Electronic
Research Archive

https://www.aimspress.com/journal/era

ERA, 33(10): 6298–6321.
DOI: 10.3934/era.2025278
Received: 30 August 2025
Revised: 22 September 2025
Accepted: 09 October 2025
Published: 24 October 2025

Research article

Efficient energy-stable second-order predictor-corrector SAV scheme for the Cahn-Hilliard equation: algorithm, analysis, and computation

Shimin Lin[1], **Chi Fui William Ni**[2], **Jun Zhang**[3,*] **and Pengtao Yue**[4]

[1] School of Science, Jimei University, Xiamen 361021, China
[2] The Pennington School, Pennington, NJ 08534, USA
[3] Computational Mathematics Research Center, Guizhou University of Finance and Economics, Guiyang 550025, China
[4] Department of Mathematics, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061, USA

* **Correspondence:** Email: jzhang@mail.gufe.efu.cn.

Abstract: The Cahn-Hilliard equation plays a central role in modeling phase separation processes in complex systems, including alloys, polymers, and biological materials. Numerical schemes for this equation must balance efficiency, stability, and accuracy in order to capture the rich dynamics of interfacial evolution. In this work, we developed a new second-order predictor–corrector scheme within the scalar auxiliary variable (SAV) framework, combined with Crank–Nicolson (CN) time discretization. The proposed method is linear, uniquely solvable, and unconditionally energy stable, while also providing rigorous error estimates. Computational experiments demonstrated that the new scheme not only maintains second-order temporal accuracy for relatively large time steps, but also yields smaller numerical errors compared to standard SAV-CN methods. These results highlight both the theoretical advantages and practical potential of the predictor–corrector SAV approach for advancing accurate and efficient simulations of phase-field models in science and engineering.

Keywords: SAV; IEQ; predictor-corrector; time-discrete; second-order; error estimates

1. Introduction

The Cahn–Hilliard equation, originally introduced by Cahn and Hilliard [1] to model phase separation and coarsening dynamics, has since become a classical framework in mathematical physics, with extensive applications in areas such as complex fluids, polymer physics, and soft matter (cf. [2–7]). Owing to its fundamental role, the development of efficient numerical methods for the Cahn–Hilliard equation has been recognized as a problem of considerable importance. A distinctive characteristic of the equation is its energy dissipation property, and consequently the design of numerical schemes that preserve this property at the discrete level has received significant attention. The principal challenge in such constructions arises from the discretization of the nonlinear potential. A fully implicit treatment of the nonlinear terms results in a nonlinear system at each time step, incurring substantial computational cost, whereas a fully explicit treatment leads to severe time-step restrictions and, in most cases, the loss of the energy dissipation property. These challenges have motivated the development of a variety of energy-stable schemes intended to overcome the limitations of straightforward implicit or explicit discretizations.

Several classes of such energy-stable schemes have been proposed in the literature, including (but not limited to) the convex splitting method [8, 9], the stabilized explicit method [10–12], and the invariant energy quadratization (IEQ) method together with its variant, the scalar auxiliary variable (SAV) method [13–15]. Furthermore, several studies have introduced improved schemes based on IEQ or SAV to address certain limitations inherent in the original formulations. For instance, Jiang et al. [16] proposed the relaxed-SAV (RSAV) method, which employs a relaxation technique to effectively penalize numerical errors in the auxiliary variables while retaining all the advantageous properties of the baseline SAV approach. Liu et al. [17] presented a novel technique to correct the modified energy of the SAV method. This led to the construction of several high-order implicit-explicit schemes based on the proposed energy-optimized SAV (EOP-SAV) approaches for dissipative systems. These schemes offer improved accuracy, simplified calculations, and are proven to be unconditionally energy stable. For the hydrodynamically coupled binary phase-field crystal model, Tan and colleagues [18] devised a second-order scheme and corrected the energy through a relaxation technique applied to the exponential scalar auxiliary variable (E-SAV). This scheme is not only highly efficient but also demonstrates enhanced energy consistency. The convex splitting approach guarantees unconditional energy stability and unique solvability; however, the decomposition of nonlinear terms into convex and concave parts leads to nonlinear systems, which significantly increase the computational burden, particularly for long-time simulations. The stabilized explicit approach is straightforward to implement and requires solving only constant-coefficient equations at each step, but since nonlinear terms are treated fully explicitly, additional stabilization terms must be introduced. As a result, the method is only conditionally energy stable and often requires Lipschitz-type restrictions on the double-well potential. To relax or remove this restriction, improved stabilization strategies have been proposed, although these approaches remain conditionally stable. It is worth emphasizing that other effective energy-stable strategies also exist, and the above methods represent only some of the most widely adopted in practice.

The IEQ and SAV approaches were introduced to address some of the drawbacks of earlier schemes. The central idea is to reformulate the nonlinear potential into a quadratic form by introducing auxiliary variables, which leads to linear schemes that are unconditionally energy stable and can be extended to higher-order temporal discretizations. These methods significantly reduce the complexity of implementation compared to convex splitting schemes and avoid the stringent restrictions often required in stabilized explicit formulations. Nonetheless, certain limitations remain. The IEQ method typically requires the nonlinear potential to be bounded from below, and the quadratic reformulation results in variable-coefficient equations, which may increase computational cost. Although the SAV approach alleviates this difficulty by modifying the auxiliary variable so that only constant-coefficient equations are solved—thereby improving efficiency while retaining unconditional stability, both IEQ- and SAV-type schemes still rely on semi-implicit treatments of the quadratic terms. This reliance inevitably introduces truncation errors, which may contaminate the numerical solution when large time steps are employed.

To improve the accuracy of SAV-type schemes while retaining their favorable properties, we propose in this work a new predictor–corrector SAV scheme. The approach is illustrated using the Cahn–Hilliard equation, which serves as a prototypical model for gradient flow systems. The key idea is to correct the extrapolation error of the quadratic terms through a predictor–corrector procedure, thereby enhancing the accuracy of the potential approximation even for relatively large time steps. In addition, a linear stabilizing term is incorporated to suppress oscillations caused by the nonlinear terms. The resulting schemes are linear, unconditionally energy stable, and uniquely solvable. Under suitable assumptions, we further establish uniform L^∞-bounds for the numerical solutions, and rigorous error analysis demonstrates that the schemes achieve second-order temporal accuracy. A series of numerical experiments is presented to confirm the theoretical findings and to illustrate the robustness and effectiveness of the proposed methods.

The remainder of this article is organized as follows. In Section 2, the SAV formulation of the Cahn–Hilliard equation is briefly reviewed. Section 3 presents the proposed predictor–corrector SAV schemes, together with analyses of their unconditional energy stability, unique solvability, and convergence properties. Numerical experiments illustrating the accuracy and efficiency of the schemes are reported in Section 4. Finally, conclusions and perspectives are given in Section 5.

2. The Cahn-Hilliard equation and the SAV approach

2.1. Preliminaries and notation

Throughout this article, $(\cdot,\cdot)$ denotes the L^2- inner product with associated norm $\|\cdot\|$. For $s \in \mathbb{Z}$, $\|\cdot\|_s$ denotes the norm in the Sobolev space $H^s(\Omega)$. We further define the periodic function space with zero mean value by

$$H_{per}(\Omega) = \{\phi \ \ is \ \ periodic, \phi \in H^1(\Omega), \ \ and \ \ \int_\Omega \phi dx = 0\}.$$

This functional setting is natural for the Cahn–Hilliard equation with periodic boundary conditions, since the solution is mass-conserving, i.e., $\int_\Omega \phi(t), dx = \int_\Omega \phi(0)dx$.

For the time discretization, we adopt the following notations for averages and extrapolations. Given a discrete sequence $(\bullet)^n$, we define the midpoint average by

$$(\bullet)^{n+\frac{1}{2}} = \tfrac{1}{2}(\bullet)^n + \tfrac{1}{2}(\bullet)^{n+1}. \tag{2.1}$$

In predictor–corrector iterations, we also use

$$(\bullet)_{i+1}^{n+\frac{1}{2}} = \tfrac{1}{2}(\bullet)^n + \tfrac{1}{2}(\bullet)_{i+1}^{n+1}, \tag{2.2}$$

where the subscript i denotes the iteration index within a time step.

For extrapolations, we employ the standard second-order formulas

$$(\bar{\bullet})^{,n+\frac{1}{2}} = \tfrac{3}{2}(\bullet)^n - \tfrac{1}{2}(\bullet)^{n-1} \tag{2.3}$$

and

$$(\bar{\bullet})^{,n+1} = 2(\bullet)^n - (\bullet)^{n-1}, \tag{2.4}$$

with analogous definitions for other quantities. In particular, $(\bar{\bullet})^{,n+\frac{1}{2}}$ is used in CN-type schemes, while $(\bar{\bullet})^{,n+1}$ is employed in second-order backward differential (BDF2)-type schemes. These extrapolations ensure second-order temporal accuracy while keeping the resulting schemes fully linear at each step.

Finally, for ease of notation, we will use C (possibly with subscripts) to denote generic positive constants that are independent of the discretization parameters, but may vary from line to line.

2.2. The Cahn-Hilliard equation

We begin with the free energy functional

$$E(\phi) = \int_\Omega \left(\tfrac{1}{2}|\nabla\phi|^2 + F(\phi)\right) dx, \tag{2.5}$$

where $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ denotes the computational domain. The first term represents the interfacial energy, while the potential $F(\phi)$ describes the bulk contribution. By the variational principle [3], one obtains the classical Cahn–Hilliard equation in the form

$$\begin{cases} \phi_t = \Delta w, \\ w = \delta E/\delta\phi = -\Delta\phi + f(\phi), \end{cases} \tag{2.6}$$

where $f(\phi) = F'(\phi)$. The system is considered with the initial condition $\phi|_{t=0} = \phi_0$ and periodic boundary conditions.

A well-known property of the Cahn–Hilliard equation is its energy dissipation law. Taking the L^2- inner product of the first equation in (2.6) with $-w$, and of the second equation with ϕ_t, yields

$$\frac{d}{dt}E(\phi) = -\int_\Omega |\nabla w|^2 dx, \tag{2.7}$$

which indicates that the free energy decays monotonically in time. This intrinsic energy dissipation plays a crucial role in the physical consistency of the model, and therefore numerical schemes that preserve this property at the discrete level are of particular interest.

2.3. Standard second-order SAV schemes

To construct linear, unconditionally energy-stable schemes, the SAV approach was introduced [13,14]. The main idea is to reformulate the nonlinear part of the free energy into a quadratic form by introducing an auxiliary variable. Specifically, we define

$$q(\phi) = \sqrt{\int_\Omega F(\phi) - \frac{\gamma}{2}\phi^2 dx + B}, \quad g(\phi) = \frac{f(\phi) - \gamma\phi}{\sqrt{\int_\Omega F(\phi) - \frac{\gamma}{2}\phi^2 dx + B}}, \tag{2.8}$$

where $\gamma > 0$ is a given constant and B is chosen sufficiently large so that the square root is well-defined and strictly positive. Here $q(\phi)$ serves as the scalar auxiliary variable, while $g(\phi)$ is a normalized function that couples the original nonlinear term with $q(\phi)$.

