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Research article

Transnational investment strategies for DC pension plan under inflation and model ambiguity

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Abstract: This paper studies the optimal portfolio decisions of participants in defined contribution (DC) pension plans who are able to invest their wealth in transnational securities. More specifically, pension participants can allocate their investments across cash, bonds, domestic stocks, foreign stocks, inflation-indexed instruments, and exchange rate futures. Furthermore, we assume that pension managers face ambiguity regarding the distribution of foreign asset prices. In this context, by employing dynamic programming and the "relative entropy penalty" method, the paper derives robust optimal portfolio strategies for DC pension plan participants, accompanied by a verification theorem. Additionally, we explore two specific scenarios: the optimal investment strategy for pension managers under ambiguity neutrality, and the utility loss incurred by ambiguity-averse fund managers who misapply the optimal investment strategy. Our analysis is illustrated through numerical examples.

Keywords: DC pension plan; optimal strategy; transnational investment; ambiguity aversion; inflation

1. Introduction

As the global economy and healthcare systems advance, leading to increased life expectancy, considerable attention has been paid to pension portfolio management aimed at improving post-retirement living standards. Currently, there are two primary types of pension plans worldwide. One is the defined benefit (DB) pension plan, where participants are not required to bear the investment risk associated with their pensions. Upon retirement, they receive a fixed income, with both investment risk and longevity risk borne by the plan sponsor (typically an employer). The other type is the DC pension plan, where participants contribute a percentage of their pre-retirement salary and generate returns by investing their pension assets in capital markets. In contrast to DB plans, DC plans effectively alleviate pressure on social security systems by shifting investment risk from employers to plan members. In recent years, a growing number of countries have transitioned from DB to DC pension plans. This paper aims to explore the asset allocation of DC pension plans.

In recent years, a large body of literature has examined the portfolio decisions of DC pension plans. Some researchers have approached the issue by focusing on different utility functions and analyzing the problem of maximizing expected terminal utility. For instance, Gao et al. [1] examined the optimal investment and benefit adjustment problem within collective DC pension plans integrated with long-term care insurance. Specifically, their research aimed to maximize the expected utility of total benefits and terminal wealth under the hyperbolic absolute risk aversion (HARA) utility function. Doreleijers and Christoph [2] analytically examined the optimal investment problem for a regret-averse investor. By extending the CRRA utility framework to incorporate regret and rejoicing, they proposed a multiplicative regret-rejoicing utility function. Li et al. [3] considered an α robust optimal investment problem for a DC pension plan with uncertainty about jump and diffusion risks in a mean variance framework. Additionally, some works have studied the complexity and multiple constraint characteristics faced by pension investment in actual operation. For example, Wang et al. [4] investigated a robust optimal portfolio choice problem for a DC pension plan member, who worries about model ambiguity and aims to seek robust optimal investment strategies. Wang and Jia [5] investigated equilibrium investment strategies for a DC pension plan member who faces random risk preferences. Dong and Zheng [6] investigated the optimal investment problem for a DC pension fund manager, considering loss aversion (characterized by an S-shaped utility function) alongside trading constraints and VaR constraints. They solved the problem by applying the concavification and dual control methods, deriving a closed-form expression for the optimal terminal wealth in terms of a controlled dual state variable.

According to Markowitz's portfolio theory, diversification is an effective strategy for mitigating investment risk in pension funds, and the extent of risk diversification is closely linked to the correlation among underlying assets. However, the existence of a certain degree of convergence in the price fluctuations of financial assets within the same economy weakens the effectiveness of pension managers' attempts to reduce risk through diversification. Levyand Sarnat [7] was the first to theoretically examine risk reduction through global investment diversification. By analyzing the mean and variance of stock market returns across 28 countries from 1951 to 1967, an empirical methodology for determining optimal cross-country portfolios was proposed. Jorion [8] treated foreign investment returns as a separate asset class in the U.S. market between 1978 and 1988, and investigated how the risk and return characteristics of foreign stocks and bonds affect the U.S. securities market after accounting for exchange rate risk. More recently, Guo et al. [9] studied the optimal strategy for proportional reinsurance companies investing in foreign securities, and found that, under the assumption that insurers' utility follows a CRRA utility function, insurers investing in foreign markets achieve higher utility than those limited to domestic markets.

Investing in foreign securities provides investors with dual benefits: on one hand, they can gain dividends from foreign financial markets; on the other hand, by investing in markets relatively independent of the domestic market, investors can reduce investment risks [7, 8]. However, unlike domestic market investments, investing in foreign securities involves not only general financial risks but also risks such as exchange rate risk. Additionally, due to differences in understanding of the political and economic environments of the invested countries, pension managers face ambiguity regarding the price distribution of foreign securities. In the presence of ambiguity about the price distribution of securities in a portfolio, Anderson et al. [10] was the first to incorporate ambiguity in asset price distributions within securities markets, having proposed corresponding robust optimization strategies. Uppal and Wang [11] further extended this by considering model distribution ambiguity in intertemporal portfolio problems and analyzing the relationship between the joint distribution of security returns and marginal distribution ambiguity. The classical framework for studying ambiguity in asset price distributions is the robust control approach based on Girsanov transformation, proposed by Maenhout [12]. This framework uses a "relative entropy penalty" method to characterize the relationship between investors reference measures and alternative measures. Subsequent studies, building on the framework of Maenhout [12], have explored asset price distribution ambiguity in various financial contexts (see [13–18]).

This paper considers the scenario where DC pension managers have the option to invest separately in domestic and foreign equity markets. Given the typically long-term nature of pension investments, we assume that interest rates in financial markets are stochastic, enabling managers to hedge against interest rate risk by holding zero-coupon bonds in those markets. Similarly, we postulate that managers can mitigate inflation risk by investing in inflation-linked bonds. Additionally, while the contribution process of pension participants is exposed to market risk, this risk can be hedged using financial market assets. Moreover, recognizing that managers must exchange local and foreign currencies during the investment process, this paper incorporates exchange rate risk into the analysis. Under the above assumptions regarding background risks, pension managers are considered to invest in domestic zero-coupon bonds, inflation-linked bonds, and stocks, as well as foreign zero-coupon bonds and stocks, while hedging exchange rate risk through exchange rate futures. Given that pension managers are less familiar with foreign markets compared to domestic ones, we assume ambiguity in the price distribution of foreign assets and that managers exhibit ambiguity aversion. Against this backdrop, we derive a robust optimal control strategy with an entropy penalty, analyzing the optimal strategy under both ambiguity aversion and ambiguity neutrality. Finally, we also consider the utility loss incurred when an ambiguity-averse manager adopts a suboptimal strategy.

The innovations of this paper are as follows: First, we formulate a robust optimal investment problem for DC pension plan participants that incorporates transnational investments. Second, we comprehensively account for interest rate risk, stock market risk, exchange rate risk, inflation risk, and ambiguity related to foreign markets in both bond and equity markets. Third, we derive both the optimal strategy and an explicit solution for the value function. Lastly, through numerical examples, we analyze the impact of foreign asset ambiguity and utility loss on the investment strategy.

The rest of this paper is organized as follows: Section 2 formulates the optimal investment problem for the ambiguity aversion DC pension plan for investors who can invest their wealth in domestic and foreign assets. The explicit solutions for value function and optimal strategy are presented in Section 3. Section 4 considers two special cases: no model ambiguity and no foreign asset investment. Section 5 investigates the impact of changes in market parameters on optimal strategies and utility levels through numerical simulation. Section 6 concludes the paper, and Section A gives the proofs of main theorems.

2. Problem formulation

In this section, we model the optimization problem for the DC pension plan investor who can participate in both domestic and foreign financial markets. Suppose that the stochastic variables involved in domestic and foreign financial markets are defined in two complete probability spaces, $(\Omega^d, \mathcal{F}^d, P^d)$ and $(\Omega^f, \mathcal{F}^f, P^f)$, respectively. Here, $\mathcal{F}_t^f$ and $\mathcal{F}_t^f$ are two right continuous filtrations, represented by the standard Brownian motions $W_1(t) = \left(W_r^d(t), W_H(t), W_s^d(t), W_I(t)\right)$ and $W_2(t) = (W_r^f(t), W_s^f(t))$, respectively, with $W_1(t)$ and $W_2(t)$ being assumed as independent. In addition, let

$$(\Omega, \mathcal{F}, P) = (\Omega^d \times \Omega^f, \mathcal{F}^d \times \mathcal{F}^f, P^d \times P^f),$$

be a new probability space, that satisfies $P(A \times B) = P^d(A) \times P^f(B)$ for any $A \in \mathcal{F}^d$, $B \in \mathcal{F}^f$. In addition, assume that there are no transaction costs and taxes in the financial market.

2.1. The wealth process

The exchange rate is a key factor for investors when they invest their assets in foreign markets. Therefore, before establishing a cross-border portfolio of assets, we first follow Guoet al. [9] to give the driving equation of exchange rates:

$$\frac{dH(t)}{H(t)} = \left(r^d(t) - r^f(t)\right)dt + \sigma_H\left(\lambda_H dt + dW_H(t)\right), \tag{2.1}$$

where σ_H, λ_H are positive constants, λ_H is the corresponding risk premium with Brownian motion $W_H(t)$, and $r^l(t)$ with $l \in \{d, f\}$ represents the domestic and foreign interest rate process that evolves according to

$$dr^l(t) = a^l\left(b^l - r^l(t)\right)dt - \sqrt{k_1^l r^l(t) + k_2^l}dW_r^l(t), \quad r^l(0) = r_0^l, \tag{2.2}$$

where a^l, b^l, k_1^l, k_2^l are positive constants, b^l denotes the mean reversion level of $r^l(t)$, and a^l denotes the reversion speed.

Remark 2.1. *The model (2.2) reduces the Vasicek interest rate model when $k_1 = 0$ and CIR interest rate model when $k_2 = 0$ with the condition $2a^l > b^l$ being satisfied to ensure $r(t) > 0$ (see [19]).*

Assume the DC pension plan investor can invest in the financial market that consists of cash, bond, a domestic stock, a foreign stock, inflation index, and exchange rate futures.

The price of cash is given by

$$\frac{dS_0(t)}{S_0(t)} = r^d(t)dt. \tag{2.3}$$

As one of the important tools to hedge interest rates, bonds are characterized by a face value of 1, paid on the maturity date, and no interest is paid before the maturity date. Under the risk neutral measure, the price process of a zero-coupon bond with a maturity date s , denoted by $B^l(t, s)$ at the time of t, satisfies (see [20])

$$\begin{cases} \frac{dB^l(t,s)}{B^l(t,s)} = r^l(t)dt + h^l(s-t)\sqrt{k_1^l r^l(t) + k_2^l}\left(\lambda_r^l \sqrt{k_1^l r^l(t) + k_2^l}dt + dW_r^l(t)\right), \\ B^l(s, s) = 1, \end{cases} \tag{2.4}$$

where $h^l(t) = \frac{2(e^{m^l t}-1)}{m^l-(b^l-k_1^l\lambda_r^l)+e^{m^l t}(m^l+b^l-k_1^l\lambda_r^l)}$ with $m^l = \sqrt{(b^l - k_1^l\lambda_r^l)^2 + 2k_1^l}$, and $\lambda_r^l\sqrt{k_1^l r^l(t) + k_2^l}$ is the risk premium of Brownian motion $W_r^l(t)$. After some simple calculations, the explicit solution of $B^l(t, s)$ can be obtained as

$$B^l(t, s) = e^{-h^l(s-t)r^l(t)+h_1^l(s-t)}, \tag{2.5}$$

where

$$h_1^l(t) = \frac{k_2^l}{k_1^l}t - \frac{k_2^l}{k_1^l}h^l(t) + \left(a^l + \frac{k_2^l}{k_1^l}b^l\right)\frac{2}{k_1^l}\log\left\{\frac{2m^l e^{t(m^l+b^l-k_1^l\lambda_r^l)/2}}{m^l - (b^l - k_1^l\lambda_r^l) + e^{m^l t}(m^l + b^l - \lambda_r^l k_1^l)}\right\}.$$

But a zero-coupon bond with any maturity date can't be easily obtained in the real financial market, so we follow Boulier et al. [21] to consider that the pension manager can invest in a rolling bond with a constant remaining date of $K = T_1$ or $K = T_2$. The rolling bond satisfies the following stochastic differential equation (SDE)

$$\frac{dB^l_K(t)}{B^l_K(t)} = r^l(t)dt + \sigma^l_B(K)\sqrt{k^l_1 r^l(t) + k^l_2}\Big(\lambda^l_r\sqrt{k^l_1 r^l(t) + k^l_2}dt + dW^l_r(t)\Big), \tag{2.6}$$

with $(l, K) = \{(d, T_1), (K, T_2)\}$.

