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Research article

The boundedness on $BMO^{\beta}_{L_\alpha}$ of maximal operators for semigroups related to Laguerre operator

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Abstract: For $\alpha > -\frac{1}{2}$, the Laguerre differential operator is defined as

$$L_\alpha = \frac{1}{2}\Big(- \frac{d^2}{dx^2} + x^2 + \frac{1}{x^2}(\alpha^2 - \frac{1}{4})\Big),\ x \in (0, \infty).$$

For sufficiently good function f, the maximal functions associated with heat and Poisson semigroups are defined by

$$Tf(x) = \sup_{t>0} |T_t f(x)|,\ x \in (0, \infty),$$

where $\{T_t\}_{t>0}$ is the heat semigroup $\{e^{-tL_\alpha}\}_{t>0}$ or Poisson semigroup $\{e^{-t\sqrt{L_\alpha}}\}_{t>0}$ related to the Laguerre differential operator L_α. In this paper, we first established a $T1$ criterion for the boundedness of the γ-Laguerre-Calderón-Zygmund operator on $BMO^{\beta}_{L_\alpha}((0,\infty))$ $(0 \le \beta \le 1)$ spaces related to the Laguerre differential operator L_α. As applications, using this $T1$ criterion, we proved the boundedness on $BMO^{\beta}_{L_\alpha}((0,\infty))$ $(0 \le \beta \le 1)$ of the maximal operators for semigroups related to the Laguerre differential operator L_α.

Keywords: Laguerre operators; heat semigroup; Poisson semigroup; T1 theorem; Campanato type spaces

1. Introduction

In the research of harmonic analysis and partial differential equations, the regularity estimates for the second-order differential operators play an important role and have been studied extensively by many scholars. Sobolev and Schauder estimates for the second-order differential operators are fundamental results in this context and can be interpreted as the boundedness of Hölder spaces under negative powers of differential operators.

It is well-known that the classical Hölder space $C^\alpha(\mathbb{R}^n)$ can be identified with the Campanato space $BMO^\alpha(\mathbb{R}^n)$; see [1]. Bongioanni et al. [2] obtained the analogous results for the Schrödinger operator $H := -\Delta + |x|^2$. They identified the Hölder space related to the Schrödinger operator with a Campanato-type $BMO^\alpha_H(\mathbb{R}^n)$ space. The authors derived Hölder regularity estimates for time-independent Schrödinger operators via the $T1$ theorem method, and gave some applications [3]. Recently, Wang et al. studied the regularity of fractional heat semigroup associated with the Schrödinger operator $H := -\Delta + V$, for which the nonnegative potential V satisfies the reverse Hölder inequality [4].

For $n \in \mathbb{N}$ and $\alpha > -1$, the Laguerre function of Hermite type φ_α on $(0, +\infty)$ is defined as

$$\varphi^\alpha_n(y) = \left(\frac{\Gamma(n+1)}{\Gamma(n+1+\alpha)}\right)^{1/2} e^{-\frac{y^2}{2}} y^\alpha L^\alpha_n(y^2)(2y)^{1/2}, \ y \in (0, +\infty),$$

where $L^\alpha_n(x)$ represents the Laguerre polynomial of degree n and order α [5]. It is well-known that for every $\alpha > -1$, the system $\{\varphi^\alpha_n\}^\infty_{n=0}$ forms an orthonormal basis of $L^2((0, \infty))$. Furthermore, these functions are eigenfunctions of the Laguerre differential operator

$$L_\alpha = \frac{1}{2}\Big(- \frac{d^2}{dy^2} + y^2 + \frac{\alpha^2 - \frac{1}{4}}{y^2}\Big)$$

satisfying $L_\alpha \varphi^\alpha_n = (2n+\alpha+1)\varphi^\alpha_n$, and L_α can be extended to a positive self-adjoint operator on $L^2((0, \infty))$ by specifying a suitable domain of definition; see [6].

This paper is devoted to studying the boundedness of maximal operators of semigroups associated with the Laguerre operator $L_\alpha(\alpha > -1/2)$ on the Campanato-type spaces via the $T1$ theorem. Inspired by the work of Stinga and collaborators in [3], we first establish a simple $T1$ criterion of the γ-Laguerre-Calderón-Zygmund operator T given in Definition 1.1 to be bounded on Campanato-type spaces $BMO^\beta_{L_\alpha}((0, \infty))$ related to the Laguerre operator, and then use this $T1$ criterion to obtain the boundedness of maximal operators of semigroups associated with the Laguerre operator on $BMO^\beta_{L_\alpha}((0, \infty))$.