With these definitions, the Cahn–Hilliard system (2.6) can be equivalently written as

$$\begin{cases} \phi_t = \Delta w, \\ w = -\Delta\phi + \gamma\phi + qg, \\ q_t = \dfrac{1}{2}\displaystyle\int_\Omega g\phi_t dx, \end{cases} \tag{2.9}$$

together with the initial conditions $\phi|_{t=0} = \phi_0$ and $q|_{t=0} = \sqrt{\int_\Omega F(\phi_0) - \frac{\gamma}{2}\phi_0^2 dx + B}$. The boundary conditions are taken to be periodic, consistent with (2.6).

The reformulated system (2.9) inherits an energy dissipation structure similar to (2.6). Indeed, taking the L^2- inner product of the first equation with w and of the second equation with ϕ_t, and multiplying the third equation by $2q$, one obtains

$$\frac{d}{dt}E(\phi, q) = -\int_\Omega |\nabla w|^2 dx, \tag{2.10}$$

where the modified energy is given by

$$E(\phi, q) = \int_\Omega (\frac{1}{2}|\nabla\phi|^2 + \frac{\gamma}{2}|\phi|^2)dx + |q|^2 - B. \tag{2.11}$$

It is straightforward to verify that the transformed energy (2.11) is mathematically equivalent to the original free energy (2.5) at the continuous level. This reformulation provides the foundation for constructing linear, unconditionally energy-stable numerical schemes for the Cahn–Hilliard equation.

Remark 2.1. *The SAV reformulation replaces the nonlinear potential contribution with a quadratic auxiliary variable, which allows the construction of linear schemes that preserve the energy dissipation law unconditionally. Compared with fully implicit or convex-splitting discretizations, the resulting schemes avoid solving nonlinear systems while retaining stability. Moreover, in contrast to stabilized explicit methods, the SAV framework does not require restrictive assumptions on the potential function. These features make SAV-based methods particularly attractive for designing efficient, robust, and higher-order time discretizations of gradient flow models.*

Building upon this reformulation, several standard time discretization strategies have been developed in the SAV framework. In particular, second-order schemes based on the CN method and the backward differentiation formula of order two (BDF2) are widely used in practice. We present them here as reference schemes, which will serve as a basis for the predictor–corrector variants introduced in the next section.

Scheme 3.1: Standard SAV-CN scheme

Given $\phi^{n-1}, w^{n-1}, q^{n-1}$ and ϕ^n, w^n, q^n, we can obtain $\phi^{n+1}, w^{n+1}, q^{n+1}$ via

$$\begin{cases} \dfrac{\phi^{n+1} - \phi^n}{\delta t} = \Delta w^{n+\frac{1}{2}}, \\ w^{n+\frac{1}{2}} = -\Delta\phi^{n+\frac{1}{2}} + \gamma\phi^{n+\frac{1}{2}} + g(\bar{\phi}^{n+\frac{1}{2}})q^{n+\frac{1}{2}}, \\ \dfrac{q^{n+1} - q^n}{\delta t} = \dfrac{1}{2}\displaystyle\int_\Omega g(\bar{\phi}^{n+\frac{1}{2}})\frac{\phi^{n+1} - \phi^n}{\delta t}dx. \end{cases} \tag{2.12}$$

Scheme 3.2: Standard SAV-BDF2 scheme

Given $\phi^{n-1}, w^{n-1}, q^{n-1}$ and ϕ^n, w^n, q^n, we can obtain $\phi^{n+1}, w^{n+1}, q^{n+1}$ via

$$\begin{cases} \dfrac{3\phi^{n+1} - 4\phi^n + \phi^{n-1}}{2\delta t} = \Delta w^{n+1}, \\ w^{n+1} = -\Delta\phi^{n+1} + \gamma\phi^{n+1} + g(\bar{\phi}^{n+1})q^{n+1}, \\ \dfrac{3q^{n+1} - 4q^n + q^{n-1}}{2\delta t} = \dfrac{1}{2}\displaystyle\int_\Omega g(\bar{\phi}^{n+1})\dfrac{3\phi^{n+1} - 4\phi^n + \phi^{n-1}}{2\delta t}dx. \end{cases} \tag{2.13}$$

In both schemes above, the nonlinear term $g(\phi)$ is evaluated at extrapolated values of ϕ in order to maintain linearity of the discrete system. In particular, in the SAV–CN scheme, the quantity $\bar{\phi}^{n+\frac{1}{2}}$ denotes the second-order extrapolation

$$\bar{\phi}^{n+\frac{1}{2}} = \tfrac{3}{2}\phi^n - \tfrac{1}{2}\phi^{n-1}, \tag{2.14}$$

while in the SAV–BDF2 scheme, the extrapolated value is given by

$$\bar{\phi}^{n+1} = 2\phi^n - \phi^{n-1}. \tag{2.15}$$

This extrapolation strategy is standard in SAV schemes, as it ensures second-order temporal accuracy while keeping the scheme fully linear at each time step. However, the use of extrapolation may introduce additional errors, especially for large time steps. This observation serves as the motivation for the predictor–corrector modifications to be introduced in the next section.

3. The predictor-corrector SAV scheme

The standard SAV–CN and SAV–BDF2 schemes introduced in the previous section achieve second-order accuracy while preserving unconditional energy stability. However, both schemes rely on extrapolation of the nonlinear term, e.g., $\bar{\phi}^{n+\frac{1}{2}}$ or $\bar{\phi}^{n+1}$, in order to maintain linearity. Although this extrapolation strategy is widely used, it may introduce additional errors, especially for relatively large time steps. To reduce these errors and improve the accuracy of SAV schemes, we propose a predictor–corrector modification.

The basic idea is as follows: In the prediction step, an approximation ϕ_*^{n+1} is generated by an efficient extrapolation or iterative strategy. In the correction step, this predicted value is then incorporated into the computation of the nonlinear term $g(\phi)$, thereby replacing the standard extrapolated values $\bar{\phi}^{n+\frac{1}{2}}$ or $\bar{\phi}^{n+1}$. This procedure corrects the extrapolation error and improves the accuracy of the scheme without sacrificing linearity or stability.

Scheme 3.3 (Predictor–corrector SAV–CN)

Given $\phi^{n-1}, w^{n-1}, q^{n-1}$ and ϕ^n, w^n, q^n, the solution $(\phi^{n+1}, w^{n+1}, q^{n+1})$ is obtained in two steps:

Step 1 (Predictor): A predicted value ϕ_*^{n+1} is obtained by a fixed-point iteration. Starting from the initial guess $\phi_0^{n+1} = 2\phi^n - \phi^{n-1}$, we compute ϕ_{i+1}^{n+1} successively for $i = 0, 1, \cdots, M-1$:

$$\begin{cases} \dfrac{\phi_{i+1}^{n+1} - \phi^n}{\delta t} = \Delta w^{n+\frac{1}{2}}, \\ w^{n+\frac{1}{2}} = -\Delta\phi_{i+1}^{n+\frac{1}{2}} + \gamma\phi_{i+1}^{n+\frac{1}{2}} + g(\dfrac{\phi_i^{n+1} + \phi^n}{2})q_{i+1}^{n+\frac{1}{2}}, \\ \dfrac{q_{i+1}^{n+1} - q^n}{\delta t} = \dfrac{1}{2}\displaystyle\int_\Omega g(\dfrac{\phi_i^{n+1} + \phi^n}{2})\dfrac{\phi_{i+1}^{n+1} - \phi^n}{\delta t}dx, \end{cases} \tag{3.1}$$

until convergence, i.e., $\|\phi_{i+1}^{n+1} - \phi_i^{n+1}\| \leq \varepsilon_0$, where ϵ_0 is a prescribed tolerance, or the maximum iteration number M is reached. Finally, we set

$$\phi_*^{n+1} = \phi_M^{n+1}. \tag{3.2}$$

Step 2 (Corrector): In the correction step, the predicted value ϕ_*^{n+1} is used in place of the standard extrapolated values when evaluating the nonlinear term. We then compute ϕ^{n+1} by solving

$$\begin{cases} \dfrac{\phi^{n+1} - \phi^n}{\delta t} = \Delta w^{n+\frac{1}{2}}, \\ w^{n+\frac{1}{2}} = -\Delta\phi^{n+\frac{1}{2}} + \gamma\phi^{n+\frac{1}{2}} + g(\dfrac{\phi_*^{n+1} + \phi^n}{2})q^{n+\frac{1}{2}}, \\ \dfrac{q^{n+1} - q^n}{\delta t} = \dfrac{1}{2}\displaystyle\int_\Omega g(\dfrac{\phi_*^{n+1} + \phi^n}{2})\dfrac{\phi^{n+1} - \phi^n}{\delta t}dx. \end{cases} \tag{3.3}$$

After presenting Scheme 3.3, we next investigate its energy stability. One of the main advantages of the SAV framework is that it enables the construction of schemes which preserve the energy dissipation law at the discrete level. The following lemma shows that the corrector step of Scheme 3.3 inherits this important property.

Lemma 3.1. *The corrector step of the predictor–corrector SAV–CN Scheme 3.3 is unconditionally energy stable. In particular, the following discrete energy dissipation law holds:*

$$E(\phi^{n+1}, q^{n+1}) + \delta t\|\nabla(-\Delta\phi^{n+\frac{1}{2}} + \gamma\phi^{n+\frac{1}{2}} + g(\frac{\phi_*^{n+1} + \phi^n}{2})q^{n+\frac{1}{2}})\|^2 = E(\phi^n, q^n), \tag{3.4}$$

where

$$E(\phi^{n+1}, q^{n+1}) = \frac{1}{2}\|\nabla\phi^{n+1}\|^2 + |q^{n+1}|^2.$$

Moreover, for the prediction step, it holds that

$$E(\phi_{i+1}^{n+1}, q_{i+1}^{n+1}) \leq E(\phi^n, q^n). \tag{3.5}$$

Proof. Taking the L^2- inner product of the first equation in (3.3) with $2\delta t w^{n+\frac{1}{2}}$, we obtain

$$2(\phi^{n+1} - \phi^n, w^{n+\frac{1}{2}}) = -2\delta t\|\nabla w^{n+\frac{1}{2}}\|^2.$$

Taking the L^2- inner product of the second equation in (3.3) with $2(\phi^{n+1}-\phi^n)$, we obtain

$$2(w^{n+\frac{1}{2}},\phi^{n+1}-\phi^n)=\|\nabla\phi^{n+1}\|^2-\|\nabla\phi^n\|^2+\gamma(\|\phi^{n+1}\|^2-\|\phi^n\|^2)$$
$$+(q^{n+1}+q^n)(g(\frac{\phi_*^{n+1}+\phi^n}{2}),\phi^{n+1}-\phi^n).$$