As in Lioui and Poncet [22], it is necessary to convert foreign rolling bonds into the domestic currency through the exchange rate, that is, the price of foreign rolling bonds denominated in domestic currency $\tilde{B}^f_K(t) = B^f_K(t)\cdot H(t)$. By Itô formula, we can get the dynamic equation of foreign rolling bonds priced in domestic currency as

$$\begin{aligned}\frac{dB^f_K(t)}{B^f_K(t)} &= r^d(t)dt + \sigma^f_B(T_2)\sqrt{k^f_1 r^f(t) + k^f_2}\Big(\lambda^f_r\sqrt{k^f_1 r^f(t) + k^f_2}dt \\ &\quad + dW^f_r(t)\Big) + \sigma_H\Big(\lambda_H dt + dW_H(t)\Big).\end{aligned} \tag{2.7}$$

In cross-border investment, exchange rate risk is the primary investment risk considered by the pension manager, and in this paper, they hedge such a risk by purchasing exchange rate futures. Similar to [23], $F_{T_3}(t)$ represents the futures price of the exchange rate with the maturity date T_3, and $H^F_{T_3}(t)$ represents the forward price of the exchange rate with the maturity date T_3. Then according to [23], we have

$$F_{T_3}(t) = H^F_{T_3}(t)\exp\Big[\int_t^{T_3}\sigma^d_B(T_3 - s)^2(k^d_1 r^d(s) + k^d_2)ds\Big], \tag{2.8}$$

Moreover, by

$$H^F_{T_3}(t)\cdot B^d(t, T_3) = B^f(t, T_3)\cdot H(t), \tag{2.9}$$

we can get $H^F_{T_3}(t)$; from Eq (2.8), we can further obtain the exchange rate futures price $F_{T_3}(t)$. According to Eqs (2.2) and (2.6)–(2.9), by Itô formula, the driving equation of exchange rate futures can be obtained as (for more detailed derivation, see [23])

$$\begin{aligned}\frac{dF_{T_3}(t)}{F_{T_3}(t)} &= \sigma_H\Big(\lambda_H dt + dW_H(t)\Big) \\ &\quad -\sigma^d_B(T_3)\sqrt{k^d_1 r^d(t) + k^d_2}\Big(\lambda^d_r\sqrt{k^d_1 r^d(t) + k^d_2}dt + dW^d_r(t)\Big) \\ &\quad +\sigma^f_B(T_3)\sqrt{k^f_1 r^f(t) + k^f_2}\Big(\lambda^f_r\sqrt{k^f_1 r^f(t) + k^f_2}dt + dW^f_r(t)\Big).\end{aligned} \tag{2.10}$$

Moreover, the pension manager invests both in domestic and foreign stock markets, and we assume that the dynamic process of stocks follows the SDE

$$\begin{aligned}\frac{dS^l(t)}{S^l(t)} &= r^l(t)dt + \sigma^l_{s1}\sqrt{k^l_1 r^l(t) + k^l_2}\Big(\lambda^l_r\sqrt{k^l_1 r^l(t) + k^l_2}dt + dW^l_r(t)\Big) \\ &\quad + \sigma^l_{s2}\Big(\lambda^l_s dt + dW^l_s(t)\Big),\end{aligned} \tag{2.11}$$

where σ_{s1}^{l}, σ_{s2}^{l} are positive constants, which describe the volatility level of the stock market, and λ_s^l is a non-negative constant, which represents the market price of the risk $W_s^l(t)$. Similar to the case of bonds, the dynamic process of foreign stock prices after considering exchange rate changes follows (for symbolic simplicity, we still write as $S^f(t)$)

$$\begin{aligned}\frac{dS^f(t)}{S^f(t)} &= r^d(t)dt + \sigma_H\Big(\lambda_H dt + dW_H(t)\Big) + \sigma_{s1}^f\sqrt{k_1^f r^f(t) + k_2^f} \\ &\quad \Big(\lambda_r^f\sqrt{k_1^f r^f(t) + k_2^f}dt \ + dW_r^f(t)\Big) + \sigma_{s2}^f\Big(\lambda_s^f dt + dW_s^f(t)\Big).\end{aligned} \tag{2.12}$$

In the long-term investment process of pension, the inflation risk is an important investment risk that pension managers need to consider. Similar to Guan and Liang [24], consider the continuous time inflation index model as

$$\begin{cases}\frac{dI(t)}{I(t)} = \Big(r^d(t) - r_r(t)\Big)dt + \sigma_{I1}\sqrt{k_1^d r^d(t) + k_2^d}\Big(\lambda_r^d\sqrt{k_1^d r^d(t) + k_2^d}dt \\ \qquad + dW_r^d(t)\Big) + \sigma_{I2}\Big(\lambda_I dt + dW_I(t)\Big), \\ I(0) = I_0,\end{cases} \tag{2.13}$$

where, $r_r(t)$ represents the real interest rate of the domestic market, being a deterministic function of time t, σ_{I1}, σ_{I2} are positive constants, that describe the volatility level of the inflation index, and λ_I is a non-negative constant, that represents the market price of the risk $W_I(t)$. In order to hedge the inflation risk in the domestic market, the pension manager considers buying inflation bonds, and its dynamic process satisfies (for the detailed derivation process, see [24]).

$$\begin{aligned}\frac{dP(t)}{P(t)} &= r^d(t)dt + \sigma_{I1}\sqrt{k_1^d r^d(t) + k_2^d}\Big(\lambda_r^d\sqrt{k_1^d r^d(t) + k_2^d}dt + dW_r^d(t)\Big) \\ &\quad + \sigma_{I2}\Big(\lambda_I dt + dW_I(t)\Big).\end{aligned} \tag{2.14}$$

Finally, in the whole accumulation phrase, the participants of the pension plan will pay contributions to the pension account regularly, and we assume that the contribution process of participants is influenced by domestic interest rates, inflation risk, and the stock market. Its dynamic process follows

$$\begin{cases}\frac{dC(t)}{C(t)} = \eta dt + \sigma_{c1}\sqrt{k_1^d r^d(t) + k_2^d}\Big(\lambda_r^d\sqrt{k_1^d r^d(t) + k_2^d}dt + dW_r^d(t)\Big) \\ \qquad + \sigma_{c2}\Big(\lambda_s^d dt + dW_s^d(t)\Big) + \sigma_{c3}\Big(\lambda_I dt + dW_I(t)\Big), \\ C(0) = c_0.\end{cases} \tag{2.15}$$

where σ_{c1}, σ_{c2}, σ_{c3} are positive constants, that describe the volatility level of the contribution process.

After finishing the financial market modeling, consider the pension manager's strategy as follows: the amount invested in cash assets is $\mu_0(t)$, the amount in rolling bonds is $\mu_B^l(t)$, the amount in stocks is $\mu_s^l(t)$, the amount in inflation is $\mu_p(t)$, and the amount in exchange rate futures is $\mu_F(t)$. Then the wealth process of pensions is satisfied as

$$\begin{cases}dX(t) = C(t)dt + \mu_0(t)\frac{dS_0(t)}{S_0(t)} + \mu_B^d(t)\frac{dB_K^d(t)}{B_K^d(t)} + \mu_B^f(t)\frac{dB_K^f(t)}{B_K^f(t)} \\ \qquad + \mu_s^d(t)\frac{dS^d(t)}{S^d(t)} + \mu_s^f(t)\frac{dS^f(t)}{S^f(t)} + \mu_F(t)\frac{dF_{T_3}(t)}{F_{T_3}(t)} + \mu_p(t)\frac{dP(t)}{P(t)}, \\ X(0) = x_0.\end{cases} \tag{2.16}$$

After substituting (2.3), (2.6), (2.7), (2.10)–(2.12) and (2.14) into the above equation, the driving equation of the pension wealth process can be expressed as

$$dX(t) = C(t)dt + r^d(t)X(t)dt + \bar{\boldsymbol{\mu}}^\top(\boldsymbol{t})\boldsymbol{\sigma}(\boldsymbol{t})\Big(\boldsymbol{\xi}(\boldsymbol{t})dt + d\boldsymbol{W}(\boldsymbol{t})\Big), \tag{2.17}$$

where

$$\boldsymbol{\sigma}(\boldsymbol{t}) = \begin{pmatrix} \sigma_B^d(T_1)\Delta_1(t) & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_B^f(T_2)\Delta_2(t) & \sigma_H & 0 & 0 & 0 \\ -\sigma_B^d(T_3)\Delta_1(t) & \sigma_B^f(T_2)\Delta_2(t) & \sigma_H & 0 & 0 & 0 \\ \sigma_{s1}\Delta_1(t) & 0 & 0 & \sigma_{s2}^d & 0 & 0 \\ 0 & \sigma_{s1}^f\Delta_2(t) & \sigma_H & 0 & \sigma_{s2}^f & 0 \\ \sigma_{I1}\Delta_1(t) & 0 & 0 & 0 & 0 & \sigma_{I2} \end{pmatrix},$$

$$\bar{\boldsymbol{\mu}}(\boldsymbol{t}) = \left(\mu_B^d(t), \mu_B^f(t), \mu_F(t), \mu_s^d(t), \mu_s^f(t), \mu_p(t)\right)^\top,$$

$$\boldsymbol{\xi}(\boldsymbol{t}) = (\lambda_r^d\Delta_1(t), \lambda_r^f\Delta_2(t), \lambda_H, \lambda_s^d, \lambda_s^f, \lambda_I)^\top,$$

$$\boldsymbol{W}(\boldsymbol{t}) = \left(W_r^d(t), W_r^f(t), W_H(t), W_s^d(t), W_s^f(t), W_I(t)\right)^\top,$$

$$\Delta_1(t) \triangleq \sqrt{k_1^d r^d(t) + k_2^d}, \qquad \Delta_2(t) \triangleq \sqrt{k_1^f r^f(t) + k_2^f}.$$

And $\bar{\boldsymbol{\mu}}(\boldsymbol{t})$ is an admissible strategy with the definition given as follows:

Let $\mathcal{O} := \mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}\times\mathbb{R}$, $\mathcal{G} := [0,T]\times\mathcal{O}$. For any fixed $t\in[0,T]$, we call the portfolio strategy $\bar{\boldsymbol{\mu}}(\boldsymbol{t})$ an admissible strategy, if it meets the following conditions:

(i) $\forall t\in[0,T], \bar{\boldsymbol{\mu}}(\boldsymbol{t})$ is $\mathcal{F}_t$ progressively measurable;

(ii) For any initial value $(t, x, r^d, r^f, I, c)\in\mathcal{G}$, the SDE (2.17) has a unique solution;

(iii) $\mathbb{E}[\int_0^T \left(\bar{\boldsymbol{\mu}}(\boldsymbol{t})^\top\boldsymbol{\sigma}(\boldsymbol{t})\boldsymbol{\sigma}(\boldsymbol{t})^\top\bar{\boldsymbol{\mu}}(\boldsymbol{t})\right)dt] < \infty$.

We denote the set of all admissible strategies by $\mathcal{A}$.

2.2. The optimization problem

The aim of the investor is to find the best strategy such that

$$\sup_{\bar{\boldsymbol{\mu}}\in\mathcal{A}} E^Q_{t,x,r^d,r^f,I}\left[U(\frac{X(T)}{I(T)})\right], \tag{2.18}$$

where U is the constant relative risk aversion (CRRA) utility function defined as

$$U(\frac{x}{I}) = \frac{(\frac{x}{I})^{1-\gamma}}{1-\gamma}. \tag{2.19}$$

The parameter γ (with $\gamma > 0$) measures the degree of relative risk aversion that is implicit in the utility function. The CRRA utility function is a classical utility function used in optimal portfolio strategy models (see, for instance, [2, 24, 25]).

Since the DC pension plan will receive contributions from participants regularly during the whole accumulation phase, it is not a self-financing process. We first convert it into a self-financing process. To do this, consider a virtual derivative whose payment at time s is $C(s)$, and the price at time t is $D(t,s)$ (the maturity of the derivative is $s>t$), then $D(t,s)$ satisfies the following partial differential equation

$$\begin{cases} D_t + D_r[a^d(b^d - r^d)] + D_c\mu C + \frac{1}{2}D_{rr}(k_1^d r^d + k_2^d) + \frac{1}{2}D_{cc}[\sigma_{c1}^2 C^2(k_1^d r^d + k_2^d) \\ +\sigma_{c2}^2 C^2 + \sigma_{c3}^2 C^2] - D_{rc}C\sigma_{c1}(k_1^d r^d + k_2^d) = r^d D - D_r\lambda_r^d(k_1^d r^d + k_2^d), \\ D(s,s) = C(s). \end{cases}$$

Solving the above partial differential equation, it is not difficult to obtain the explicit solution of $D(t,s)$

$$D(t,s) = C(t)e^{f_1(s-t)+f_2(s-t)r^d(t)},$$

where

$$f_2(t) = \frac{\theta_1\theta_2[e^{-\frac{k_1^d}{2}(\theta_1-\theta_2)t} - 1]}{\theta_1 e^{-\frac{k_1^d}{2}(\theta_1-\theta_2)t} - \theta_2},$$

$$f_1(t) = \int_0^t [\frac{1}{2}f_2^2(s)k_2^d - (a^d b^d + \lambda_r^d k_2^d - k_2\sigma_{c1})f_2(s) + \lambda_r^d k_2^d \sigma_{c1} - \mu]ds,$$

where $\theta_{1,2} = \frac{a^d + k_1^d\sigma_{c1} - k_1^d\lambda_r^d \mp \sqrt{(a^d + k_1^d\sigma_{c1} - k_1^d\lambda_r^d)^2 + 2k_1^d}}{k_1^d}$. Moreover, $D(t,s)$ satisfies the backward stochastic differential equation (BSDE)

$$\begin{cases} \frac{dD(t,s)}{D(t,s)} = r^d(t)dt + \left(\sigma_{c1} - f_2(t)\right)\sqrt{k_1^d r^d(t) + k_2^d}\left(\lambda_r^d\sqrt{k_1^d r^d(t) + k_2^d}dt \right. \\ \qquad \left. +dW_r^d(t)\right) + \sigma_{c2}\left(\lambda_s^d dt + dW_s^d(t)\right) + \sigma_{c3}\left(\lambda_I dt + dW_I(t)\right), \\ D(s,s) = C(s). \end{cases}$$

Let $F(t,T) = \int_t^T D(t,s)ds$, then $F(t,T)$ denotes the value at t of all contributions received by the participants from time t to the end of the accumulation period, and we have

$$\begin{aligned} dF(t,T) &= -C(t)dt + r^d(t)F(t,T)dt + \int_t^T D(t,s)\left(\sigma_{c1} - f_2(s-t)\right)ds \\ &\quad \sqrt{k_1^d r^d(t) + k_2^d}\left(\lambda_r^d\sqrt{k_1^d r^d(t) + k_2^d}dt + dW_r^d(t)\right) + \sigma_{c2}F(t,T) \\ &\quad \left(\lambda_s^d dt + dW_s^d(t)\right) + \sigma_{c3}F(t,T)\left(\lambda_I dt + dW_I(t)\right). \end{aligned} \tag{2.20}$$

Apparently, we have

$$\begin{aligned} \frac{dF(t,T) + C(t)dt}{F(t,T)} &= r^d(t)dt + \frac{\int_t^T D(t,s)(\sigma_{c1} - f_2(s-t))ds}{F(t,T)}\sqrt{k_1^d r^d(t) + k_2^d} \\ &\quad \left(\lambda_r^d\sqrt{k_1^d r^d(t) + k_2^d}dt + dW_r^d(t)\right) + \sigma_{c2}\left(\lambda_s^d dt + dW_s^d(t)\right) \\ &\quad +\sigma_{c3}\left(\lambda_I dt + dW_I(t)\right). \end{aligned} \tag{2.21}$$

