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ERA, 33(10): 6267–6297.
DOI: 10.3934/era.2025277
Received: 19 July 2025
Accepted: 24 September 2025
Revised: 14 September 2025
Published: 24 October 2025

Research article

Analysis of a diffusion epidemic SIR model with saturated treatment in a heterogeneous environment

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Abstract: The spread of infectious diseases is profoundly influenced by spatial heterogeneity and the availability of medical resources. While reaction-diffusion epidemic models have been extensively studied, there has been little research on SIR models that incorporate a spatially heterogeneous standard incidence rate and a saturated treatment function, which is crucial for modeling the limited capacity of health care systems. To address this, we proposed a novel diffusive SIR epidemic model in a heterogeneous environment. Methodologically, we defined the basic reproduction number R_0 and employed Lyapunov functions, the theory of monotone dynamical systems, and asymptotic analysis to investigate the dynamics. Our key results showed that if $R_0 < 1$, the disease-free equilibrium is globally asymptotically stable, implying epidemic extinction. If $R_0 > 1$, the disease becomes uniformly persistent, and an endemic equilibrium exists. Furthermore, we derived the asymptotic profiles of the endemic equilibrium as the diffusion rates of susceptible or infected populations approached zero. Numerical simulations not only validated our theoretical findings but also demonstrated that increasing medical resources or reducing spatial heterogeneity can effectively lower the infection peak and help control the disease. These results provide theoretical guidance for designing effective public health policies.

Keywords: diffusive epidemic system; basic reproduction number; spatial heterogeneity; the saturated treatment function

1. Introduction

The transmission dynamics of infectious diseases have long been a core research focus at the intersection of public health and applied mathematics. Their evolutionary processes are not only influenced by the transmission characteristics of pathogens themselves but also closely linked to

spatial environmental heterogeneity and medical resource supply capacity. With the acceleration of globalization and urbanization, cross-regional population mobility has become frequent, and the contradiction between the spatial spread patterns of infectious diseases and the limited nature of medical resources has become increasingly prominent. Traditional epidemic models based on the assumption of a homogeneous environment can no longer accurately depict the evolutionary characteristics of real-world epidemics. Therefore, constructing a diffusion-type epidemic model that integrates spatial heterogeneity, a saturated treatment mechanism, and population dynamic constraints has become a key requirement for analyzing complex epidemic patterns and supporting the formulation of prevention and control policies.

In the pioneering work of Kermack and McKendrick [1], the bilinear function βSI was applied represent the mass action infection mechanism in an SIR (susceptible-infected-recovered) epidemic system, where β is the constant and stands for the rate of disease transmission. Following this study, Jong et al. [2] proposed the standard incidence infection mechanism function $\beta\frac{SI}{S+I}$ to describe the transmission of epidemic diseases. Currently, the academic community has developed various forms such as nonlinear transmission functions, bilinear incidence rates, and standard incidence rates to adapt to the transmission characteristics of different infectious diseases.

In the course of epidemiological investigation, the epidemic system modeled by differential equations plays an important part in analyzing the infectious disease dynamics [3]. It is gradually known that the environmental factors considered in ODEs are homogeneous, and spatial heterogeneity (position, age, sex, etc) and individual movement are significant factors that affect the spread of disease [4–6]. Spatial heterogeneity can produce complex disease dynamics [7, 8]. Thus, more and more attention has been given to the epidemic models that have been conducted to study the impacts of diffusion and spatial heterogeneity on the transmission and control of diseases (see [5–10] and references therein). In epidemic systems, the transmission function is important in analyzing disease dynamics. Therefore, the transmission rate in the incidence function is not limited to the constant form. For example, Han et al. [8] proposed an SIRS reaction-diffusion model with a standard incidence rate that varies with location x in the spatial heterogeneity environment, which is as follows:

$$\begin{cases} \dfrac{\partial S}{\partial t} - d_S\Delta S = \Lambda(x) - \beta(x)\dfrac{SI}{S+I} - \mu(x)S + \gamma(x)R, & x \in \Omega, t > 0, \\ \dfrac{\partial I}{\partial t} - d_I\Delta I = \beta(x)\dfrac{SI}{S+I} - [\delta(x) + \mu(x) + \alpha(x)]I, & x \in \Omega, t > 0, \\ \dfrac{\partial R}{\partial t} - d_R\Delta R = \delta(x)I - [\mu(x) + \gamma(x)]R, & x \in \Omega, t > 0, \\ \dfrac{\partial S}{\partial \vec{\nu}} = \dfrac{\partial I}{\partial \vec{\nu}} = \dfrac{\partial R}{\partial \vec{\nu}}0, & x \in \partial\Omega, t > 0, \\ S(y,0) = S_0(y) \geq 0, I(y,0) = I_0(y) \geq, \not\equiv 0, & y \in Y, \end{cases} \tag{1.1}$$

where $S(x,t)$, $I(x,t)$, and $R(x,t)$ represent the densities of susceptible, infected, and recovered individuals, respectively, at location x and time t; the positive constants d_S, d_I, and d_R stand for the migration (movement) coefficients for the susceptible, infected, and recovered populations, respectively; $\Lambda, \beta, \mu, \gamma, \delta$, and α, are positive H$\ddot{o}$lder continuous functions on $\Omega \times [0, \infty)$. Λ is the recruitment rate of susceptible individuals corresponding to births and immigration; μ is the natural mortality rate; β and δ represent the rates of disease transmission and recovery, respectively; α

accounts for the disease-related death rate, and γ is the rate at which the recovered population return to the susceptible class and does not acquire immunity. The habitat $\Omega \in R^N (N \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, and the homogeneous Neumann boundary conditions mean that no population flux crosses the boundary $\partial\Omega$. With respect to (1.1), the researchers obtained the the uniform bounds of solutions and derive the threshold dynamics in terms of the basic reproduction number R_0. This study provides an important paradigm for the analysis of epidemic models under spatial heterogeneity, but it still does not cover the impact of medical intervention measures, especially under the constraints of limited medical resources. In [9, 10], the authers further verified that for the diffusion SIS and SIR models with spatially dependent standard incidence rates, the global stability of its steady-state solution can be rigorously proven through eigenvalue analysis and comparison principles. These findings collectively indicate that spatial heterogeneity is not simply an "environmental disturbance term", but one of the core variables that determine the evolutionary trajectory of the epidemic.

Since the treatment is an important factor in the control of the epidemic diseases [11], the reasonable portrayal of the "treatment function" has become an important direction for infectious disease models to be closer to reality. Many researchers studied diffusive epidemic models with different treatment functions in the spatial heterogeneity environment. Considering that the saturated treatment rate is more reasonable than the linear treatment function because the processing rate no longer increases as saturation is reached [12, 13]. This saturation effect is particularly prominent in large-scale epidemics such as influenza A and COVID–19. Therefore, incorporating a saturation treatment function into modeling is of greater practical significance. However, researchers have mostly focused on spatially uniform or simple diffusion scenarios, and research on the SIR model that simultaneously incorporates spatially heterogeneous transmission rates and saturation treatment mechanisms remains limited [14–22].

To our best knowledge, there is little literature on the dynamics of the diffusive SIR epidemic models considering the logistic growth of the susceptible, based on considering the saturated treatment function, standard incidence rate that varies with location y in the spatial heterogeneity environment. Therefore, we want to make a contribution in this aspect. The corresponding specific system is as follows:

$$\begin{cases} \dfrac{\partial S}{\partial t} - d_S \Delta S = \Lambda - \mu S - \beta(y)\dfrac{SI}{S+I}, & y \in Y, t > 0, \\ \dfrac{\partial I}{\partial t} - d_I \Delta I = \beta(y)\dfrac{SI}{S+I} - (\mu + d + \sigma)I - \dfrac{\gamma I}{1+\varepsilon I}, & y \in Y, t > 0, \\ \dfrac{\partial R}{\partial t} - d_R \Delta R = \sigma I + \dfrac{\gamma I}{1+\varepsilon I} - \mu R, & y \in Y, t > 0, \\ \dfrac{\partial S}{\partial \vec{n}}(y,t) = \dfrac{\partial I}{\partial \vec{n}}(y,t) = \dfrac{\partial R}{\partial \vec{n}}(y,t) = 0, & y \in \partial Y, t > 0, \\ S(y,0) = S_0(y) \geq 0, I(y,0) = I_0(y) \geq, \not\equiv 0, R(y,0) = R_0(y) \geq 0, & y \in Y, \end{cases} \tag{1.2}$$

among them, y, t, Y, ∂Y, $\vec{n}$, $S(y,t)$, $I(y,t)$, d_S, d_I, d_R, $\beta(y)$, and $\beta(y)\dfrac{SI}{S+I}$ have the same epidemiological interpretation as that of system (1.1); $R(y,t)$ represents the density of individuals that are recovered at location y and time t; μ, d, and σ stand for the natural death rate, the mortality rate

due to disease and the natural recovery rate of infected people, respectively; Λ is a constant recruitment of susceptible individuals; Linear function $\Lambda - \mu S$ represents the linear growth of susceptible individuals and has been used extensively in [23, 24]; $\frac{\gamma I}{1+\varepsilon I}$ denotes the saturated treatment; ε stands for the saturation constant used to measure the impact of delayed treatment in infected individuals; $\gamma \geq 0$ represents the largest medical resource that can be provided to each individual per unit of time; parameters μ, d, Λ, ε, and σ are positive; and S_0, I_0 and R_0 are Hölder continuous function on $\bar{Y}$. Because the first two equations of system (1.2) do not include the variable R, we leave the third equation out and put emphasis on the subsystem (1.3), which is

$$\begin{cases} \dfrac{\partial S}{\partial t} - d_S \Delta S = \Lambda - \mu S - \beta(y)\dfrac{SI}{S+I}, & y \in Y, t > 0, \\ \dfrac{\partial I}{\partial t} - d_I \Delta I = \beta(y)\dfrac{SI}{S+I} - \theta I - \dfrac{\gamma I}{1+\varepsilon I}, & y \in Y, t > 0, \\ S(y,0) = S_0(y) \geq 0, I(y,0) = I_0(y) \geq, \not\equiv 0, & y \in Y, \\ \dfrac{\partial S}{\partial \vec{n}}(y,t) = \dfrac{\partial I}{\partial \vec{n}}(y,t) = 0, & y \in \partial Y, t > 0, \end{cases} \tag{1.3}$$

and $\theta = \mu + d + \sigma$ can be on behalf of the total clearance rate of infected individuals.

The remaining parts are organized as follows: Section 2 is dedicated to discussing the existence and uniform boundedness of solutions. In Section 3, the conditions of determining whether the disease-free equilibrium is stable, and whether the endemic equilibrium exists and is consistent and persistent are obtained. Moreover, the asymptotic distribution of endemic equilibria are established. In Section 4, numerical simulation is carried out by Matlab to support theoretical results. This work ends with the discussions and conclusions.

