,1}\mathrm{d}x_2\mathrm{d}\eta - \mu\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\eta}u_{,1\eta}\mathrm{d}x_2\mathrm{d}\eta,
\tag{2.2}
$$

where ω is an arbitrary positive constant which will be defined later.

Proof: Multiplying both sides of (1.6) by $\exp(-\omega\eta)u_{,\eta}$ and integrating, we obtain

$$
\begin{aligned}
0 = {} & \int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\eta}(\rho_1 u_{,\eta\eta} - u_{,\alpha\alpha} - \mu u_{,\alpha\alpha\eta} - a v_{,\alpha\alpha})\mathrm{d}A\mathrm{d}\eta \\
= {} & \frac{\rho_1\omega}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\eta}^2\mathrm{d}A\mathrm{d}\eta + \frac{\rho_1}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)u_{,t}^2\mathrm{d}A \\
& + \int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\alpha\eta}u_{,\alpha}\mathrm{d}A\mathrm{d}\eta + \int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\eta}u_{,1}\mathrm{d}x_2\mathrm{d}\eta \\
& + \mu\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\alpha\eta}u_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta + \mu\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\eta}u_{,1\eta}\mathrm{d}x_2\mathrm{d}\eta \\
& + a\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\eta}v_{,\alpha\alpha}\mathrm{d}A\mathrm{d}\eta \\
= {} & \frac{\rho_1\omega}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\eta}^2\mathrm{d}A\mathrm{d}\eta + \frac{\rho_1}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)u_{,t}^2\mathrm{d}A \\
& + \frac{\omega}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\alpha}u_{,\alpha}\mathrm{d}A\mathrm{d}\eta + \frac{1}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)u_{,\alpha}u_{,\alpha}\mathrm{d}A \\
& + \int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\eta}u_{,1}\mathrm{d}x_2\mathrm{d}\eta + \mu\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\alpha\eta}u_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta \\
& + \mu\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\eta}u_{,1\eta}\mathrm{d}x_2\mathrm{d}\eta + a\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\eta}v_{,\alpha\alpha}\mathrm{d}A\mathrm{d}\eta.
\end{aligned}
\tag{2.3}
$$

If we define a function,

$$F_1(z,t)=\frac{\rho_1\omega}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\eta}^2\mathrm{d}A\mathrm{d}\eta+\frac{\rho_1}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)u_{,t}^2\mathrm{d}A$$
$$+\frac{\omega}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\alpha}u_{,\alpha}\mathrm{d}A\mathrm{d}\eta+\frac{1}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)u_{,\alpha}u_{,\alpha}\mathrm{d}A$$
$$+\mu\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\alpha\eta}u_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta+a\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\eta}v_{,\alpha\alpha}\mathrm{d}A\mathrm{d}\eta.$$

Inserting (2.1) into (2.3), $F_1(z,t)$ can be written as (2.2).

Proposition 2.2: Let (u,v) be a classical solution (the solution is smooth and differentiable) of the initial boundary value problems (1.6)–(1.14), and we define a function

$$F_2(z,t)=\frac{\rho_2\omega}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\eta}^2\mathrm{d}A\mathrm{d}\eta+\frac{\rho_2}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)v_{,t}^2\mathrm{d}A$$
$$+\frac{r\omega}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}A\mathrm{d}\eta+\frac{r}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}A \tag{2.4}$$
$$-a\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\eta}u_{,\alpha}\mathrm{d}A\mathrm{d}\eta+\frac{m^2}{k}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\eta}v_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta.$$

$F_2(z,t)$ can also be expressed as

$$F_2(z,t)=-r\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\alpha\eta}v_{,\alpha1}\mathrm{d}x_2\mathrm{d}\eta+r\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\eta}v_{,1\beta\beta}\mathrm{d}x_2\mathrm{d}\eta$$
$$+a\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\eta}u_{,1}\mathrm{d}x_2\mathrm{d}\eta-\frac{m^2}{k}\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\eta}v_{,1\eta}\mathrm{d}x_2\mathrm{d}\eta. \tag{2.5}$$