Proposition 2.1. *The cash flow $C(s)$ and $F(t,T)$ on $[t,T]$ can be hedged with domestic stock, rolling bond, inflation bond, and cash account in the financial market, which means*

$$\frac{dF(t,T) + C(t)dt}{F(t,T)} = \mu_0^F(t)\frac{dS_0(t)}{S_0(t)} + \mu_b^F(t)\frac{dB_K^d(t)}{B_K^d(t)} + \mu_s^F(t)\frac{dS^d(t)}{S^d(t)} + \mu_p^F(t)\frac{dP(t)}{P(t)},$$

where

$$
\begin{cases}
\mu_0^F(t) = 1 - \mu_b^F(t) - \mu_s^F(t) - \mu_p^F(t),\\
\mu_b^F(t) = \frac{\int_t^T D(t,s)(\sigma_{c1} - f_2(s-t))ds}{F(t,T)\sigma_B^d(T_1)} - \frac{\sigma_{s1}^d \sigma_{c2}}{\sigma_{s2}^d \sigma_B^d(T_1)} - \frac{\sigma_{I1}\sigma_{c3}}{\sigma_{I2}\sigma_B^d(T_1)},\\
\mu_p^F(t) = \frac{\sigma_{c3}}{\sigma_{I2}},\\
\mu_s^F(t) = \frac{\sigma_{c2}}{\sigma_{s2}}.
\end{cases}
\tag{2.22}
$$

Proof. According to Eq (2.21) and the driving equations for domestic cash account, rolling bond, stock, and inflation bond (2.3), (2.6), (2.11), and (2.14), it is not difficult to draw these conclusions. □

Let $Y(t) = X(t) + F(t,T)$ (clearly there is $Y(T) = X(T)$), then from the above proposition, we have

$$
\begin{aligned}
dY(t) &= dX(t) + dF(t,T)\\
&= r^d(t)\big(X(t) + F(t,T)\big)dt + \big(\mu_B^d(t) + \mu_b^F(t)F(t,T)\big)\frac{dB_K^d(t)}{B_K^d(t)}\\
&+ \big(\mu_s^d(t) + \mu_s^F(t)F(t,T)\big)\frac{dS^d(t)}{S^d(t)} + \big(\mu_p^F(t)F(t,T) + \mu_P(t)\big)\frac{P(t)}{P(t)}\\
&+ \mu_B^f(t)\frac{dB_K^f(t)}{B_K^f(t)} + \mu_F(t)\frac{dF(t)}{F(t)} + \mu_s^f(t)\frac{dS^f(t)}{S^f(t)}.
\end{aligned}
$$

If we let

$$
\boldsymbol{\mu}(t) = \bar{\boldsymbol{\mu}}(t) + \big(\mu_b^F F(t,T), 0, 0, \mu_s^F F(t,T), 0, \mu_p^F F(t,T)\big)^\top, \tag{2.23}
$$

then

$$
dY(t) = r^d(t)Y(t)dt + \boldsymbol{\mu}^\top(t)\boldsymbol{\sigma}(t)[\boldsymbol{\xi}(t)dt + d\boldsymbol{W}(t)]. \tag{2.24}
$$

Thus, the optimization problem (2.18) is equivalent to

$$
\sup_{\boldsymbol{\mu}(t)\in\mathcal{A}} E^Q_{t,y,r^d,r^f,I}\left[U\left(\frac{Y(T)}{I(T)}\right)\right], \tag{2.25}
$$

whose wealth state variable is in the self-financing market. Furthermore, we call the new strategy $\boldsymbol{\mu}(t)$ an admissible strategy if and only if the corresponding strategy $\bar{\boldsymbol{\mu}}(t)$ is an admissible strategy in Eq (2.17); for ease of expression, when $\boldsymbol{\mu}(t)$ is an admissible strategy, we still write as $\boldsymbol{\mu}(t) \in \mathcal{A}$.

2.3. The asset price distribution ambiguity in the foreign market

Compared with the domestic market, pension managers are not familiar with foreign markets, which is mainly reflected in the lag of information acquisition and the differences in social and cultural environment and ideology. Therefore, even in the face of the same information, the feedback from domestic and foreign markets will be different, so only limited information can be used to analyze the distribution of foreign asset prices. The true distribution of security prices cannot be fully understood. In order to understand the portfolio decisions of pension managers under limited information, it is assumed that the probability distribution of prices obtained by pension managers in the domestic financial market is deterministic, while the probability measure of foreign financial markets is uncertain. As mentioned above, pension managers are fuzzy-averse. Based on the above assumptions, this paper discusses the optimal portfolio strategy of DC pension managers according to the robust optimal control method including entropy penalty strategy from [12].

In robust optimization theory, the measure of asset price distribution obtained in advance based on limited information is generally called a reference measure. In the actual investment process, on the other hand, the pension manager realises that the reference measure they receive is not necessarily the true market measure, and therefore considers a set of probability measures Q that are equivalent to the reference measure P, and bounds the "distance" between the reference measure P and the equivalent measure Q by establishing an entropy penalty. This section investigates the DC pension portfolio strategy while the pension manager is ambiguity-averse by introducing a probability measure Q that is equivalent to P. Since the pension manager is familiar with the domestic market, we assume that the manager knows the distribution of domestic financial securities prices, thus the probability measure Q is equivalent to the reference measure P and does not change the probability measure of the domestic market, i.e.,

$$\mathcal{Q} := \{Q \mid Q \sim P\}, \tag{2.26}$$

where Q will not change the distribution of domestic asset prices*.

Because $Q \in \mathcal{Q}$ is equivalent to P, and Q will not change the distribution of $W_1(t)$, then we can write as $Q = P^d \times Q^f$†. According to Girsanov theorem, Q and Q^f satisfy

$$\frac{dQ}{dP}(W_2[0,t]) = \frac{dQ_f}{dP_f}(W_2[0,t]) = \Lambda(t), \qquad t \geq 0, \tag{2.27}$$

where

$$\begin{aligned}\Lambda(t) = \exp\{&\int_0^t \phi_1(s)dW_r^f(s) - \frac{1}{2}\int_0^t \phi_1^2(s)ds \\ &+ \int_0^t \phi_2(s)dW_s^f(s) - \frac{1}{2}\int_0^t \phi_2^2(s)ds\}.\end{aligned} \tag{2.28}$$

If $\Lambda(t)$ satisfies the conditions: 1) $\Lambda(t)$ is progressively measurable w.r.t. $\{\mathcal{F}_t\}_{t\in[0,T]}$; 2) $\mathbb{E}^P[\exp(\frac{1}{2}\int_0^T(\phi_1^2(s) + \phi_2^2(s))ds)] < \infty$, then $\Lambda(t)$ is a $P-martingale$ w.r.t. $\{\mathcal{F}_t\}_{t\in[0,T]}$. Thus, we can rewrite $\mathcal{Q}$ as

$$\mathcal{Q} := \{Q \mid Q = P^d \times Q^f\},$$

and Q^f follows (2.27).

According to the Gosanov theorem, the Brownian motion of foreign financial assets can be rewritten as $W_r^f(t)$, $W_S^f(t)$ under the measure Q; then we have

$$dW_r^f(t) = \phi_1(t)dt + dW_r^{Q,f}(t),$$

$$dW_s^f(t) = \phi_2(t)dt + dW_s^{Q,f}(t),$$

where $W_r^{Q,f}(t)$, $W_S^{Q,f}(t)$ represents Brownian motion under probability measure Q. Therefore, the driving equation of foreign rolling bond and stock market can be rewritten as

$$\frac{dB_k^f(t)}{B_k^f(t)} = r^d(t)dt + \sigma_B^f(T_2)\sqrt{k_1^f r^f(t) + k_2^f}\Big(\lambda_r^f\sqrt{k_1^f r^f(t) + k_2^f}dt + \phi_1(t)$$

*Similarly to [26, 27], here we assume that $\mathcal{Q}$ is non-empty.

†In [26, 27], the author discussed the life cycle problem with individual mortality rate (which is without ambiguity) and asset price process (which has ambiguity).

$$+ dW_r^{Q,f}(t)\Big) + \sigma_H\Big(\lambda_H dt + dW_H(t)\Big), \tag{2.29}$$

$$\frac{dS^f(t)}{S^f(t)} = r^d(t)dt + \sigma_H\Big(\lambda_H dt + dW_H(t)\Big) + \sigma_{s2}^f\Big(\lambda_s^f dt + \phi_2(t)dW_s^{Q,f}(t)\Big)$$
$$+ \sigma_{s1}^f\sqrt{k_1^f r^f(t) + k_2^f}\Big(\lambda_r^f\sqrt{k_1^f r^f(t) + k_2^f}dt + \phi_1(t) + dW_r^{Q,f}(t)\Big). \tag{2.30}$$

Thus, the wealth process of the DC pension can be rewritten as

$$dY(t) = r^d(t)Y(t)dt + \boldsymbol{\mu}^\top(t)\boldsymbol{\sigma}(\boldsymbol{t})[\boldsymbol{\xi}(\boldsymbol{t})dt + \boldsymbol{\phi}(\boldsymbol{t})dt + d\mathbf{W}^{\boldsymbol{Q}}(\boldsymbol{t})], \tag{2.31}$$

where $\boldsymbol{\phi}(\boldsymbol{t}) = (0, \phi_1(t), 0, 0, \phi_2(t), 0)^\top$.

Considering that the reference measure P is the most reasonable estimate of the probability measure made by the pension manager based on the limited information they observed, and also that it has some reference value, in order to measure the degree of deviation of the probability measure Q from the reference measure P, similarly to Maenhout [12], this section considers using the relative entropy penalty method to describe the degree of deviation of the probability measure Q from the reference measure P. The relative entropy between probability measures Q and P is defined as follows

$$\begin{aligned}
&H_{[0,t]}(Q\|P) = E^Q_{[0,t]}[\log\frac{dQ}{dP}]\\
&= E^Q[\int_0^t \phi_1(s)dW_r^f - \frac{1}{2}\int_0^t \phi_1^2(s)ds + \int_0^t \phi_2(s)dW_s^f - \frac{1}{2}\int_0^t \phi_2^2(s)ds]\\
&= E^Q[\int_0^t \phi_1(s)dW_r^{Q,f} + \frac{1}{2}\int_0^t \phi_1^2(s)ds + \int_0^t \phi_2(s)dW_s^{Q,f} + \frac{1}{2}\int_0^t \phi_2^2(s)ds].\\
&= E^Q[\frac{1}{2}\int_0^t \phi_1^2(s)ds + \frac{1}{2}\int_0^t \phi_2^2(s)ds].
\end{aligned}$$

Therefore, we consider using $\frac{1}{2}\phi_1^2(s) + \frac{1}{2}\phi_2^2(s)$ to describe the relative entropy between the probability measure Q and P. The higher the value is, the greater the degree of deviation of the probability measure Q from the reference measure P, and the more cautious the pension manager is about the reliability of the reference measure.

Under the ambiguity theory, problem (2.25) becomes

$$\begin{aligned}
J^{\boldsymbol{\mu}}(t, y, r^d, r^f, I) \;=\; & E^Q_{t,y,r^d,r^f,I}\Big\{U\Big(\frac{Y(T)}{I(T)}\Big) + \int_t^T \Big[\frac{\phi_1^2(s)}{2\Psi_1(s, Y(s), r^d(s), r^f(s), I(s))}\\
&+\frac{\phi_2^2(s)}{2\Psi_2(s, Y(s), r^d(s), r^f(s), I(s))}\Big]ds\Big\},
\end{aligned} \tag{2.32}$$

where

$$\Psi_i(t, y, r^d, r^f, I) = \frac{\beta_i}{(1-\gamma)J(t, y, r^d, r^f, I)}, \quad (i = 1, 2). \tag{2.33}$$

And the value function of the pension manager with ambiguity aversion is

$$V(t, y, r^d, r^f, I) = \sup_{\boldsymbol{\mu}(\boldsymbol{t})\in\mathcal{A}} \inf_{Q\in\mathcal{Q}} J^{\boldsymbol{\mu},Q}(t, y, r^d, r^f, I), \tag{2.34}$$

that is, the pension manager considers the optimal portfolio strategy under the worst case scenario.

Remark 2.2. *Since the pension manager usually has different degrees of ambiguity about foreign bond and stock prices (generally, pension managers are more confident about the distribution of bond prices than stock, mainly because the conditional variance of bond market returns is generally lower than that of the stock market, see [28]), this paper uses* Ψ_1, Ψ_2 *to depict the different levels of ambiguity aversion of the pension manager. When the ambiguity aversion of the pension manager is stronger (i.e.,* Ψ_1, Ψ_2 *is larger), it indicates that the pension manager has less confidence in the reference measure, which allows the probability measure Q to deviate from the reference measure P with less penalty. Conversely, when the penalty item* Ψ_1, Ψ_2 *is smaller, for example,* $\Psi_1 = \Psi_2 = 0$, *the pension manager has greater confidence in the reference measure, and any deviation from the reference measure will be severely punished, at which point the objective function will degenerate into a ambiguity-neutral case.*

3. Closed-form solutions

This section derives the closed-form solutions of optimal strategies and value function. To this end, we first denote the Hamiltonian as

$$\begin{aligned}\mathcal{A}^{\boldsymbol{\mu},\phi} f(t,y,I,r^d,r^f) \;=\;& f_t + f_y[r^d y + \boldsymbol{\mu}^\top \boldsymbol{\sigma}(\boldsymbol{t})(\boldsymbol{\xi}(\boldsymbol{t}) + \boldsymbol{\phi}(\boldsymbol{t}))] + I f_I[r^d + \sigma_{I1}\lambda_r^d \Delta_1^2(t) \\ & -r_r^d + \sigma_{I2}\lambda_I] + f_{r^d}[a^d(b^d - r^d)] + f_{r^f}[a^f(b^f - r^f) \\ & -\boldsymbol{\Delta}^\top(\boldsymbol{t})\boldsymbol{\phi}(\boldsymbol{t})] + \frac{1}{2} f_{yy}\boldsymbol{\mu}^\top \boldsymbol{\sigma}(\boldsymbol{t})\boldsymbol{\sigma}^\top(\boldsymbol{t})\boldsymbol{\mu} + \frac{1}{2} f_{II} I^2(\sigma_{I1}^2\Delta_1^2 \\ & +\sigma_{I2}^2) + \frac{1}{2} f_{r^d r^d}\Delta_1^2(t) + \frac{1}{2} f_{r^f r^f}\Delta_2^2(t) - f_{yr^d}\Delta_1(t)\boldsymbol{\mu}^\top \boldsymbol{\sigma}(\boldsymbol{t})\boldsymbol{e_1} \\ & +f_{yI}\boldsymbol{\mu}^\top \boldsymbol{\sigma}(\boldsymbol{t})\boldsymbol{\sigma_I}(\boldsymbol{t}) I - f_{yr^f}\Delta_2(t)\boldsymbol{\mu}^\top \boldsymbol{\sigma}(\boldsymbol{t})\boldsymbol{e_2} - f_{Ir^d}\Delta_1^2\sigma_{I1} I,\end{aligned} \tag{3.1}$$

where $\boldsymbol{e_i}(i = 1, 2)$ denotes a 6×1 column vector, where the i-th element is 1, and the remaining elements are 0, $\boldsymbol{\Delta(t)} = (0, \Delta_2(t), 0, 0, 0, 0)^\top$, $\boldsymbol{\sigma_I(t)} = (\sigma_{I1}\Delta_1(t), 0, 0, 0, 0, \sigma_{I2})^\top$. The corresponding HJB equation (see [29] for the exact derivation process) is

$$\begin{aligned}&\sup_{\boldsymbol{\mu(t)}\in\mathcal{A}} \inf_{Q\in Q}\{\mathcal{A}^{\boldsymbol{\mu},\boldsymbol{\phi}} V(t,y,I,r^d,r^f) + \frac{\phi_1^2(t)}{2\Psi_1(t,y,r^d,r^f,I)} \\ &+ \frac{\phi_2^2(t)}{2\Psi_2(t,y,r^d,r^f,I)}\} = 0,\end{aligned} \tag{3.2}$$

with the boundary condition satisfying $V(T,y,I,r^d,r^f) = U(y/I)$.