2. Existence and uniformly boundedness of solutions

For the purpose of the following research, the following marks are made in this paper:

- $\beta^* \triangleq \max_{y\in\bar{Y}} \beta(y)$;
- Banach space $X \triangleq C(\bar{Y})$ and $X^2 \triangleq X \times X$, their norms can be separately equipped with $\|\varphi\|_X \triangleq \sup_{y\in Y} |\varphi(y)|$, and $\| (\varphi_1, \varphi_2)^T \|_{X^2} = \|\varphi_1\|_X + \|\varphi_2\|_X$, where T represents the transpose of a vector;
- Define A_1, A_2 be two linear operators on X and A be linear operator on X^2, and A, A_1, A_2 satisfy

$$A(\varphi_1, \varphi_2) = \begin{pmatrix} A_1\varphi_1 \\ A_2\varphi_2 \end{pmatrix} = \begin{pmatrix} d_S\varphi_1 \\ d_I\varphi_2 \end{pmatrix}, \quad D(A) = D(A_1) \times D(A_2),$$

$$D(A_i) = \left\{ \varphi \in X : \Delta\varphi \in X, \ \left.\frac{\partial\varphi}{\partial\vec{n}}\right|_{\partial Y} = 0 \right\} \ (i = 1, 2),$$

  then $\left\{e^{tA}\right\}_{t\geq 0}$ and $\left\{e^{tA_i}\right\}_{t\geq 0}$ are separately the strongly continuous semigroup by A and A_i generating. A denotes the infinitesimal generator of $\left\{e^{tA}\right\}_{t\geq 0}$ in X^2. A_i stand for the infinitesimal generators of $\left\{e^{tA_i}\right\}_{t\geq 0}$ in X;
- X_+ is the positive cone of X and $X_+^2 = X_+ \times X_+ \subset X^2$;

- For every $(\varphi_1, \varphi_2) \in X_+^2$, there is a nonlinear operator F on X^2 satisfying

$$F(\varphi_1, \varphi_2) = \begin{pmatrix} \Lambda - \mu\varphi_1 - \beta\dfrac{\varphi_1\varphi_2}{\varphi_1 + \varphi_2} \\ \beta\dfrac{\varphi_1\varphi_2}{\varphi_1 + \varphi_2} - \theta\varphi_2 - \dfrac{\gamma\varphi_2}{1 + \varepsilon\varphi_2} \end{pmatrix}.$$

Therefore, system (1.3) can be expressed by the following form

$$\frac{d}{dt}u(t) = Au(t) + F(u(t)), \quad u(t) = \begin{pmatrix} S(\cdot, t) \\ I(\cdot, t) \end{pmatrix}, \quad u(0) = \begin{pmatrix} S_0(\cdot) \\ I_0(\cdot) \end{pmatrix}, \tag{2.1}$$

On X_+^2, F is Lipschitz continuous [25]. Furthermore, from Theorem 3.3.3 in [26], the following claim can be obtained.

Lemma 2.1. *For every $(S_0, I_0)^T \in X_+^2$, there is a maximum interval $[0, T_0)$. On this interval, the only solution of model (2.1) exists and*

$$u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}F(u(s))ds.$$

According to the strong maximum principle of parabolic equation and Hopf lemma, the proof that the solution are positive can be obtained.

Theorem 2.2. *If (S, I) is a solution of system (1.3), then there is a positive constant C_1 depending on the initial value, such that (S, I) satisfies*

$$\|S(\cdot, t)\|_{L^\infty(Y)} + \|I(\cdot, t)\|_{L^\infty(Y)} \le C_1, \quad \forall t \ge 0. \tag{2.2}$$

In addition, there is a positive constant C_2 that does not depend on the initial value such that for one large time $T > 0$,

$$\|S(\cdot, t)\|_{L^\infty(Y)} + \|I(\cdot, t)\|_{L^\infty(Y)} \le C_2, \quad \forall t \ge T. \tag{2.3}$$

Proof.

$$U(t) = \int_Y [S(y, t) + I(y, t)]dy.$$

By system (1.3), there is

$$\begin{aligned} \frac{dU(t)}{dt} &= \int_Y \frac{\partial S}{\partial t}dy + \int_Y \frac{\partial I}{\partial t}dy \\ &= \int_Y (d_S\Delta S + \Lambda - \mu S)dy + \int_Y d_I\Delta I dy - \int_Y \left[\theta I + \frac{\gamma I}{1 + \varepsilon I}\right]dy \\ &= \int_Y (d_S\Delta S + d_I\Delta I)dy + \int_Y \Lambda dy - \int_Y (\mu S + \theta I)dy - \int_Y \frac{\gamma I}{1 + \varepsilon I}dy. \end{aligned}$$

By the divergence theorem, one can deduce

$$\int_Y (d_S\Delta S + d_I\Delta I)dy = d_S\int_{\partial Y}\frac{\partial S}{\partial \vec{n}}dy + d_I\int_{\partial Y}\frac{\partial I}{\partial \vec{n}}dy = 0,$$

thus we have

$$\begin{aligned}\frac{dU(t)}{dt} &= \int_Y \Lambda dy - \int_Y (\mu S + \theta I) dy - \int_Y \frac{\gamma I}{1+\varepsilon I} dy \\ &\leq \int_Y \Lambda dy - \int_Y (\mu S + \theta I) dy \\ &\leq \Lambda |Y| - \mu U(t).\end{aligned}$$

Therefore,

$$\frac{dU(t)}{dt} + \mu U(t) \leq \int_Y \Lambda dy = \Lambda |Y|,$$

by the Gronwall inequality, we can obtain

$$U(t) \leq U(0)e^{-\mu t} + \frac{|Y|\Lambda}{\mu}(1 - e^{-\mu t}). \tag{2.4}$$

Further, by Lemma 2.1 in [27] and $S > 0$, $I > 0$, Eq (2.2) can be established. In addition, from Eq (2.4), we have

$$\limsup_{t\to\infty} U(t) \leq \frac{|Y|\Lambda}{\mu},$$

and it is independent of the initial value. Applying Lemma 2.1 in [27] again, Eq (2.3) can be obtained.

3. The basic reproduction number

As $I = 0$, by system (1.3), we consider the following ellipse problem:

$$-d_S \Delta S = \Lambda - \mu S, \quad y \in Y; \quad \frac{\partial S}{\partial \vec{n}} = 0, \quad y \in \partial Y. \tag{3.1}$$

From [28], it has a unique positive solution $\tilde{S}$. Then, $(\tilde{S}, 0)$ is the solution of system (1.3), called disease-free equilibrium, denoted as E_0.

Next, the basic reproduction number is defined based on the method in document [8]. Linearizing system (1.3) at disease-free equilibrium E_0, we get:

$$\begin{cases} \dfrac{\partial S}{\partial t}(y,t) = d_S \Delta S(y,t) - \mu S(y,t) - \beta(y) I(y,t), & y \in Y,\ t > 0, \\ \dfrac{\partial I}{\partial t}(y,t) = d_I \Delta I(y,t) + (\beta(y) - \theta - \gamma) I, & y \in Y,\ t > 0, \\ \dfrac{\partial S}{\partial \vec{n}}(y,t) = \dfrac{\partial I}{\partial \vec{n}}(y,t) = 0, & y \in \partial Y,\ t > 0. \end{cases} \tag{3.2}$$

Substitute $S(y,t) = e^{-\lambda t}\phi(y)$, $I(y,t) = e^{-\lambda t}\psi(y)$, $\lambda \in \mathbb{C}$ into (3.2), and then be divided on both sides by $e^{-\lambda t}$, the following characteristic problem can be obtained:

$$\begin{cases} d_S \Delta\phi(y) - \mu\phi(y) - \beta(y)\psi(y) + \lambda\phi(y) = 0, & y \in Y, \\ d_I \Delta\psi(y) + (\beta(y) - \theta - \gamma)\psi(y) + \lambda\psi(y) = 0, & y \in Y, \\ \dfrac{\partial\phi}{\partial\vec{n}}(y) = \dfrac{\partial\psi}{\partial\vec{n}}(y) = 0, & y \in \partial Y. \end{cases} \tag{3.3}$$

Similar to the discussion in [8] and [25], there is a least eigenvalue λ^*:

$$\lambda^* = \inf\left\{\int_Y \left[d_I|\nabla\varphi(y)|^2 + (\theta+\gamma-\beta(y))\varphi(y)^2\right]dy : \varphi \in W^{1,2}(Y),\ \int_Y \varphi(y)^2 dy = 1\right\},$$

and a corresponding positive eigenfunction ψ^*:

$$d_I\Delta\psi^*(y) + (\beta(y)-\theta-\gamma)\psi^*(y) + \lambda^*\psi^*(y) = 0,\ y \in Y;\ \frac{\partial\psi^*}{\partial\vec{n}}(y) = 0,\ y \in \partial Y. \tag{3.4}$$

Thus, the basic reproduction number can be defined as follows:

$$\mathcal{R}_0 = \sup_{\varphi\in W^{1,2}(Y),\ \varphi\neq 0}\left\{\frac{\int_Y \beta(y)\varphi(y)^2 dy}{\int_Y \left[d_I|\nabla\varphi(y)|^2 + (\theta+\gamma)\varphi(y)^2\right]dy}\right\}. \tag{3.5}$$

Thus, similar to papers [7, 8], the following conclusions about $\mathcal{R}_0$ can be obtained.

Lemma 3.1. *$\mathcal{R}_0$ satisfies the following conclusions:*

(a) $\mathcal{R}_0$ is a positive monotonically decreasing function about $d_I \geq 0$;
(b) $\mathcal{R}_0 \to \frac{\beta^*}{\theta+\gamma}$, as $d_I \to 0$;
(c) $\mathcal{R}_0 \to \frac{\int_Y \beta(y)dy}{(\theta+\gamma)|Y|}$, as $d_I \to \infty$;
(d) $\mathcal{R}_0 > 1$, as $\lambda^* < 0$; $\mathcal{R}_0 = 1$, as $\lambda^* = 0$; and $\mathcal{R}_0 < 1$, as $\lambda^* > 0$;
(e) if $\beta^* < \theta+\gamma$, then for every $d_I \geq 0$, $\mathcal{R}_0 < 1$;
(f) if $\int_Y \beta(y)dy < (\theta+\gamma)|Y|$ and $\beta^* > \theta+\gamma$, then there is $d_I^* \in (0,\infty)$, as $0 \leq d_I < d_I^*$, $\mathcal{R}_0 > 1$ and as $d_I > d_I^*$, $\mathcal{R}_0 < 1$;
(g) if $\int_Y \beta(y)dy \geq (\theta+\gamma)|Y|$, then $\forall d_I \geq 0$, $\mathcal{R}_0 > 1$.

4. Disease-free equilibrium and endemic equilibrium

4.1. Stability of disease-free equilibrium

It is known from the above section that the system has a disease-free equilibrium E_0, and we next discuss its local stability and global stability.

Theorem 4.1. *If $\mathcal{R}_0 < 1$, then E_0 is locally asymptotically stable. If $\mathcal{R}_0 > 1$, then E_0 is unstable.*

Proof. Suppose $\mathcal{R}_0 < 1$, we will prove that E_0 is locally asymptotically stable. That is, for every solution (λ, ϕ, ψ) of Eq (3.3), if at least one of ϕ or ψ is not identically zero, then $Re(\lambda) > 0$. Next, by contradiction, assume that there is a solution (λ, ϕ, ψ) of (3.3), and at least one of ϕ or ψ not identically zero, such that $Re(\lambda) \leq 0$.

As $\mathcal{R}_0 < 1$ and (λ, ϕ, ψ) is a solution of Eq (3.3), if $\psi \equiv 0,\ \phi \not\equiv 0$, then by Eq (3.3), there is

$$d_S\Delta\phi(y) - \mu\phi(y) + \lambda\phi(y) = 0,\ y \in Y;\ \frac{\partial\phi}{\partial\vec{n}}(y) = 0,\ y \in \partial Y.$$

This means $\lambda - \mu \geq 0$. Therefore, $\lambda \geq \mu > 0$, which contradicts $Re(\lambda) \leq 0$; if $\phi \equiv 0$, $\psi \not\equiv 0$, by Eq (3.3), then there is

$$d_I \Delta\psi(y) + (\beta(y) - \theta - \gamma)\psi(y) + \lambda\psi(y) = 0,\ y \in Y;\ \frac{\partial\psi}{\partial\vec{n}}(y) = 0,\ y \in \partial Y.$$

For $d_I\Delta + (\beta(y) - \theta - \gamma)$ is a self-adjoint operator, so λ is real and not positive, and $\lambda^* \leq \lambda \leq 0$, and then by the condition (d) of Lemma 3.1, we have $\mathcal{R}_0 \geq 1$, which contradicts $\mathcal{R}_0 < 1$. Thus, $Re(\lambda) > 0$, the disease-free equilibrium E_0 is linearly stable. By [26], E_0 is locally asymptotically stable.

Suppose that $\mathcal{R}_0 > 1$. We are going to prove that E_0 is linearly unstable.