Multiplying by $2\delta t(q^{n+1}+q^n)$ on both sides of the third equation in (3.3), we have

$$2|q^{n+1}|^2-2|q^n|^2=(q^{n+1}+q^n)\int_\Omega g(\frac{\phi_*^{n+1}+\phi^n}{2})(\phi^{n+1}-\phi^n)dx.$$

Combining the above equation together, we arrive at

$$\|\nabla\phi^{n+1}\|^2-\|\nabla\phi^n\|^2+2|q^{n+1}|^2-2|q^n|^2=-2\delta t\|\nabla w^{n+\frac{1}{2}}\|^2.$$

Thus we derive (3.4). Finally, by applying the same arguments to the predictor step (3.1), we obtain the inequality (3.5). This completes the proof. □

Lemma 3.2. *The predictor–corrector SAV–CN Scheme 3.3 is uniquely solvable.*

Proof. From the third equation of (3.3), we get

$$q^{n+1}=q^n+\frac{1}{2}(g(\frac{\phi_*^{n+1}+\phi^n}{2}),\phi^{n+1})-\frac{1}{2}(g(\frac{\phi_*^{n+1}+\phi^n}{2}),\phi^n). \tag{3.6}$$

Substituting this expression into the first two equations of (3.3), the corrector step can be equivalently rewritten in the form

$$\begin{cases}\phi^{n+1}-\frac{1}{2}\delta t\Delta w^{n+1}=Q_1,\\ P(\phi^{n+1})-w^{n+1}=Q_2,\end{cases} \tag{3.7}$$

where

$$\begin{cases}Q_1=\phi^n+\frac{1}{2}\delta t\Delta w^n,\\ Q_2=\Delta\phi^n+g(\frac{\phi_*^{n+1}+\phi^n}{2})\Big(\frac{1}{2}(g(\frac{\phi_*^{n+1}+\phi^n}{2}),\phi^n)-2q^n\Big)+w^n,\\ P(\phi)=-\Delta\phi+\frac{1}{2}g(\frac{\phi_*^{n+1}+\phi^n}{2})(g(\frac{\phi_*^{n+1}+\phi^n}{2}),\phi).\end{cases} \tag{3.8}$$

Thus, the pair (ϕ^{n+1},w^{n+1}) can be solved directly from (3.7). After ϕ^{n+1} is obtained, the value of q^{n+1} follows automatically from (3.6). Moreover, when a test function φ is applied to $P(\phi)$, we have

$$(P(\phi),\varphi)=(\nabla\phi,\nabla\varphi)+\frac{1}{2}(g(\frac{\phi_*^{n+1}+\phi^n}{2}),\varphi)(g(\frac{\phi_*^{n+1}+\phi^n}{2}),\phi)=(\varphi,P(\phi)). \tag{3.9}$$

Then, the linear operator $P(\phi)$ is self-adjoint.

Furthermore, if $\int_\Omega\phi dx=0$, we have

$$(P(\phi),\phi)=\|\nabla\phi\|^2+\frac{1}{2}(g(\frac{\phi_*^{n+1}+\phi^n}{2}),\phi)^2\geq C_\Omega\|\phi\|_1^2. \tag{3.10}$$

Thus the operator $P(\phi)$ is positive definite.

Then, taking the L^2- inner product of the first equation in (3.3) with 1, we derive the conservation of mass:

$$\int_\Omega \phi^{n+1}dx = \int_\Omega \phi^n dx = \cdots = \int_\Omega \phi^0 dx. \tag{3.11}$$

Set $v_\phi = \frac{1}{|\Omega|}\int_\Omega \phi^0 dx$, $v_w = \frac{1}{|\Omega|}\int_\Omega w^{n+1}dx$, and define the zero-mean variables

$$\widehat{\phi}^{n+1} = \phi^{n+1} - v_\phi, \ \ \widehat{w}^{n+1} = w^{n+1} - v_w. \tag{3.12}$$

With these notations, system (3.7) can be equivalently written in the weak form

$$\begin{cases}(\phi,\mu) + \dfrac{1}{2}\delta t(\nabla w, \nabla \mu) = (Q_3, \mu),\\ (P(\phi),\psi) - (w,\psi) = (Q_4,\psi),\end{cases} \tag{3.13}$$

for $\mu, \psi \in H^1(\Omega)$.

We denote this system as

$$(\mathcal{L}X, Y) = (\mathcal{B}, Y), \tag{3.14}$$

where $X = (w,\phi)$, $Y = (\mu,\psi)$, $\mathcal{B} = (Q_3, Q_4)^T$, and $X, Y \in H^1_{per}(\Omega)$.

It is straightforward to verify the boundedness of the bilinear form:

$$(\mathcal{L}X, Y) \le C(\|\phi\|_1 + \|w\|_1)(\|\psi\|_1 + \|\mu\|). \tag{3.15}$$

Moreover, we have the coercivity estimate

$$(\mathcal{L}X, X) = \frac{\delta t}{2}\|\nabla w\|^2 + (P(\phi),\phi) \ge C(\|w\|_1^2 + \|\phi\|_1^2). \tag{3.16}$$

Therefore, by the Lax–Milgram theorem, system (3.14) admits a unique solution $(w,\phi) \in H_{per}(\Omega) \times H_{per}(\Omega)$. Consequently, linear system (3.7) has a unique solution $w^{n+1}, \phi^{n+1} \in H^1(\Omega)$. □

4. Error estimates

In this section, we derive error bounds for the proposed predictor–corrector SAV–CN scheme. Our goal is to show that the method achieves second-order convergence in time under suitable regularity assumptions on the exact solution. To this end, we first introduce the error functions by comparing the exact solution $(\phi(t_n), q(t_n))$ with the corresponding numerical approximations (ϕ^n, q^n). In addition, several local truncation errors will be defined in order to measure the consistency of the scheme. These auxiliary quantities will play a key role in the subsequent analysis.

We denote

$$\begin{cases} e^{n+1}_{i+1,\phi} = \phi(t_{n+1}) - \phi^{n+1}_{i+1}, e^n_\phi = \phi(t_n) - \phi^n,\\ e^{n+1}_{i+1,q} = q(t_{n+1}) - q^{n+1}_{i+1}, e^n_q = q(t_n) - q^n.\end{cases} \tag{4.1}$$

Furthermore, we define the local truncation errors as

$$\begin{aligned}
R_1^n &:= \frac{\phi(t_{n+1}) - \phi(t_n)}{\delta t} - \phi_t(t_{n+\frac{1}{2}}), R_2^n := \frac{\phi(t_{n+1}) + \phi(t_n)}{2} - \phi(t_{n+\frac{1}{2}}),\\
R_3^n &:= \frac{q(t_{n+1}) - q(t_n)}{\delta t} - q_t(t_{n+\frac{1}{2}}), R_4^n := \frac{q(t_{n+1}) + q(t_n)}{2} - q(t_{n+\frac{1}{2}}),\\
R_5^n &:= \frac{w(t_{n+1}) + w(t_n)}{2} - w(t_{n+\frac{1}{2}}).
\end{aligned}$$

By Taylor expansion, these quantities satisfy the bounds

$$\|R_1^n\| \le C\delta t^2,\ \ \|R_2^n\| \le C\delta t^2,\ \ \|R_3^n\| \le C\delta t^2,\ \ \|R_4^n\| \le C\delta t^2,\ \ \|R_5^n\| \le C\delta t^2. \tag{4.2}$$

These estimates show that the local truncation errors are of order $O(\delta t^2)$, which is consistent with the expected second-order accuracy of the scheme. In the following, we prepare several auxiliary lemmas which will be crucial in establishing the stability and boundedness properties needed for the error analysis. The first two lemmas provide uniform bounds for the nonlinear functions involved in the SAV formulation.

Lemma 4.1. *Suppose that (a) $F \in C^2(-\infty, +\infty)$; (b) $\int_\Omega (F(y) - \frac{\gamma}{2}y^2)dy > -A$; (c) there exists a positive constant C_1 such that*

$$\max_{n\le k} \{\|\phi(t_n)\|_{L^\infty}, \|\phi^n\|_{L^\infty}, \|\phi_i^{n+1}\|_{L^\infty}\} \le C_1. \tag{4.3}$$

Then the following bounds hold:

$$\max_{n\le k} \left\{\|E_2(\chi^n)\|_{L^\infty}, \|f_1(\chi^n)\|_{L^\infty}, \|f_1'(\chi^n)\|_{L^\infty}, \|\sqrt{E_2(\phi) + B}\|_{L^\infty}\right\} \le C_2, \tag{4.4}$$

where $\chi^n = \varepsilon_1\phi(t_n) + \varepsilon_2\phi^n + \varepsilon_3\phi_i^{n+1}, \varepsilon_1, \varepsilon_2, \varepsilon_3 \in [0, 1]$, $E_2(y) = \int_\Omega F(y) - \frac{\gamma}{2}y^2 dy$, $f_1(y) = f(y) - \gamma y$. Furthermore,

$$\|g(\phi(t_n)) - g(\phi^n)\| \le C_3\|\phi(t_n) - \phi^n\|, \tag{4.5}$$

where C_3 depends only on C_1, C_2, A, and B.

Proof. From assumption (c), χ^n is uniformly bounded, and together with assumption (a), we obtain (4.4). Inequality (4.5) then follows by applying the mean value theorem:

$$\begin{aligned}
|g(\phi(t_n)) - g(\phi^n)| &= \Big|\frac{f_1(\phi(t_n))}{\sqrt{E_2(\phi(t_n)) + B}} - \frac{f_1(\phi^n)}{\sqrt{E_2(\phi^n) + B}}\Big|\\
&= \Big|\frac{f_1(\phi(t_n))\sqrt{E_2(\phi^n) + B} - f_1(\phi^n)\sqrt{E_2(\phi(t_n)) + B}}{\sqrt{E_2(\phi(t_n)) + B}\sqrt{E_2(\phi^n) + B}}\Big|\\
&\le \Big|\frac{f_1(\phi(t_n))\Big(\sqrt{E_2(\phi^n) + B} - \sqrt{E_2(\phi(t_n)) + B}\Big)}{\sqrt{E_2(\phi(t_n)) + B}\sqrt{E_2(\phi^n) + B}}\Big|\\
&\quad + \Big|\frac{\sqrt{E_2(\phi(t_n)) + B}\Big(f_1(\phi(t_n)) - f_1(\phi^n)\Big)}{\sqrt{E_2(\phi(t_n)) + B}\sqrt{E_2(\phi^n) + B}}\Big| \le C|\phi(t_n) - \phi^n|.
\end{aligned}$$