Proof: Multiplying both sides of (1.7) by $\exp(-\omega\eta)v_{,\eta}$ and integrating, we obtain

$$0=\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\eta}(\rho_2v_{,\eta\eta}+rv_{,\alpha\alpha\beta\beta}+au_{,\alpha\alpha}-\frac{m^2}{k}v_{,\alpha\alpha\eta})\mathrm{d}A\mathrm{d}\eta. \tag{2.6}$$

The first term on the right side of (2.6) can be written as

$$\rho_2\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\eta}v_{,\eta\eta}\mathrm{d}A\mathrm{d}\eta=\frac{\rho_2\omega}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\eta}^2\mathrm{d}A\mathrm{d}\eta$$
$$+\frac{\rho_2}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)v_{,t}^2\mathrm{d}A. \tag{2.7}$$

The second term on the right side of (2.6) can be written as

$$\begin{aligned}
&r\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\eta}v_{,\alpha\alpha\beta\beta}\mathrm{d}A\mathrm{d}\eta\\
&=-r\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\eta}v_{,\alpha\beta\beta}\mathrm{d}A\mathrm{d}\eta-r\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\eta}v_{,1\beta\beta}\mathrm{d}x_2\mathrm{d}\eta\\
&=r\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\beta\eta}v_{,\alpha\beta}\mathrm{d}A\mathrm{d}\eta+r\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\alpha\eta}v_{,\alpha 1}\mathrm{d}x_2\mathrm{d}\eta\\
&-r\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\eta}v_{,1\beta\beta}\mathrm{d}x_2\mathrm{d}\eta\\
&=\frac{r\omega}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}A\mathrm{d}\eta-r\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\eta}v_{,1\beta\beta}\mathrm{d}x_2\mathrm{d}\eta\\
&+\frac{r}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}A+r\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\alpha\eta}v_{,\alpha 1}\mathrm{d}x_2\mathrm{d}\eta.
\end{aligned}\tag{2.8}$$

The third term on the right side of (2.6) can be written as

$$\begin{aligned}
a\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\eta}u_{,\alpha\alpha}\mathrm{d}A\mathrm{d}\eta=&-a\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\eta}u_{,\alpha}\mathrm{d}A\mathrm{d}\eta\\
&-a\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\eta}u_{,1}\mathrm{d}x_2\mathrm{d}\eta.
\end{aligned}\tag{2.9}$$

The last term on the right side of (2.6) can be written as

$$\begin{aligned}
-\frac{m^2}{k}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\eta}v_{,\alpha\alpha\eta}\mathrm{d}A\mathrm{d}\eta=&\frac{m^2}{k}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\eta}v_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta\\
&+\frac{m^2}{k}\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\eta}v_{,1\eta}\mathrm{d}x_2\mathrm{d}\eta
\end{aligned}\tag{2.10}$$

We define a new function $F_2(z,t)$ as

$$\begin{aligned}
F_2(z,t)=&\frac{\rho_2\omega}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\eta}^2\mathrm{d}A\mathrm{d}\eta+\frac{\rho_2}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)v_{,t}^2\mathrm{d}A\\
&+\frac{r\omega}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}A\mathrm{d}\eta+\frac{r}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}A\\
&-a\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\eta}u_{,\alpha}\mathrm{d}A\mathrm{d}\eta+\frac{m^2}{k}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\eta}v_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta.
\end{aligned}\tag{2.11}$$

A combination of (2.6)–(2.11) gives the desired result (2.5).