Note (λ^*, ψ^*) meets Eq (3.4) and $\psi^* > 0$ on Y. So, by condition (d) of Lemma 3.1, $\lambda^* < 0$. Hence, the inhomogeneous linear equation

$$d_S \Delta\phi(y) + (\lambda^* - \mu)\phi(y) - \beta(y)\psi^*(y) = 0,\ y \in Y$$

has a unique positive solution ϕ^*, which satisfies $\frac{\partial\phi^*}{\partial\vec{n}}(y) = 0,\ y \in \partial Y$. So $(\lambda^*, \phi^*, \psi^*)$ is a solution of Eq (3.3). Moreover, $\psi^* > 0$, $\lambda^* < 0$, which means that E_0 is linearly unstable. By [26], E_0 is unstable.

Theorem 4.2. *If*

$$\beta^* < \theta, \tag{4.1}$$

one can deduce that $\mathcal{R}_0 < 1$; further, as $d_S,\ d_I > 0$ or $d_S = d_I = 0$, the disease-free equilibrium E_0 is globally asymptotically stable.

Proof. First, by conditions (e) of Lemma 3.1 and condition (4.1), we have the following inequality: $\beta^* < \theta \leq \theta + \gamma$, then for every $d_I \geq 0$, $\mathcal{R}_0 < 1$. By Theorem 4.1, E_0 is locally asymptotically stable.

On the other hand, considering the case $d_S,\ d_I > 0$, construct the following Lyapunov function:

$$L_1(t) = \frac{1}{2}\int_Y I^2(y,t)dy.$$

Then

$$\begin{aligned}
\frac{dL_1(t)}{dt} &= \int_Y I(y,t)\frac{\partial I(y,t)}{\partial t}dy \\
&= d_I\int_Y I\Delta I dy + \int_Y I^2\left(\frac{\beta(y)S}{S+I} - \theta - \frac{\gamma}{1+\varepsilon I}\right)dy \\
&\leq d_I\int_Y I\Delta I dy + \int_Y I^2\left(\frac{\beta(y)S}{S+I} - \theta\right)dy \\
&= d_I\int_Y I\Delta I dy + \int_Y I^2\left(\frac{\beta(y)S - \theta S - \theta I}{S+I}\right)dy \\
&\leq d_I\int_Y I\Delta I dy + \int_Y I^2\left(\frac{(\beta^* - \theta)S}{S+I}\right)dy.
\end{aligned}$$

Due to

$$0 = \int_{\partial Y} I\frac{\partial I}{\partial\vec{n}}ds = \int_Y \mathrm{div}(I\nabla I)dy = \int_Y \left(I\Delta I + |\nabla I|^2\right)dy,$$

there is

$$\int_Y I\Delta I dy = -\int_Y |\nabla I|^2 dy.$$

So

$$\frac{dL_1(t)}{dt} \le -d_I \int_Y |\nabla I|^2 dy + \int_Y I^2 \left(\frac{(\beta^* - \theta)\, S}{S + I}\right) dy \le 0,$$

and $\frac{dL_1(t)}{dt} = 0$, if and only if $I = 0$. One can deduce that as $t \to \infty$, on $L^2(Y)\times L^2(Y)$, $(S(y,t), I(y,t)) \to E_0$. By the similar method of Theorem 4.1 of [5], one can deduce that $t \to \infty$, on $L^\infty(Y) \times L^\infty(Y)$, $(S(y,t), I(y,t)) \to E_0$, so E_0 is globally asymptotically stable.

As $d_S = d_I = 0$, y can be regarded as fixed. We consider the following system:

$$\begin{cases} \frac{dS}{dt} &= \Lambda - \mu S - \beta(y)\frac{S I}{S + I}, \\ \frac{dI}{dt} &= \beta(y)\frac{S I}{S + I} - \theta I - \frac{\gamma I}{1 + \varepsilon I}. \end{cases}$$

Construct Lyapunov function

$$L_2(t) = \frac{1}{2} I^2(y,t).$$

Then

$$\begin{aligned} \frac{dL_2(t)}{dt} &= I(y,t)\frac{dI(y,t)}{dt} = I^2\left(\beta(y)\frac{S}{S + I} - \theta - \frac{\gamma}{1 + \varepsilon I}\right) \\ &\le \frac{I^2}{S + I}(\beta(y)S - \theta(S + I)) \le -\theta\frac{I^2}{S + I}\left(\left(1 - \frac{\beta^*}{\theta}\right)S + I\right) \le 0, \end{aligned}$$

and $\frac{dL_1(t)}{dt} = 0$, if and only if $I = 0$. Similar to the previous discussion, because $\mathcal{R}_0 < 1$, y can be regarded as fixed, and E_0 is globally asymptotically stable by the LaSalle invariant set principle. The proof is finished.

In order to study the global stability of disease-free equilibrium E_0 when one diffusion rate is positive and the other is zero, Kuratowski measure k of non-compactness is introduced and defined as

$$k(B) = inf\,\{\delta > 0 : B \text{ has a finite cover of diameter} < \delta\},$$

where B is a bounded set.

Set $\Phi(t) : X^{2+} \to X^{2+}$ represent the semiflow of model (1.3), and B denotes any bounded set belonged to X^{2+}, there are the following results.

Lemma 4.3. *Suppose (4.1) holds. If $d_S > 0$, $d_I = 0$, then $\Phi(t) : X^{2+} \to X^{2+}, t \ge 0$ is $k-$contraction on X^{2+} and for $B \subset X^{2+}$,*

$$k(\Phi(t)B) \le e^{(\beta^* - \theta)t} k(B),$$

among them, $0 < e^{(\beta^* - \theta)t} < 1$.

Proof. When $d_I = 0$, system (1.3) can be expressed as

$$\begin{pmatrix} \frac{\partial S}{\partial t}(y,t) - d_S \Delta S(y,t) \\ \frac{\partial I}{\partial t}(y,t) \end{pmatrix} = J(E_0)\begin{pmatrix} S \\ I \end{pmatrix} + \begin{pmatrix} F_1(y,S,I) \\ F_2(y,S,I) \end{pmatrix}, \tag{4.2}$$

where

$$J(E_0) = \begin{pmatrix} -\mu & -\beta(y) \\ 0 & \beta(y) - \theta - \gamma \end{pmatrix}$$

stands for its corresponding Jacobian matrix; $(F_1(y,S,I), F_2(y,S,I))^T$ is its relevant nonlinear terms. For each $\phi(\cdot) = (\phi_1(\cdot), \phi_2(\cdot)) \in X^{2+}$, system (4.2) have the following initial criteria:

$$S(y,0) = \phi_1(y), \quad I(y,0) = \phi_2(y), \quad y \in Y,$$

and the associated semiflow can be defined as:

$$\Phi(t)\phi = (S(\cdot,t,\phi),\ I(\cdot,t,\phi)), \quad \forall \phi \in X^{2+},\ t \geq 0.$$

One can see that $I(\cdot,t,\phi)$ satisfies:

$$\begin{cases} \dfrac{\partial I}{\partial t}(y,t) = (\beta(y) - \theta - \gamma)I(y,t,\phi) + F_2(y,S,I), \quad t > 0,\ y \in Y, \\ I(y,0) = \phi_2(y), \quad y \in Y. \end{cases} \tag{4.3}$$

Notice that $\frac{\partial I}{\partial t}(y,t)$ also meets:

$$\begin{aligned} \frac{\partial I}{\partial t}(y,t) &= -\theta I(y,t,\phi) + \frac{\beta(y)S(y,t,\phi)I(y,t,\phi)}{S(y,t,\phi) + I(y,t,\phi)} - \frac{\gamma I(y,t,\phi)}{1 + \varepsilon I(y,t,\phi)} \\ &\leq (\beta(y) - \theta)I(y,t,\phi) + \beta^* S(y,t,\phi), \end{aligned}$$

where $t > 0,\ y \in Y$. Then,

$$I(\cdot,t,\phi) = T_2(t)\phi_2 + \int_0^t T_2(t-s)F_2(y,S(\cdot,s,\phi))ds, \tag{4.4}$$

where $T_2(t) : C(\bar{Y},\mathbb{R}) \to C(\bar{Y},\mathbb{R})$ and $T_2(t) = e^{(\beta(\cdot)-\theta-\gamma)t}$. For $\forall \phi \in X^{2+}$, set the linear operator $\mathbb{L}(t)$ and the nonlinear operator $\mathbb{N}(t)$ be as follows:

$$\mathbb{L}(t)\phi = (0, T_2(t)\phi_2),$$

$$\mathbb{N}(t)\phi = \left(S(\cdot,t,\phi), \int_0^t T_2(t-s)F_2(y,S(\cdot,s,\phi),\ I(\cdot,s,\phi))ds\right).$$

Thus, for $\forall \phi \in X^{2+}$,

$$\begin{aligned} &\mathbb{L}(t)\phi \leq \left(0, e^{(\beta(\cdot)-\theta)t}\phi_2\right), \\ &\mathbb{N}(t)\phi \leq \left(S(\cdot,t,\phi), \int_0^t e^{(\beta(\cdot)-\theta)(t-s)}\beta^* S(\cdot,s,\phi)ds\right). \end{aligned}$$

It can be obtained from the above discussion that

$$\Phi(t)\phi = \mathbb{L}(t)\phi + \mathbb{N}(t)\phi, \quad \forall \phi \in X^{2+},\ t \geq 0.$$

Let $T_1(t) : C(\bar{Y}, \mathbb{R}) \to C(\bar{Y}, \mathbb{R})$ be the C_0 semigroups related with $d_S\Delta - 1 - \beta(\cdot)$ under Neumann boundary conditions. According to the compactness of $T_1(t)$, by (4.2) and (4.3), $\mathbb{N}(t) : X^{2+} \to X^{2+}$, $\forall t > 0$ is compact. There is $t > 0$ and $k(\mathbb{N}(t)B) = 0$ because

$$\|\mathbb{L}(t)\phi\| \le \left\|e^{(\beta(\cdot)-\theta)t}\phi_2\right\| \le e^{(\beta^*-\theta)t}\|\phi\|,$$

one can deduce

$$\|\mathbb{L}(t)\| \le e^{(\beta^*-\theta)t}, \quad \forall t > 0.$$

Thus,

$$k(\Phi(t)B) \le k(\mathbb{L}(t)B) + k(\mathbb{N}(t)B) \le \|\mathbb{L}(t)\|k(B) + 0 \le e^{(\beta^*-\theta)t}k(B), \quad \forall t > 0,$$

and by the condition (4.1), $0 < e^{(\beta^*-\theta)t} < 1$. So the proof of lemma is done.

Lemma 4.4. *If $d_S = 0$, $d_I > 0$, then, for $\forall t \ge 0$, on X^{2+}, $\Phi(t) : X^{2+} \to X^{2+}$ is the $k-$ contraction and*

$$k(\Phi(t)B) \le e^{-\mu t}k(B),$$

Proof. Set $u = S - \tilde{S}$. When $d_S = 0$, system (1.3) becomes

$$\begin{aligned} &\frac{\partial u}{\partial t} = \Lambda - \mu(u + \tilde{S}) - \beta(y)\frac{(u + \tilde{S})}{u + \tilde{S} + I} = -\mu u - \beta(y)\frac{(u + \tilde{S})I}{u + \tilde{S} + I}, \\ &\frac{\partial I}{\partial t} - d_I\Delta I = \beta(y)\frac{(u + \tilde{S})I}{u + \tilde{S} + I} - \theta I - \frac{\gamma I}{1 + \varepsilon I}. \end{aligned} \tag{4.5}$$

Set $T_1(t) = e^{-\mu t}$ and $T_2(t) : C(\bar{Y}, \mathbb{R}) \to C(\bar{Y}, \mathbb{R})$ as the C_0 semigroup subject to $d_I\Delta - \theta$ under the Newmann boundary conditions.