Let $\alpha > -1/2$, and the auxiliary function ρ_{L_α} related to the Laguerre operator L_α is defined as

$$\rho_{L_\alpha}(x) = \frac{1}{8} \min\left(x, \frac{1}{x}\right), \ x > 0. \tag{1.1}$$

Definition 1.1. Set $0 \le \gamma < 1$, $1 < p \le q < \infty$, $\frac{1}{q} = \frac{1}{p} - \gamma$. Let T be a bounded linear operator from $L^p(0, \infty)$ into $L^q(0, \infty)$ such that

$$Tf(x) = \int_0^\infty K(x, y)f(y)dy, f \in L^p_c((0, \infty)) \text{ and a.e. x} \notin \text{supp(f)}.$$

Then, T is a γ-Laguerre-Calderón-Zygmund operator if there exists a constant $C > 0$ and regularity exponent $\sigma > 0$,

(i) $|K(x, y)| \le \frac{C}{|x-y|^{1-\gamma}} e^{-|x-y|^2}$, for all $x, y \in (0, \infty)$ with $x \ne y$,

(ii) $|K(x, y) - K(x, z)| + |K(y, x) - K(z, x)| \le C \frac{|y-z|^\sigma}{|x-y|^{1+\sigma-\gamma}}$, when $|x - y| > 2|y - z|$.

Remark 1.1. The γ-Laguerre-Calderón-Zygmund operator is also the classical Calderón-Zygmund operator.

Our first main result is the following $T1$-type theorem for the Laguerre-Calderón-Zygmund operator T, given in Definition 1.1, concerning its boundedness on $BMO^{\beta}_{L\alpha}((0,\infty))$ related to the Laguerre operator L_α. For the definition and properties of $BMO^{\beta}_{L_\alpha}((0,\infty))$, we refer to Definition 2.1.

Theorem 1.2. *($T1$ criterion for $BMO^{\beta}_{L_\alpha}((0,\infty))$ ($0<\beta<1$). Let T be a γ-Laguerre-Calderón-Zygmund operator given in Definition 1.1, where the real number $\gamma>0$ and smoothness exponent δ satisfies $\beta+\gamma<min\{1,\delta\}((0,\infty))$. Then the operator T is bounded from $BMO^{\beta}_{L_\alpha}((0,\infty))$ into $BMO^{\beta+\gamma}_{L_\alpha}((0,\infty))$ if and only if there exists a constant $C>0$ such that, for every ball $B=B(x,r), x\in(0,\infty)$ and $0<r\le\frac{1}{2}\rho_{L_\alpha}(x)$,*

(i) $\displaystyle\frac{1}{|B(x,\rho_{L_\alpha}(x))|}\int_{B(x,\rho_{L_\alpha}(x))}|T1(y)|dy\le C,$

(ii) $\displaystyle\left(\frac{\rho_{L_\alpha}(x)}{r}\right)^{\beta}\frac{1}{|B|^{1+\gamma}}\int_B|T1(y)-(T1)_B|dy\le C,$

where $(T1)_B=\frac{1}{|B|}\int_B T1(y)dy$, the auxiliary function $\rho_{L_\alpha}(x)$ is defined in (1.1), and the $BMO^{\beta}_{L_\alpha}((0,\infty))$ space is defined in Definition 2.1.

Remark 1.2. Some further comments on Theorem 1.2:

(i) Theorem 1.2 also holds for vector-valued setting. For the Hermite operator case, see Remark 1.1 in [8]; for the Schrödinger operators case, see [3].

(ii) Suppose that $T1$ is a bounded function in $(0,\infty)$. Then $T1$ satisfies the first condition of Theorem 1.2. The second condition of Theorem 1.2 is fulfilled whenever there exists $0<\alpha\le1$ such that $|T1(x)-T1(y)|\le C|x-y|^{\alpha}$ for $x,y\in(0,\infty)$. For example, if $\nabla T1\in L^{\infty}(0,\infty)$, then the condition (ii) holds.

For endpoint case $\beta=0$, we get the following theorem.

Theorem 1.3. *($T1$ criterion for $BMO_{L_\alpha}((0,\infty))$). Let T be a γ-Laguerre-Calderón-Zygmund operator given in Definition 1.1, $0\le\gamma<min\{1,\delta\}$, with smoothness exponent δ. Then T is bounded from $BMO_{L_\alpha}((0,\infty))$ into $BMO^{\gamma}_{L_\alpha}((0,\infty))$ if and only if there exists a constant $C>0$ such that, for every ball $B=B(x,r)$, $x\in(0,\infty)$ and $0<r\le\frac{1}{2}\rho_{L_\alpha}(x)$,*

(i) $\displaystyle\frac{1}{|B(x,\rho_{L_\alpha}(x))|}\int_{B(x,\rho_{L_\alpha}(x))}|T1(y)|dy\le C;$

(ii) $\displaystyle\log\left(\frac{\rho_{L_\alpha}(x)}{r}\right)\frac{1}{|B|^{1+\gamma}}\int_B|T1(y)-(T1)_B|dy\le C,$

where $(T1)_B=\frac{1}{|B|}\int_B T1(y)dy$ and $\rho_{L_\alpha}(x)$ is defined in (1.1).