Proposition 2.3: Let (u,v) be the classical solution (the solution is smooth and differentiable) of the initial boundary value problems (1.6)–(1.14), and we define a function

$$F(z,t)=F_1(z,t)+F_2(z,t).$$

We have

$$
\begin{aligned}
F(z,t) &= \frac{\rho_1\omega}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\eta}^2\mathrm{d}A\mathrm{d}\eta + \frac{\rho_1}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)u_{,t}^2\mathrm{d}A \\
&+ \frac{\omega}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\alpha}u_{,\alpha}\mathrm{d}A\mathrm{d}\eta + \frac{1}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)u_{,\alpha}u_{,\alpha}\mathrm{d}A \\
&+ \mu\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\alpha\eta}u_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta + a\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\eta}v_{,\alpha\alpha}\mathrm{d}A\mathrm{d}\eta \\
&+ \frac{\rho_2\omega}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\eta}^2\mathrm{d}A\mathrm{d}\eta + \frac{\rho_2}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)v_{,t}^2\mathrm{d}A \\
&+ \frac{r\omega}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}A\mathrm{d}\eta + \frac{r}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}A \\
&- a\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\eta}u_{,\alpha}\mathrm{d}A\mathrm{d}\eta + \frac{m^2}{k}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\eta}v_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta.
\end{aligned}
\tag{2.12}
$$

$F(z,t)$ can also be expressed as

$$
\begin{aligned}
F(z,t) = &-\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\eta}u_{,1}\mathrm{d}x_2\mathrm{d}\eta - \mu\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\eta}u_{,1\eta}\mathrm{d}x_2\mathrm{d}\eta \\
&- r\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\alpha\eta}v_{,\alpha 1}\mathrm{d}x_2\mathrm{d}\eta + r\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\eta}v_{,1\beta\beta}\mathrm{d}x_2\mathrm{d}\eta \\
&+ a\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\eta}u_{,1}\mathrm{d}x_2\mathrm{d}\eta - \frac{m^2}{k}\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\eta}v_{,1\eta}\mathrm{d}x_2\mathrm{d}\eta.
\end{aligned}
\tag{2.13}
$$

Proof: Combining (2.1) and (2.4), we have the desired result (2.12).

Combining (2.2) and (2.5), we have the desired result (2.13).

Proposition 2.4: Let (u, v) be the classical solution (the solution is smooth and differentiable) of the initial boundary value problems (1.6)–(1.14), and we have

$$
\begin{aligned}
\int_z^\infty F(\xi,t)\mathrm{d}\xi = &-\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\eta}u_{,1}\mathrm{d}A\mathrm{d}\eta - \mu\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\eta}u_{,1\eta}\mathrm{d}A\mathrm{d}\eta \\
&- r\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\eta}v_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta + r\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\eta}v_{,\beta\beta}\mathrm{d}x_2\mathrm{d}\eta \\
&- r\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,1\eta}v_{,\beta\beta}\mathrm{d}A\mathrm{d}\eta + a\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\eta}u_{,1}\mathrm{d}A\mathrm{d}\eta \\
&- \frac{m^2}{k}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\eta}v_{,1\eta}\mathrm{d}A\mathrm{d}\eta.
\end{aligned}
\tag{2.14}
$$

Proof: In (2.13), the term $r\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\eta}v_{,1\beta\beta}\mathrm{d}x_2\mathrm{d}\eta$ can be rewritten as

$$
\begin{aligned}
r\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\eta}v_{,1\beta\beta}\mathrm{d}x_2\mathrm{d}\eta = &\frac{\partial}{\partial z}\left[r\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\eta}v_{,\beta\beta}\mathrm{d}x_2\mathrm{d}\eta\right] \\
&- r\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,1\eta}v_{,\beta\beta}\mathrm{d}x_2\mathrm{d}\eta.
\end{aligned}
\tag{2.15}
$$

Inserting (2.15) into (2.13), and integrating (2.13), we can obtain the result (2.14).

Proposition 2.5: Let (u, v) be the classical solution (the solution is smooth and differentiable) of the initial boundary value problems (1.6)–(1.14), and we have

$$\int_z^\infty F(\xi,t)\mathrm{d}\xi \le \lambda_1\left(-\frac{\partial F(z,t)}{\partial z}\right) + \lambda_2 F(z,t), \tag{2.16}$$

where λ_1 and λ_2 are positive constants.