This concludes the proof. □

Lemma 4.2. *Under the assumptions (a) $F(y) \in C^2(-\infty, +\infty)$; (b) $E_2(y) > -A$, $\forall y \in \mathbb{R}$; and (c) if there exists a positive constant C_4 such that*

$$\max_{n\le k} \{\|\phi(t_n)\|_{L^\infty}, \|\phi^n\|_{L^\infty}, \|\phi_i^{n+1}\|_{L^\infty}\} \le C_4,$$

then the following estimates hold:

$$\max_{n\le k} \left\{\|E_2(\chi^n)\|_{L^\infty}, \|f_1(\chi^n)\|_{L^\infty}, \|f_1'(\chi^n)\|_{L^\infty}, \|f_1''(\chi^n)\|_{L^\infty}, \|\sqrt{E_2(\phi)+B}\|_{L^\infty}\right\} \le C_5,$$
$$\|\nabla g(\phi(t_n)) - \nabla g(\phi^n)\| \le C_6\|\phi(t_n) - \phi^n\|_1.$$

Lemma 4.3. *Denote $\{u^n\}_{n=0}^{N-1}$ as a sequence of functions on Ω, and then we have*

$$\|u^{n+1}\| \le \sum_{m=0}^{n} \|u^{m+1} + u^m\| + \|u^0\|.$$

In order to analyze the error estimates, we impose the following regularity assumptions on the exact solution of the Cahn-Hilliard equation:

$$\begin{cases} \phi \in L^\infty(0,T;H^2(\Omega)) \cap L^\infty(0,T;W^{1,\infty}(\Omega)), \\ \phi_t \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^\infty\Omega), \\ q \in L^\infty(0,T;W^{1,\infty}(\Omega)), q_{tt}, \phi_{tt} \in L^2(0,T;L^2(\Omega)), w \in L^\infty(0,T;H^2(\Omega)). \end{cases} \tag{4.6}$$

We also define ν such that

$$\nu = \max_{0\le t\le T} \|\phi(t)\|_{L^\infty} + 1. \tag{4.7}$$

Lemma 4.4. *Assume $E_2(y) > -A$ for all $y \in (-\infty, +\infty)$, $F(y) \in C^3(-\infty, +\infty)$, system (2.9) has a unique solution ϕ, which satisfies the assumptions (4.6), and there exists $\tau > 0$, for $\delta t < \tau$, where the numerical solution of (3.1)–(3.3) is bounded as follows:*

$$\|\phi^n\|_{L^\infty} \le \nu, \|\phi_{i+1}^{n+1}\|_{L^\infty} \le \nu, \quad n = 0, 1, \cdots, K = T/\delta t, 0 \le i \le M-1. \tag{4.8}$$

Proof. We proceed by mathematical induction. For $n = 0, i = 0$, subtracting (3.1) from (2.9), we obtain

$$\left(\frac{e_{1,\phi}^1 - e_\phi^0}{\delta t}, \varphi\right) = -(\nabla e_w^{\frac{1}{2}}, \nabla\varphi) + (R_1^0 - \Delta R_5^0, \varphi), \tag{4.9}$$

$$\begin{aligned}(e_w^{\frac{1}{2}}, \theta) = {}& (R_5^0 + \Delta R_2^0 - \gamma R_2^0, \theta) + \gamma\left(\frac{e_{1,\phi}^1 + e_\phi^0}{2}, \theta\right) + \left(\nabla\frac{e_{1,\phi}^1 + e_\phi^0}{2}, \nabla\theta\right) \\ & + (g(\phi_{\frac{1}{2}})q(t_{\frac{1}{2}}) - g(\phi^0)q_1^{\frac{1}{2}}, \theta),\end{aligned} \tag{4.10}$$

$$\frac{e_{1,q}^1 - e_q^0}{\delta t} = R_3^0 + \frac{1}{2}\int_\Omega g(t_{\frac{1}{2}})\phi_t(t_{\frac{1}{2}}) - g(\phi^0)\frac{\phi_1^1 - \phi^0}{\delta t} dx. \tag{4.11}$$

By choosing $\varphi = \delta t e_{1,\phi}^1$ and $\varphi = 2\delta t e_w^{\frac{1}{2}}$, respectively, in (4.9), we obtain

$$\begin{aligned} \|e_{1,\phi}^1\|^2 &= -\delta t(\nabla e_w^{\frac{1}{2}}, \nabla e_{1,\phi}^1) + \delta t(R_1^0 - \Delta R_5^0, e_{1,\phi}^1), \\ 2(e_{1,\phi}^1, e_w^{\frac{1}{2}}) &= -2\delta t\|\nabla e_w^{\frac{1}{2}}\|^2 + 2\delta t(R_1^0 - \Delta R_5^0, e_w^{\frac{1}{2}}). \end{aligned} \tag{4.12}$$

By choosing $\theta = 2e^1_{1,\phi}$ and $\theta = 2\delta t e_w^{\frac{1}{2}}$, respectively, in (4.10), we obtain

$$\begin{aligned}
2(e_w^{\frac{1}{2}}, e^1_{1,\phi}) &= 2(R_5^0 + \Delta R_2^n - \gamma R_2^0, e^1_{1,\phi}) + \gamma\|e^1_{1,\phi}\|^2 + \|\nabla e^1_{1,\phi}\|^2 \\
&\quad + 2(g(\phi(t_{\frac{1}{2}}))q(t_{\frac{1}{2}}) - g(\phi^0)q_1^{\frac{1}{2}}, e^1_{1,\phi}), \\
2\delta t\|e_w^{\frac{1}{2}}\|^2 &= 2\delta t(R_5^0 + \Delta R_2^n - \gamma R_2^0, e_w^{\frac{1}{2}}) + \gamma\delta t(e^1_{1,\phi}, e_w^{\frac{1}{2}}) + \delta t(\nabla e^1_{1,\phi}, \nabla e_w^{\frac{1}{2}}) \\
&\quad + 2\delta t(g(\phi(t_{\frac{1}{2}}))q(t_{\frac{1}{2}}) - g(\phi^0)q_1^{\frac{1}{2}}, e_w^{\frac{1}{2}}).
\end{aligned}$$

Multiplying both sides of (4.11) by $2e^1_{1,q}$, we obtain

$$2|e^1_{1,q}|^2 = 2\delta t R_3^0 e^1_{1,q} + e^1_{1,q}\delta t\int_\Omega g(\phi_{\frac{1}{2}})\phi_t(t_{\frac{1}{2}}) - g(\phi^0)\frac{\phi_1^1 - \phi^0}{\delta t}dx.$$

By adding the above equations together, we arrive at

$$\begin{aligned}
&\|e^1_{1,\phi}\|_1^2 + \gamma\|e^1_{1,\phi}\|^2 + 2|e^1_{1,q}|^2 + 2\delta t\|e_w^{\frac{1}{2}}\|_1^2 \\
&\quad = \delta t(R_1^0 - \Delta R_5^0, e^1_{1,\phi}) + 2\delta t(R_1^0 - R_5^0, e_w^{\frac{1}{2}}) + 2\delta t R_3^0 e^1_{1,q} \\
&\qquad - 2(\Delta R_2^0 + R_5 - \gamma R_2^0, e^1_{1,\phi}) + 2\delta t(R_5^0 + \Delta R_2^0 - \gamma R_2^0, e_w^{\frac{1}{2}}) \\
&\qquad - 2(g(\phi(t_{\frac{1}{2}}))q(t_{\frac{1}{2}}) - g(\phi^0)q_1^{\frac{1}{2}}, e^1_{1,\phi}) + 2\delta t(g(\phi(t_{\frac{1}{2}}))q(t_{\frac{1}{2}}) - g(\phi^0)q_1^{\frac{1}{2}}, e_w^{\frac{1}{2}}) \\
&\qquad + e^1_{1,q}\delta t\int_\Omega g(\phi(t_{\frac{1}{2}}))\phi_t(t_{\frac{1}{2}}) - g(\phi^0)\frac{\phi_1^1 - \phi^0}{\delta t}dx.
\end{aligned} \tag{4.13}$$

Applying the Cauchy-Schwarz and Young inequalities, we obtain

$$\begin{aligned}
&\delta t(R_1^0 - \Delta R_5^0, e^1_{1,\phi}) \le C\delta t^6 + \gamma\|e^1_{1,\phi}\|^2, \\
&2\delta t(R_1^0 - R_5^0, e_w^{\frac{1}{2}}) \le C\delta t^5 + \frac{\delta t}{3}\|e_w^{\frac{1}{2}}\|^2, \\
&2\delta t R_3^0 e^1_{1,q} \le C\delta t^5 + \delta t|e^1_{1,q}|^2, \\
&2\delta t(R_5^0 + \Delta R_2^0 - \gamma R_5^0, e_w^{\frac{1}{2}}) \le C\delta t^5 + \frac{\delta t}{3}\|e_w^{\frac{1}{2}}\|^2, \\
&-2(\Delta R_2^0 + R_5^0 - \gamma R_2^0, e^1_{1,\phi}) = -2\delta t(\Delta R_2^0 + R_5^0 - \gamma R_5^0, R_1^0 - \Delta R_5^0 + \Delta e_w^{\frac{1}{2}}) \\
&\qquad\qquad \le C\delta t^5 + \frac{\delta t}{2}\|\nabla e_w^{\frac{1}{2}}\|^2.
\end{aligned}$$

For the nonlinear terms, we derive

$$\begin{aligned}
&-2(g(\phi(t_{\frac{1}{2}}))q(t_{\frac{1}{2}}) - g(\phi^0)q_1^{\frac{1}{2}}, e^1_{1,\phi}) + e^1_{1,q}\delta t\int_\Omega g(\phi(t_{\frac{1}{2}}))\phi_t(t_{\frac{1}{2}}) - g(\phi^0)\frac{\phi_1^1 - \phi^0}{\delta t}dx \\
&\quad = -2\delta t(q(t_{\frac{1}{2}})(g(\phi(t_{\frac{1}{2}})) - g(\phi^0)) + g(\phi^0)(-R_4^0 + \frac{e^1_{1,q}}{2}), \frac{e^1_{1,\phi} - e^0_\phi}{\delta t}) \\
&\qquad + e^1_{1,q}\delta t\int_\Omega \phi_t(t_{\frac{1}{2}})(g(\phi(t_{\frac{1}{2}})) - g(\phi^0)) + g(\phi^0)(-R_1^0 + \frac{e^1_{1,\phi} - e^0_\phi}{\delta t})dx
\end{aligned}$$

$$
\begin{aligned}
&= -2\delta t(q(t_{\frac{1}{2}})(g(\phi(t_{\frac{1}{2}})) - g(\phi^0)) - g(\phi^0)R_4^0, R_1^0 - \Delta R_5^0 + \Delta e_w^{\frac{1}{2}}) \\
&\quad + e_{1,q}^1 \delta t \int_\Omega \phi_t(t_{\frac{1}{2}})(g(\phi(t_{\frac{1}{2}})) - g(\phi^0)) - g(\phi^0)R_1^0 dx \\
&\leq C\delta t(\delta t^2 + |e_{1,q}^1|^2) + \frac{\delta t}{2}\|\nabla e_w^{\frac{1}{2}}\|^2,
\end{aligned}
$$

and

$$
2\delta t(g(\phi(t_{\frac{1}{2}}))q(t_{\frac{1}{2}}) - g(\phi^0)q_1^{\frac{1}{2}}, e_w^{\frac{1}{2}}) \leq C\delta t(\delta t^2 + |e_{1,q}^1|^2) + \frac{1}{3}\delta t\|e_w^{\frac{1}{2}}\|^2.
$$