Set the linear operator $\mathbb{L}(t)$ and the nonlinear operator $\mathbb{N}(t)$ as follows:

$$\mathbb{L}(t)\phi = (T_1(t)\phi_1, 0),$$

$$\mathbb{N}(t)\phi = \left(\int_0^t T_1(t-s)F(y, u(\cdot, s, \phi), I(\cdot, s, \phi))s, I(\cdot, t, \phi)\right),$$

among them,

$$F(y, u, I) = -\beta(y)\frac{(u + \tilde{S})I}{u + \tilde{S} + I} \le -\beta(y)\frac{uI}{u + \tilde{S} + I}.$$

Recall that

$$\|\mathbb{N}(t)\phi\| \le \max\left(\int_0^t \beta^*\|u(\cdot, s, \phi)\|\|I(\cdot, s, \phi)\|s, \|I(\cdot, t, \phi)\|\right). \tag{4.6}$$

Thus,

$$u(\cdot, t, \phi) = T_1(t)\phi_1 + \int_0^t T_1(t-s)F(y, u(\cdot, s, \phi),\ I(\cdot, s, \phi))ds. \tag{4.7}$$

It follows from the above discussion that:

$$\Phi(t)\phi = \mathbb{L}(t)\phi + \mathbb{N}(t)\phi, \quad \forall \phi \in X^{2+},\ t \ge 0.$$

Since $T_2(t)$ is compact, combining (4.6) and (4.7), one can deduce that $\mathbb{N}(t)$ is compact. Thus, $\forall B \subset X^{2+}$ and $t > 0$, there is $k(\mathbb{N}(t), B) = 0$. Because

$$\|\mathbb{L}(t)\phi\| \leq \left\|e^{-\mu t}\phi_1\right\| \leq e^{-\mu t}\|\phi\|,$$

we get

$$\|\mathbb{L}(t)\| \leq e^{-\mu t}, \ \forall t > 0.$$

Thus, for each bounded set $B \subset X^{2+}$, there is

$$k(\Phi(t)B) \leq k(\mathbb{L}(t)B) + k(\mathbb{N}(t)B) \leq \|\mathbb{L}(t)\|k(B) + 0 \leq e^{-\mu t}k(B), \ \forall t > 0.$$

The lemma is proved until now.

Theorem 4.5. *Suppose (4.1) holds. If $d_S,\ d_I \geq 0$, then the disease-free equilibrium E_0 is globally asymptotically stable.*

Proof. As $d_S,\ d_I > 0$ or $d_S = d_I = 0$, it follows from Theorem 4.2 that E_0 is globally asymptotically stable.

As $d_S > 0,\ d_I = 0$, by Theorem 2.2 and Lemma 4.3, Φ is the point of dissipation. The positive orbits that belong to the bounded subsets of X^{2+} are bounded, and on X^{2+}, $\Phi(t)$ is $k-$ contracting. Thus, according to Theorem 1.3.7 in [29], $\Phi(t)$ has a related global attractor attracted B. The case that $d_S = 0,\ d_I > 0$ can be discussed in a similar way.

It can be inferred from the above discussion that the disease can be eradicated as the basic reproduction number is small enough or the recovery or treatment rate is large enough. However, the system may be more complicated as condition (4.1) is not established.

4.2. Consistent persistence and existence of the endemic equilibrium

In this subsection, the criteria for determining the consistent persistence of the solution of the problem (1.3) is established, and the criteria of existing the endemic equilibrium for model (1.3) is obtained.

Theorem 4.6. *If*

$$\beta(y) > \theta + \gamma, \quad y \in \bar{Y}, \tag{4.8}$$

then $\eta \in R^+$, which does not rely on the initial value, such that the solution (S, I) of model (1.3) consistently meets

$$\liminf_{t\to\infty} S(y,t) \geq \eta, \quad \liminf_{t\to\infty} I(y,t) \geq \eta, \quad y \in \bar{Y}.$$

As a result, the disease persists uniformly. Moreover, $\mathcal{R}_0 > 1$, and there is at least one endemic equilibrium of system (1.3).

Proof. Set $\mathbf{X} = C(\bar{Y}, \mathbb{R}^2_+)$, $\|\varphi\| = \max_{y\in\bar{Y}} |\varphi(y)|$ and define:

$$\mathbf{X_0} = \{\varphi = (\varphi_1, \varphi_2) \in \mathbf{X} : \ \varphi_2(y) \not\equiv 0\},$$
$$\partial\mathbf{X_0} = \mathbf{X}\backslash\mathbf{X_0} = \{\varphi \in \mathbf{X} : \ \varphi_2(y) \equiv 0\}.$$

Set $[\Phi(t)\varphi](y) = (S(y,t,\varphi),\ I(y,t,\varphi))$ is the corresponding unique solution of system (1.3), and $(S_0, I_0) = \varphi$, $\varphi \in \mathbf{X}$ is a given initial state.

Let

$$M_\partial = \{\varphi \in \partial\mathbf{X_0} : \Phi(t)\varphi \in \partial\mathbf{X_0},\ \forall t \geq 0\},$$

and $\omega(\varphi)$ is ω limit set of $\gamma^+(\varphi) = \{\Phi(t)\varphi : t \geq 0\}$. For $t \geq 0$, $\Phi(t)\mathbf{X_0} \subset \mathbf{X_0}$ holds. If $\varphi \in \partial\mathbf{X_0}$, then there is one only solution $I(y,t,\varphi) \equiv 0$. By [28], one can deduce that $\lim_{t\to\infty} S(y,t,\varphi) \equiv \tilde{S},\ y \in \bar{Y}$. Therefore, for $\forall\varphi \in M_\partial$, $\omega(\varphi) = \{E_0\}$.

Next, we will prove that there is a normal number η_0, satisfying

$$\limsup_{t\to\infty} d(\Phi(t)\varphi, E_0) = \limsup_{t\to\infty} \|\Phi(t)\varphi - E_0\| \geq \eta_0, \quad \forall\varphi \in \mathbf{X_0}. \tag{4.9}$$

From (4.8), there is a small enough number η_0, such that

$$\frac{\beta(y)(\tilde{S} - \eta_0)}{\tilde{S} + 2\eta_0} - (\theta + \gamma) > 0.$$

Assume $\limsup_{t\to\infty} d(\Phi(t)\varphi, E_0) < \eta_0$, for some $\varphi \in \mathbf{X_0}$. If $d(\Phi(t)\varphi, E_0) < \eta_0,\ \forall t \geq 0$, and use $\Phi(t_0)\varphi$ as a new initial state, then we can get

$$\begin{aligned} &0 < \tilde{S} - \eta_0 < S(y,t) < \tilde{S} + \eta_0, \quad \forall t \geq t_0,\ y \in \bar{Y},\\ &0 < I(y,t) < \eta_0, \quad \forall t \geq t_0,\ y \in \bar{Y}. \end{aligned}$$

From the above conditions, the second expression of system (1.3) can be obtained:

$$\begin{aligned} \frac{\partial I}{\partial t} &\geq d_I\Delta I + \left[\frac{\beta(y)(\tilde{S} - \eta_0)}{\tilde{S} + 2\eta_0} - (\theta + \gamma)\right] I\\ &\geq d_I\Delta I + n_0 I, \quad \forall t \geq t_0,\ y \in \bar{Y}. \end{aligned}$$

By conditions (a) and (c) of Lemma 3.1 and condition (4.8), one can get for $d_S, d_I \geq 0$,

$$\mathcal{R}_0 > \frac{\int_Y \beta(y)dy}{(\theta + \gamma)|Y|} > 1.$$

Now, $\lambda^* < 0$. By the continuity of the main characteristic value, we can know if η_0 is small enough to make $\lambda^*(\eta_0) < 0$, the corresponding positive characteristic function is $\psi^*(\eta_0)$. Because $I(y,t_0) > 0$, one can find a small enough positive constant c^*, such that $I(y,t_0) \geq c^*\psi^*(\eta_0)$. We can observe $c^*e^{-\lambda^*(\eta_0)(t-t_0)}\psi^*(\eta_0)$ is the solution of the following systems

$$\begin{cases} \dfrac{\partial w}{\partial t} = d_I\Delta w + \left[\dfrac{\beta(y)(\tilde{S} - \eta_0)}{\tilde{S} + 2\eta_0} - (\theta + \gamma)\right] w, & y \in Y,\ t > t_0,\\ \dfrac{\partial w}{\partial \vec{n}} = 0, & y \in \partial Y,\ t > t_0,\\ w(y,t_0) = c^*\psi^*(\eta_0), & y \in Y. \end{cases}$$

Through the comparison principle, one can obtain $I(y,t) \geq c^*e^{-\lambda^*(\eta_0)(t-t_0)}\psi^*(\eta_0) \to \infty$ on $\bar{Y}$ as $t \to \infty$. This contradicts Theorem 2.2, so (4.9) holds.

Applying Theorem 3 in [30], by $p(\varphi) = \min_{y\in\bar{Y}} \varphi_2(y)$, it can be inferred that

$$\liminf_{t\to\infty} p(\Phi(t)\varphi) \geq \eta_1, \quad \forall \varphi \in \mathbf{X_0},$$

where η_1 is a positive constant. Then

$$\liminf_{t\to\infty} \min_{y\in\bar{Y}} I(y,t,\varphi) \geq \eta_1, \quad \forall \varphi \in \mathbf{X_0}.$$

That is to say, for $\varphi \in \mathbf{X_0}$, there is $T_1 = T_1(\varphi) > 0$ such that

$$I(y,t,\varphi) \geq \eta_1, \quad \forall t \geq T_1,\ y \in \bar{Y}.$$

Based on this conclusion and doing a similar discussion with [31] for system (1.3), it can be concluded that:

$$S(y,t,\varphi) \geq \eta_2, \quad \forall t \geq T_2,\ y \in \bar{Y},$$

where η_2 is a positive constant and $T_2 \geq T_1$. Set $\eta = \min\{\eta_1,\ \eta_2\}$, then the consistent persistence holds. According to Theorem 4.5 of [31], it can be obtained that under the condition (4.8), there is at least one endemic equilibrium of system (1.3). The proof of the theorem is finished.

4.3. The asymptotic profiles of the endemic equilibria

Next, we will consider the asymptotic profiles of the endemic equilibria of (1.3) based on the assumption that condition (4.8) holds. Let (S, I) represent an endemic equilibrium of system (1.3), so (S, I) satisfies the following elliptic problem

$$\begin{cases} -d_S \Delta S = \Lambda - \mu S - \beta(y)\dfrac{SI}{S+I}, & y \in Y, \\ -d_I \Delta I = \beta(y)\dfrac{SI}{S+I} - \theta I - \dfrac{\gamma I}{1+\varepsilon I}, & y \in Y, \\ \dfrac{\partial S}{\partial \vec{n}}(y,t) = \dfrac{\partial I}{\partial \vec{n}}(y,t) = 0, & y \in \partial Y. \end{cases} \tag{4.10}$$

By adding and integrating the first two expressions of (4.10), we get

$$\int_Y (\Lambda - \mu S - \theta I - \frac{\gamma I}{1+\varepsilon I}) dy = 0.$$

Then

$$\int_Y (\mu S + \theta I + \frac{\gamma I}{1+\varepsilon I}) dy = \int_Y \Lambda dy \leq \Lambda |Y|.$$

There are

$$\int_Y S\, dy \leq \frac{\Lambda |Y|}{\mu},$$

$$\int_Y (\theta I + \frac{\gamma I}{1+\varepsilon I}) dy \leq \Lambda |Y|. \tag{4.11}$$

For

$$\int_Y \theta I dy \leq \int_Y (\theta I + \frac{\gamma I}{1+\varepsilon I}) dy. \tag{4.12}$$

Due to (4.11) and (4.12), there is

$$\int_Y I dy \le \frac{\Lambda |Y|}{\theta}. \tag{4.13}$$

As $d_S \to 0$, the following result about the asymptotic profile of the endemic equilibrium can be obtained.