Notice that, by tracking down the exact constant in the proof, we can see that Theorem 1.3 can be viewed as the limiting case of Theorem 1.2. This phenomenon is analogous to known results for other operators. For the Schrödinger operator case due to [3] and for the Laguerre operator, see [7].

As a by-product of our main results, we also characterize the pointwise multipliers on $BMO^{\beta}_{L_\alpha}((0,\infty))$. That is, we obtain the following pointwise multiplier theorem.

Corollary 1. *Assume that g is a measurable function on $(0,\infty)$. We define the multiplier operator $T_g(f) := fg$. Then*

(i) T_g is a bounded operator in $BMO^{\beta}_{L_\alpha}((0,\infty))$ for $0<\beta<1$ if and only if $g\in L^{\infty}(0,\infty)$, and there exists a constant $C>0$ such that, for all ball $B=B(x_0,r_0)$, $x_0\in(0,\infty)$ and $0<r_0<\frac{1}{2}\rho_{L_\alpha}(x_0)$,

$$\left(\frac{\rho_{L_\alpha}(x_0)}{r_0}\right)^{\beta}\frac{1}{|B|}\int_B |g(y)-g_B|dy \le C.$$

(ii) T_g is a bounded operator in $BMO_{L_\alpha}((0,\infty))$ if and only if $g\in L^{\infty}((0,\infty))$ and there exists a constant $C>0$ such that, for all ball $B=B(x_0,r_0)$, $x_0\in(0,\infty)$ and $0<r_0<\frac{1}{2}\rho_{L_\alpha}(x_0)$,

$$\log\left(\frac{\rho_{L_\alpha}(x_0)}{r_0}\right)\frac{1}{|B|}\int_B |g(y)-g_B|dy \le C.$$

Remark 1.3. If $g\in C^{0,\alpha}((0,\infty))\cap L^{\infty}((0,\infty))$, $0<\alpha\le 1$, then T_g is bounded on $BMO_{L_\alpha}((0,\infty))$. Moreover, if for some γ-Laguerre-Calderón-Zygmund operator T we have that $T1$ defines a pointwise multiplier in $BMO^{\beta}_{L_\alpha}((0,\infty))$, then Corollary 1 and Theorems 1.2 and 1.3 imply that T is a bounded operator on $BMO^{\alpha}_{L_\alpha}((0,\infty))$.

Before introducing some applications, we first review some recent works about $T1$ theorem. Betancor et al. [8] established a $T1$ criterion for Calderón-Zygmund operators on $BMO_H(\mathbb{R}^n)$ related to the Hermite operator $H=-\Delta+|x|^2$. They then utilized this $T1$ criterion to obtain the $BMO_H(\mathbb{R}^n)$-boundedness of several singular integral operators associated with H, including maximal operators, Littlewood-Paley g-functions, Riesz transforms, and variation operators. Ma et al. [3] established an analogous $T1$ criterion for Schrödinger operator. They obtained the boundedness on Campanato-type space $BMO^{\alpha}_{H}(\mathbb{R}^n)$ of maximal operators, square functions, Laplace transform-type multipliers, negative powers, and Riesz transforms. The authors provided necessary and sufficient conditions in terms of $T1$ criteria for generalized Calderón-Zygmund-type operators to be bounded on $H^p_L(\mathbb{R}^n)$ and $BMO_L(\mathbb{R}^n)$ with respect to the Schrödinger operator $L=-\Delta+V$ with nonnegative potential V, which satisfies the reverse Hölder inequality. Wang et al. [4] studied the boundedness of the operator generated by the fractional semigroup related to the Schrödinger operators on Campanato-type space via the $T1$ theorem. More recently, Fan et al. proved the boundedness of variation operators for semigroups related to the Laguerre operator on $BMO_{L_\alpha}((0,\infty))$; see [7]. Ma et al. [9] established the oscillation of the Poisson semigroup associated with the parabolic Bessel operator by the vector-valued Calderón-Zygmund theorem. Ye et al. [10] also established the oscillation of the Poisson semigroup associated with a parabolic Bessel operator. Xiao and Li [11] studied the oscillation of Poisson semigroup related to discrete Laplacian by the discrete vector-valued Calderón-Zygmund theorem. For square functions and potential spaces associated with the discrete Laplacian, see [12].

Now, we turn to our applications. We shall show the boundedness of the maximal operators generated by semigroups associated with the Laguerre operator on $BMO^{\beta}_{L_\alpha}((0,\infty))$. We have the following theorem.