For (3.2), applying the first-order condition to $\boldsymbol{\phi(t)}$ gives

$$\boldsymbol{\phi}^*(\boldsymbol{t}) = \frac{V_{r^f}}{(1-\gamma)V}\boldsymbol{\beta\Delta}(\boldsymbol{t}) - \frac{V_y}{(1-\gamma)V}\boldsymbol{\beta\sigma}^\top(\boldsymbol{t})\boldsymbol{\mu}(\boldsymbol{t}), \tag{3.3}$$

where $\boldsymbol{\beta} = diag(0, \beta_1, 0, 0, \beta_2, 0)^\top$. Substitute $\boldsymbol{\phi}^*(\boldsymbol{t})$ into the HJB equation (2.33), then we have

$$\begin{aligned}&\sup_{\boldsymbol{\mu(t)}\in\mathcal{A}} \{V_y[r^d y + \boldsymbol{\mu}^\top\boldsymbol{\xi}(\boldsymbol{t})\boldsymbol{\sigma}(\boldsymbol{t}) + \frac{V_{r^f}}{(1-\gamma)V}\boldsymbol{\mu}^\top\boldsymbol{\sigma}(\boldsymbol{t})\boldsymbol{\beta\Delta}(\boldsymbol{t}) - \frac{V_y}{(1-\gamma)V}\boldsymbol{\mu}^\top\boldsymbol{\sigma}(\boldsymbol{t})\boldsymbol{\beta\sigma}^\top(\boldsymbol{t})\boldsymbol{\mu}] \\ &+ V_t + I V_I[r^d - r_r + \sigma_{I1}\lambda_r^d\Delta_1^2(t) + \sigma_{I2}\lambda_I] + V_{r^d}[a^d(b^d - r^d)] + V_{r^f}[a^f(b^f - r^f)]\end{aligned}$$

$$
\begin{aligned}
&- \frac{V_{r^f}}{(1-\gamma)V}\boldsymbol{\Delta}^\top(t)\boldsymbol{\beta}\boldsymbol{\Delta}(t) + \frac{V_y}{(1-\gamma)V}\boldsymbol{\Delta}^\top(t)\boldsymbol{\beta}\boldsymbol{\sigma}^\top(t)\boldsymbol{\mu} + \frac{1}{2}V_{yy}\boldsymbol{\mu}^\top\boldsymbol{\sigma}(t)\boldsymbol{\sigma}^\top(t)\boldsymbol{\mu} \\
&+ \frac{1}{2}V_{II}I^2(\sigma_{I1}^2\Delta_1^2(t) + \sigma_{I2}^2) + \frac{1}{2}V_{r^dr^d}\Delta_1^2(t) + \frac{1}{2}V_{r^fr^f}\Delta_2^2(t) - V_{yr^d}\Delta_1(t)\boldsymbol{\mu}^\top\boldsymbol{\sigma}(t)\boldsymbol{e_1} \\
&+ V_{yI}\boldsymbol{\mu}^\top\boldsymbol{\sigma}(t)\boldsymbol{\sigma_I}(t)I - V_{yr^f}\Delta_2(t)\boldsymbol{\mu}^\top\boldsymbol{\sigma}(t)\boldsymbol{e_2} - V_{Ir^d}\Delta_1^2(t)\sigma_{I1}I + \frac{1}{2}[V_{r^f}\boldsymbol{\Delta}^\top(t) \\
&- V_y\boldsymbol{\mu}^\top\boldsymbol{\sigma}(t)][\frac{V_{r^f}}{(1-\gamma)V}\boldsymbol{\beta}\boldsymbol{\Delta}(t) - \frac{V_y}{(1-\gamma)V}\boldsymbol{\beta}\boldsymbol{\sigma}^\top(t)\boldsymbol{\mu}] = 0.
\end{aligned} \tag{3.4}
$$

Using the first-order condition to the above equation with respect to $\boldsymbol{\mu}(t)$, then

$$
\begin{aligned}
\boldsymbol{\mu}^*(t) \;=\; & \Big(\frac{V_y^2\boldsymbol{\beta}\boldsymbol{\sigma}^\top(t)}{(1-\gamma)V} - V_{yy}\boldsymbol{\sigma}^\top(t)\Big)^{-1}[V_y\boldsymbol{\xi}(t) + \frac{V_yV_{r^f}}{(1-\gamma)V}\boldsymbol{\beta}\boldsymbol{\Delta}(t) \\
& -V_{yr^d}\Delta_1(t)\boldsymbol{e_1} - V_{yr^f}\Delta_2(t)\boldsymbol{e_2} + V_{yI}I\boldsymbol{\sigma_I}(t)].
\end{aligned} \tag{3.5}
$$

To solve (3.4), we guess that the form of V satisfies

$$
V(t,y,I,r^d,r^f) = \frac{1}{1-\gamma}(\frac{y}{I})^{1-\gamma}h(t,r^d,r^f), \tag{3.6}
$$

with

$$
h(t,r^d,r^f) = \exp\{r^dq_1(t) + r^fq_2(t) + q_3(t)\}.
$$

By taking (3.6) into (3.5), then we can rewrite $\boldsymbol{\mu}^*(t)$ into

$$
\begin{aligned}
\boldsymbol{\mu}^*(t) \;=\; & y(\boldsymbol{\sigma^{-1}}(t))^\top(\boldsymbol{\beta} + \gamma\boldsymbol{I_{6\times6}})^{-1}[\boldsymbol{\xi}(t) + \frac{q_2(t)}{1-\gamma}\boldsymbol{\beta}\boldsymbol{\Delta}(t) - q_1(t)\Delta_1(t)\boldsymbol{e_1} \\
& -q_2(t)\Delta_2(t)\boldsymbol{e_2} - (1-\gamma)\boldsymbol{\sigma_I}],
\end{aligned} \tag{3.7}
$$

then substituting the value function (3.6) into (3.4) gives

$$
\begin{aligned}
& r^dq_1'(t) + r^fq_2'(t) + q_3'(t) + (1-\gamma)(\boldsymbol{\mu}^*)^\top\boldsymbol{\sigma}(t)\boldsymbol{\xi}(t) + \frac{q_2(t)}{2}(\boldsymbol{\mu}^*)^\top\boldsymbol{\sigma}(t)\boldsymbol{\beta}\boldsymbol{\Delta}(t) + 2(1-\gamma)r_r \\
& - \frac{1-\gamma}{2}(\boldsymbol{\mu}^*)^\top\boldsymbol{\sigma}(t)\boldsymbol{\beta}\boldsymbol{\sigma}^\top(t)\boldsymbol{\mu}^* - (1-\gamma)\sigma_{I1}\lambda_r^d\Delta_1^2(t) - (1-\gamma)\sigma_{I2}\lambda_I + q_1(t)[a^d(b^d - r^d)] \\
& + q_2(t)[a^f(b^f - r^f) - \frac{q_2^2(t)}{2(1-\gamma)}\boldsymbol{\Delta}^\top(t)\boldsymbol{\beta}\boldsymbol{\Delta}(t) + \frac{q_2(t)}{2}\boldsymbol{\Delta}^\top(t)\boldsymbol{\beta}\boldsymbol{\sigma}^\top(t)\boldsymbol{\mu}^* + \frac{1}{2}q_1^2(t)\Delta_1^2(t) \\
& - \frac{\gamma(1-\gamma)^2(2-\gamma)}{2}(\sigma_{I1}^2\Delta_1^2(t) + \sigma_{I2}^2)(\boldsymbol{\mu}^*)^\top\boldsymbol{\sigma}(t)\boldsymbol{\sigma}^\top(t)\boldsymbol{\mu}^* + \frac{1}{2}q_2^2(t)\Delta_2^2(t) \\
& - (1-\gamma)q_1(t)\Delta_1(t)(\boldsymbol{\mu}^*)^\top\boldsymbol{\sigma}(t)\boldsymbol{e_1} - (\boldsymbol{\mu}^*)^\top\boldsymbol{\sigma}(t)\boldsymbol{\sigma_I}(t) - (1-\gamma)q_2(t)\Delta_2(t)(\boldsymbol{\mu}^*)^\top\boldsymbol{\sigma}(t)\boldsymbol{e_2} \\
& + (1-\gamma)q_1(t)\Delta_1^2(t)\sigma_{I1} + [\frac{q_2(t)}{1-\gamma}\boldsymbol{\Delta}(t)^\top - \frac{(\boldsymbol{\mu}^*)^\top\boldsymbol{\sigma}(t)}{y}][\frac{q_2(t)}{1-\gamma}\boldsymbol{\beta}\boldsymbol{\Delta}(t) - \frac{\boldsymbol{\beta}\boldsymbol{\sigma}^\top(t)\boldsymbol{\mu}^*}{y}] = 0.
\end{aligned}
$$

Remark 3.1. *Here, we do not substitute $\boldsymbol{\mu}^*(t)$ directly into the HJB equation, because in the actual derivation, it is easier to deal with $(\boldsymbol{\mu}^*(t))^\top\boldsymbol{\sigma}(t)$ or $\boldsymbol{\sigma}^\top(t)(\boldsymbol{\mu}^*(t))^\top$.*

By separating the variables, we can get the following three ordinary differential equations:

$$
\begin{cases}
q_1'(t) + \frac{k_1^d}{2\gamma}q_1^2(t) - [a^d - (1-\gamma)\sigma_{I1}k_2^d]q_1(t) + \frac{(1-\gamma)k_1^d(\lambda_r^d-\sigma_{I1})^2}{2\gamma} = 0, \\
q_1(T) = 0,
\end{cases} \tag{3.8}
$$

$$\begin{cases} q_2'(t) + \frac{k_1^f(1-\beta_1-\gamma)}{2(1-\gamma)(\beta_1+\gamma)} q_2^2(t) + [\frac{\beta_1+\gamma-1}{\gamma+\beta_1}\lambda_r^f k_1^f - a^f] q_2(t) + \frac{1-\gamma}{2\gamma} k_1^d (\lambda_r^d)^2 \\ + \frac{(1-\gamma)k_1^f(\lambda_r^f)^2}{2(\beta_1+\gamma)} = 0, \\ q_2(T) = 0, \end{cases} \tag{3.9}$$

$$\begin{cases} q_3'(t) + \frac{k_2^d}{2\gamma} q_1^2(t) + (a^d b^d - \frac{1-\gamma}{\gamma} k_2^d(\lambda_r^d - \sigma_{I1})) q_1(t) - \frac{(\beta_1-1+\gamma)k_2^f}{2(\gamma+\beta_1)(1-\gamma)} q_2^2(t) \\ +[a^f b^f + k_2^f \lambda_r^f - \frac{k_2^f \lambda_r^f}{\beta_1+\gamma}] q_2(t) + G = 0, \\ q_3(T) = 0, \end{cases} \tag{3.10}$$

where $G = \frac{1-\gamma}{2\gamma}[\lambda_Q^2 + (\lambda_s^d)^2] + \frac{1-\gamma}{2(\beta_1+\gamma)} k_2^f (\lambda_r^f)^2 + \frac{1-\gamma}{2(\beta_2+\gamma)} (\lambda_s^f)^2 + (1-\gamma) r_r^d - (1-\gamma)[\sigma_{I1}\lambda_r^d k_2^d + \sigma_{I2}\lambda_I] + \frac{\gamma(1-\gamma)}{2}(\sigma_{I1}^2 k_2^d + \sigma_{I2}^2) + \frac{1-\gamma}{2\gamma} k_2^d [\lambda_r^d - (1-\gamma)\sigma_{I1}]^2 + \frac{1-\gamma}{2\gamma}[\lambda_I - (1-\gamma)\sigma_{I2}]^2$.