Summing the above equations, we obtain

$$
\|e_{1,\phi}^1\|_1^2 + 2|e_{1,q}^1| + \delta t\|e_w^{\frac{1}{2}}\|_1^2 \leq C\delta t^3 + C_7\delta t|e_{1,q}^1|^2.
$$

If $C_7\delta t \leq 1$, we have

$$
\|e_{1,\phi}^1\|_1^2 + |e_{1,q}^1| + \delta t\|e_w^{\frac{1}{2}}\|_1^2 \leq C\delta t^3.
$$

From the second equation of (3.1) with $n = 0, i = 0$, it follows that

$$
\begin{aligned}
\|\Delta\phi_1^{\frac{1}{2}}\|^2 &\leq 3\Big(\|w^{\frac{1}{2}}\|^2 + C_\Omega\|g(\phi^0)\|_{L^\infty}^2|q_1^{\frac{1}{2}}|^2 + \gamma\|\phi_1^{\frac{1}{2}}\|^2\Big) \\
&\leq C\Big(\|w(t_1)\|^2 + \|w(t_0)\|^2 + \|e_w^1\|^2 + |e_{1,q}^1|^2 + |q(t_1)|^2 \\
&\qquad + |q(t_0)|^2 + \|\phi(t_1)\|^2 + \|\phi(t_0)\|^2 + \|e_{1,\phi}^1\|^2\Big) \leq C.
\end{aligned}
$$

Therefore, we get

$$
\|\Delta e_{1,\phi}^1\|^2 \leq 2\|\Delta\phi_1^1\|^2 + 2\|\Delta\phi(t_1)\|^2 \leq C.
$$

Thus, we derive

$$
\begin{aligned}
\|\phi_1^1\|_{L^\infty} &\leq \|e_{1,\phi}^1\|_{L^\infty} + \|\phi(t_1)\|_{L^\infty} \\
&\leq C\|e_{1,\phi}^1\|_1^{\frac{1}{2}}\|e_{1,\phi}^1\|_2^{\frac{1}{2}} + \|\phi(t_1)\|_{L^\infty} \\
&\leq C_8\delta t^{\frac{3}{4}} + \|\phi(t_1)\|_{L^\infty}.
\end{aligned}
$$

If $C_8\delta t^{\frac{3}{4}} \leq 1$, we obtain

$$
\|\phi_1^1\|_{L^\infty} \leq 1 + \|\phi(t_1)\|_{L^\infty} \leq \nu. \tag{4.14}
$$

If $i = j$, we assume that $\|\phi_j^1\|_{L^\infty} \leq \nu$. We then show that the estimate $\|\phi_{j+1}^1\|_{L^\infty} \leq \nu$ still holds.

Subtracting (3.1) from (2.9) at $t_{1/2}$, and applying the Cauchy–Schwarz and Young inequalities, together with Lemmas 4.1 and 4.2, we obtain

$$
\|e_{j+1,\phi}^1\|_1^2 + 2|e_{j+1,q}^1|^2 + \delta t\|e_w^{\frac{1}{2}}\|_1^2 \leq C\delta t^5 + C_9\delta t\|e_{j,\phi}^1\|_1^2 + C_{10}\delta t|e_{j+1,q}^1|^2. \tag{4.15}
$$

If $C_{10}\delta t \le 1$, the above inequality becomes

$$||e^1_{j+1,\phi}||^2_1 + |e^1_{j+1,q}|^2 + \delta t||e_w^{\frac{1}{2}}||^2_1 \le C\delta t^5 + C_9\delta t||e^1_{j,\phi}||^2_1. \tag{4.16}$$

It is easy to find that

$$||e^1_{j+1,\phi}||^2_1 + |e^1_{j+1,q}|^2 + \delta t||e_w^{\frac{1}{2}}||^2_1 \le a^j||e^1_{1,\phi}||^2_1 + C\delta t^5\Big(\frac{1-a^j}{1-a}\Big), \tag{4.17}$$

where $a = C_9\delta t$. If $a < 1$,

$$||e^1_{j+1,\phi}||^2_1 + |e^1_{j+1,q}|^2 + \delta t||e_w^{\frac{1}{2}}||^2_1 \le C\delta t^3. \tag{4.18}$$

From (3.1), we find

$$\begin{aligned}
||\Delta\phi^{\frac{1}{2}}_{j+1}||^2 &\le 3\Big(||w^{\frac{1}{2}}||^2 + C_\Omega||g(\phi^{\frac{1}{2}}_j)||^2_{L^\infty}|q^{\frac{1}{2}}_{j+1}|^2 + \gamma||\phi^{\frac{1}{2}}_{j+1}||^2\Big)\\
&\le C\Big(||w(t_1)||^2 + ||w(t_0)||^2 + ||e^1_w||^2 + |e^1_{j+1,q}|^2 + |q(t_1)|^2\\
&\quad + |q(t_0)|^2 + ||\phi(t_1)||^2 + ||\phi(t_0)||^2 + ||e^1_{j+1,\phi}||^2\Big) \le C.
\end{aligned}$$

Then

$$||\Delta e^1_{j+1,\phi}||^2 \le 2||\Delta\phi^1_{j+1}||^2 + 2||\Delta\phi(t_1)||^2 \le C.$$

Thus, we have

$$\begin{aligned}
||\phi^1_{j+1}||_{L^\infty} &\le ||e^1_{j+1,\phi}||_{L^\infty} + ||\phi(t_1)||_{L^\infty}\\
&\le C||e^1_{j+1}||^{\frac{1}{2}}_1||e^1_{j+1}||^{\frac{1}{2}}_2 + ||\phi(t_1)||_{L^\infty}\\
&\le C_{11}\delta t^{\frac{3}{4}} + ||\phi(t_1)||_{L^\infty}.
\end{aligned}$$

If $C_{11}\delta t^{\frac{3}{4}} \le 1$, we derive

$$||\phi^1_{j+1}||_{L^\infty} \le \nu. \tag{4.19}$$

Suppose that $||\phi^k||_{L^\infty} \le \nu$, $||\phi^{k+1}_{i+1}||_{L^\infty} \le \nu, \forall 0 \le i \le M-1$. we now show that the following estimates $||\phi^{k+1}||_{L^\infty} \le \nu, ||\phi^{k+2}_{i+1}||_{L^\infty} \le \nu$ still hold.

First, for the case $i = 0$, by subtracting (3.1) from (2.9) at $t_{n+\frac{1}{2}}$, we derive

$$\begin{aligned}
\frac{e^{n+1}_{1,\phi} - e^n_\phi}{\delta t} &= \Delta e_w^{n+\frac{1}{2}} + R^n_1 - \Delta R^n_5,\\
e_w^{n+\frac{1}{2}} &= R^n_5 + \Delta R^n_2 - \gamma R^n_2 + \gamma\frac{e^1_{1,\phi} + e^0_\phi}{2} - \Delta\frac{e^{n+1}_{1,\phi} + e^n_\phi}{2}\\
&\quad + g(\phi(t_{n+\frac{1}{2}}))q(t_{n+\frac{1}{2}}) - g(\frac{\phi^{n+1}_0 + \phi^n}{2})q_1^{n+\frac{1}{2}},\\
\frac{e^{n+1}_{1,q} - e^n_q}{\delta t} &= R^n_3 + \frac{1}{2}\int_\Omega g(\phi(t_{n+\frac{1}{2}}))\phi_t(t_{n+\frac{1}{2}}) - g(\frac{\phi^{n+1}_0 + \phi^n}{2})\frac{\phi^{n+1}_1 - \phi^n}{\delta t}dx.
\end{aligned}$$

Following the same arguments as in the previous proof, we obtain

$$\begin{aligned}&\|e_{1,\phi}^{n+1}\|_1^2 - \|e_\phi^n\|_1^2 - \gamma\|e_\phi^n\|^2 + 2|e_{1,q}^{n+1}|^2 - 2|e_q^n|^2 + \delta t\|e_w^{n+\frac{1}{2}}\|_1^2\\ &\leq C_{12}\delta t|e_{1,q}^{n+1}|^2 + C\delta t(\delta t^4 + \|e_\phi^n\|_1^2 + \|e_\phi^{n-1}\|_1^2 + |e_q^n|^2).\end{aligned}$$

If $C_{12}\delta t \leq 1$, we get

$$\|e_{1,\phi}^{n+1}\|_1^2 \leq \|e_\phi^n\|_1^2 + \gamma\|e_\phi^n\|^2 + 2|e_q^n|^2 + C\delta t(\delta t^4 + \|e_\phi^n\|_1^2 + \|e_\phi^{n-1}\|_1^2 + |e_q^n|^2). \tag{4.20}$$

Similarly, we derive

$$\begin{aligned}&\|e_{i+1,\phi}^{n+1}\|_1^2 - \|e_\phi^n\|_1^2 - \gamma\|e_\phi^n\|^2 + 2|e_{i+1,q}^{n+1}|^2 - 2|e_q^n|^2 + \delta t\|e_w^{n+\frac{1}{2}}\|_1^2\\ &\leq C_{13}\delta t|e_{i+1,q}^{n+1}|^2 + C\delta t(\delta t^4 + \|e_{i,\phi}^{n+1}\|_1^2 + \|e_\phi^n\|_1^2 + |e_q^n|^2).\end{aligned}$$

If $C_{13}\delta t \leq 1$, we arrive at

$$\|e_{i+1,\phi}^{n+1}\|_1^2 \leq C_{14}\delta t\|e_{i,\phi}^{n+1}\|^2 + C(\delta t^5 + \|e_\phi^n\|_1^2 + |e_q^n|^2).$$

It is easy to see that

$$\|e_{i+1,\phi}^{n+1}\|_1^2 \leq b^i\|e_{1,\phi}^{n+1}\|^2 + C(\delta t^5 + \|e_\phi^n\|_1^2 + |e_q^n|^2)\frac{1-b^i}{1-b},$$

with $b = C_{14}\delta t$.