Theorem 4.7. *Under the assumption (4.8) is true. If $d_I > 0$, $d_S \to 0$, then each positive solution (S_{d_S}, I_{d_S}) of system (4.10) satisfies*

$$(S_{d_S}, I_{d_S}) \to (S^*, I^*),$$

where

$$S^*(y) = G(y, I^*(y)) = \frac{1}{2}\Big\{\Lambda - (\mu + \beta(y))I^* + \sqrt{[\Lambda - (\mu + \beta(y))I^*]^2 + 4\Lambda I^*}\Big\},$$

and I^* is a positive solution of the following problem

$$\begin{cases} -d_I \Delta I^* = \beta(y)\dfrac{S^* I^*}{S^* + I^*} - \theta I^* - \dfrac{\gamma I^*}{1 + \varepsilon I^*}, & y \in Y, \\ \dfrac{\partial I^*}{\partial \vec{n}}(y, t) = 0, & y \in \partial Y. \end{cases} \tag{4.14}$$

Proof. First, we will prove the positiveness of the solutions for system (4.14). Rewrite the Eq (4.14) as:

$$-d_I \Delta I^* - c(y) I^* = 0,$$

where

$$c(y) = \beta(y)\frac{S^*}{S^* + I^*} - \theta - \frac{\gamma}{1 + \varepsilon I^*}, y \in Y.$$

Combined with condition (4.8)

$$\beta(y)\frac{S^*}{S^* + I^*} \ge \beta(y)\frac{S^*}{S^* + \max_{\bar{Y}} I^*} > \theta + \gamma \ge \theta + \frac{\gamma}{1 + \varepsilon I^*}$$

thus, $c(y) > 0$ for all $y \in \bar{Y}$, satisfying the conditions of the strong maximum principle. Application of the Strong Maximum Principle for elliptic equations with Neumann boundary conditions and Hopf's Lemma, we infer $I^* > 0$ for all $y \in \bar{Y}$. The positiveness of $S^*(y)$ is directly guaranteed by its definition:

$$S^*(y) = \frac{1}{2}\Big\{\Lambda - (\mu + \beta(y))I^* + \sqrt{[\Lambda - (\mu + \beta(y))I^*]^2 + 4\Lambda I^*}\Big\},$$

here, $\Lambda > 0, I^* > 0$, $\sqrt{[\Lambda - (\mu + \beta(y))I^*]^2 + 4\Lambda I^*} \ge 0$. Then any solutions (S^*, I^*) of (4.14) are strictly positive in $\bar{Y}$.

Under the hypothesis (4.8) holds, by Theorem 4.6, $\mathcal{R}_0 > 1$ and model (4.10) has at least one solution as $d_S,\ d_I \ge 0$.

Step 1: Estimation of the upper bound of I. Note that I satisfies

$$\begin{cases} \Delta I + \left[\beta(y)\dfrac{S}{d_I(S + I)} - \dfrac{\theta}{d_I} - \dfrac{\gamma}{d_I(1 + \varepsilon)}\right] I = 0, & y \in Y, \\ \dfrac{\partial I}{\partial \vec{n}}(y, t) = 0, & y \in \partial Y. \end{cases}$$

From the Harnack inequality [32] of the elliptic equation, it can be obtained that

$$\max_{\bar{Y}} I(y) \le C \min_{\bar{Y}} I(y). \tag{4.15}$$

Thereafter, C will be a positive constant that does not depend on $d_S > 0$, which may change during the proof. From (4.13) and (4.15), there is

$$\max_{\bar{Y}} I(y) \le C \min_{\bar{Y}} I(y) \le \frac{C}{|Y|} \int_Y I dy \le C. \tag{4.16}$$

Step 2: Convergence of I. Obviously, I satisfies

$$\begin{cases} -d_I \Delta I + \theta I = \beta(y) \dfrac{S}{S+I} I - \dfrac{\gamma}{1+\varepsilon I} I, & y \in Y, \\ \dfrac{\partial I}{\partial \vec{n}}(y,t) = 0, & y \in \partial Y. \end{cases}$$

By Theorem 2.2, $\|S\|_{L^\infty(Y)} \le C$ can be available and

$$\|S\|_{L^p(Y)} \le |Y|^{1/p} \|S\|_{L^\infty(Y)} \le C.$$

Through (4.16) and the above inequality, we can get:

$$\left\| \frac{\beta S}{S+I} I - \frac{\gamma}{1+\varepsilon I} I \right\|_{L^p(Y)} \le C, \quad p > 1.$$

From the standard L^p- estimate of the elliptic equation [33], it can be known that:

$$\|I\|_{W^{2,p}(Y)} \le C, \quad p > 1.$$

Making p be sufficiently large, one can get

$$\|I\|_{C^{1+\alpha}(\bar{Y})} \le C, \quad 0 < \alpha < 1.$$

Hence, there is a subsequence $d_S \to 0$ denoted $d_n = d_{S,n}$, which satisfies $d_n \to 0$ as $n \to \infty$. A corresponding positive solution sequence $(S_n, I_n) = (S_{d_{S,n}}, I_{d_{S,n}})$ of model (4.10) with $d_S = d_{S,n}$, such that

$$I_n \to I^*, \quad n \to \infty,$$

where $0 \le I^* \in C^1(\bar{Y})$. By (4.15), $I^* = 0$ or $I^* > 0$. Suppose $I^* = 0$, then

$$I_n \to 0, \quad n \to \infty. \tag{4.17}$$

That is to say, for any $\epsilon > 0$, there is sufficiently large n satisfying $0 \le I_n(\cdot) \le \epsilon$. From the first expression of (4.10), S_n meets

$$-d_n \Delta S_n \le \Lambda - \mu S_n, \ y \in Y; \ \frac{\partial S_n}{\partial \vec{n}}(y) = 0, \ y \in \partial Y,$$

$$-d_n \Delta S_n \ge \Lambda - \mu S_n - \epsilon\beta(y), \ y \in Y; \ \frac{\partial S_n}{\partial \vec{n}}(y) = 0, \ y \in \partial Y.$$

Consider the following two auxiliary systems:

$$-d_n\Delta u = \Lambda - \mu u,\ y \in Y;\ \frac{\partial u}{\partial \vec{n}}(y) = 0,\ y \in \partial Y, \tag{4.18}$$

and

$$-d_n\Delta v = \Lambda - \mu v - \epsilon\beta(y),\ y \in Y;\ \frac{\partial v}{\partial \vec{n}}(y) = 0,\ y \in \partial Y. \tag{4.19}$$

Use u_n and v_n denote the only positive solutions of (4.18) and (4.19). By simply discussing the upper solution, the lower solution and the uniqueness of solution, we have

$$v_n \le S_n \le u_n. \tag{4.20}$$

By the method in [34], it can be obtained that

$$\lim_{n\to\infty} u_n(y) = \frac{\Lambda}{\mu}, \quad \lim_{n\to\infty} v_n(y) = \frac{\Lambda - \epsilon\beta(y)}{\mu}.$$

In (4.20), let $n \to \infty$, so there is

$$\frac{\Lambda - \epsilon\beta(y)}{\mu} \le \liminf_{n\to\infty} S_n(y) \le \limsup_{n\to\infty} S_n(y) \le \frac{\Lambda}{\mu}.$$

By the arbitrariness of ϵ,

$$S_n(y) \to \frac{\Lambda}{\mu}. \tag{4.21}$$

Next, we consider the second expression of system (4.10) satisfied by I_n

$$-d_I\Delta I_n = \beta(y)\frac{S_n I_n}{S_n + I_n} - \theta I_n - \frac{\gamma I_n}{1 + \varepsilon I_n}, y \in Y;\ \frac{\partial I_n}{\partial \vec{n}}(y) = 0, y \in \partial Y. \tag{4.22}$$

Define

$$\tilde{I}_n(y) := \frac{I_n(y)}{\|I_n\|_{L^\infty(Y)}}.$$

For $n \ge 1, \|\tilde{I}_n(y)\|_{L^\infty(Y)} = 1$. $\tilde{I}_n$ satisfies

$$-d_I\Delta\tilde{I}_n = \left[\beta(y)\frac{S_n}{S_n + I_n} - \theta - \frac{\gamma}{1 + \varepsilon I_n}\right]\tilde{I}_n, y \in Y;\ \frac{\partial \tilde{I}_n}{\partial \vec{n}}(y) = 0, y \in \partial Y. \tag{4.23}$$

By a compactness discussion of the elliptic equation, after passing to a further subsequence if necessary, assume that

$$\tilde{I}_n \to \tilde{I}, \quad n \to \infty,$$

where $0 \le \tilde{I} \in C^1(\bar{Y}), \|\tilde{I}\|_{L^\infty(Y)} = 1$. By (4.17), (4.21) and (4.23), $\tilde{I}$ satisfies

$$-d_I\Delta\tilde{I} = [\beta(y) - \theta - \gamma]\,\tilde{I},\ y \in Y;\ \frac{\partial \tilde{I}}{\partial \vec{n}}(y) = 0,\ y \in \partial Y.$$

Through the inequality [32], $\tilde{I} > 0$ is available in $\bar{Y}$. From the uniqueness of the principal eigenvalue, we can get the principal eigenvalue $\lambda^* = 0$ of (3.4). Because of the condition (d) of Lemma 3.1, this contradicts $\mathcal{R}_0 > 1$. Thus, for $\bar{Y}$,

$$I^* > 0,\ I_n \to I^*, \quad n \to \infty. \tag{4.24}$$

Step 3: Convergence of S. Consider the first expression of (4.10) satisfied by S_n:

$$-d_n\Delta S_n = \Lambda - \mu S_n - \beta(y)\frac{S_nI_n}{S_n+I_n},\ y\in Y;\ \frac{\partial S_n}{\partial\vec{n}}(y)=0,\ y\in\partial Y.$$

According to (4.24), for any small $\epsilon>0$, sufficiently large n, there is

$$\begin{aligned}\Lambda-\mu S_n-\beta(y)\frac{S_nI_n}{S_n+I_n}&\le \Lambda-\mu S_n-\beta(y)\frac{S_n(I^*-\epsilon)}{S_n+(I^*-\epsilon)},\\ &=\frac{(g_+^{1,\epsilon}(y,I^*(y))-S_n)(S_n-g_-^{1,\epsilon}(y,I^*(y)))}{S_n+(I^*-\epsilon)},\end{aligned}$$

where

$$\begin{aligned}g_\pm^{1,\epsilon}(y,I^*(y))=&\frac{1}{2}\Big\{\Lambda-(\mu+\beta(y))(I^*-\epsilon)\\ &\pm\sqrt{[\Lambda-(\mu+\beta(y))(I^*-\epsilon)]^2+4\Lambda(I^*-\epsilon)}\Big\}.\end{aligned}$$

Consider the following auxiliary questions

$$\begin{cases}-d_nw=\dfrac{(g_+^{1,\epsilon}(y,I^*(y))-w)(w-g_-^{1,\epsilon}(y,I^*(y)))}{w+(I^*-\epsilon)}, & y\in Y,\\ \dfrac{\partial w}{\partial\vec{n}}(y,t)=0, & y\in\partial Y.\end{cases}\tag{4.25}$$

With $g_+^{1,\epsilon}>0$ and $g_-^{1,\epsilon}<0$ on $\bar{Y}$. Furthermore, S_n is the lower solution of (4.25). Any large enough constant C on $\bar{Y}$ meeting $S_n\le C$ is its super solution. Thus, for (4.25) has at least one positive solution w_n satisfied $S_n\le w_n\le C$ on $\bar{Y}$.