Proof: Differentiating (2.12) with respect to z, we obtain

$$\begin{aligned}
-\frac{\partial F(z,t)}{\partial z} &= \frac{\rho_1\omega}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\eta}^2\mathrm{d}x_2\mathrm{d}\eta + \frac{\rho_1}{2}\int_{L_z}\exp(-\omega t)u_{,t}^2\mathrm{d}x_2\\
&+\frac{\omega}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\alpha}u_{,\alpha}\mathrm{d}x_2\mathrm{d}\eta + \frac{1}{2}\int_{L_z}\exp(-\omega t)u_{,\alpha}u_{,\alpha}\mathrm{d}x_2\\
&+\mu\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\alpha\eta}u_{,\alpha\eta}\mathrm{d}x_2\mathrm{d}\eta + a\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\eta}v_{,\alpha\alpha}\mathrm{d}x_2\mathrm{d}\eta\\
&+\frac{\rho_2\omega}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\eta}^2\mathrm{d}x_2\mathrm{d}\eta + \frac{\rho_2}{2}\int_{L_z}\exp(-\omega t)v_{,t}^2\mathrm{d}x_2\\
&+\frac{r\omega}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}x_2\mathrm{d}\eta + \frac{r}{2}\int_{L_z}\exp(-\omega t)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}x_2\\
&-a\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\alpha\eta}u_{,\alpha}\mathrm{d}x_2\mathrm{d}\eta + \frac{m^2}{k}\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\alpha\eta}v_{,\alpha\eta}\mathrm{d}x_2\mathrm{d}\eta.
\end{aligned} \tag{2.17}$$

In the following discussions, we will use the following Schwarz inequality:

$$\int_0^t\int_{L_z}|ab|\,\mathrm{d}x_2\mathrm{d}\eta \le \frac{\epsilon_1}{2}\int_0^t\int_{L_z}a^2\mathrm{d}x_2\mathrm{d}\eta + \frac{\epsilon_2}{2}\int_0^t\int_{L_z}b^2\mathrm{d}x_2\mathrm{d}\eta,$$

where ϵ_1 and ϵ_2 are arbitrary positive constants.

Using the Schwarz inequality, we have

$$\begin{aligned}
\left|a\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\eta}v_{,\alpha\alpha}\mathrm{d}x_2\mathrm{d}\eta\right| &\le \frac{a}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\eta}^2\mathrm{d}x_2\mathrm{d}\eta\\
&+\frac{a}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}x_2\mathrm{d}\eta,
\end{aligned} \tag{2.18}$$

and

$$\begin{aligned}
\left|a\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\alpha\eta}u_{,\alpha}\mathrm{d}x_2\mathrm{d}\eta\right| &\le \frac{ka^2}{2m^2}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\alpha}u_{,\alpha}\mathrm{d}x_2\mathrm{d}\eta\\
&+\frac{m^2}{2k}\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\alpha\eta}v_{,\alpha\eta}\mathrm{d}x_2\mathrm{d}\eta.
\end{aligned} \tag{2.19}$$

Inserting (2.18) and (2.19) into (2.17), we have

$$
\begin{aligned}
-\frac{\partial F(z,t)}{\partial z} \geq & \left(\frac{\rho_1\omega}{2}-\frac{a}{2}\right)\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\eta}^2\mathrm{d}x_2\mathrm{d}\eta+\frac{\rho_1}{2}\int_{L_z}\exp(-\omega t)u_{,t}^2\mathrm{d}x_2\\
&+\left(\frac{\omega}{2}-\frac{ka^2}{2m^2}\right)\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\alpha}u_{,\alpha}\mathrm{d}x_2\mathrm{d}\eta+\frac{1}{2}\int_{L_z}\exp(-\omega t)u_{,\alpha}u_{,\alpha}\mathrm{d}x_2\\
&+\mu\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\alpha\eta}u_{,\alpha\eta}\mathrm{d}x_2\mathrm{d}\eta+\frac{\rho_2\omega}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\eta}^2\mathrm{d}x_2\mathrm{d}\eta\\
&+\frac{\rho_2}{2}\int_{L_z}\exp(-\omega t)v_{,t}^2\mathrm{d}x_2+\left(\frac{r\omega}{2}-\frac{a}{2}\right)\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}x_2\mathrm{d}\eta\\
&+\frac{r}{2}\int_{L_z}\exp(-\omega t)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}x_2+\frac{m^2}{2k}\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\alpha\eta}v_{,\alpha\eta}\mathrm{d}x_2\mathrm{d}\eta.
\end{aligned}
\tag{2.20}
$$