If $b < 1$, we get

$$\|e_{i+1,\phi}^{n+1}\|_1^2 \leq \|e_{1,\phi}^{n+1}\|^2 + C(\delta t^5 + \|e_\phi^n\|_1^2 + |e_q^n|^2). \tag{4.21}$$

Similarly, by subtracting (3.3) from (2.9) and combining the result with (4.20)–(4.21), we obtain

$$\begin{aligned}&\|e_\phi^{n+1}\|_1^2 - \|e_\phi^n\|_1^2 + \gamma(\|e_\phi^{n+1}\|^2 - \|e_\phi^n\|^2) + 2(|e_q^{n+1}|^2 - |e_q^n|^2) + \delta t\|e_w^{n+\frac{1}{2}}\|_1^2\\ &\leq C\delta t^5 + C\delta t(\|e_{*,\phi}^{n+1}\|_1^2 + \|e_\phi^n\|_1^2 + |e_q^{n+1}|^2)\\ &\leq C\delta t^5 + C\delta t(|e_q^{n+1}|^2 + \|e_\phi^n\|_1^2 + \|e_\phi^{n-1}\|_1^2 + |e_q^n|^2).\end{aligned}$$

Summing the above equation over $n = 0, \cdots, k$, we obtain

$$\|e_\phi^{k+1}\|_1^2 + |e_q^{k+1}|^2 + \delta t\sum_{n=1}^{k}\|e_w^{n+\frac{1}{2}}\|^2 \leq C\sum_{n=0}^{k}(\|e_\phi^n\|_1^2 + |e_q^{n+1}|^2).$$

By applying the discrete Gronwall inequality, we obtain

$$\|e_\phi^{k+1}\|_1^2 + |e_q^{k+1}|^2 + \delta t\sum_{n=1}^{k}\|e_w^{n+\frac{1}{2}}\|_1^2 \leq C\delta t^4.$$

From the second equation of (2.9) and (3.3), we derive

$$\|\Delta e_\phi^{n+\frac{1}{2}}\|^2 \leq C\Big(\|\Delta R_2^n\|^2 + \|R_2^n\|^2 + \|R_5^n\|^2 + \|e_\phi^{n+\frac{1}{2}}\|^2 + \|e_w^{n+\frac{1}{2}}\|^2$$

$$+ \|g(\phi(t_{n+\frac{1}{2}}))q(t_{n+\frac{1}{2}}) - g(\frac{\phi_*^{n+1} + \phi^n}{2})q^{n+\frac{1}{2}}\|^2\Big)$$
$$\leq C\delta t^4.$$

From Lemma 4.3, we derive

$$\|\Delta e_\phi^{k+1}\| \leq \sum_{n=0}^{k} \|\Delta e_\phi^{n+1} + \Delta e_\phi^n\| + \|\Delta e_\phi^0\| \leq C\delta t. \tag{4.22}$$

On the other hand, we also have

$$\begin{aligned}
\|\phi^{k+1}\|_{L^\infty} &\leq \|e_\phi^{k+1}\|_{L^\infty} + \|\phi(t_{k+1})\|_{L^\infty} \\
&\leq C\|e_\phi^{k+1}\|_1^{\frac{1}{2}}\|e_\phi^{k+1}\|_2^{\frac{1}{2}} + \|\phi(t_{k+1})\|_{L^\infty} \\
&\leq C_{15}\delta t^{\frac{3}{2}} + \|\phi(t_{k+1})\|_{L^\infty}.
\end{aligned}$$

If $C_{15}\delta t^{\frac{3}{2}} \leq 1$, we get

$$\|\phi^{k+1}\|_{L^\infty} \leq 1 + \|\phi(t_{k+1})\|_{L^\infty} \leq \nu.$$

Next, we prove $\|\phi_{i+1}^{k+2}\|_{L^\infty} \leq \nu, 0 \leq i \leq M-1$. By subtracting (3.1) from (2.9) at $t_{k+\frac{3}{2}}, i = 0$, we derive

$$\|e_{1,\phi}^{k+2}\|_1^2 + |e_{1,q}^{k+2}|^2 + \delta t\|e_w^{k+\frac{3}{2}}\|_1^2 \leq C\delta t^4. \tag{4.23}$$

By subtracting (3.5) from (2.11) at $t_{k+\frac{3}{2}}, i = 0$, we derive

$$\begin{aligned}
\|\Delta e_{1,\phi}^{k+\frac{3}{2}}\|^2 &\leq C\Big(\|\Delta R_2^{k+1}\|^2 + \|R_2^{k+1}\|^2 + \|R_5^{k+1}\|^2 + \|e_{1,\phi}^{k+\frac{3}{2}}\|^2 + \|e_w^{k+\frac{3}{2}}\|^2 \\
&\quad + \|g(\phi(t_{k+\frac{3}{2}}))q(t_{k+\frac{3}{2}}) - g(\frac{\phi_0^{k+2} + \phi^{k+1}}{2})q_1^{k+\frac{3}{2}}\|^2\Big) \\
&\leq C\delta t^{\frac{9}{4}}.
\end{aligned}$$

Combining the above inequality with (4.22), we get

$$\|\Delta e_{1,\phi}^{k+2}\| \leq C\delta t.$$

Thus, we derive

$$\begin{aligned}
\|\phi_1^{k+2}\|_{L^\infty} &\leq \|e_{1,\phi}^{k+2}\|_{L^\infty} + \|\phi(t_{k+2})\|_{L^\infty} \\
&\leq C\|e_{1,\phi}^{k+2}\|_1^{\frac{1}{2}}\|e_{1,\phi}^{k+2}\|_2^{\frac{1}{2}} + \|\phi(t_{k+2})\|_{L^\infty} \\
&\leq C_{16}\delta t^{\frac{3}{2}} + \|\phi(t_{k+2})\|_{L^\infty}.
\end{aligned}$$

If $C_{16}\delta t^{\frac{3}{2}} \leq 1$, we have

$$\|\phi_1^{k+2}\|_{L^\infty} \leq 1 + \|\phi(t_{k+2})\|_{L^\infty} \leq \nu. \tag{4.24}$$

Suppose $\|\phi_i^{k+2}\|_{L^\infty} \le \nu$, and then we will prove $\|\phi_{i+1}^{k+2}\|_{L^\infty} \le \nu$ still holds. Subtracting (3.1) from (2.9) at $t_{k+\frac{3}{2}}$, we derive

$$\begin{aligned}\|e_{i+1,\phi}^{k+2}\|_1^2 - \|e_\phi^{k+1}\|_1^2 - \gamma\|e_\phi^{k+1}\|^2 + 2\|e_{i+1,q}^{k+2}\|^2 - 2|e_q^{k+1}|^2 + \delta t\|e_w^{k+\frac{3}{2}}\|_1^2 \\ \le C_{17}\delta t\|e_{i+1,q}^{k+2}\|^2 + C\delta t(\delta t^4 + \|e_{i,\phi}^{k+2}\|_1^2 + \|e_\phi^{k+1}\|_1^2 + |e_q^{k+1}|^2).\end{aligned}$$

If $C_{17}\delta t \le 1$, we get

$$\|e_{i+1,\phi}^{k+2}\|_1^2 + |e_{i+1,q}^{k+2}|^2 + \delta t\|e_w^{k+\frac{3}{2}}\|_1^2 \le C_{18}\delta t\|e_{i,\phi}^{k+2}\|^2 + C\delta t^4.$$

It is easy to see that

$$\|e_{i+1,\phi}^{k+2}\|_1^2 + |e_{i+1,q}^{k+2}|^2 + \delta t\|e_w^{k+\frac{3}{2}}\|_1^2 \le d^m\|e_{1,\phi}^{k+2}\|^2 + C\delta t^4\frac{1-d^i}{1-d},$$

with $d = C_{18}\delta t$. If $d < 1$, we get

$$\|e_{i+1,\phi}^{k+2}\|_1^2 + |e_{i+1,q}^{k+2}|^2 + \delta t\|e_w^{k+\frac{3}{2}}\|_1^2 \le C\delta t^4.$$

Note that

$$\begin{aligned}\|\Delta e_{i+1,\phi}^{k+2} + \Delta e_\phi^{k+1}\|^2 \le{}& C\Big(\|\Delta R_2^{k+1}\|^2 + \|R_2^{k+1}\|^2 + \|R_5^{k+1}\|^2 + \|e_{i+1,\phi}^{k+\frac{3}{2}}\|^2 + \|e_w^{k+\frac{3}{2}}\|^2 \\ &+ 2\|g(\phi(t_{k+\frac{3}{2}}))q(t_{k+\frac{3}{2}}) - g(\frac{\phi_i^{k+2}+\phi^{k+1}}{2})q_{i+1}^{k+\frac{3}{2}}\|^2\Big) \\ \le{}& C\delta t^{\frac{9}{4}}.\end{aligned}$$

Hence, we derive

$$\|\Delta e_{i+1,\phi}^{k+2}\| \le C\delta t.$$

Furthermore, we obtain

$$\begin{aligned}\|\phi_{i+1,\phi}^{k+2}\|_{L^\infty} &\le \|e_{i+1,\phi}^{k+2}\|_{L^\infty} + \|\phi(t_{k+2})\|_{L^\infty} \\ &\le C_\Omega\|e_{i+1,\phi}^{k+2}\|_1^{\frac{1}{2}}\|e_{i+1,\phi}^{k+2}\|_2^{\frac{1}{2}} + \|\phi(t_{k+2})\|_{L^\infty} \\ &\le C_{19}\delta t^{\frac{3}{2}} + \|\phi(t_{k+2})\|_{L^\infty}.\end{aligned}$$

If $C_{19}\delta t^{\frac{3}{2}} \le 1$, we derive

$$\|\phi_{i+1}^{k+2}\|_{L^\infty} \le 1 + \|\phi(t_{k+2})\|_{L^\infty} \le \nu.$$

This concludes the proof. □

Remark 4.1. *Note that the proposed prediction strategy can be extended to other predictor schemes. In particular, for a given preconditioned algorithm, if the associated preconditioner is uniformly bounded in the L^∞-norm, then the corrected solution can also be shown to remain uniformly bounded in the L^∞-norm.*

Theorem 4.1. *Under the assumption of Lemma 4.4, we obtain*

$$\|\phi(t_{k+1}) - \phi^{k+1}\| + |q(t_{k+1}) - q^{k+1}| + \delta t \sum_{n=0}^{k} \|w(t_{n+\frac{1}{2}}) - w^{n+\frac{1}{2}}\| \leq C\delta t^2.$$

Proof. Subtracting (3.1) from (2.9), we obtain

$$\begin{aligned}
&\frac{e_{i+1,\phi}^{n+1} - e_\phi^n}{\delta t} = \Delta e_w^{n+\frac{1}{2}} + R_1^n - \Delta R_5^n,\\
&e_w^{n+\frac{1}{2}} = R_5^n + \Delta R_2^n - \gamma R_2^n + \gamma \frac{e_{i+1,\phi}^1 + e_\phi^0}{2} - \Delta \frac{e_{i+1,\phi}^{n+1} + e_\phi^n}{2}\\
&\qquad + g(\phi(t_{n+\frac{1}{2}}))q(t_{n+\frac{1}{2}}) - g(\frac{\phi_i^{n+1} + \phi^n}{2})q_1^{n+\frac{1}{2}},\\
&\frac{e_{i+1,q}^{n+1} - e_q^n}{\delta t} = R_3^n + \frac{1}{2}\int_\Omega g(\phi(t_{n+\frac{1}{2}}))\phi_t(t_{n+\frac{1}{2}}) - g(\frac{\phi_i^{n+1} + \phi^n}{2})\frac{\phi_{i+1}^{n+1} - \phi^n}{\delta t} dx.
\end{aligned}$$