Note that the above ordinary differential equations are Riccati equations, which can be solved as (see [15] for more detail)

$$q_1(t) = \frac{\lambda_1\lambda_2(e^{\sqrt{\Delta_{q1}}(T-t)} - 1)}{\lambda_1 e^{\sqrt{\Delta_{q1}}(T-t)} - \lambda_2}, \tag{3.11}$$

$$q_2(t) = \frac{\widetilde{\lambda}_1\widetilde{\lambda}_2(e^{\sqrt{\Delta_{q2}}(T-t)} - 1)}{\widetilde{\lambda}_1 e^{\sqrt{\Delta_{q2}}(T-t)} - \widetilde{\lambda}_2}, \tag{3.12}$$

$$\begin{aligned} q_3(t) = \int_t^T & \{\frac{k_2^d}{2\gamma} q_1^2(s) + \left(a^d b^d - \frac{1-\gamma}{\gamma} k_2^d(\lambda_r^d - \sigma_{I1})\right) q_1(s) \\ & - \frac{(\beta_1 - 1 + \gamma)k_2^f}{2(\gamma+\beta_1)(1-\gamma)} q_2^2(s) + [a^f b^f + k_2^f\lambda_r^f - \frac{k_2^f\lambda_r^f}{\beta_1+\gamma}] q_2(s) + G\} ds, \end{aligned} \tag{3.13}$$

where

$$\lambda_{1,2} = \frac{\gamma[a^d - (1-\gamma)\sigma_{I1}k_1^d \pm \sqrt{\Delta_{q1}}]}{k_1^d},$$

$$\Delta_{q1} = [(1-\gamma)\sigma_{I1}k_1^d - a^d]^2 - \frac{1-\gamma}{\gamma^2}(k_1^d)^2(\lambda_r^d - \sigma_{I1})^2,$$

$$\widetilde{\lambda}_{1,2} = \frac{(1-\gamma)[a^f(\beta_1+\gamma) - k_1^f\lambda_r^f(\beta_1+\gamma-1) \pm (\beta_1+\gamma)\sqrt{\Delta_{q2}}]}{k_1^f(1-\beta_1-\gamma)},$$

$$\Delta_{q2} = [a^f - k_1^f\lambda_r^f\frac{\beta_1+\gamma-1}{\beta_1+\gamma}]^2 - \frac{(1-\beta_1-\gamma)(k_1^f\lambda_r^f)^2}{(\beta_1+\gamma)^2}.$$

Remark 3.2. *Since we assume that the risk-aversion coefficient of the pension manager is $\gamma > 1$, it is obvious that $\Delta_{q1} > 0$, $\Delta_{q2} > 0$.*

Summarizing the above, we draw the main conclusions of this paper, which give the optimal strategy and the explicit solution of the value function, as follows:

Theorem 3.1. *For the robust optimization problem* (3.2) *under CRRA utility function, the explicit solution of V satisfies*

$$V(t, y, I, r^d, r^f) = \frac{1}{1-\gamma}\left(\frac{y}{I}\right)^{1-\gamma} e^{r^d q_1(t) + r^f q_2(t) + q_3(t)}, \tag{3.14}$$

and the optimal strategy is given by

$$\begin{aligned}\boldsymbol{\mu}^*(\boldsymbol{t}) &= Y^{\boldsymbol{\mu}^*}(t)(\boldsymbol{\sigma}^{-1}(\boldsymbol{t}))^\top(\boldsymbol{\beta}+\gamma \boldsymbol{I}_{6\times 6})^{-1}[\boldsymbol{\xi}(\boldsymbol{t}) + \frac{q_2(t)}{1-\gamma}\boldsymbol{\beta}\boldsymbol{\Delta}(\boldsymbol{t}) \\ &\quad -q_1(t)\Delta_1(t)\boldsymbol{e_1} - q_2(t)\Delta_2(t)\boldsymbol{e_2} - (1-\gamma)\boldsymbol{\sigma_I}(\boldsymbol{t})],\end{aligned} \tag{3.15}$$

with the corresponding worst case scenario probability measure being described by

$$\begin{aligned}\boldsymbol{\phi}^*(\boldsymbol{t}) &= \frac{q_2(t)}{1-\gamma}\boldsymbol{\beta}\boldsymbol{\Delta}(\boldsymbol{t}) - \boldsymbol{\beta}(\boldsymbol{\beta}+\gamma \boldsymbol{I}_{6\times 6})^{-1}[\boldsymbol{\xi}(\boldsymbol{t}) + \frac{q_2(t)}{1-\gamma}\boldsymbol{\Delta}(\boldsymbol{t}) \\ &\quad -q_1(t)\Delta_1(t)\boldsymbol{e_1} - q_2(t)\Delta_2(t)\boldsymbol{e_2} - (1-\gamma)\boldsymbol{\sigma_I}(\boldsymbol{t})],\end{aligned} \tag{3.16}$$

here $q_1(t)$, $q_2(t)$, $q_3(t)$ as in (3.11)–(3.13).

Remark 3.3. *The candidate optimal strategy here consists of two parts: one consists of $(\boldsymbol{\sigma}^{-1}(\boldsymbol{t}))^\top(\boldsymbol{\beta}+\gamma I_{6\times 6})^{-1}\boldsymbol{\xi}(\boldsymbol{t})$, which is similar to the Sharp ratio in the classic Merton strategy, while the remaining part can be considered as a hedging item, indicating a hedge against risks arising from interest rates, inflation, and exchange rates during the investment process.*

After the above derivation, we have obtained the candidate optimal value function and candidate optimal strategy for the pension manager with ambiguity aversion characteristics when their objective function is like (3.2). In order to show that the solution of the HJB equation and the candidate optimal strategy given by Theorem 3.1 are indeed the optimal solution to the problem, we need to verify them by a verification theorem. Before such verification theorem, we first need to show that the definition of the worst case equivalent probability measure given by the theorem is well defined, that is, to show that the Radon-Nikodym derivative $\Lambda^{\boldsymbol{\phi}^*}(t)$ of the equivalent probability measure Q with respect to the reference measure P is a P martingale, under $\boldsymbol{\phi}^*(\boldsymbol{t})$ given in Eq (3.16). Obviously, $\Lambda^{\boldsymbol{\phi}^*}(t)$ is progressively measurable with $\mathcal{F}_t$, so in order to prove that $\Lambda^{\boldsymbol{\phi}^*}(t)$ is a P martingale, it is sufficient to show that it satisfies the Novikov condition.

Lemma 3.1. *If the condition*

$$\frac{\beta_1^4 q_2^2(0) + \beta_1^2(\lambda_r^f)^2(1-\gamma)^2}{2(\beta_1+\gamma)^2(1-\gamma)^2} \le \frac{a^f}{2(k_1^f)^2}, \tag{3.17}$$

is satisfied, then $\boldsymbol{\phi}^*(\boldsymbol{t})$ given by Eq (3.16) satisfies the Novikov condition

$$\mathbb{E}^P[\exp\left(\frac{1}{2}\int_0^T(\phi_1^2(s)+\phi_2^2(s))ds\right)] < \infty.$$

Proof. See the proof of Lemma 3.1 in the Appendix of this paper. □

Now, we can give the corresponding verification theorem for Theorem 3.1.

Theorem 3.2. *(Verification theorem) For problem (2.34), if there exists a solution that satisfies the corresponding HJB equation (3.2) and the boundary condition $V(T, y, I, r^d, r^f) = U(y/I)$, satisfies the conditions (3.17), and*

(i) $16N_1(s) + 128M_1^2(s) \le \frac{a^d}{2(k_1^d)^2}$ *when* $k_1^d \neq 0$,

(*ii*) $16N_2(s) + 128M_2^2(s) \leq \frac{a^f}{2(k_1^f)^2}$ *when* $k_1^f \neq 0$,
then the solution of the HJB equation given by (3.2) is the optimal solution of problem (2.34), and the corresponding candidate strategy is the optimal strategy. Here, $M_1(s)$, $M_2(s)$, $N_1(s)$, $N_2(s)$ *as in (A.5)–(A.7).*

Proof. See the proof of Theorem 3.2 in the Appendix of this paper. □

Finally, using the conclusion of Proposition 2.1, Theorems 3.1 and 3.2, we can obtain the optimal strategy of the initial problem.

Corollary 3.1. *While the wealth process of pension fund follows (2.16), according to the Proposition 2.1, Theorem 3.1 and Eq (2.23), the optimal strategy of the initial problem is*

$$\bar{\boldsymbol{\mu}}^*(\boldsymbol{t}) = \boldsymbol{\mu}^*(\boldsymbol{t}) - (\mu_b^F(t)F(t,T), 0, 0, \mu_s^F(t)F(t,T), 0, \mu_p^F(t)F(t,T))^\top, \tag{3.18}$$

where $\boldsymbol{\mu}^*(\boldsymbol{t})$ *is according to (3.15), and* $\mu_b^F(t)$, $\mu_s^F(t)$, $\mu_p^F(t)$ *are as in (2.22).*

4. Two special cases

4.1. The case in which the pension manager is ambiguity neutral

In order to further compare the impact of ambiguity in the distribution of foreign asset prices and the impact of pension managers' ambiguity aversion characteristics on the optimal strategies, this subsection considers the ambiguity-neutral pension manager's portfolio decisions. Under this circumstance, the ambiguity-neutral pension manager no longer considers the ambiguity of the distribution of foreign asset prices, and believes that the reference probability measure P can truly reflect the trajectory of asset prices. In this case, the optimization problem (2.32) degenerates into a general optimal portfolio problem. While the financial market is as in (2.1)–(2.14), and the wealth process of the pension is as in Eq (2.24), the objective function of the ambiguity-neutral pension manager is

$$\tilde{J}^{\tilde{\boldsymbol{\mu}}}(t, y, r^d, r^f, I) = \mathbb{E}_{t,y,r^d,r^f,I}[U(\frac{Y(T)}{I(T)})],$$

and the corresponding value function is

$$\widetilde{V}(t, y, r^d, r^f, I) = \sup_{\tilde{\boldsymbol{\mu}} \in \mathcal{A}} J^{\tilde{\mu}}(t, y, r^d, r^f, I). \tag{4.1}$$

Similar to the previous section, we can obtain the optimal strategy $\tilde{\boldsymbol{\mu}}(\boldsymbol{t})$ and the corresponding value function for ambiguity-neutral pension manager.

Theorem 4.1. *When the pension manager is ambiguity neutral, and the wealth process of the pension follows (2.24), the utility of the pension manager is characterized by the CRRA utility function, and the corresponding value function is*

$$\widetilde{V}(t, y, I, r^d, r^f) = \frac{1}{1-\gamma}(\frac{y}{I})^{1-\gamma} e^{r^d p_1(t) + r^f p_2(t)}, \tag{4.2}$$

and the optimal strategy is

$$\tilde{\boldsymbol{\mu}}^*(\boldsymbol{t}) = Y^{\boldsymbol{\mu}^*}(t)(\boldsymbol{\sigma}^{-1}(\boldsymbol{t}))^\top[\boldsymbol{\xi}(\boldsymbol{t}) - p_1(t)\Delta_1(t)\boldsymbol{e_1} - p_2(t)\Delta_2(t)\boldsymbol{e_2} - (1-\gamma)\boldsymbol{\sigma_I}(\boldsymbol{t})], \tag{4.3}$$

where

$$p_1(t) = q_1(t), \tag{4.4}$$

$$p_2(t) = \frac{\eta_1\eta_2(e^{\sqrt{\Delta_{p2}}(T-t)} - 1)}{\eta_1 e^{\sqrt{\Delta_{p2}}(T-t)} - \eta_2}, \tag{4.5}$$

here $\eta_{1,2} = \frac{a^f\gamma + k_1^f\lambda_r^f(1-\gamma)\pm\gamma\sqrt{\Delta_{p2}}}{k_1^f}$, $\Delta_{p2} = (a^f + \frac{1-\gamma}{\gamma}k_1^f\lambda_r^f)^2 - \frac{(1-\gamma)(k_1^f\lambda_r^f)^2}{\gamma^2}$.

Proof. The proof is similar to that of pension managers with ambiguity aversion characteristics, so we omit it here. □

Remark 4.1. *The optimal strategy for the ambiguity-neutral pension manager is actually the optimal strategy for the ambiguity-aversion pension manager when the ambiguity aversion coefficient is* $\beta_1 = 0$, $\beta_2 = 0$.

Remark 4.2. *If foreign securities markets are not considered here, then the model degenerates into the case in [20].*

Similarly, the optimal strategy for the ambiguity-neutral pension manager for the initial problem is

Corollary 4.1. *While the wealth process of the pension manager as in (2.16), according to Proposition 2.1, Theorem 4.1, and Eq (2.23), the optimal strategy for the initial problem is*

$$\bar{\boldsymbol{\mu}}^*(\boldsymbol{t}) = \tilde{\boldsymbol{\mu}}^*(\boldsymbol{t}) - (\mu_b^F(t)F(t,T), 0, 0, \mu_s^F(t)F(t,T), 0, \mu_p^F(t)F(t,T))^\top, \tag{4.6}$$

where $\tilde{\boldsymbol{\mu}}^*(\boldsymbol{t})$ *as in (4.3), and* $\mu_b^F(t)$, $\mu_s^F(t)$, $\mu_p^F(t)$ *as in (2.22).*

Consider a special case where the pension manager with ambiguity aversion does not adopt the optimal strategy $\mu^*(t)$ given in Theorem 3.1, but instead incorrectly adopts the ambiguity-neutral pension manager's optimal strategy $\tilde{\mu}^*(t)$ (which is also known as the suboptimal strategy), i.e., utility loss occurs. While the pension manager with ambiguity aversion adopts the suboptimal strategy, then the corresponding value function is

$$\begin{aligned}\widehat{V}(t, y, r^d, r^f, I) = \inf_{Q\in\mathcal{Q}} E^Q_{\{t,y,r^d,r^f,I\}}\{U(\frac{Y^{\tilde{\boldsymbol{u}}^*}(T)}{I(T)}) + \int_t^T [\frac{\phi_1^2(s)}{2\Psi_1(s, Y(s), r^d(s), r^f(s), I(s))} \\ + \frac{\phi_2^2(s)}{2\Psi_2(s, Y(s), r^d(s), r^f(s), I(s))}]ds\},\end{aligned} \tag{4.7}$$

where, $\Psi_i(t, y, r^d, r^f, I) = \frac{\beta_i}{(1-\gamma)\hat{V}^{\tilde{\boldsymbol{\mu}}^*}(t,y,r^d,r^f,I)}$, $(i = 1, 2)$. Similarly, we can derive the optimal strategy and value function for the pension manager in this scenario as follows:

Theorem 4.2. *When a ambiguity aversion pension manager adopts the optimal strategy of a ambiguity neutral pension manager, then the corresponding value function is*

$$\hat{V}(t,y.I.r^d,r^f)=\frac{1}{1-\gamma}(\frac{y}{I})^{1-\gamma}e^{r^d g_1(t)+r^f g_2(t)+g_3(t)},\tag{4.8}$$

here

$$g_1(t)=\frac{\zeta_1\zeta_2(e^{\sqrt{\Delta_{g1}}(T-t)}-1)}{\zeta_1 e^{\sqrt{\Delta_{g1}}(T-t)}-\zeta_2},\tag{4.9}$$

$$g_2(t)=\frac{\widetilde{\zeta_1}\widetilde{\zeta_2}(e^{\sqrt{\Delta_{g2}}(T-t)}-1)}{\widetilde{\zeta_1}e^{\sqrt{\Delta_{g2}}(T-t)}-\widetilde{\zeta_2}},\tag{4.10}$$

$$\begin{aligned}g_3(t)=&\int_t^T\{[a^d b^d+\frac{k_2^d(1-\gamma)(\sigma_{I1}-\lambda_r^d)}{\gamma}]g_1(s)+\frac{k_2^d}{2\gamma}g_1^2(s)-[\frac{k_2^f}{2\gamma}+\beta_1k_2^f\\&(\frac{1-\gamma}{2\gamma^2}+\frac{1}{2(1-\gamma)})]g_2^2(s)+[a^f b^f+k_2^f\lambda_r^f\frac{1-\gamma}{\gamma^2}]g_2(s)+G_2\}ds,\end{aligned}\tag{4.11}$$

$$\zeta_{1,2}=\frac{\gamma a^d-k_1^d(1-\gamma)(\sigma_{I1}-\lambda_r^d)\pm\sqrt{\Delta_{g1}}}{k_1^d},$$

$$\Delta_{g1}=[k_1^d(\sigma_{I1}-\lambda_r^d)\frac{1-\gamma}{\gamma}-a^d]^2-\frac{4G_1}{1-\gamma},$$

$$\tilde{\zeta}_{1,2}=\frac{\gamma(1-\gamma)[\gamma a^f+(1-\gamma)k_1^f\lambda_r^f+\beta_1\lambda_r^f k_1^f]\pm\sqrt{\Delta_{g2}}}{k_1^f(\gamma(1-\gamma)-\beta_1)},$$

$$\Delta_{g2}=[a^f+\frac{1-\gamma}{\gamma}k_1^f\lambda_r^f+\frac{\beta_1k_1^f\lambda_r^f}{\gamma}]^2-\frac{(k_1^f\lambda_r^f)^2(1-\gamma-\gamma\beta_1)(1-\gamma+2\beta_1)}{\gamma^2(1-\gamma)},$$

$G_1=2(\lambda_r^d-\sigma_{I1})\lambda_r^d-(\lambda_r^d-(1-\gamma)\sigma_{I1})^2+\gamma\sigma_{I1}^2(2-\gamma)-2(1-\gamma)\sigma_{I1}(\lambda_r^d-(1-\gamma)\sigma_{I1})$, $G_2=\frac{(2-\gamma)(\sigma_{I1}^2(k_2^d)^2+\sigma_{I2}^2)}{2}+r_r-\sigma_{I1}\lambda_r^d k_2^d-\sigma_{I2}\lambda_I-\frac{\beta_2(\lambda_s^f)^2+(\beta_1-\gamma)k_2^f(\lambda_r^f)^2}{2\gamma^2}+\frac{\lambda_H^2+(\lambda_s^d)^2+(\lambda_s^f)^2}{2\gamma}+\frac{k_2^d((\lambda_r^d)^2-(1-\gamma)^2\sigma_{I1}^2)}{2\gamma}+\frac{(\lambda_I-(1-\gamma)\sigma_{I2})^2}{\gamma}$,

Proof. The proof is similar to that of pension manager with ambiguity aversion characteristics, so we omit here. □

Similarly to Branger et al. [30], In this paper, we define the utility loss of pension manager as

$$\begin{aligned}UL:&=1-\frac{V(t,y,r^d,r^f,I)}{\hat{V}(t,y,r^d,r^f,I)}\\&=1-\exp((q_1(t)-g_1(t))r^d+(q_2(t)-g_2(t))r^f+q_3(t)-g_3(t)).\end{aligned}\tag{4.12}$$

4.2. The case when there is no investment in foreign securities market

As mentioned earlier, investing in foreign capital markets can effectively reduce investment risks and obtain benefits from foreign financial markets. However, due to insufficient knowledge of the political, economic, and other fields of the investment country, investors may face a situation where the distribution of foreign asset prices is ambiguous. According to the entropy penalty method based on robust control theory, as investors' ambiguity aversion increases, investing in foreign securities may no longer be a wise choice for them. In this section, we consider a situation where the pension manager only invests in the domestic securities market (i.e., in rolling bonds, domestic stocks, and inflationary bonds), and the distribution of domestic asset prices is determined. Under this circumstances, the wealth process of the pension follows:

$$dX_1(t) = C(t)dt + r^d(t)X_1(t)dt + \bar{\boldsymbol{\pi}}^\top(\boldsymbol{t})\boldsymbol{\sigma_1}(\boldsymbol{t})\Big(\boldsymbol{\xi_1}(\boldsymbol{t})dt + d\boldsymbol{B}(\boldsymbol{t})\Big), \tag{4.13}$$

where

$$\boldsymbol{\sigma_1}(\boldsymbol{t}) = \begin{pmatrix} \sigma_B^d(T_1)\Delta_1(t) & 0 & 0 \\ \sigma_{s1}^d\Delta_1(t) & \sigma_{s2}^d & 0 \\ \sigma_{I1}\Delta_1(t) & 0 & \sigma_{I2} \end{pmatrix}, \tag{4.14}$$

$$\bar{\boldsymbol{\pi}}^\top(\boldsymbol{t}) = \Big(\pi_B^d(t), \pi_s^d(t), \pi_p(t)\Big)^\top, \qquad \boldsymbol{\xi_1}(\boldsymbol{t}) = (\lambda_r^d\Delta_1(t), \lambda_s^d, \lambda_I)^\top,$$

$$\boldsymbol{B}(\boldsymbol{t}) = \Big(W_r^d(t), W_s^d(t), W_I(t)\Big)^\top, \qquad \Delta_1(t) \triangleq \sqrt{k_1^d r^d(t) + k_2^d}.$$

Using the auxiliary process in Section 2, we can further obtain the self-financing form

$$Y_1(t) = X_1(t) + F(t, T). \tag{4.15}$$

Let

$$\boldsymbol{\pi}(\boldsymbol{t}) = \bar{\boldsymbol{\pi}}(\boldsymbol{t}) + \Big(\pi_b^F F(t,T), \pi_s^F F(t,T), \pi_p^F F(t,T)\Big)^\top, \tag{4.16}$$

then the corresponding auxiliary process is

$$dY_1(t) = r^d(t)Y_1(t)dt + \boldsymbol{\pi}^\top(\boldsymbol{t})\boldsymbol{\sigma_1}(\boldsymbol{t})[\boldsymbol{\xi_1}(\boldsymbol{t})dt + d\boldsymbol{B}(\boldsymbol{t})]. \tag{4.17}$$

Theorem 4.3. *We only consider that the pension manager invested in the domestic market, and is ambiguity neutral; then the value function of the pension manager is*

$$V_1(t, y_1, I, r^d) = \frac{1}{1-\gamma}(\frac{y_1}{I})^{1-\gamma} e^{r^d h_1(t) + h_2(t)}, \tag{4.18}$$

the optimal strategy is

$$\boldsymbol{\pi}^*(\boldsymbol{t}) = (\boldsymbol{\sigma}_{\boldsymbol{1}}^\top(\boldsymbol{t}))^{-1}\frac{Y_1^*(t)}{\gamma}[\boldsymbol{\xi_1}(\boldsymbol{t}) - h_1(t)\Delta_1(t)\boldsymbol{e_1}^\top - (1-\gamma)\boldsymbol{\sigma}_{\boldsymbol{1I}}^\top(\boldsymbol{t})], \tag{4.19}$$

where

$$h_1(t) = \frac{\rho_1\rho_2(e^{\sqrt{\Delta_{h1}}(T-t)} - 1)}{\rho_1 e^{\sqrt{\Delta_{\rho_1}}(T-t)} - \rho_2}, \tag{4.20}$$

$$\begin{aligned} h_2(t) = \int_t^T &\{[a^d b^d + \frac{k_2^d(1-\gamma)(\sigma_{I1} - \lambda_r^d)}{\gamma}]h_1(s) + \frac{k_2^d}{2\gamma}h_1^2(s) + [\frac{1-\gamma}{2\gamma}(\sigma_{I1}^2 k_2^d \\ &+ \sigma_{I2}^2 + (\lambda_r^d)^2 k_2^d + (\lambda_s^d)^2 + \lambda_I^2) + (1-\gamma)(r_r) - \frac{1-\gamma}{2\gamma}\lambda_r^d(\sigma_{I1}k_2^d + \sigma_{I2})]\}ds, \end{aligned} \tag{4.21}$$

$$\rho_{1,2} = \frac{\gamma a^d - k_1^d(1-\gamma)(\sigma_{I1} + (1-\gamma)k_1^d\lambda_r^d) \pm \sqrt{\Delta_{h1}}}{k_1^d},$$

$$\Delta_{h1} = [k_1^d(\sigma_{I1} - \lambda_r^d)\frac{1-\gamma}{\gamma} - a^d]^2 - (k_1^d)^2(\sigma_{I1} - \lambda_r^d)^2\frac{1-\gamma}{\gamma^2},$$

$$\boldsymbol{e}_1^\top = (1,0,0), \qquad \boldsymbol{\sigma}_{1I}^\top(\boldsymbol{t}) = (\sigma_{I1}\Delta_1(t), 0, \sigma_{I2}).$$

Proof. The proof is similar to that of the pension manager with ambiguity-aversion characteristics, so we omit it here. □

5. Numerical simulation

To investigate the portfolio strategy of pension managers with ambiguity aversion, whose utility preferences follow a CRRA utility function, this section examines how parameter changes affect the portfolio strategy through numerical simulation. The parameters employed in the model are based on the annualized benchmark values presented in Table 1‡.

Table 1. Model parameters.

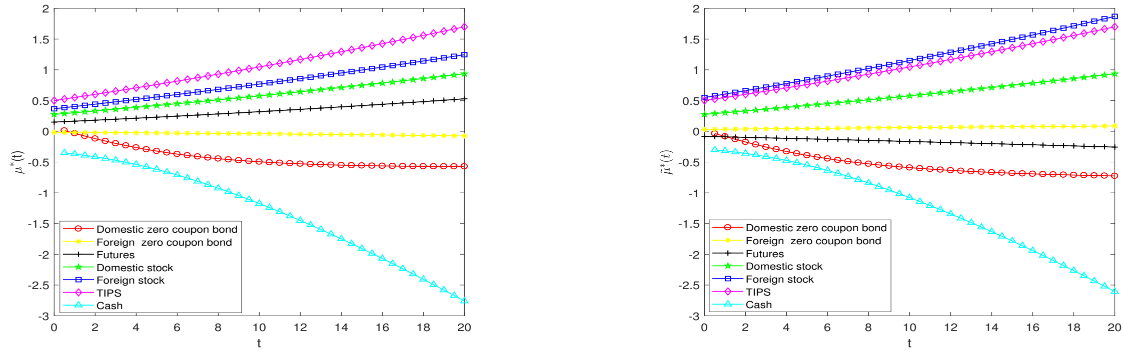
x_0	c_0	T	T_1	T_2	T_3	b^d	a^d	a^f	b^f	μ	γ
1	0.1	20	10	10	1	0.024	0.02	0.03	0.05	0.02	2
r_0^d	r_0^f	r_r	k_1^d	k_2^d	k_1^f	k_2^f	σ_{I1}	σ_{I2}	σ_{s1}^d	σ_{s2}^d	σ_q
0.03	0.05	0.025	0.073	0	0.06	0	0.08	0.08	0.02	0.2	0.05
σ_{s1}^f	σ_{s2}^f	λ_r^f	λ_r^d	λ_s^f	λ_s^d	λ_I	λ_H	β_1	β_2		
0.2	0.1	0.4	0.3	0.11	0.11	0	0.05	1	1		

Example 5.1. *Figure 1 depicts the optimal portfolio strategies of pension managers with ambiguity aversion and ambiguity neutrality, under a time horizon of $T = 20$. It shows that managers with different ambiguity preferences adopt similar strategies: both allocate the majority of their investments to domestic and foreign stocks and inflation-linked bonds, while also considering short-selling cash and domestic bonds. However, compared to the ambiguity-averse manager in Figure* 1(*a*)*, the ambiguity-neutral manager in Figure* 1(*b*) *holds approximately* 50% *more foreign stocks and about* 20% *more foreign bonds at the terminal time. In fact, throughout the entire investment period, the ambiguity-neutral manager consistently allocates a higher proportion of their portfolio to foreign stocks and bonds than the ambiguity-averse manager. This indicates that pension managers' ambiguity preferences can significantly influence optimal portfolio decisions.*

Example 5.2. *Figure 2 analyzes the impact of ambiguity aversion coefficients β_1, β_2 on pension managers' investment strategies for foreign bonds and stocks, where β_1, β_2 depicts the degree of ambiguity aversion toward the price distributions of foreign bonds and stocks. In Figure* 2(*a*)*, as β_1 ($\beta_1 = 1, 5, 10$) increases, the proportion of foreign bonds in the portfolio of ambiguity-averse managers decreases by 59% (from $\beta_1 = 1$ to $\beta_1 = 5$) and 35% (from $\beta_1 = 5$ to $\beta_1 = 10$), respectively. This indicates that as managers' tolerance for ambiguity in foreign bond price distributions diminishes, their*

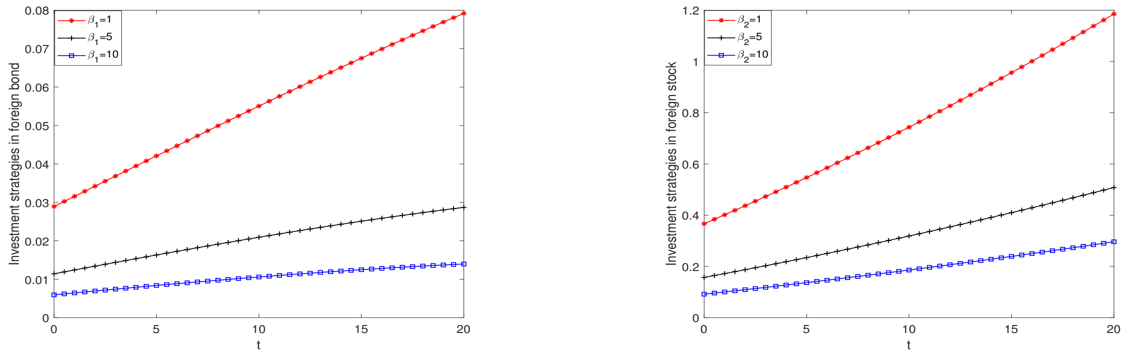
‡The market parameters in Table 1 are primarily sourced from [9] and [25].

willingness to invest in foreign bonds also declines. Similarly, as shown in Figure 2(*b*)*, with the increase of* β_2 *(*$\beta_2 = 1, 5, 10$*), pension manager's exposure to foreign stock decreases by 54% (from* $\beta_2 = 1$ *to* $\beta_2 = 5$*), and 41% (from* $\beta_2 = 5$ *to* $\beta_2 = 10$*).*



(a) Optimal strategy of ambiguity-aversion pension manager (b) Optimal strategy of ambiguity-neutral pension manager

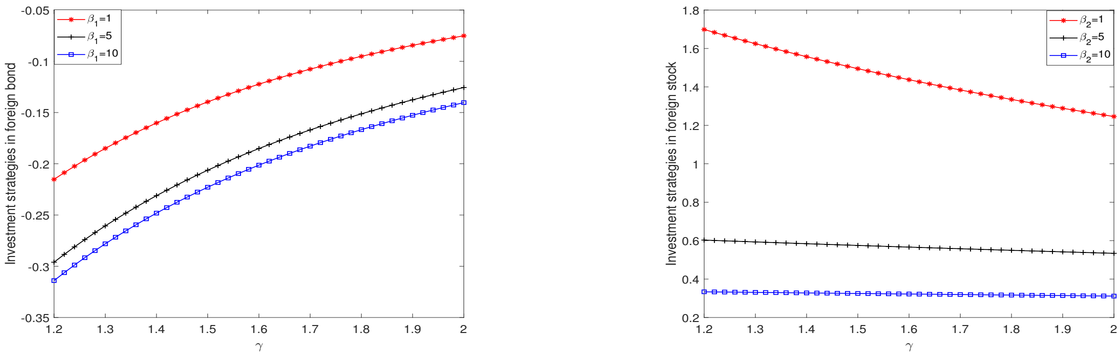
Figure 1. Changes in the optimal strategy of the pension manager vs. time t in two scenarios.