Theorem 1.4. *Let $0\le\beta\le 1$. The maximal operators generated by the heat semigroup $\{W^{L_\alpha}_t\}_{t>0}$ and Poisson semigroup $\{P^{L_\alpha}_t\}_{t>0}$ related to the Laguerre operator are bounded from $BMO^{\beta}_{L_\alpha}((0,\infty))$ into itself.*

Note that the authors in [13] proved that the maximal operators of the heat semigroup and Poisson semigroup, as well as the Littlewood-Paley g-functions of the heat and Poisson semigroups related to the Laguerre operator, are bounded on $BMO_{L_\alpha}((0,\infty))$ space.

Here, we also review recent progress in harmonic analysis associated with the Laguerre operator. Betancor et al. [14] studied a transference principle between the Laguerre and Hermite settings, and obtained some new properties of the Laguerre operators. Dziubański [15] studied the Hardy space $H^1_{L_\alpha}((0,\infty))$ related to the Laguerre operator L_α for $\alpha > -1/2$ and used the maximal function related to the heat-diffusion semigroup generated by L_α and atomic decompositions to characterize this Hardy space. In the sequel, Betancor et al. [16] characterized the Hardy space associated with certain Laguerre expansions by means of the Laguerre-Riesz transform. Betancor et al. [17] studied the L^p-boundedness of area Littlewood-Paley g-functions associated with Hermite and Laguerre operators. Dziubański et al. [18] studied the BMO spaces related to Schrödinger operators with potentials satisfying a reverse Hölder inequality. Cha and Liu [19] studied the $\mathrm{BMO}_{L_\alpha}((0,\infty))$ space related to L_α for $\alpha > -1/2$, which is identified as the dual space of $H^1_{L_\alpha}((0,\infty))$ associated with L_α. They characterized the $\mathrm{BMO}_{L_\alpha}((0,\infty))$ by Carleson measures related to appropriate square functions, and obtained the boundedness on $\mathrm{BMO}_{L_\alpha}((0,\infty))$ of the fractional integral operator and the Riesz transform related to the Laguerre operator. Dong and Liu in [20] established the boundedness of the Riesz transform associated with the Laguerre operator on $\mathrm{BMO}_{\mathrm{L}_\alpha}((0,\infty))$. Wang and Li in [21] considered the fractional integral of variable exponent space associated with the Schrödinger operators. For more references, see [22–25].

The outline of this paper is as follows: in Section 2, we introduce the definition and some properties of $BMO^\beta_{L_\alpha}((0,\infty))$. In Section 3, we show the proofs of Theorems 1.2 and 1.3. In Section 4, we establish the boundedness of the maximal operators generated by semigroups associated with the Laguerre operator L_α on $BMO^\beta_{L_\alpha}((0,\infty))$.

Throughout this paper, we denote by C and c suitable positive constants that may change at each occurrence. We will repeatedly use the inequality $t^\alpha e^{-\beta t} \le C$, $\alpha \ge 0, \beta > 0$.

2. The space $BMO^\beta_{L_\alpha}((0,\infty)), 0 \le \beta \le 1$

In this section, we first introduce the definition of $BMO^\beta_{L_\alpha}((0,\infty))$ related to the Laguerre operator L_α for $\alpha > -\frac{1}{2}$, and then give some properties that will be used frequently later; see e.g., [20].

We first introduce several notations. Denote by $B_r(x) = B(x,r)$ a ball with center x and radius r in $(0,\infty)$, B^* denotes $B_{2r}(x)$; $x \sim y$ means $x \le Cy$ and $y \le Cx$ for some constant.

Definition 2.1. Let $\alpha > -\frac{1}{2}$ and $0 \le \beta \le 1$. Denote by $B_r(x)$ a ball with center x and radius r in $(0,\infty)$. We say a locally integrable function f on $(0,\infty)$ belongs to $BMO^\beta_{L_\alpha}((0,\infty))$ if there exists a constant $C > 0$ independent of r and x such that

(i) $\displaystyle \frac{1}{|B_r(x)|}\int_{B_r(x)} |f(x) - f_{B_r(x)}|dx \le C|B_r(x)|^\beta$, for every ball $B_r(x)$ in $(0,\infty)$,

(ii) $\displaystyle \frac{1}{|B_r(x)|}\int_{B_r(x)} |f(x)|dx \le C|B_r(x)|^\beta$, for $r \ge \rho_{L_\alpha}(x)$,

where $f_{B_r(x)} = \frac{1}{|B_r(x)|}\int_{B_r(x)} |f(x)|dx$, and the auxiliary function $\rho_{L_\alpha}(x)$ is defined in (1.1). The norm $\|f\|_{BMO^\beta_{L_\alpha}}$ of f is defined as the minimum $C > 0$ such that (i) and (ii) above hold. Specially, we have

$BMO^0_{L_\alpha}((0,\infty)) = BMO_{L_\alpha}((0,\infty))$, where $BMO_{L_\alpha}((0,\infty))$ is the BMO space related to the Laguerre operator, see [19].