Since ω is an arbitrary positive constant, if we choose $\omega \geq \max\left\{\frac{2a}{\rho_1}, \frac{2ka^2}{m^2}, \frac{2a}{r}\right\}$, we have

$$
-\frac{\partial F(z,t)}{\partial z} \geq 0.
$$

Let us define

$$
\begin{aligned}
E(z,t) = & \left(\frac{\rho_1\omega}{2}-\frac{a}{2}\right)\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\eta}^2\mathrm{d}A\mathrm{d}\eta+\frac{\rho_1}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)u_{,t}^2\mathrm{d}A\\
&+\left(\frac{\omega}{2}-\frac{ka^2}{2m^2}\right)\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\alpha}u_{,\alpha}\mathrm{d}A\mathrm{d}\eta+\frac{1}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)u_{,\alpha}u_{,\alpha}\mathrm{d}A\\
&+\mu\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\alpha\eta}u_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta+\frac{\rho_2\omega}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\eta}^2\mathrm{d}A\mathrm{d}\eta\\
&+\frac{\rho_2}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)v_{,t}^2\mathrm{d}A+\left(\frac{r\omega}{2}-\frac{a}{2}\right)\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}A\mathrm{d}\eta\\
&+\frac{r}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}A+\frac{m^2}{2k}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\eta}v_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta.
\end{aligned}
\tag{2.21}
$$

Using the similar method as in deriving (2.20), we can get

$$
F(z,t) \geq E(z,t). \tag{2.22}
$$

In the following discussions, we will obtain an integral differential inequality for the energy $F(z,t)$. Using the Schwarz inequality, we have

$$
\begin{aligned}
\left|-\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\eta}u_{,1}\mathrm{d}A\mathrm{d}\eta\right| \leq & \frac{1}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\eta}^2\mathrm{d}A\mathrm{d}\eta\\
&+\frac{1}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,1}^2\mathrm{d}A\mathrm{d}\eta.
\end{aligned}
\tag{2.23}
$$

$$\left|\mu\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\eta}u_{,1\eta}\mathrm{d}A\mathrm{d}\eta\right|\le\frac{\mu}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\eta}^2\mathrm{d}A\mathrm{d}\eta+\frac{\mu}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,1\eta}^2\mathrm{d}A\mathrm{d}\eta.\tag{2.24}$$

$$\left|r\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,1\eta}v_{,\beta\beta}\mathrm{d}A\mathrm{d}\eta\right|\le\frac{r}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,1\eta}^2\mathrm{d}A\mathrm{d}\eta+\frac{r}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\beta\beta}^2\mathrm{d}A\mathrm{d}\eta.\tag{2.25}$$

$$\left|a\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\eta}u_{,1}\mathrm{d}A\mathrm{d}\eta\right|\le\frac{a}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\eta}^2\mathrm{d}A\mathrm{d}\eta+\frac{a}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,1}^2\mathrm{d}A\mathrm{d}\eta.\tag{2.26}$$

$$\left|\frac{m^2}{k}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\eta}v_{,1\eta}\mathrm{d}A\mathrm{d}\eta\right|\le\frac{m^2}{2k}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\eta}^2\mathrm{d}A\mathrm{d}\eta+\frac{m^2}{2k}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,1\eta}^2\mathrm{d}A\mathrm{d}\eta.\tag{2.27}$$

$$\left|r\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\eta}v_{,\beta\beta}\mathrm{d}x_2\mathrm{d}\eta\right|\le\frac{r}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\eta}^2\mathrm{d}x_2\mathrm{d}\eta+\frac{r}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\beta\beta}^2\mathrm{d}x_2\mathrm{d}\eta.\tag{2.28}$$