(a) The impact of ambiguity aversion coefficient β_1 on the investment in foreign bonds (b) The impact of ambiguity aversion coefficient β_2 on the investment in foreign stocks

Figure 2. The impact of ambiguity aversion coefficient $\beta_i (i = 1, 2)$ on the investment in foreign assets.

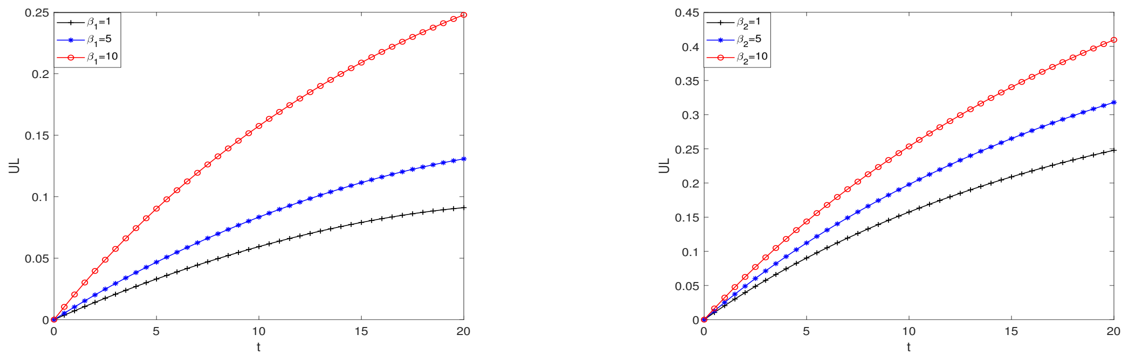
Example 5.3. *Figure 3 illustrates the investment strategy of the ambiguity-aversion pension manager in foreign bonds and stocks under the interaction of ambiguity aversion (portrayed by* β_i*) and risk aversion (portrayed by* γ_i*). Similar to the conclusion in Figure 2, as* β_i *increases, the pension managers' exposure to both foreign bonds and stocks decreases. Notably, the proportion of foreign bonds in managers' portfolios in Figure* 3(*a*) *increases with higher* γ*, which contradicts the findings in general literature. However, a closer look at Figure* 3(*b*) *reveals that as* γ *rises, managers significantly reduce their exposure to foreign stocks. Most of this reduced exposure is shifted to cash accounts, with a portion reallocated to foreign bonds (spillover effect), leading to an increase in managers' foreign bond holdings. (In the financial market parameter settings of this section, the volatility of foreign stocks exceeds that of bonds.)*



(a) Optimal strategy on foreign bonds (b) Optimal strategy on foreign stocks

Figure 3. When β_i has different values, the investment proportion of foreign bonds and stocks vs. risk-aversion coefficient γ.

Example 5.4. *Figure 4 illustrates the impact of changes in ambiguity aversion parameters on pension managers' utility loss. Utility loss here refers to the loss incurred when ambiguity-averse managers mistakenly adopt ambiguity-neutral investment strategies (i.e., suboptimal strategies). In Figure* 4(*a*)*, as managers' ambiguity aversion toward foreign bonds increases, the utility loss from adopting suboptimal strategies rises gradually. When the ambiguity aversion parameter for foreign bonds is* $\beta_1 = 10$*, the utility loss reaches* 25% *over a* 20*-year investment horizon. In Figure* 4(*b*)*, the utility loss caused by the ambiguity aversion parameter* β_2 *for foreign stocks is similar to that in Figure* 4(*a*)*. As the ambiguity aversion parameter increases, the pension manager's utility loss gradually increase, but the magnitude of the increase is significantly smaller than that in the case of Figure* 4(*a*)*. When the pension manager's ambiguity aversion parameter for foreign stock is* $\beta_2 = 10$*, then the utility loss of the pension manager reaches* 41% *over a* 20*-years investment horizon.*



(a) Ambiguity aversion coefficient β_1 (b) Ambiguity-aversion coefficient β_2

Figure 4. Utility loss (UL) vs. time t under different ambiguity-aversion coefficient β_i.

Example 5.5. *Figure 5 considers the effect of the risk-aversion parameter on the pension manager's utility loss. The study found that with the same risk-aversion parameter, the utility loss of the pension manager increases with time* t*. On the other hand, with an increase in the risk- aversion parameter* γ

($\gamma = 2, 3, 4$), then the degree of utility loss of the pension manager increases. At a time horizon $T = 20$, the utility loss increases by 450% from $UL = 0.13$ at $\gamma = 2$ to $UL = 0.72$ at $\gamma = 4$. This phenomenon indicates that when an ambiguity-aversion pension manager adopts a suboptimal investment strategy, investors who are more sensitive to risk will suffer more severe utility losses.

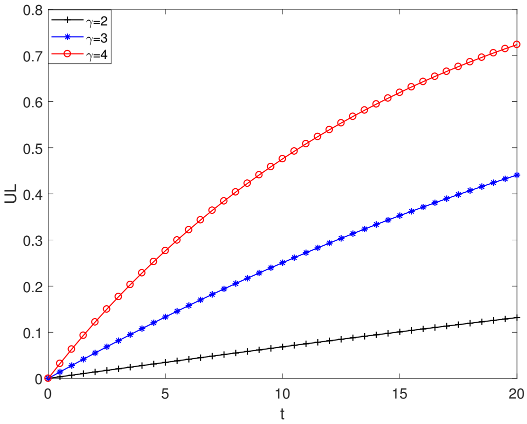


Figure 5. The utility loss (UL) vs. time t under different risk-aversion coefficients γ.

6. Conclusions

This paper examines the portfolio decision problem of a DC pension fund that can invest in foreign securities. Pension funds are exposed to domestic inflation risks, domestic and foreign interest rate risks, domestic and foreign securities price risks, and exchange rate risks. Additionally, we consider that pension managers exhibit both risk aversion and ambiguity aversion, with ambiguity surrounding the price distribution of foreign assets. First, the problem is transformed into a self-financing one via an auxiliary process, thereby reducing the model's complexity. Second, the model is solved using dynamic programming, yielding candidate solutions, which are then verified as optimal via a verification theorem. Finally, the paper discusses the optimal investment strategy for ambiguity-neutral pension managers, as well as the utility loss incurred when ambiguity-averse managers incorrectly adopt the ambiguity-neutral optimal strategy. The analysis of these issues helps pension managers formulate rational investment strategies based on their own ambiguity aversion when investing in overseas securities and related assets.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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A. Appendix

A.1. The proof of Lemma 3.1

Before proving that the Novikov condition is satisfied, we first give a lemma for $q_1(t)$, $q_2(t)$.

Lemma A.1. *In Theorem 3.2, $q_1(t)$ and $q_2(t)$ given by (3.8) and (3.9) are monotonically increasing on* $[0, T]$*, thus we have*

$$q_1(0) \le q_1(t) \le q_1(T) = 0, \qquad q_2(0) \le q_2(t) \le q_2(T) = 0.$$

Proof. First, taking the derivative to $q_1(t)$, we have

$$q_1'(t) = \frac{\sqrt{\Delta_{q1}}\lambda_1\lambda_2 e^{\sqrt{\Delta_{q1}}(T-t)}(\lambda_2 - \lambda_1)}{(\lambda_1 e^{\sqrt{\Delta_{q1}}(T-t)} - \lambda_2)^2},$$

for the numerator of this formula, clearly we have $\lambda_2 - \lambda_1 \le 0$, and we can easily prove that $\lambda_1\lambda_2 \le 0$, thus $q_1(t)$ is monotonically increasing on $[0, T]$. Moreover, because $q_1(T) = 0$, then we have $q_1(0) \le q_1(t) \le q_1(T) = 0$. Similarly, we can prove the properties of $q_2(t)$. □

Now, we can prove that $\boldsymbol{\phi}^*(t)$ in Lemma 3.1 satisfies the Novikov condition, and we already know that

$$\phi_1^*(t) = \frac{\beta_1 q_2(t) - \beta_1\lambda_r^f(1-\gamma)}{(\beta_1+\gamma)(1-\gamma)}\sqrt{k_1^f r^f(t) + k_2^f},$$

$$\phi_2^*(t) = -\frac{\lambda_s^f\beta_2}{\beta_2+\gamma},$$

then we have

$$\phi_1^2(t) + \phi_2^2(t) = [\frac{\beta_1 q_2(t) - \beta_1\lambda_r^f(1-\gamma)}{(\beta_1+\gamma)(1-\gamma)}]^2(k_1^f r^f(t) + k_2^f) + \frac{(\lambda_s^f\beta_2)^2}{(\beta_2+\gamma)^2}.$$

The following discussion will focus on $k_1^f = 0$, $k_1^f \ne 0$ (in the following expression, K represents a positive constant, but the value of K in different expressions may vary).

1) As $k_1^f = 0$, then

$$\begin{aligned}\frac{1}{2}\phi_1^2(t) + \frac{1}{2}\phi_2^2(t) &= \frac{1}{2}[\frac{\beta_1 q_2(t) - \beta_1\lambda_r^f(1-\gamma)}{(\beta_1+\gamma)(1-\gamma)}]^2 k_2^f + \frac{(\lambda_s^f\beta_2)^2}{2(\beta_2+\gamma)^2}\\ &= K + \frac{\beta_1^2 q_2^2(t) + \beta_1^2(\lambda_r^f)^2(1-\gamma)^2 - 2\beta_1^2\lambda_r^f(1-\gamma)q_2(t)}{2(\beta_1+\gamma)^2(1-\gamma)^2}k_2^f\\ &\le K + \frac{\beta_1^2 q_2^2(0) + \beta_1^2(\lambda_r^f)^2(1-\gamma)^2}{2(\beta_1+\gamma)^2(1-\gamma)^2}k_2^f\\ &\le K.\end{aligned}$$

The first inequality is derived from the monotonically increasing property of $q_2(t)$, and $\gamma > 1, q_2(T) = 0$. From the above inequality, it is obvious that

$$E^P[\exp\{\int_0^T (\frac{1}{2}\phi_1^*(s)^2 + \frac{1}{2}\phi_2^*(s)^2)ds\}] < \infty.$$

2) As $k_1^f \neq 0$, let $R^f(t) = k_1^f r^f(t) + k_2^f$, then

$$dR^f(t) = (k_1^f a^f b^f + k_2^f a^f - a^f R^f(t))dt - k_1^f \sqrt{R^f(t)}dW_r^f(t),$$

further, we have

$$\begin{aligned}
\frac{1}{2}\phi_1^2(t) + \frac{1}{2}\phi_2^2(t) &= \frac{1}{2}[\frac{\beta_1 q_2(t) - \beta_1\lambda_r^f(1-\gamma)}{(\beta_1+\gamma)(1-\gamma)}]^2(k_1^f r^f(t) + k_2^f) + \frac{(\lambda_s^f\beta_2)^2}{2(\beta_2+\gamma)^2} \\
&= K + \frac{\beta_1^2 q_2^2(t) - 2\beta_1^2(\lambda_r^f)(1-\gamma)q_2(t) + \beta_1^2(\lambda_r^f)^2(1-\gamma)^2}{2(\beta_1+\gamma)^2(1-\gamma)^2}R^f(t) \\
&\leq K + \frac{\beta_1^2 q_2^2(0) + \beta_1^2(\lambda_r^f)^2(1-\gamma)^2}{2(\beta_1+\gamma)^2(1-\gamma)^2}R^f(t) \\
&\leq K + \frac{a^f}{2(k_1^f)^2}.
\end{aligned}$$

The first inequality can be obtained by referring to 1), and the second inequality refers to the Theorem 5.1 in [31]. Thus, as $k_1^f \neq 0$, we have

$$E^p[\exp\{\int_0^T (\frac{1}{2}(\phi_1^*(s))^2 + \frac{1}{2}(\phi_2^*(s))^2)ds\}] < \infty.$$

A.2. The proof of Theorem 3.2

We try to prove this part by using the Corollary 1.2 in Taksar and Zeng [31]. According to Corollary 1.2, to prove the verification theorem, it is only necessary to prove the following equivalent conclusions.