Thanks to the John-Nirenberg inequality, it can be seen that if in (i) and (ii) L^1-norms are replaced by L^p-norms for $1 < p < \infty$, then the space $BMO^\beta_{L_\alpha}((0,\infty))$ does not change, and equivalent norms appear. In this case, the conditions are expressed as:

(i)$_\mathrm{p}$ $\left(\frac{1}{|B_r(x)|}\int_{B_r(x)}|f(x) - f_{B_r(x)}|^p dx\right)^{\frac{1}{p}} \le C|B_r(x)|^\beta$, for every ball $B_r(x)$ in $(0,\infty)$,

(ii)$_\mathrm{p}$ $\left(\frac{1}{|B_r(x)|}\int_{B_r(x)}|f(x)|^p dx\right)^{\frac{1}{p}} \le C|B_r(x)|^\beta$, for $r \ge \rho_{L_\alpha}(x)$.

Note that if (ii) (resp. $(ii)_p$) above is true for some ball B, then (i) (resp. $(i)_p$) holds for the same ball, so we might ask for (i) (resp. $(i)_p$) only for balls with radii smaller that $\rho_{L_\alpha}(x)$. If $\beta > 1$, then the space $BMO^\beta_{L_\alpha}((0,\infty))$ only contains constant functions. Hence, the restriction $\beta < 1$ in the definition above is necessary.

Lemma 2.2. *(According to [18], Lemma 1) Let $\rho_{L_\alpha}(x)$ denote the auxiliary function defined in* (1.1)*. Assume that $x_0 = 1, x_j = x_{j+1} + \rho_{L_\alpha}(x_{j-1})$ for $j > 1$, and $x_j = x_{j+1} - \rho_{L_\alpha}(x_{j+1})$ for $j < 1$. We define the family of "critical balls" of $Q = \{Q_k\}_{k=-\infty}^{\infty}$, where $Q_k := \{x \in (0,\infty) : |x - x_k| < \rho_{L_\alpha}(x_k)\}$. Then*

1) $\cup_{k=-\infty}^{\infty} Q_k = (0,\infty)$;

2) For every $k \in Z$, $Q_k \cap Q_j = \emptyset$ provided that $j \notin \{k-1, k, k+1\}$;

3) For any $y_0 \in (0,\infty)$, at most three balls in Q have nonempty intersection with $Q(y_0, \rho_{L_\alpha(y_0)})$.

It is not hard to check that for every $Q_R(x) \subseteq (0,\infty)$ with $R \ge \rho_{L_\alpha}(y_0)$, there exists a constant $C > 0$ such that $|Q_R(x)| \le \sum_{Q_k\in Q, Q_k\cap Q_R(x)\neq\emptyset}|Q_k| \le C|Q_R(x)|$.

Lemma 2.3. *([20], p346) Let $\alpha > -\frac{1}{2}$ and $0 \le \beta \le 1$. An operator S defined on $BMO^\beta_{L_\alpha}((0,\infty))$ is bounded from $BMO^\beta_{L_\alpha}((0,\infty))$ into $BMO^{\beta+\gamma}_{L_\alpha}((0,\infty))$, $\beta + \gamma \le 1$, $\gamma > 0$, if there exists a constant $C > 0$ such that for every $f \in BMO^\beta_{L_\alpha}((0,\infty))$ and $k \in \mathbb{N}$,*

(A_k) $\frac{1}{|Q_k|^{1+\beta+\gamma}}\int_{Q_k}|Sf(x)|dx \le C\|f\|_{BMO^\beta_{L_\alpha}}$,

(B_k) $\|Sf\|_{BMO^{\beta+\gamma}(Q_k^*)} \le C\|f\|_{BMO^\beta_{L_\alpha}}$,

where $BMO^\beta(Q_k^*)$ denotes the usual BMO^β space (see [1]) on the ball Q_k^*.

Lemma 2.4. *Let $\rho_{L_\alpha}(x)$ denote the auxiliary function defined in* (1.1)*. Assume that $B = B(x,r)$ with $r < \rho_{L_\alpha}(x)$. Then*

(1) (see [26], p.141) If $f \in BMO_{L_\alpha}((0,\infty))$, $\alpha > -\frac{1}{2}$, then

$$|f_B| \le C\left(1 + \log\frac{\rho_{L_\alpha}(x)}{r}\right)\|f\|_{BMO_{L_\alpha}}.$$

(2) If $f \in BMO^\beta_{L_\alpha}((0,\infty))$, $\alpha > -\frac{1}{2}$, $0 < \beta \le 1$, then $|f_B| \le C_\beta\rho_{L_\alpha}(x)\|f\|_{BMO_{L_\alpha}}$.