By a similar discussion with Lemma 2.4 of [34], any positive solution $\bar{w}_n$ of (4.25) satisfies

$$\bar{w}_n\to g_+^{1,\epsilon}(y,I^*(y)),\quad n\to\infty.$$

$S_n\le w_n\le C$ on $\bar{Y}$, there is

$$\limsup_{n\to\infty}S_n(y)\le g_+^{1,\epsilon}(y,I^*(y)).\tag{4.26}$$

Furthermore, by (4.24), for sufficiently large n, there is

$$\begin{aligned}\Lambda-\mu S_n-\beta(y)\frac{S_nI_n}{S_n+I_n}&\ge \Lambda-\mu S_n-\beta(y)\frac{S_n(I^*+\epsilon)}{S_n+(I^*+\epsilon)},\\ &=\frac{(g_+^{2,\epsilon}(y,I^*(y))-S_n)(S_n-g_-^{2,\epsilon}(y,I^*(y)))}{S_n+(I^*+\epsilon)},\end{aligned}$$

where

$$g_\pm^{1,\epsilon}(y,I^*(y))=\frac{1}{2}\Big\{\Lambda-(\mu+\beta(y))(I^*+\epsilon)\pm\sqrt{[\Lambda-(\mu+\beta(y))(I^*+\epsilon)]^2+4\Lambda(I^*+\epsilon)}\Big\}.$$

Consider the following auxiliary questions:

$$\begin{cases}-d_nw=\dfrac{(g_+^{2,\epsilon}(y,I^*(y))-w)(w-g_-^{2,\epsilon}(y,I^*(y)))}{w+(I^*+\epsilon)}, & y\in Y,\\ \dfrac{\partial w}{\partial\vec{n}}(y,t)=0, & y\in\partial Y.\end{cases}$$

Similar to the previous discussion, we further obtain

$$\liminf_{n\to\infty} S_n(y) \geq g_+^{1,\epsilon}(y, I^*(y)). \tag{4.27}$$

By the arbitrariness of ϵ,

$$\lim_{\epsilon\to 0} g_+^{1,\epsilon}(y, I^*(y)) = \lim_{\epsilon\to 0} g_+^{2,\ \epsilon}(y, I^*(y)) = G(y, I^*(y)).$$

By (4.26) and (4.27),

$$S_n(y) \to G(y, I^*(y)),\ y \in \bar{Y}, \quad n \to \infty.$$

From (4.22), we know I^* satisfies (4.14). The proof is done.

Remark 4.8. *d_S describes the mobility of susceptible individuals within the spatial domain Y. When $d_S \to 0$, it means that susceptible individuals hardly migrate in space and move only around their initial locations. For example, during a pandemic, when people are strictly confined to their homes, or in specific areas such as nursing homes where population mobility is extremely low, or due to geographical barriers such as mountains or blockades that restrict the cross-regional movement of susceptible individuals. At this stage, the spatial distribution of susceptible individuals is primarily determined by the local dynamics of "local recruitment local infection local natural death", than by spatial redistribution due to diffusion.*

Next, we will discuss the asymptotic profile of the endemic equilibria as $d_I \to 0$.

Theorem 4.9. *Under the assumption (4.8) is true. If $d_S > 0$, $d_I \to 0$, each positive solution (S_{d_I}, I_{d_I}) of (4.10) satisfies*

$$(S_{d_I}, I_{d_I}) \to (S_*, I_*),$$

where

$$I_*(y) = \tilde{G}(y, S_*(y)) = \frac{(\beta(y) - \theta)\varepsilon S_* - \theta - \gamma + \sqrt{D}}{2\theta\varepsilon} > 0, \tag{4.28}$$

and

$$D = [(\beta(y) - \theta)\varepsilon S_* - \theta - \gamma]^2 + 4\theta\varepsilon[\beta(y) - \theta - \gamma]S_*.$$

S_* is a positive solution of the following problem

$$\begin{cases} -d_S \Delta S_* = \Lambda - \mu S_* - \beta(y)\dfrac{S_* I_*}{S_* + I_*}, & y \in Y, \\ \dfrac{\partial S_*}{\partial \vec{n}}(y, t) = 0, & y \in \partial Y. \end{cases} \tag{4.29}$$

Proof. C will be used to represent a positive constant that does not depend on $d_I > 0$, which may vary during the proof.

Step 1: Lower bound of S. Let $S(y_0) = \min_{y\in\bar{Y}} S(y)$. From the first expression of (4.10) and Proposition 2.2 in [35], one can deduce that

$$\Lambda - S(y_0) - \mu\beta(y_0)\frac{S(y_0)I(y_0)}{S(y_0) + I(y_0)} \leq 0, \tag{4.30}$$

which implies that

$$\Lambda \le \mu S(y_0) + \beta(y_0)\frac{I(y_0)}{S(y_0)+I(y_0)}S(y_0) \le \mu S(y_0) + \beta^* S(y_0),$$

and

$$S(y) \ge S(y_0) \ge \frac{\Lambda}{\mu+\beta^*} > 0, \quad y \in \bar{Y}. \tag{4.31}$$

Step 2: For small q, the $W^{1,q}-$ bound of S. The first expression of (4.10) becomes

$$\begin{cases} -d_S \Delta S + \left[\mu + \beta(y)\frac{I}{S+I}\right]S = \Lambda, & y \in Y, \\ \frac{\partial S}{\partial \vec{n}}(y,t) = 0, & y \in \partial Y. \end{cases} \tag{4.32}$$

It is easy to know that

$$\int_Y \Lambda dy \le C. \tag{4.33}$$

Combining (4.32) with (4.33), from Lemma 2.2 in [36], using L^1- estimate of the elliptic equation, we get

$$\|S\|_{W^{1,q}(Y)} \le C, \quad \forall q \in [1,\ N/(N-1)) \quad (\text{or, if } N=1,\ \forall q \in [1,\ \infty)). \tag{4.34}$$

Step 3: For any p, L^p- bound of S and I. Using (4.34) and Sobolev embedding $W^{1,q} \hookrightarrow L^{p_1}$, $q \in [1,\ N/(N-1))$, we get

$$\|S\|_{L^{p_1}(Y)} \le C, \quad \forall 1 < p_1 \le \frac{Nq}{N-q}.$$

Since q is close to $N/(N-1)$, the above equation becomes

$$\|S\|_{L^{p_1}(Y)} \le C, \quad \forall 1 < p_1 \le \frac{N}{N-2}. \tag{4.35}$$

If $N \le 2$, then for $\forall 1 < p_1 < \infty$, (4.35) is established.

Multiplying the second expression of (4.10) by $I^k (k > 0)$ and integrating, there is

$$0 \le d_I k \int_Y I^{k-1} |\nabla I|^2 dy = \int_Y \beta(y)\frac{S I^{k+1}}{S+I}dy - \theta \int_Y I^{k+1}dy - \gamma \int_Y \frac{I^{k+1}}{1+\varepsilon I}dy.$$

Thus, one can derive that

$$\theta \int_Y I^{k+1}dy \le \theta \int_Y I^{k+1}dy + \gamma \int_Y \frac{I^{k+1}}{1+\varepsilon I}dy \le \int_Y \beta(y)\frac{I}{S+I}S I^k dy \le \int_Y \beta^* S I^k dy,$$

which means

$$\theta \int_Y I^{k+1}dy \le \int_Y \beta^* S I^k dy, \quad \forall k > 0. \tag{4.36}$$

Make $k_1 = 1/q_1,\ q_1 = 1 + 1/(p_1 - 1)$ meet $1/q_1 + 1/p_1 = 1$. From (4.13), (4.35), (4.36), and Hölder inequality, there is

$$\theta \int_Y I^{k_1+1}dy \le \int_Y \beta^* S I^{k_1} dy \le \beta^* \left(\int_Y S^{p_1} dy\right)^{1/p_1} \left(\int_Y I dy\right)^{1/q_1} \le C.$$

Hence,

$$\|I\|_{L^{k_1+1}(Y)} \le C. \tag{4.37}$$

Next, making $k_2 = (k_1+1)/q_1 = 1/q_1 + 1/q_1^2$, by (4.35)–(4.37) and Hölder inequality, we have

$$\theta \int_Y I^{k_2+1} dy \le \int_Y \beta^* S I^{k_2} dy \le \beta^* \left(\int_Y S^{p_1} dy \right)^{1/p_1} \left(\int_Y I^{k_2 q_1} dy \right)^{1/q_1} \le C.$$

Therefore,

$$\|I\|_{L^{k_2+1}(Y)} \le C. \tag{4.38}$$

This step can be continued, as shown in Theorem 5.2 in [5]

$$\|S\|_{L^p(Y)},\ \|I\|_{L^p(Y)} \le C, \quad \forall 1 \le p < \infty. \tag{4.39}$$

Step 4: Convergence of S and I. By (4.32) and (4.39), there is

$$\|S\|_{W^{2,p}(Y)} \le C, \quad \forall 1 < p < \infty.$$

Thus, for a large enough number p, the standard embedding theory makes sure that there is a sequence $d_I \to 0$. Let $d_m = d_{I,m}$, as $m \to \infty$, $d_m \to 0$. The corresponding positive solutions sequence of (4.10), $(S_m, I_m) = (S_{d_{I,m}}, I_{d_{I,m}})$, $d_I = d_{I,m}$, such that

$$S_m \to S_*, \quad m \to \infty, \tag{4.40}$$

where $S_* \in C^1(\bar{Y})$.

From (4.31), $S_* > 0$ holds on $\bar{Y}$. I_m satisfies

$$\begin{cases} -d_m \Delta I_m = \dfrac{H(y, I_m(y))}{(S_* + I_m)(1+\varepsilon I_m)} I_m, & y \in Y, \\ \dfrac{\partial I_m}{\partial \vec{n}}(y,t) = 0, & y \in \partial Y, \end{cases} \tag{4.41}$$

where

$$H(y, I) = -\theta\varepsilon I^2 + [(\beta(y) - \theta)\varepsilon S_* - \theta - \gamma] I + [\beta(y) - \theta - \gamma] S_*.$$

From (4.40), for any given $\epsilon_1 > 0$, there is

$$H(y, I) < -\theta\varepsilon I^2 + [(\beta(y) - \theta)\varepsilon(S_* + \epsilon_1) - \theta - \gamma] I + [\beta(y) - \theta - \gamma](S_* + \epsilon_1) = H_{\epsilon_1}(y, I).$$

Therefore, for all sufficiently large m,

$$\frac{H(y, I_m(y))}{(S_* + I_m)(1+\varepsilon I_m)} I_m \le \frac{H_{\epsilon_1}(y, I)}{(S_* + I_m)(1+\varepsilon I_m)} I_m. \tag{4.42}$$

To emphasize the I dependence, we fixed y with $H_{\epsilon_1}(I)$ instead of $H_{\epsilon_1}(y, I)$. Then, for every $y \in Y$, according to (4.8), there is

$$-\theta\varepsilon < 0, \quad [\beta(y) - \theta - \gamma](S_* + \epsilon_1) > 0,$$

which means the quadratic equation $H_{\epsilon_1}(I) = 0$ exists a unique positive solution, marked as $G_{\epsilon_1}(y, S_*(y))$. Further, let $H_{\epsilon_1}(I) = [G_{\epsilon_1}(y, S_*(y)) - I]\tilde{H}_{\epsilon_1}(I)$, where $\tilde{H}_{\epsilon_1}(I) > 0$. For $y \in Y$ and $I > 0$, there is

$$\frac{H_{\epsilon_1}(y, I)}{(S_* + I_m)(1 + \varepsilon I_m)} I_m = \frac{[G_{\epsilon_1}(y, S_*(y)) - I_m]\tilde{H}_{\epsilon_1}(I_m)}{(S_* + I_m)(1 + \varepsilon I_m)} I_m. \tag{4.43}$$

For sufficiently large m, we study the auxiliary questions

$$\begin{cases} -d_m w = \dfrac{[G_{\epsilon_1}(y, S_*(y)) - w]\tilde{H}_{\epsilon_1}(w)}{(S_* + w)(1 + \varepsilon w)} w, & y \in Y, \\ \dfrac{\partial w}{\partial \vec{n}}(y, t) = 0, & y \in \partial Y. \end{cases} \tag{4.44}$$

Through (4.42) and (4.43), we know I_m is a lower solution of (4.44). For all large enough constant C, it is a super solution of (4.44). Therefore, there is at least one positive solution w_m meeting $I_m \le w_m \le C$. By having a similar discussion with Lemma $A.1$ in [37] and Lemma 5.1 in [38], there are $G_{\epsilon_1 > 0}$ and $\tilde{H}_{\epsilon_1} > 0$ at the same time. Hence, each positive solution of (4.44) meets

$$\bar{w}_m \to G_{\epsilon_1}(y, S_*(y)), \quad m \to \infty.$$

United with $I_m \le w_m \le C$, we can obtain

$$\limsup_{m \to \infty} I_m(y) \le G_{\epsilon_1}(y, S_*(y)). \tag{4.45}$$

Through a similar discussion, it can be available that

$$\liminf_{n \to \infty} I_m(y) \ge G^{\epsilon_1}(y, S_*(y)), \tag{4.46}$$

where $G^{\epsilon_1}(y, S_*(y))$ is the only positive solution of $H^{\epsilon_1}(I) = 0$ and

$$H^{\epsilon_1}(I) = H^{\epsilon_1}(y, I) = -\theta\varepsilon I^2 + [(\beta(y) - \theta)\varepsilon(S_* - \epsilon_1) - \theta - \gamma]I + [\beta(y) - \theta - \gamma](S_* - \epsilon_1).$$

By the arbitrariness of ϵ_1, there is

$$\lim_{\epsilon_1 \to 0} G_{\epsilon_1}(y, S_*(y)) = \lim_{\epsilon_1 \to 0} G^{\epsilon_1}(y, S_*(y)) = \tilde{G}(y, S_*(y)),$$

where $\tilde{G}(y, S_*(y))$ is the unique solution of $H(y, I) = 0$. By (4.45) and (4.46),

$$I_m(y) \to \tilde{G}(y, S_*(y)), \quad m \to \infty.$$

S_* meets (4.29). The proof is finished.