(1) $\boldsymbol{\mu^*(t)}$ is an admissible strategy;

(2) $E^{Q^*}\{\sup_{t\in[0,T]} \mid J(t, Y^{\pi^*}, r^f(t), r^d(t), I(t)) \mid^4\} < \infty$;

(3) $E^{Q^*}\{\sup_{t\in[0,T]} \mid \frac{(\phi_1^*(t))^2}{2\Psi_1(t,Y^{\mu^*}(t),r^d(t),r^f(t),I(t))} + \frac{(\phi_2^*(t))^2}{2\Psi_2(t,Y^{\mu^*}(t),r^d(t),r^f(t),I(t))} \mid^2\} < \infty$.

For (1), according to Definition 2.1, the condition (*i*) in Definition 2.1 is obviously valid, while (*ii*) and (*iii*) can be obtained from (2). So, only need to prove (2) and (3).

For the proof of (2), let $Z(t) = \frac{Y^*(t)}{I(t)}$, then we have $\frac{dZ(t)}{Z(t)} = \frac{dY^*(t)}{Y^*(t)} - \frac{dI(t)}{I(t)} - \frac{dY^*(t)}{Y^*(t)} \cdot \frac{dI(t)}{I(t)}$. Thus

$$\begin{aligned}
\frac{dZ(t)}{Z(t)} &= \frac{1}{Y^*(t)}\{r^d(t)Y^*(t)dt + \mu^\top(t)\sigma_t[\Lambda_t dt + \boldsymbol{\phi^*(t)}dt + d\boldsymbol{W^{Q^*}(t)}]\} \\
&- \{(r^d(t) - r_r(t))dt + \sigma_{I1}\Delta_1(t)(\lambda_r^d\Delta_1(t)dt + dW_r^d(t)) + \sigma_{I2}(\lambda_I dt + dW_I(t))\} \\
&- \frac{1}{\gamma}\{\sigma_{I1}[\lambda_r^d - q_1(t) - (1-\gamma)\sigma_{I1}]\Delta_1(t) + \sigma_{I2}(\lambda_I - (1-\gamma)\sigma_{I1})\}.
\end{aligned} \tag{A.1}$$

Substitute $\mu^*(t)$ into the above equation, and after some simple transformations, we get the explicit expression of $Z(t)$.

$$
\begin{aligned}
Z(t) = Z(0)\exp\{&\int_0^t A_1(s)ds + A_2 W_H(t) + A_3 W_s^d(t) + A_4 W_s^{f,Q^*}(t) + A_5 W_I(t)\\
&+ \int_0^t M_1(s)\sqrt{k_1^d r^d(t) + k_2^d}\,dW_r^d(s) + \int_0^t M_2(s)\sqrt{k_1^f r^f(t) + k_2^f}\,dW_r^{f,Q^*}(s)\\
&+ \int_0^t N_1(s)(k_1^d r^d(t) + k_2^d)ds + \int_0^t N_2(s)(k_1^f r^f(t) + k_2^f)ds,
\end{aligned} \tag{A.2}
$$

where

$$
\begin{aligned}
A_1(s) = r_r(s) + \frac{\lambda_H^2}{\gamma} + \frac{(\lambda_s^d)^2}{\gamma} + \frac{(\lambda_s^f)^2}{\gamma+\beta_2} - \frac{\beta_2(\lambda_s^f)^2}{(\gamma+\beta_2)^2} - \sigma_{I2}\lambda_I\\
+ \frac{(2\lambda_I - \sigma_{I2})(\lambda_I - (1-\gamma)\sigma_{I2})}{\gamma};
\end{aligned} \tag{A.3}
$$

$$
A_2 = \frac{\lambda_Q}{\gamma},\ A_3 = \frac{\lambda_s^d}{\gamma},\ A_4 = \frac{\lambda_s^f}{\gamma+\beta_2},\ A_5 = \frac{\lambda_I - (1-\gamma)\sigma_{I2}}{\gamma} - \sigma_{I2}; \tag{A.4}
$$

$$
M_1(s) = \frac{\lambda_r^d - q_1(s) - (1-\gamma)\sigma_{I1}}{\gamma} - \sigma_{I1},\ M_2(s) = \frac{\lambda_r^f + \frac{\beta_1 q_2(s)}{1-\gamma} - q_2(s)}{\beta_1+\gamma}; \tag{A.5}
$$

$$
N_1(s) = \frac{\lambda_r^d - \sigma_{I1}}{\gamma}[\lambda_r^d - q_1(s) - (1-\gamma)\sigma_{I1}] - \sigma_{I1}\lambda_r^d; \tag{A.6}
$$

$$
N_2(s) = \frac{(\lambda_r^f\gamma(1-\gamma) + \beta_1 q_2(s))(\lambda_r^f + \frac{\beta_1 q_2(s)}{1-\gamma} - q_2(s))}{(\beta_1+\gamma)^2(1-\gamma)}; \tag{A.7}
$$

thus

$$
\begin{aligned}
&|\ J(t, Y^{\mu^*}(t), r^d(t), r^f(t), I(t))\ |^4\\
=&\left|\ \frac{1}{(1-\gamma)^4}(Z^{\mu^*}(t))^4 \exp\{4q_1(t)r^d(t) + 4q_2(t)r^f(t) + 4q_3(t)\}\ \right|\\
\le& K(Z^{\mu^*}(t))^4\\
=& KZ(0)\exp\{\int_0^t 4A_1(s)ds + 4A_2 W_H(t) + 4A_3 W_s^d(t) + 4A_4 W_s^{f,Q^*}(t) + 4A_5 W_I(t)\\
&+ \int_0^t 4M_1(s)\sqrt{k_1^d r^d(t) + k_2^d}\,dW_r^d(s) + \int_0^t 4M_2(s)\sqrt{k_1^f r^f(t) + k_2^f}\,dW_r^{f,Q^*}(s)\\
&+ \int_0^t 4N_1(s)(k_1^d r^d(t) + k_2^d)ds + \int_0^t 4N_2(s)(k_1^f r^f(t) + k_2^f)ds\},
\end{aligned}
$$

here, the first inequality holds according to the conclusion in Lemma A.1.

In the following, we only consider the case $k_1^d = 0$, $k_1^f = 0$ or $k_1^d \neq 0$, $k_1^f \neq 0$; other cases can be similarly proved.

(1) As $k_1^d = k_1^f = 0$,

$$
\begin{aligned}
&E^{Q^*} \mid J(t, Y^{\mu^*}(t), r^d(t), r^f(t), I(t)) \mid^4 \\
&\leq KE^{Q^*}[\exp\{\int_0^t 4A_1(s)ds + 4A_2W_H(t) + 4A_3W_s^d(t) + 4A_4W_s^{f,Q^*}(t) \\
&+ 4A_5W_I(t) + \int_0^t 4M_1(s)\sqrt{k_2^d}dW_r^d(s) + \int_0^t 4M_2(s)\sqrt{k_2^f}dW_r^{f,Q^*}(s) \\
&+ \int_0^t 4N_1(s)k_2^d ds + \int_0^t 4N_2(s)k_2^f ds\}] \\
&\leq KE^{Q^*}[\exp\{\int_0^t 4M_1(s)\sqrt{k_2^d}dW_r^d(s) + \int_0^t 4M_2(s)\sqrt{k_2^f}dW_r^{f,Q^*}(s) \\
&+ \int_0^t 4N_1(s)k_2^d ds + \int_0^t 4N_2(s)k_2^f ds\}] \\
&\leq KE^{Q^*}[\exp\{\int_0^t 4M_1(s)\sqrt{k_2^d}dW_r^d(s) + \int_0^t 4M_2(s)\sqrt{k_2^f}dW_r^{f,Q^*}(s) < \infty.
\end{aligned}
$$

The first inequality obviously holds, and the second inequality holds because $A_1(s)$ is a bounded, deterministic function on $[0, t]$. In addition, we clearly have
$\exp\{\int_0^t KdW(s)$: K is a constant, $W(s)$ is a standard brownian motion$\}$ = $\underbrace{\exp(2\int_0^t K^2 ds)}_{\text{Bounded}} \times$

$\underbrace{\exp\{\int_0^t KdW(s) - \int_0^t 2K^2 ds\}}_{\text{Martingale}},$

thus we have $E^{Q^*}[\exp\{\int_0^t KdW(s)\}] < \infty$. For the third inequality, it can be obtained from Lemma A.1. The fourth inequality holds because, for the defined function $M_1(s)$, $M_2(s)$, we have

$$
\begin{aligned}
\exp\{\int_0^t 4M_1(s)\sqrt{k_2^d}dW_r^d(s)\} &= \underbrace{\exp\{\int_0^t 8M_1^2(s)k_2^d ds\}}_{\text{Bounded}} \\
&\times \underbrace{\exp\{\int_0^t 4M_1(s)\sqrt{k_2^d}dW_r^d(s) - \int_0^t 8M_1^2(s)k_2^d ds\}}_{\text{Martingale}},
\end{aligned}
\tag{A.8}
$$

(2) As $k_1^d \neq 0$, $k_1^f \neq 0$, consider stochastic process $R^l(t) = k_1^l r^l(t) + k_2^l$, where $l = (d, f)$. Then

$$
dR^l(t) = (k_1^l a^l b^l + k_2^l a^l - a^l R^l(t))dt - k_1^l \sqrt{R^l(t)}dW_r^{l,Q^*}(t),
$$

thus

$$
\begin{aligned}
& E^{Q^*} \mid J(t, Y^{\mu^*}(t), r^d(t), r^f(t), I(t)) \mid^4 \\
& \leq K E^{Q^*}[\exp\{\int_0^t 4M_1(s)\sqrt{R^d(s)}dW_r^d(s) + \int_0^t 4M_2(s)\sqrt{R^f(s)}dW_r^{f,Q^*}(s) \\
& + \int_0^t 4N_1(s)R^d(s)ds + \int_0^t 4N_2(s)R^f(s)ds\}] \\
& \leq K E^{Q^*}[\exp\{\int_0^t 8M_1(s)\sqrt{R^d(s)}dW_r^d(s) + \int_0^t 8N_1(s)R^d(s)ds\}]^{\frac{1}{2}} \\
& \times E^{Q^*}[\exp\{\int_0^t 8M_2(s)\sqrt{R^f(s)}dW_r^{f,Q^*}(s) + \int_0^t 8N_2(s)R^f(s)ds\}]^{\frac{1}{2}}.
\end{aligned}
$$

The first inequality is similar to the proof in case (1), considering

$$
\begin{aligned}
& \exp\{\int_0^t 8M_1(s)\sqrt{R^d(s)}dW_r^d(s) + \int_0^t 8N_1(s)R^d(s)ds\} \\
& \quad = \underbrace{\exp\{\int_0^t [8N_1(s) + 64M_1^2(s)]R^d(s)ds\}}_{H} \\
& \times \underbrace{\exp\{-\int_0^t 64M_1^2(s)R^d(s)ds + \int_0^t 8M_1(s)\sqrt{R^d(s)}dW_r^d(s)\}}_{L},
\end{aligned}
\tag{A.9}
$$

and for the L part, we have

$$
\begin{aligned}
E^{Q^*}(L^2) &= E^{Q^*}[\exp\{-\int_0^t 128M_1^2(s)R^d(s)ds \\
&\quad + \int_0^t 16M_1(s)\sqrt{R^d(s)}dW_r^d(s)\}] < \infty.
\end{aligned}
\tag{A.10}
$$

This is because $M_1(s)$ is bounded in $[0, T]$, so L^2 is a martingale (see Lemma 4.3 in [31]). For the H part, we have

$$
E^{Q^*}(H^2) = E^{Q^*}[\exp\{\int_0^t [16N_1(s) + 128M_1^2(s)]R^d(s)ds\}]. \tag{A.11}
$$

If $16N_1(s) + 128M_1^2(s) \leq \frac{a^d}{2(k_1^d)^2}$ is satisfied (according to the assumption, the condition is obviously satisfied), then $E^{Q^*}(H^2) < \infty$ (see Theorem 5.1 in [31]), thus

$$
\begin{aligned}
& E^{Q^*}[\exp\{\int_0^t 8M_1(s)\sqrt{R^d(s)}dW_r^d(s) + \int_0^t 8N_1(s)R^d(s)ds\}]^{\frac{1}{2}} \\
& \quad \leq (E^{Q^*}(H^2) \cdot E^{Q^*}(L^2))^{\frac{1}{4}} < \infty.
\end{aligned}
\tag{A.12}
$$

Similarly, we can prove that if $16N_2(s) + 128M_2^2(s) \leq \frac{a^f}{2(k_1^f)^2}$ is satisfied, then we have

$$
E^{Q^*}[\exp\{\int_0^t 8M_2(s)\sqrt{R^f(s)}dW_r^{f,Q^*}(s) + \int_0^t 8N_2(s)R^f(s)ds\}]^{\frac{1}{2}} < \infty. \tag{A.13}
$$

Based on the above process, we have proved that

$$E^{Q^*}[\sup_{t\in[0,T]} \mid J(t, Y^{\pi^*}, r^f(t), r^d(t), I(t)) \mid^4] < \infty.$$

For part (3), we defined $\Sigma(t) = (\frac{(\phi_1^*(t))^2}{2\beta_1} + \frac{(\phi_2^*(t))^2}{2\beta_2})(1-\gamma)$, and it is obvious that $\Sigma(t)$ is bounded in $[0, T]$, thus we have

$$\begin{aligned}
&E^{Q^*}\{\sup_{t\in[0,T]} \mid \frac{(\phi_1^*(t))^2}{2\Psi_1(t, Y^{\mu^*}(t), r^d(t), r^f(t), I(t))} + \frac{(\phi_2^*(t))^2}{2\Psi_2(t, Y^{\mu^*}(t), r^d(t), r^f(t), I(t))} \mid^2\} \\
&= E^{Q^*}\{\sup_{t\in[0,T]} \mid \Sigma(t) \mid^2\mid J(t, Y^{\mu^*}(t), r^d(t), r^f(t), I(t)) \mid^2\} \\
&\le E^{Q^*}[\sup_{t\in[0,T]} \mid \Sigma(t) \mid^4)^{\frac{1}{2}} \times E^{Q^*}[\sup_{t\in[0,T]} \mid J(t, Y^{\mu^*}(t), r^d(t), r^f(t), I(t)) \mid^4]^{\frac{1}{2}} \\
&< \infty,
\end{aligned}$$

That is, the condition (3) is satisfied. So far, we have completed the proof of all the conditions, thus Theorem 3.2 holds.

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