Proof. (1) The detailed proof of (1) can be seen in [8].

(2) Let $2^n r < \rho_{L_\alpha}(x) < 2^{n+1} r$ with n be a positive integer. For $f \in BMO^\beta_{L_\alpha}((0,\infty))$, $\alpha > -\frac{1}{2}$, $0 < \beta \le 1$, we obtain that

$$\begin{aligned}
|f_B| &\le \frac{1}{|B|}\int_B |f(y) - f_{2B}|dy + \sum_{i=1}^{n} |f_{2^i B} - f_{2^{i+1}B}| + |f_{2^{i+1}B}| \\
&\le C\|f\|_{BMO^\beta_{L_\alpha}} |B|^\beta \sum_{i=1}^{n} (2^\beta)^i \\
&\le C|2|^\beta (2^n r)^\beta \|f\|_{BMO^\beta_{L_\alpha}} \\
&\le C_\beta \rho_{L_\alpha}(x)^\beta \|f\|_{BMO^\beta_{L_\alpha}}.
\end{aligned}$$

The proof of this lemma is completed. □

Lemma 2.5. *Let $B = B(x,r)$ with $r > 0$ and $x \in (0,\infty)$. Then, $f \in BMO^\beta_{L_\alpha}((0,\infty))$, $\alpha > -\frac{1}{2}$, $0 < \beta < 1$, if and only if f satisfies* (1) *in Definition 2.1, and for all balls Q_k in Lemma 2.2, it satisfies $|f|_{Q_k} \le C|Q_k|^\beta$.*

Proof. If $|f|_{Q_k} \le C|Q_k|^\beta$, let $B = B(x,r)$ with $x \in (0,\infty)$ and $r \ge \rho_{L_\alpha(x)}$. From Lemma 2.2, it holds that

$$\sum_{k\in K}\int_{Q_k} |f| \le (N+1)\int_{\bigcup\limits_{k\in K} Q_k} |f|,$$

where N is the constant controlling the overlapping, and $K = \{k : B \cap Q_k \ne \emptyset\}$ is finite. It is easy to know that there exists a constant C such that $Q_k \subset CB$ with $k \in K$. Then we have

$$\int_B |f| \le \sum_{k\in K}\int_{B\cap Q_k} |f| \le \sum_{k\in K}\int_{Q_k} |f| \le (N+1)\int_{\bigcup\limits_{k\in K} Q_k} |f| \le C|Q_k|^{1+\beta} \le C|B|^{1+\beta}.$$

Hence, $f \in BMO^\beta_{L_\alpha}((0,\infty))$. If $r < \rho_{L_\alpha}(x)$, by combining the above derivation and ([26], Corollary 1), we have $f \in BMO^\beta_{L_\alpha}((0,\infty))$. It is easy to check that the opposite statement holds. □

Lemma 2.6. *Let ρ_{L_α} denote the auxiliary function defined in* (1.1). *Assume that $x_0 \in (0,\infty)$ and $0 < r_0 < \rho_{L_\alpha}(x_0)$. Then, there exists a constant $C > 0$ such that for $x \in (0,\infty)$,*

(1) the function

$$g_{(x_0,r_0)}(x) =: \chi_{[0,r_0]}(|x-x_0|)\log\left(\frac{\rho_{L_\alpha}(x_0)}{r_0}\right) + \chi_{[r_0,\rho_{L_\alpha}(x_0)]}(|x-x_0|)\log\left(\frac{\rho_{L_\alpha}(x_0)}{|x-x_0|}\right)$$

belongs to $BMO_{L_\alpha}((0,\infty))$ and $\|g_{x_0,r_0}(x)\|_{BMO_{L_\alpha}} \le C$.

(2) the function

$$\begin{aligned}
h_{(x_0,r_0)}(x) =:& \chi_{[0,r_0]}(|x-x_0|)(\rho_{L_\alpha}(x_0)^\beta - r_0{}^\beta) \\
&+ \chi_{[r_0,\rho_{L_\alpha}(x_0)]}(|x-x_0|)(\rho_{L_\alpha}(x_0)^\beta - |x-x_0|^\beta)
\end{aligned}$$

belongs to $BMO^\beta_{L_\alpha}((0,\infty))$ and $\|h_{x_0,r_0}(x)\|_{BMO^\beta_{L_\alpha}} \le C$.

Proof. The proof of (1) is similar to the proof of Lemma 2.1 in [17]. The proof of (2) is similar to that of Lemma 2.5 in [3]. Here we omit these details. □

3. Proof of the main results

We divide this section into two subsections. In the first subsection, in order to prove Theorems 1.2 and 1.3, we first introduce the definition of the operator T in (3.1). Then we export an expression for Tf, where $T1$ appears, which will play an important role in our main results. In the second subsection, we first prove Theorems 1.2 and 1.3. Then we will characterize the pointwise multipliers of the spaces $BMO^{\beta}_{L_\alpha}((0,\infty))$.