From Lemma 4.4, we have

$$\begin{aligned}
&\|e_{i+1,\phi}^{n+1}\|_1^2 - \|e_\phi^n\|_1^2 - \gamma\|e_\phi^n\|^2 + 2|e_{i+1,q}^{n+1}|^2 - 2|e_q^n|^2 + \delta t\|e_w^{n+\frac{1}{2}}\|_1^2\\
&\leq C_{20}\delta t|e_{i+1,q}^{n+1}|^2 + C\delta t(\delta t^4 + \|e_\phi^n\|_1^2 + \|e_\phi^{n-1}\|_1^2 + |e_q^n|^2).
\end{aligned}$$

If $C_{20}\delta t \leq 1$, we derive

$$\|e_{i+1,\phi}^{n+1}\|_1^2 \leq \|e_\phi^n\|_1^2 + \gamma\|e_\phi^n\|^2 + 2|e_q^n|^2 + C\delta t(\delta t^4 + \|e_\phi^n\|_1^2 + \|e_\phi^{n-1}\|_1^2 + |e_q^n|^2). \tag{4.25}$$

Similarly, we derive

$$\begin{aligned}
&\|e_{i+1,\phi}^{n+1}\|_1^2 - \|e_\phi^n\|_1^2 - \gamma\|e_\phi^n\|^2 + 2|e_{i+1,q}^{n+1}|^2 - 2|e_q^n|^2 + \delta t\|e_w^{n+\frac{1}{2}}\|_1^2\\
&\leq C_{21}\delta t|e_{i+1,q}^{n+1}|^2 + C\delta t(\delta t^4 + \|e_{i,\phi}^{n+1}\|_1^2 + \|e_\phi^n\|_1^2 + |e_q^n|^2).
\end{aligned}$$

If $C_{21}\delta t \leq 1$, we obtain

$$\|e_{i+1,\phi}^{n+1}\|_1^2 \leq C_{22}\delta t\|e_{i,\phi}^{n+1}\|^2 + C(\delta t^5 + \|e_\phi^n\|_1^2 + |e_q^n|^2).$$

Note that

$$\|e_{i+1,\phi}^{n+1}\|_1^2 \leq h^i\|e_{1,\phi}^{n+1}\|^2 + C(\delta t^5 + \|e_\phi^n\|_1^2 + |e_q^n|^2)\frac{1-h^i}{1-h}$$

with $h = C_{22}\delta t$.

If $h < 1$, we get

$$\|e_{i+1,\phi}^{n+1}\|_1^2 \leq \|e_{1,\phi}^{n+1}\|^2 + C(\delta t^5 + \|e_\phi^n\|_1^2 + |e_q^n|^2). \tag{4.26}$$

Subtracting (3.3) from (2.9) and combining with (4.25) and (4.26), we obtain

$$\|e_\phi^{n+1}\|_1^2 - \|e_\phi^n\|_1^2 + \gamma(\|e_\phi^{n+1}\|^2 - \|e_\phi^n\|^2) + 2(|e_q^{n+1}|^2 - |e_q^n|^2) + \delta t\|e_w^{n+\frac{1}{2}}\|_1^2$$

$$\leq C\delta t^5 + C\delta t(|e_q^{n+1}|^2 + \|e_\phi^n\|_1^2 + \|e_\phi^{n-1}\|_1^2 + |e_q^n|^2).$$

Summing the above equation over $n = 0, \ldots, k$, we obtain

$$\|e_\phi^{k+1}\|_1^2 + |e_q^{k+1}|^2 + \delta t \sum_{n=1}^{k} \|e_w^{n+\frac{1}{2}}\|^2 \leq C \sum_{n=0}^{k} (\|e_\phi^n\|_1^2 + |e_q^{n+1}|^2).$$

By applying Gronwall's lemma, we obtain

$$\|e_\phi^{k+1}\|_1^2 + |e_q^{k+1}|^2 + \delta t \sum_{n=1}^{k} \|e_w^{n+\frac{1}{2}}\|_1^2 \leq C\delta t^4,$$

which completes the proof. □

5. Numerical examples

In this section, we present several numerical experiments to verify the theoretical results and to demonstrate the efficiency and accuracy of the proposed predictor–corrector SAV-CN scheme.

5.1. Accuracy tests

We first consider the temporal convergence. The Cahn–Hilliard equation is given by

$$\begin{cases} \phi_t = \lambda \Delta w, \\ w = -\varepsilon^2 \Delta \phi + f(\phi), \end{cases} \tag{5.1}$$

subject to a periodic boundary condition.

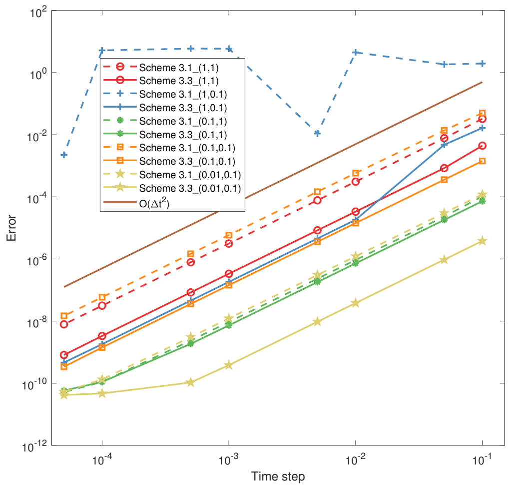
We perform the test in the square domain $\Omega = [0, 2\pi]^2$. The error between the numerical solution and the exact solution at $t = 1$ is computed for various time step sizes. Since the exact solution is not available in closed form, we use the numerical solution obtained with a sufficiently small time step $\delta t = 10^{-5}$ as the reference solution. A Fourier spectral discretization with 256^2 modes is employed, so that the spatial error is negligible compared with the temporal error.

The bulk energy functional, the initial profile of ϕ, and the parameters are chosen as follows:

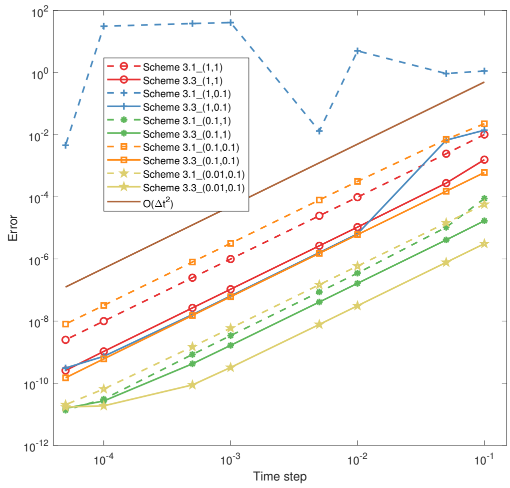
$$\begin{cases} F(\phi) = \frac{1}{4}(\phi^2 - 1)^2, \\ \phi(\mathbf{x}, t = 0) = \sin(x)\sin(y), \\ (\gamma, B, \varepsilon_0, M) = (1, 100, 10^{-12}, 500). \end{cases} \tag{5.2}$$

Figure 1 shows the L^2- and $L^\infty-$errors of ϕ, the errors of the auxiliary variable q, and the time evolution of the discrete energy (2.11) obtained by Scheme 2.3 (SAV–CN) and the proposed predictor–corrector scheme, with different parameter pairs $(\lambda, \varepsilon) = (1, 1)$, $(1, 0.1)$, $(0.1, 1)$, $(0.1, 0.1)$, and $(0.01, 0.1)$.

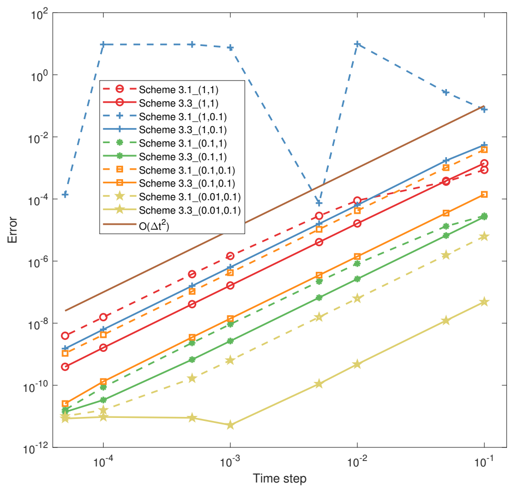
We observe that, except for the case $(\lambda, \varepsilon) = (1, 0.1)$ with the SAV–CN scheme, the numerical errors in all other cases exhibit second-order temporal accuracy. Moreover, the proposed predictor–corrector scheme consistently yields smaller errors than the SAV–CN scheme, particularly when relatively large time steps are used. The evolution of the discrete energy defined in (2.11) for different time step sizes is shown in Figure 1(d). Furthermore, the discrete energy decreases monotonically in all cases, in agreement with the theoretical prediction of energy stability.



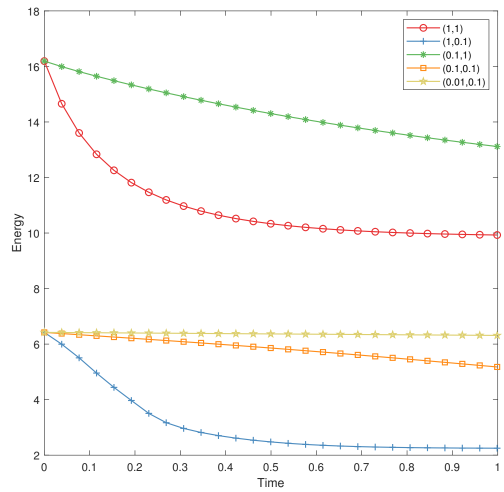
(a) L^2- error of ϕ.



(b) $L^\infty-$ error of ϕ.



(c) Error of q.



(d) Time evolution of energy.

Figure 1. Comparison of the numerical errors in ϕ and the auxiliary variable q, together with the discrete energy evolution, obtained by scheme SAV-CN (Scheme 3.1) and our proposed predictor–corrector SAV–CN scheme (Scheme 3.3) for different parameter pairs (λ, ε).

5.2. *Coarsening dynamics*

In this section, we study the long-time coarsening dynamics, which describes the process whereby small domains formed after phase separation gradually merge into larger ones, leading to a coarser structure over time. For the Cahn–Hilliard equation, this phenomenon is typically accompanied by domain coalescence and an asymptotic energy decay rate of order $O(t^{-1/3})$.

We carry out numerical simulations on the domain $[0, 2\pi]^2$ using 256^2 Fourier modes and a time step size $\delta t = 0.01$. The bulk energy functional, initial condition, and parameters are chosen as

$$\begin{cases} F(\phi) = \frac{1}{4}(\phi^2 - 1)^2, \\ \phi(\mathbf{x}, t = 0) = 10^{-3}\text{rand}(-1, 1), \\ (\gamma, B, \lambda, \varepsilon, \varepsilon_0, M) = (1, 100, 0.1, 0.05, 10^{-8}, 500). \end{cases} \tag{5.3}$$

The results are shown in Figure 2. We observe that the two phases evolve and separate over time and eventually reach a steady state. The numerical solution captures the phase separation accurately, demonstrating the effectiveness of the proposed predictor–corrector scheme. The evolution of the free energy computed by the fully discrete scheme is displayed in Figure 3. The results indicate that the energy decay rate is close to $O(t^{-1/3})$, which is consistent with the classical coarsening law reported in [19].