Combining (2.23)–(2.28) and (2.17), (2.21), we have

$$\int_z^\infty F(\xi,t)\mathrm{d}\xi\le\lambda_1\left(-\frac{\partial F(z,t)}{\partial z}\right)+\lambda_2F(z,t).\tag{2.29}$$

with $\lambda_1=\max\left\{\frac{r}{\rho_2\omega},\frac{r}{r\omega-a}\right\}$ $\lambda_2=\max\left\{\frac{1+\mu+a}{\rho_1\omega-a},\frac{(1+a)m^2}{m^2\omega-ka^2},2,\frac{kr+m^2}{m^2},\frac{r}{r\omega-a},\frac{m^2+rk}{k\rho_2\omega}\right\}$. If we choose $\omega=\max\left\{\frac{2a}{\gamma},\frac{2a}{\rho_1},\frac{2ka^2}{m^2}\right\}$, we can easily get $\lambda_1>0$ and $\lambda_2>0$.

Inequality (2.16) is the main result of this section. We will use this inequality to obtain the main result of this paper in the next section. The constants λ_1 and λ_2 play crucial roles in controlling the energy of the system. By constructing energy functions and deriving integral differential inequalities, the authors are able to estimate the decay rates of the solutions. The constants λ_1 and λ_2 enter into these estimates, influencing the bounds on the energy and other related quantities.

3. Spatial decay estimates

We can rewrite (2.16) as

$$\frac{\partial F(z,t)}{\partial z}+\frac{1}{\lambda_1}\int_z^\infty F(\xi,t)\mathrm{d}\xi\le\frac{\lambda_2}{\lambda_1}F(z,t).\tag{3.1}$$

Next, we define two functions:

$$M(z,t) = e^{-\frac{\lambda_2}{\lambda_1}z}F(z,t), \tag{3.2}$$

and

$$N(z,t) = M(z,t) + \delta\int_z^\infty e^{\frac{\lambda_2}{\lambda_1}(\xi-z)}M(\xi,t)\mathrm{d}\xi, \tag{3.3}$$

where δ is a positive constant which will be defined later.

Since it is difficult to solve (3.1), we use the form of $N(z,t)$ to solve it.

Differentiating (3.3) with respect to z, we have

$$\begin{aligned}\frac{\partial N(z,t)}{\partial z} &= \frac{\partial M(z,t)}{\partial z} - \frac{\lambda_2}{\lambda_1}\delta\int_z^\infty e^{\frac{\lambda_2}{\lambda_1}(\xi-z)}M(\xi,t)\mathrm{d}\xi - \delta M(z,t)\\ &= -\frac{\lambda_2}{\lambda_1}e^{-\frac{\lambda_2}{\lambda_1}z}F(z,t) + e^{-\frac{\lambda_2}{\lambda_1}z}\frac{\partial F(z,t)}{\partial z}\\ &- \frac{\lambda_2}{\lambda_1}\delta\int_z^\infty e^{-\frac{\lambda_2}{\lambda_1}z}F(\xi,t)\mathrm{d}\xi - \delta e^{-\frac{\lambda_2}{\lambda_1}z}F(z,t).\end{aligned} \tag{3.4}$$