Remark 4.10. *d_I describes the spatial mobility of infected individuals. When $d_I \to 0$, it means that infected individuals are strictly confined to their initial infection locations and cannot spread to other areas. For example, infected individuals are immediately isolated upon detection, severely ill patients are immobile, or patients with highly contagious diseases such as Ebola lose mobility due to the severity of their condition. In this case, the spread of the disease is confined to a "localized small area" and does not lead to cross-regional outbreaks due to the movement of infected individuals. The infection dynamics at each location are independent of other regions.*

The essence of the diffusion coefficient approaching zero is "restricting the spatial mobility of a certain group of people", while the asymptotic distribution in the equilibrium state is the result of the mutual adaptation between the "spatial dynamics of the unrestricted population" and the "local dynamics of the restricted population". This pattern is highly consistent with the logic of "restricting movement to block transmission" in actual epidemic prevention and control.

5. Numerical examples

In this section, to further investigate the model's sensitivity to key parameters, we first select the following four parameters for global sensitivity analysis: The infection rate saturation coefficient ε , the diffusion coefficient of infected individuals d_I, the maximum treatment resource γ, and the spatial heterogeneity strength B. We employed Latin Hypercube Sampling (LHS) combined with the Sobol sensitivity analysis method, using the basic reproduction number R_0 and the final number of infections I_{final} as output indicators, and calculated their first-order and total-order sensitivity indices. The results of the sensitivity analysis are shown in Figure 1.

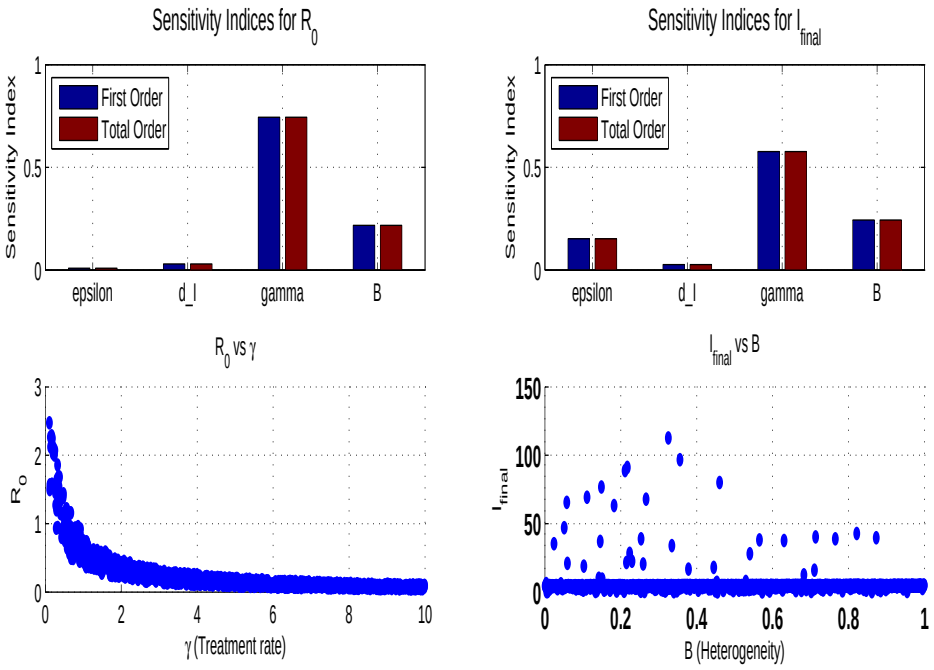


Figure 1. The upper-left and lower-right plots show the sensitivity indices, while the upper-right and lower-left plots display the scatter relationships.

The Figure 1 presents the results of a sensitivity analysis for four key parameters on the basic reproduction number R_0 and the final number of infections I_{final}. The analysis indicates that γ (maximum treatment resources) is the parameter that has the greatest impact on both R_0 and I_{final}. B (intensity of spatial heterogeneity) significantly affects the spread of the epidemic. The influence of d_I (dispersion coefficient of infected individuals) and ε (saturation coefficient) is relatively minor. The ranking of parameter sensitivity is: $\gamma > B > d_I > \varepsilon$. The results show that the increasing medical

resources (γ) is the most effective way to control the epidemic, and the reducing spatial heterogeneity (B) helps to lower the scale of infection.

Next, the numerical simulations are produced to better understand the dynamic behavior of system (1.3) under one-dimensional spatial domain $Y = [0, 1]$. Based on the insightful work in [25] and [20], the parameters are fixed as follows:

Table 1. The parameters of the numerical simulations.

Parameters	Symbol	Value	Unit
Recruitment rate	Λ	100	individuals.km^{-2}.day^{-1}
Natural death rate	μ	0.01	day^{-1}
Total removal rate	θ	0.23	day^{-1}
Maximum treatment rate	γ	0.2	day^{-1}
Treatment saturation coefficient	ε	0.5	individuals^{-1}.km^2
Diffusion coefficient (susceptible)	d_S	0.015	km^2.day^{-1}
Diffusion coefficient (infected)	d_I	0.001	km^2.day^{-1}
Transmission rate	$\beta(y)$	$0.12 - 0.107\cos(4\pi y)$	day^{-1}.individuals^{-1}.km^2

On the basis of [39], combining with system (1.3), we select the function of incidence rate as $\beta(y) = 0.12 - 0.107\cos(4\pi y)$. In this situation, $\beta^* = 0.227$ satisfies $\beta^* < \theta$ and $\mathcal{R}_0 < 1$ of Theorem 4.2. Set initial conditions

$$S_0(y) = 40, \ I_0(y) = 20. \tag{5.1}$$

We can get the following figures by Matlab.

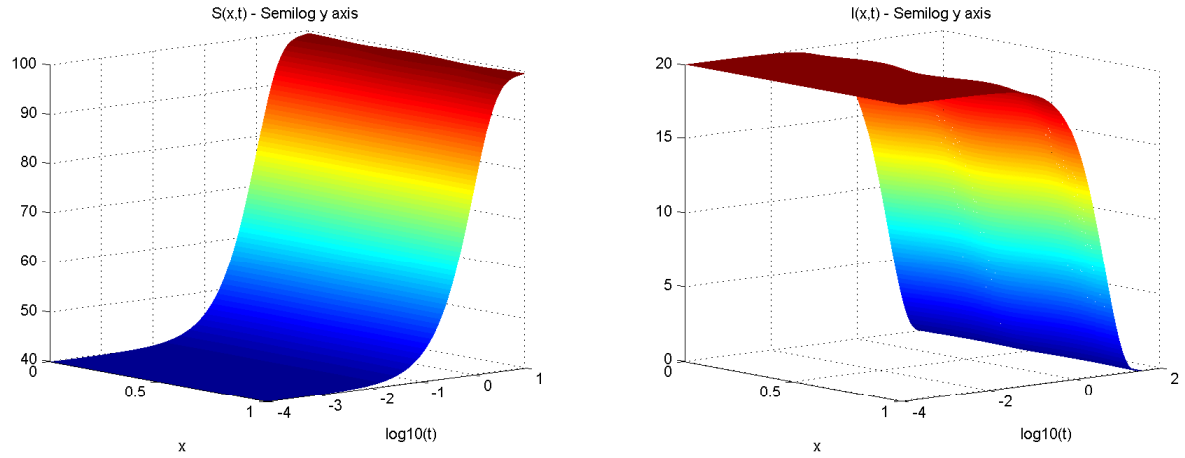


Figure 2. The time variation of the profile of S and I in the semi-logarithmic scale.

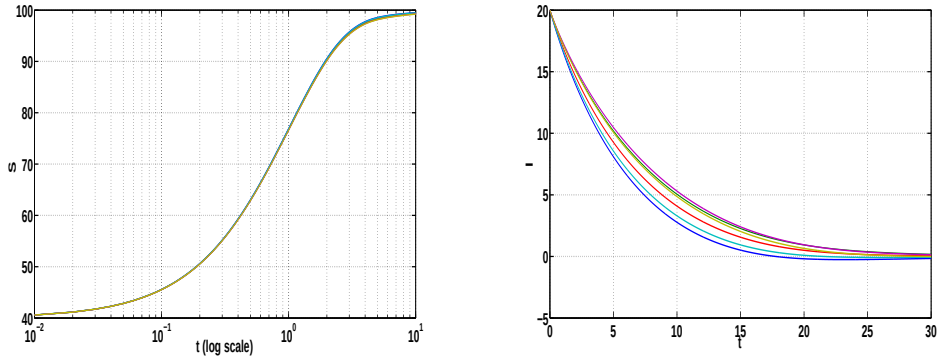
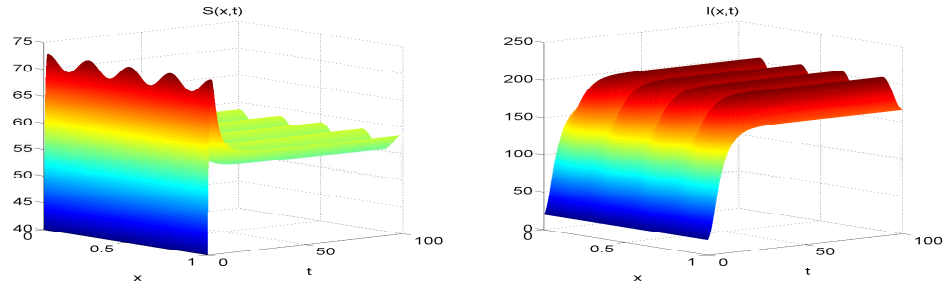


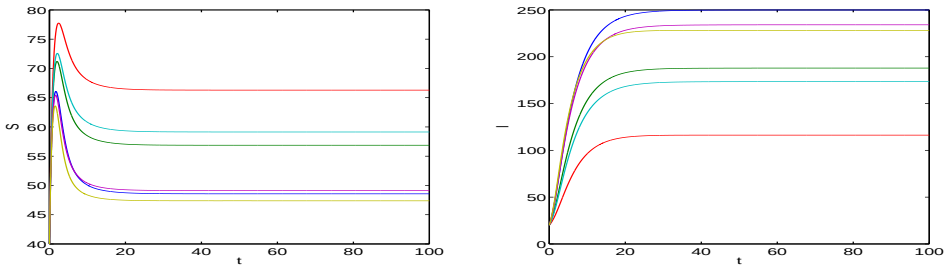
Figure 3. The asymptotic profile of $(S,\ I)$ as $t \to \infty$, S in the semi-logarithmic scale.