3.1. The operators related to Laguerre operator L_α

In this subsection, in order to prove Theorems 1.2 and 1.3, we first introduce the definition of the operator T. We denote by $L^p_c((0,\infty))$ the set of functions $f \in L^p((0,\infty))$ whose support $supp(f)$ is a compact subset of $(0,\infty)$, $1 \le p \le \infty$.

Definition of Tf for $f \in BMO^{\beta}_{L_\alpha}((0,\infty))$**,** $\alpha > -\frac{1}{2}$**,** $0 < \beta < 1$**.** Let $B_R := B(0,R)$ for every $R > 0$. Assume that $f \in L^p((0,\infty))$, we define

$$Tf(x) = T(f\chi_{B_R})(x) + T(f\chi_{B_R^c})(x) = T(f\chi_{B_R})(x) + \int_{B_R^c} K(x,y)f(y)dy. \tag{3.1}$$

Observe that the first term in the right-hand side of (3.1) makes sense since $f\chi_R \in L^p_c((0,\infty))$. The integral in the second side of (3.1) is absolutely convergent. In fact, suppose that $f \in BMO_{L_\alpha}((0,\infty))$ and $R > 1$. For every $x \in B_R$, by using Definition 1.1(1), it follows that

$$\begin{aligned}
\int_{B_R^c} |K(x,y)||f(y)|dy &\le C\sum_{j=1}^{\infty}\int_{2^jR<|y|<2^{j+1}R} \frac{e^{-|x-y|^2}}{|x-y|^{1-\gamma}}|f(y)|dy \\
&\le C\sum_{j=1}^{\infty}\frac{1}{(2^jR)^{2-\gamma}}\int_{2^jR<|y|<2^{j+1}R}|f(y)|dy \\
&\le C\sum_{j=1}^{\infty}\frac{1}{(2^jR)^{1-\beta-\gamma}}\frac{1}{(2^jR)^{1+\beta}}\int_{2^jR<|y|<2^{j+1}R}|f(y)|dy \\
&\le \frac{C}{R^{1-\beta-\gamma}}\|f\|_{BMO_{L_\alpha}}, \text{ a.e. } x \in B_R.
\end{aligned}$$

The definition of $Tf(x)$ is independent of R in the sense that if $B_R \subset B_S$ with $R < S$, then the definition using B_S coincides almost everywhere in B_R with the one just given, because in that case, for a.e. $x \in B_R$, we have

$$\begin{aligned}
T(f\chi_{B_S}) - T(f\chi_{B_R}) &= T(f\chi_{B_S\setminus B_R})(x) \\
&= \int_{B_S\setminus B_R} K(x,y)f(y)dy \\
&= \int_{B_R^c} K(x,y)f(y)dy - \int_{B_S^c} K(x,y)f(y)dy.
\end{aligned}$$

In particular, the definition just given above is equally valid for $f \equiv 1 \in BMO_{L_\alpha}((0,\infty))$.

Next we export an expression for Tf, where $T1$ appears, which will play an important role in our main results. Let $B = B(x_0, r_0)$ where $x_0 \in (0, \infty)$ and $r_0 > 0$, we have

$$f = (f - f_B)\chi_{B^{**}} + (f - f_B)\chi_{(B^{**})^c} + f_B =: f_1 + f_2 + f_3 \tag{3.2}$$

Let us choose $R > 0$ such that $B^{**} \subset B_R$. Using (3.2) and the definition of Tf given in (3.1), it is clearly that for a.e. x ∈ B**,

$$\begin{aligned} Tf(x) &= T(f\chi_{B_R})(x) + \int_{B_R^c} K(x,y)f(y)dy \\ &= T((f - f_B)\chi_{B^{**}})(x) + T((f - f_B)\chi_{B_R \setminus B^{**}})(x) + f_B T(\chi_{B_R})(x) \\ &\quad + \int_{B_R^c} K(x,y)(f(y) - f_B)dy + f_B \int_{B_R^c} K(x,y)dy \\ &= T((f - f_B)\chi_{B^{**}})(x) + \int_{(B^{**})^c} K(x,y)(f(y) - f_B)dy + f_B T1(x). \end{aligned} \tag{3.3}$$

3.2. Proof of the main results

In this section, we first prove Theorems 1.2 and 1.3. Then we will characterize the pointwise multipliers of the spaces $BMO^{\beta}_{L_\alpha}((0,\infty))$.