We can easily get

$$\begin{aligned}\frac{\partial N(z,t)}{\partial z} + \delta N(z,t) = &-\frac{\lambda_2}{\lambda_1}e^{-\frac{\lambda_2}{\lambda_1}z}F(z,t) + e^{-\frac{\lambda_2}{\lambda_1}z}\frac{\partial F(z,t)}{\partial z}\\ &- \frac{\lambda_2}{\lambda_1}\delta\int_z^\infty e^{-\frac{\lambda_2}{\lambda_1}z}F(\xi,t)\mathrm{d}\xi - \delta e^{-\frac{\lambda_2}{\lambda_1}z}F(z,t)\\ &+ \delta e^{-\frac{\lambda_2}{\lambda_1}z}F(z,t) + \delta^2\int_z^\infty e^{-\frac{\lambda_2}{\lambda_1}z}F(\xi,t)\mathrm{d}\xi.\end{aligned} \tag{3.5}$$

From (3.1), we have

$$-\frac{\lambda_2}{\lambda_1}e^{-\frac{\lambda_2}{\lambda_1}z}F(z,t) + e^{-\frac{\lambda_2}{\lambda_1}z}\frac{\partial F(z,t)}{\partial z} \le -\frac{1}{\lambda_1}e^{-\frac{\lambda_2}{\lambda_1}z}\int_z^\infty F(\xi,t)\mathrm{d}\xi. \tag{3.6}$$

By inserting (3.6) into (3.5), we get

$$\frac{\partial N(z,t)}{\partial z} + \delta N(z,t) \le \left(\delta^2 - \frac{\lambda_2}{\lambda_1}\delta - \frac{\lambda_2}{\lambda_1}\right)\int_z^\infty e^{-\frac{\lambda_2}{\lambda_1}z}F(\xi,t)\mathrm{d}\xi. \tag{3.7}$$

Let $\delta^2 - \frac{\lambda_2}{\lambda_1}\delta - \frac{\lambda_2}{\lambda_1} = 0$, and we choose $\delta_1 = \frac{\frac{\lambda_2}{\lambda_1} + \sqrt{(\frac{\lambda_2}{\lambda_1})^2 + \frac{4\lambda_2}{\lambda_1}}}{2} > 0$. We obtain the result

$$\frac{\partial N(z,t)}{\partial z} + \delta_1 N(z,t) \le 0, \tag{3.8}$$

Integrating (3.8), we obtain

$$N(z,t) \le N(0,t)e^{-\delta_1 z}. \tag{3.9}$$

A combination of (3.3) and (3.9) gives

$$M(z,t) \le N(0,t)e^{-\delta_1 z}. \tag{3.10}$$

According to the definition of $M(z,t)$ in (3.2), we have

$$F(z,t) \le N(0,t)e^{-\left(\delta_1 - \frac{\lambda_2}{\lambda_1}\right)z}. \tag{3.11}$$

We now want to give a bound for $N(0,t)$ by $F(0,t)$.
Using equations (3.3) and (3.9), we obtain

$$F(z,t) + \delta_1 \int_z^\infty F(\xi,t)\mathrm{d}\xi \le N(0,t)e^{\left(\frac{\lambda_2}{\lambda_1} - \delta_1\right)z}. \tag{3.12}$$

We rewrite inequality (3.12) as

$$-\frac{\partial}{\partial z}\left[e^{-\delta_1 z}\int_z^\infty F(\xi,t)\mathrm{d}\xi\right] \le N(0,t)e^{\left(\frac{\lambda_2}{\lambda_1} - 2\delta_1\right)z}. \tag{3.13}$$

Integrating (3.13) from 0 to ∞ , we have

$$\int_0^\infty F(\xi,t)\mathrm{d}\xi \le \frac{N(0,t)}{2\delta_1 - \frac{\lambda_2}{\lambda_1}}. \tag{3.14}$$

Using the definition of $N(0,t)$ in (3.3), we have

$$N(0,t) = F(0,t) + \delta_1 \int_0^\infty F(\xi,t)\mathrm{d}\xi. \tag{3.15}$$

Inserting (3.15) into (3.14), we have

$$\int_0^\infty F(\xi,t)\mathrm{d}\xi \le \frac{F(0,t) + \delta_1 \int_0^\infty F(\xi,t)\mathrm{d}\xi}{2\delta_1 - \frac{\lambda_2}{\lambda_1}}. \tag{3.16}$$