Using the asymptotic profile of solutions of system (1.3), we select the parameters given by (5) and initial conditions given by (5.1) as $\beta(y) = 0.12 - 0.107\cos(4\pi y)$. The solution in Figure 3 is corresponding to the disease-free equilibrium. Moreover, as the time increases, it can be seen from Figure 2 that the solution tends to the disease-free equilibrium. That is to say, the disease-free equilibrium point is globally asymptotically stable and the disease tends to extinction.

Next, keep the parameter values fixed as in Table 1 and the initial condition as (5.1). Select $\beta(y) = 0.99 - 0.559B\cos(2\pi y)$, where $0 \le B \le 1$ represents the spatial heterogeneity of transmission rate. The case of $B = 0$ corresponds to the spatially homogeneous situation. In this circumstance, $\beta(y) > \theta + \gamma$ satisfies hypotheses $\mathcal{R}_0 > 1$, and then we get the following figures as $B = 0.7$.



(a) Time variation of the profile of S and I

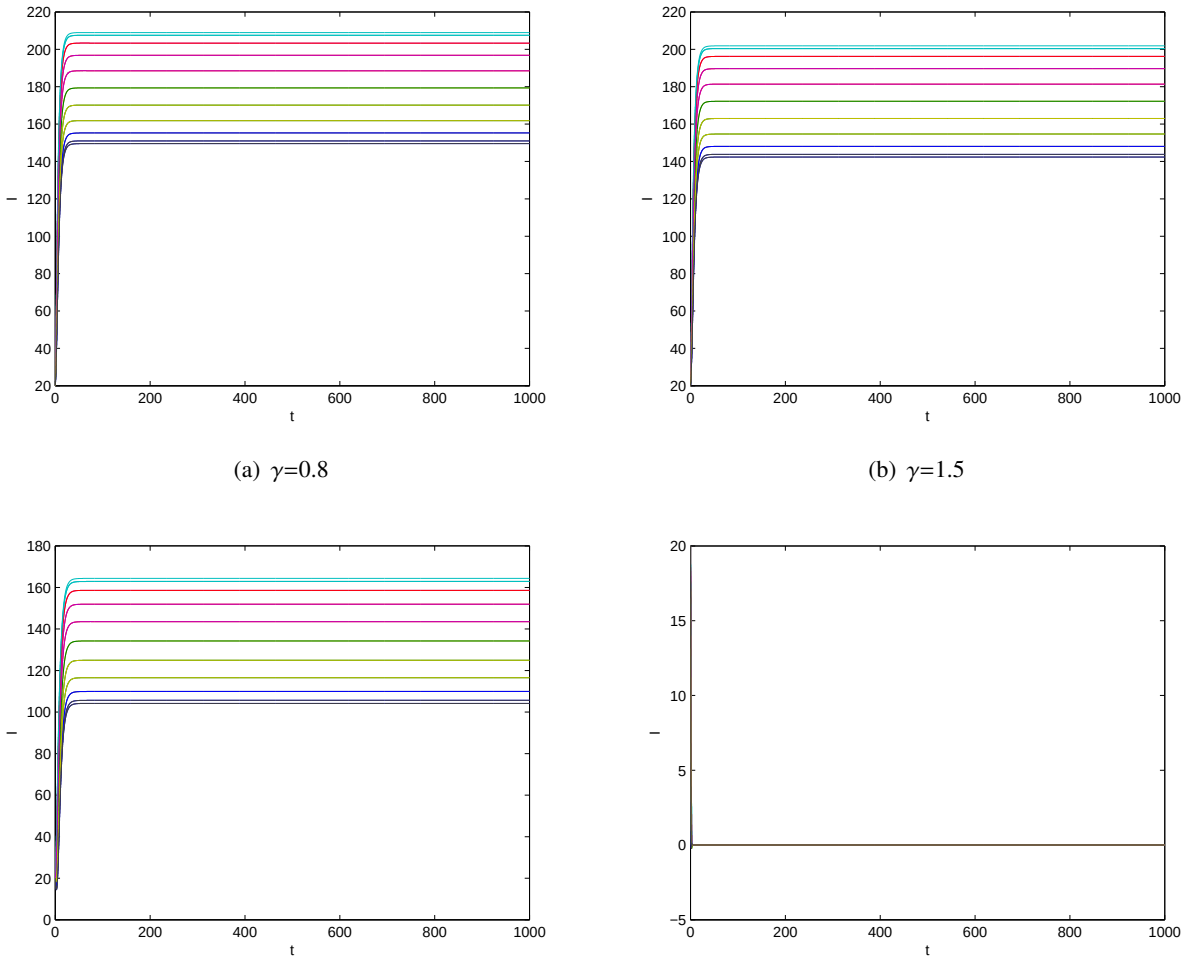


(b) The asymptotic profile of S and I as t is large enough

Figure 4. The asymptotic profile of S and I of model (1.3) with the parameters group Table 1 and the initial condition (5.1) as $\beta(y) = 0.99 - 0.559 \cdot 0.7\cos(2\pi y)$.

When $B = 0.7, \beta(y) = 0.99 - 0.559 \cdot 0.7\cos(2\pi y), \mathcal{R}_0 > 1$. Observing the distribution and change of S and I in Figure 4(b) , the solution corresponds to the endemic equilibrium, which means there is at least one endemic equilibrium with the increase of time, and the disease becomes an endemic disease and always exists. From Figure 4(a), the solutions of system (1.3) converge to a non-homogeneous endemic equilibrium.

Keeping the other parameters of Figure 4 and changing the maximum treatment resources, we can get the effects of γ on public health policies, which are given in the following figures.



(a) γ=0.8

(b) γ=1.5

(c) γ=5

(d) γ=10

Figure 5. The effects of the saturated treatment function.

Comparing Figure5(a) with Figure 5(d), as the maximum medical resource increases, from 0.2 to 0.8, 1.5, 5, and 10, the number of the infected person decreases, and gradually reduces to zero. Thus, people are vulnerable to infection. If the maximum medical resources are large enough, the disease will disappear.

Next, we continue to select $\beta(y) = 0.99 - 0.559B\cos(2\pi y)$. We will focus on the influence of variation of B, which means the change in the spatial heterogeneity on the disease prevalence.

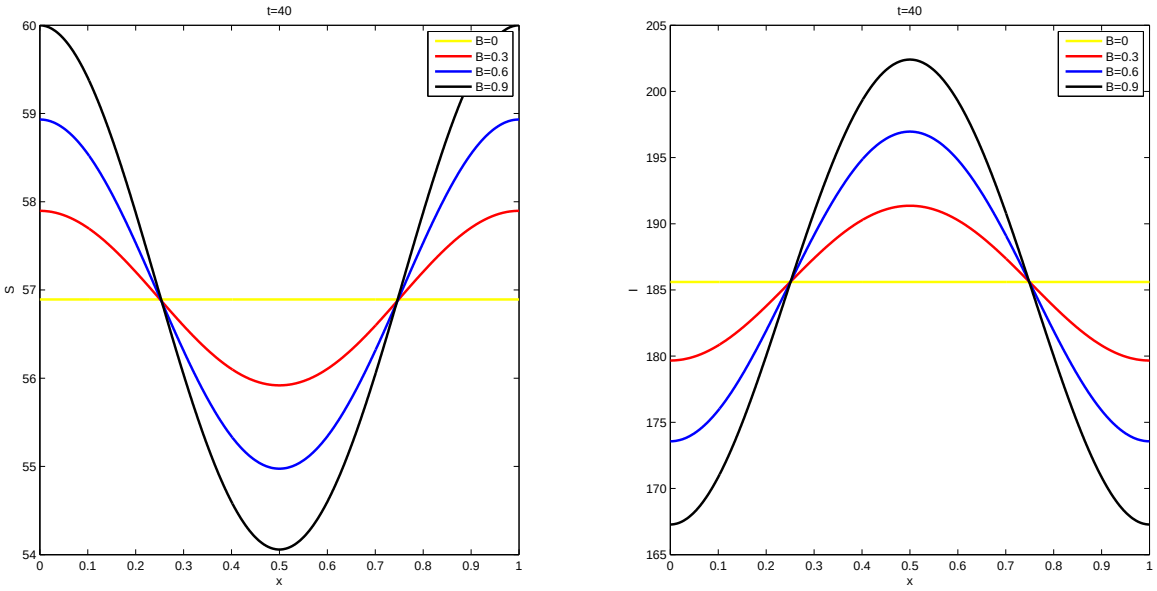


Figure 6. The change of the disease with B with selecting the parameters (Table 1), $\beta(y) = 0.99 - 0.559B\cos(2\pi y)$, initial conditions (5.1), and time $t = 40$.

As shown in Figure 6, with the increase of B, the level of the infected population in most of the middle regions has an increasing trend, and the peak value of the infected population is higher and higher. The greater the heterogeneity of the transmission rate is, the greater the heterogeneity of the susceptible population and the infected population.

6. Discussion and conclusions

In this paper, we systematically studied a diffusive SIR epidemic model with a standard incidence rate and saturated treatment function in a heterogeneous environment. Through theoretical analysis and numerical simulations, we derived several important conclusions regarding disease transmission and control, which provide insights for understanding real-world infectious disease dynamics and formulating effective public health policies. The major theoretical results of this study include: First, we defined the basic reproduction number $\mathcal{R}_0$ for the system and proved its properties with respect to the diffusion coefficient d_I of the infected population. Second, we demonstrated that when $\mathcal{R}_0 < 1$, the disease-free equilibrium is globally asymptotically stable, and the disease will become extinct. When $\mathcal{R}_0 > 1$, the disease becomes uniformly persistent, and there exists at least one endemic equilibrium. Furthermore, we analyzed the asymptotic profiles of the endemic equilibrium as the diffusion rates of susceptible or infected individuals approach zero, revealing the mechanisms by which population dispersal influences the spatial distribution of the disease.

In terms of public health policy, the numerical simulations in this study indicate that: First, increasing the maximum medical resource supply (i.e., increasing the parameter γ in the saturated treatment function) can significantly reduce the number of infected individuals and may turn an endemic situation into disease extinction. This suggests that enhancing medical investment and

optimizing resource allocation are effective measures for controlling infectious diseases. Second, reducing the spatial heterogeneity of the transmission rate (i.e., decreasing the amplitude of the function $\beta(\gamma)$) helps lower the infection peak and reduces the risk of regional outbreaks, making the epidemic more manageable and controllable. This finding implies that prevention and control strategies should emphasize inter-regional balance to avoid localized outbreaks due to disparities in resource allocation or intervention measures.

Compared to other studies, the innovation of this model lies in simultaneously incorporating standard incidence in a spatially heterogeneous environment, a saturated treatment function, and logistic growth of susceptibles, which more comprehensively reflects the transmission characteristics of real-world infectious diseases and the constraints of limited resources in control efforts. In the theoretical analysis, we combined Lyapunov functions, monotone dynamical systems theory, persistence theory, and asymptotic analysis to address the challenges posed by nonlinear terms and spatial heterogeneity.

This study has some limitations that could be addressed in future work. For instance, the model does not account for non-local temporal effects (such as incubation periods or infection delays) or the impact of stochastic disturbances or other uncertainties. Additionally, further exploration is needed for scenarios with non-constant diffusion coefficients or more complex boundary conditions. These aspects will be important directions for our future research.

In conclusion, we establish and analyze a more realistic reaction-diffusion SIR model, revealing the combined effects of spatial heterogeneity and limited medical resources on infectious disease transmission. we provide a theoretical basis and numerical support for understanding and predicting the spatiotemporal dynamics of diseases, as well as for assessing and optimizing control strategies.

Acknowledgments

This work was supported by the National Natural Science Foundation of China [grant numbers 12171135 and 11771115]; Natural Science Foundation of Hebei Province [grant numbers A2020201021 and A2019201396]; the Research Funding for High-Level Innovative Talents of Hebei University [grant number 801260201242].

Use of AI tools declaration

The writing of this article did not utilize any AI tools.

Conflict of interest

No potential conflict of interest was reported by the author(s).

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