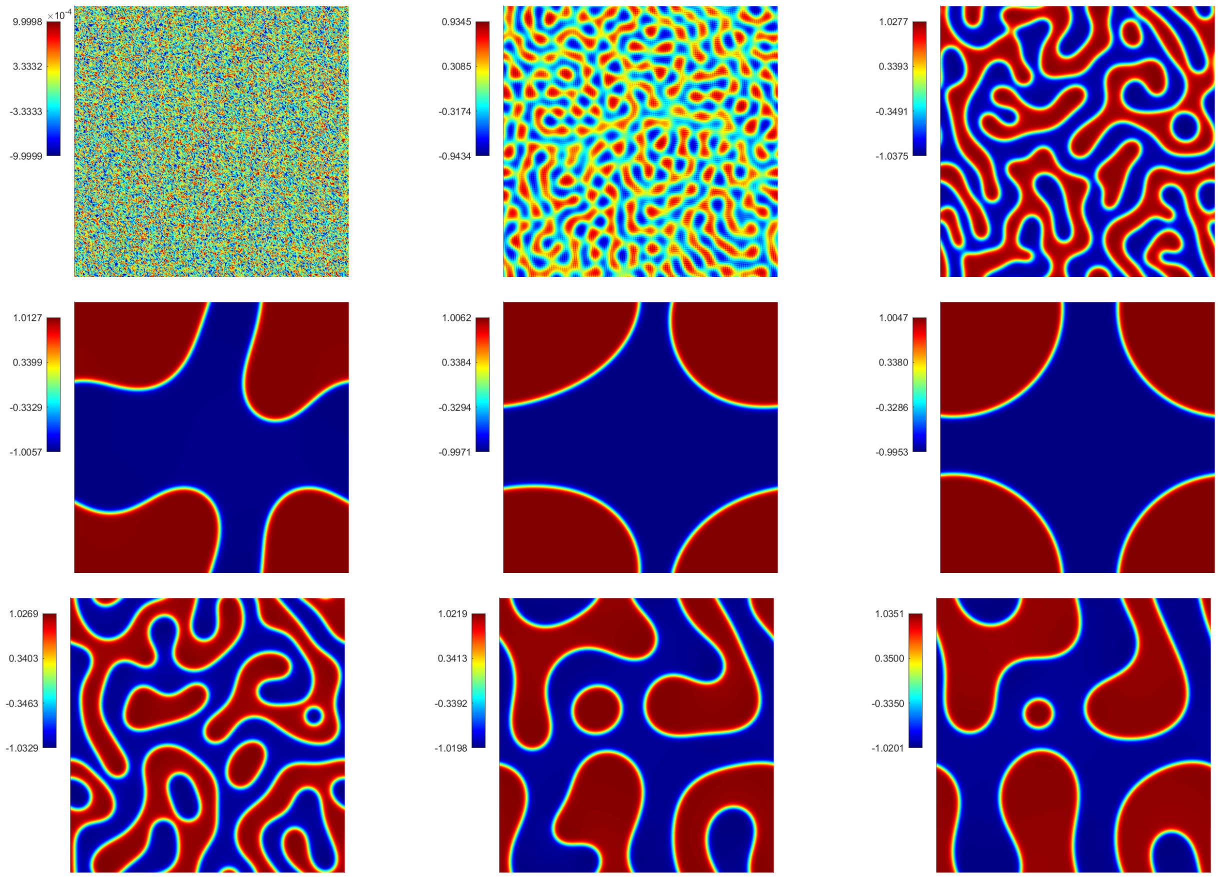


Figure 2. Snapshots of the phase field ϕ during the coarsening dynamics at times $t = 0, 1, 5, 10, 50, 100, 500, 1000, 5000$.

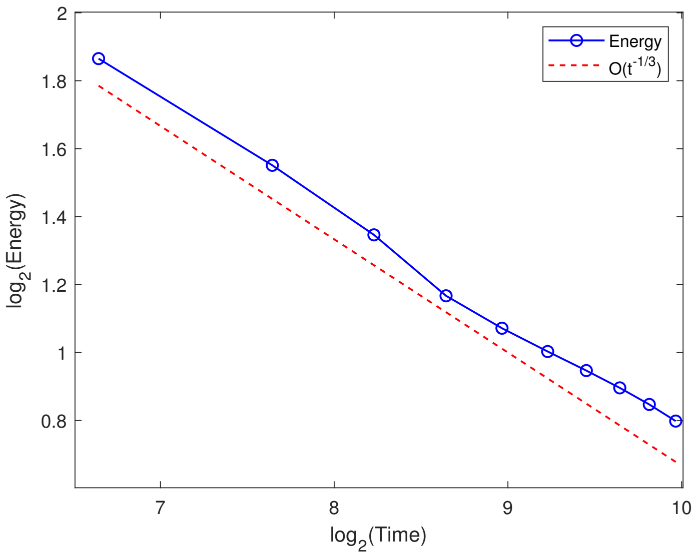


Figure 3. Time evolution of the total free energy.

6. Concluding remarks

In this paper, we have developed a predictor–corrector scheme for solving the Cahn–Hilliard equation by combining the SAV technique with predictor–corrector strategies. The resulting schemes are second-order accurate in time, linear, and unconditionally energy stable. We have provided a rigorous analysis of the energy stability of the proposed scheme and derived rigorous error estimates in the temporal direction. A series of numerical experiments have been conducted to validate the theoretical results. The simulations demonstrate that the predictor–corrector scheme exhibits superior accuracy compared with the standard SAV methods, particularly for large time step sizes. Moreover, the numerical results confirm the second-order temporal accuracy of the scheme and further verify the theoretical analysis.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The work is supported by the National Natural Science Foundation of China (No. 11901237) and the Science and Technology Foundation of Guizhou Province (No. MS[2025]230).

Conflict of interest

The authors declare there is no conflicts of interest.

References

1. J. W. Cahn, J. E. Hilliard, Free energy of a nonuniform system. I. interfacial free energy, *J. Chem. Phys.*, **28** (1958), 258–267. https://doi.org/10.1063/1.1744102
2. J. Lowengrub, L. Truskinovsky, Quasi-incompressible Cahn-Hilliard fluids and topological transitions, *Proc. R. Soc. A*, **454** (1998), 2617–2654. https://doi.org/10.1098/rspa.1998.0273
3. L. Chen, J. Shen, Applications of semi-implicit Fourier-spectral method to phase field equations, *Comput. Phys. Commun.*, **108** (1998), 147–158. https://doi.org/10.1016/S0010-4655(97)00115-X
4. L. Chen, Phase-field models for microstructure evolution, *Ann. Rev. Mater. Res.*, **32** (2002), 113–140. https://doi.org/10.1146/annurev.matsci.32.112001.132041
5. J. Eggleston, G. Mcfadden, P. Voorhees, A phase field model for highly anisotropic interfacial energy, *Physica D*, **150** (2001), 91–103. https://doi.org/10.1016/S0167-2789(00)00222-0
6. Q. Du, C. Liu, X. Wan, Simulating the deformation of vesicle membranes under elastic bending energy in three dimensions, *J. Comput. Phys.*, **212** (2006), 757–777. https://doi.org/10.1016/j.jcp.2005.07.020
7. M. Forest, Q. Wang, X. Yang, Lcp droplet dispersions: a two-phase,diffuse-interface kinetic theory and global droplet defect predictions, *Soft Matter*, **8** (2012), 9642–9660. https://doi.org/10.1039/c2sm25512j
8. C. Elliott, A. Stuart, The global dynamics of discrete semilinear parabolic equations, *SIAM J. Numer. Anal.*, **30** (1993), 1622–1663. https://doi.org/10.1137/0730084
9. D. Eyre, Unconditionally gradient stable time marching the Cahn-Hilliard equation, *MRS Online Proc. Lib.*, **529** (1998), 39–46. https://doi.org/10.1557/PROC-529-39
10. J. Zhu, L. Q. Chen, J. Shen, V. Tikare, Coarsening kinetics from a variable-mobility Cahn-Hilliard equation: application of a semi-implicit Fourier spectral method, *Phys. Rev. E*, **60** (1999), 3564–3572. https://doi.org/10.1103/PhysRevE.60.3564
11. J. Shen, X. Yang, Numerical approximations of Allen-Cahn and Cahn-Hilliard equations, *Discret. Contin. Dyn. Syst.*, **28** (2010), 1669–1691. https://doi.org/10.3934/dcds.2010.28.1669
12. R. Chen, G. Ji, X. Yang, H. Zhang, Decoupled energy stable schemes for phase-field vesicle membrane model, *J. Comput. Phys.*, **302** (2015), 509–523. https://doi.org/10.1016/j.jcp.2015.09.025
13. X. Yang, J. Zhao, Q. Wang, J. Shen, Numerical approximations for a three components cahn-hilliard phase-field model based on the invariant energy quadratization method, *Math. Models Methods Appl. Sci.*, **27** (2017), 1993–2030. https://doi.org/10.1142/S0218202517500373
14. X. Yang, Linear, first and second order, and unconditionally energy stable numerical schemes for the phase field model of homopolymer blends, *J. Comput. Phys.*, **327** (2016), 294–316. https://doi.org/10.1016/j.jcp.2016.09.029
15. X. Yang, X. He, A fully-discrete decoupled finite element method for the conserved Allen–Cahn type phase-field model of three-phase fluid flow system, *Comput. Methods Appl. Mech. Eng.*, **389** (2022). https://doi.org/10.1016/j.cma.2021.114376
16. M. Jiang, Z. Zhang, J. Zha, Improving the accuracy and consistency of the scalar auxiliary variable (SAV) method with relaxation, *J. Comput. Phys.*, **456** (2002). https://doi.org/10.1016/j.jcp.2022.110954

17. Z. Liu, Y. Zhang, X. Li, A Novel Energy-Optimized Technique of SAV-Based (EOP-SAV) Approaches for Dissipative Systems, *J. Sci. Comput.*, **101** (2024). https://doi.org/10.1007/s10915-024-02677-0

18. J. Wu, X. Zhang, Y. Wu, Z. Tan, Consistency-enhanced E-SAV BDF2 time-marching method with relaxation for the hydrodynamically-coupled binary phase-field crystal model, *Commun. Nonlinear Sci. Numer. Simul.*, **145** (2025). https://doi.org/10.1016/j.cnsns.2025.108730

19. S. Dai, Q. Du, Computational studies of coarsening rates for the Cahn-Hilliard equation with phase-dependent diffusion mobility, *J. Comput. Phys.*, **310** (2016), 85–108. https://doi.org/10.1016/j.jcp.2016.01.018

AIMS Press