Solving (3.16), we obtain

$$\int_0^\infty F(\xi,t)\mathrm{d}\xi \le \frac{F(0,t)}{\delta_1 - \frac{\lambda_2}{\lambda_1}}. \tag{3.17}$$

We thus have

$$\begin{aligned} N(0,t) &= F(0,t) + \delta_1 \int_0^\infty F(0,t)\mathrm{d}\xi \\ &\le F(0,t) + \frac{\delta_1 F(0,t)}{\delta_1 - \frac{\lambda_2}{\lambda_1}} \\ &= \frac{2\delta_1 - \frac{\lambda_2}{\lambda_1}}{\delta_1 - \frac{\lambda_2}{\lambda_1}} F(0,t). \end{aligned} \tag{3.18}$$

Inserting (3.18) into (3.11) , we obtain

$$F(z,t) \leq \frac{2\delta_1\lambda_1 - \lambda_2}{\delta_1\lambda_1 - \lambda_2} F(0,t) e^{-\left(\delta_1 - \frac{\lambda_2}{\lambda_1}\right)z}. \tag{3.19}$$

We have obtained the following main theorem.

Theorem 3.1: Let (u, v) be the classical solution (the solution is smooth and differentiable) of the initial boundary value problems (1.6)–(1.14). For the energy $E(z,t)$ defined in (2.21), we can get the decay estimates

$$E(z,t) \leq \frac{2\delta_1\lambda_1 - \lambda_2}{\delta_1\lambda_1 - \lambda_2} F(0,t) e^{-\left(\delta_1 - \frac{\lambda_2}{\lambda_1}\right)z}. \tag{3.20}$$

Note that

$$\delta_1 - \frac{\lambda_2}{\lambda_1} = \frac{1}{2}\left(-\frac{\lambda_2}{\lambda_1} + \sqrt{\left(\frac{\lambda_2}{\lambda_1}\right)^2 + \frac{4\lambda_2}{\lambda_1}}\right) > 0,$$

thanks to $\lambda_1, \lambda_2 > 0$ and

$$\frac{2\delta_1\lambda_1 - \lambda_2}{\delta_1\lambda_1 - \lambda_2} = \frac{2\delta_1 - \frac{\lambda_2}{\lambda_1}}{\delta_1 - \frac{\lambda_2}{\lambda_1}} > 0.$$

From (3.20), we can obtain the result when $z \to +\infty$, $e^{-\left(\delta_1 - \frac{\lambda_2}{\lambda_1}\right)z}$ tends to zero.

From (2.21) and (2.22), we can obtain

$$F(0,t) \geq E(0,t) > 0.$$

Inequality (3.20) shows that $E(z,t)$ can decay exponentially as the distance from the entry section tends to infinity. The result can be viewed as a version of the Saint-Venant principle.

4. Conclusions

In this paper, the authors investigate the spatial decay estimates of the solutions for the coupled system. They demonstrate that the solution can decay exponentially as the distance from the entry section tends to infinity, which aligns with the core concept of the Saint-Venant principle. This finding has significant physical implications. It suggests that the influence of the initial conditions or perturbations on the system diminishes as one moves further away from the source, reflecting a gradual weakening of the system's response with increasing spatial distance. The application of the Saint-Venant principle in this context is innovative, as it extends the principle's utility beyond its traditional domain of elastic mechanics to a more complex coupled wave-plate system. By adopting this principle, the authors are able to derive important insights into the system's behavior at large scales or long distances, which is crucial for understanding and predicting its dynamic characteristics. The result obtained in this paper provides a theoretical basis for later numerical simulations. Next, we will remove the decay assumptions on the solution at infinity. At this point, the method provided in this article will no longer be applicable, and we will proceed with further research. What's more, the structural stability for the coupled system of wave-plate type in an unbounded domain would be interesting. We will study it in another paper.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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