Proof of Theorem 1.2. First we will see that the condition on $T1$ means that T is bounded from $BMO^{\beta}_{L_\alpha}((0,\infty))$ to $BMO^{\beta+\gamma}_{L_\alpha}((0,\infty))$. In order to do this, we shall prove that there exists a constant $C > 0$ such that the conditions (A_k) and (B_k) in Lemma 2.3 hold for every $k \in N$ and $f \in BMO^{\beta}_{L_\alpha}((0,\infty))$.

We first prove the condition (A_k) in Lemma 2.3. According to (3.3) with $B = Q_k$, Q_k standing for a ball with center x_k and radius r_k, it follows that

$$\begin{aligned} Tf(x) &= T((f - f_{Q_k})\chi_{Q_k^{**}})(x) + \int_{(Q_k^{**})^c} K(x,y)(f(y) - f_{Q_k})dy + f_{Q_k} T1(x) \\ &=: I_1 + I_2 + I_3, \text{ a.e. x} \in \mathrm{Q_k}. \end{aligned} \tag{3.4}$$

We begin with I_1. As $T : L^p(0,\infty) \to L^q(0,\infty)$, where $\frac{1}{q} + \gamma = \frac{1}{p}$, thanks to Hölder's inequality and John-Nirenberg's inequality, it holds that

$$\begin{aligned} \frac{1}{|Q_k|^{1+\beta+\gamma}} \int_{Q_k} I_1 dx &\le C\Big(\frac{1}{|Q_k|^{\frac{1}{q}+\beta+\gamma}} \int_{Q_k} |T((f - f_{Q_k})\chi_{Q^{**}})(x)|^q dx\Big)^{\frac{1}{q}} \\ &\le \frac{C}{|Q_k|^{\beta}}\Big(\frac{1}{|Q_k|} \int_{Q_k^{**}} |(f(x) - f_{(Q_k)})|^p dx\Big)^{\frac{1}{p}} \\ &\le C\|f\|_{BMO_{L_\alpha}}. \end{aligned}$$

To estimate I_2, if $x \in Q_k$, then $x \sim x_k$. Applying the size condition of $K(x,y)$ in Definition 1.1, for

$\beta + \gamma < 1$, we have

$$\begin{aligned}
\frac{1}{|Q_k|^{1+\beta+\gamma}} \int_{Q_k} |I_2| dx &= \frac{1}{|Q_k|^{\beta+\gamma}} \left| \int_{(Q_k^{**})^c} K(x,y)(f(y) - f_{Q_k}) dy \right| \\
&\leq \frac{1}{|Q_k|^{\beta+\gamma}} \int_{(Q_k^{**})^c} |K(x,y)||f(y) - f_{Q_k}| dy \\
&\leq \frac{C}{|Q_k|^{\beta+\gamma}} \sum_{j=2}^{\infty} \int_{2^j r_k < |y| < 2^{j+1} r_k} \frac{1}{|x-y|^{1-\gamma}} |f(y) - f_{Q_k}| dy \\
&\leq \frac{1}{|Q_k|^{\beta+\gamma}} \sum_{j=2}^{\infty} \frac{1}{(2^j r_k)^{1-\gamma}} \int_{|y| < 2^{j+1} r_k} |f(y) - f_{Q_k}| dy \\
&\leq C \sum_{j=2}^{\infty} 2^{-j(1-\beta-\gamma)} \frac{1}{(2^j r_k)^{1+\beta}} \int_{|y| < 2^{j+1} r_k} |f(y) - f_{Q_k}| dy \\
&\leq C \sum_{j=2}^{\infty} 2^{-j(1-\beta-\gamma)} \|f\|_{BMO^{\beta}_{L_\alpha}} \\
&\leq C \|f\|_{BMO^{\beta}_{L_\alpha}}.
\end{aligned}$$

Finally, in order to prove

$$\frac{1}{|Q_k|^{1+\gamma}} \int_{Q_k} |T1(x)| dx \leq C, \tag{3.5}$$

it is enough to show that

$$\frac{1}{|Q_k|^{1+\beta+\gamma}} \int_{Q_k} |I_3| dx \leq C \|f\|_{BMO^{\beta}_{L_\alpha}}.$$

Indeed, we have

$$\frac{1}{|Q_k|^{1+\beta+\gamma}} \int_{Q_k} |I_3| dx = \frac{|f_{Q_k}|}{|Q_k|^{\beta}} \frac{1}{|Q_k|^{1+\gamma}} \int_{Q_k} |T1(x)| dx \leq C \|f\|_{BMO^{\beta}_{L_\alpha}}.$$

According to the definition of $T1$, $T1(x) = T(\chi_{Q_k^{**}})(x) + T(\chi_{(Q_k^{**})^c})(x)$, $x \in Q_k$. Hence, by T maps L^p into L^q, and applying Hölder's inequality, we derive that

$$\frac{1}{|Q_k|^{1+\gamma}} \int_{Q_k} |T(\chi_{Q_k^{**}})(x)| dx \leq \frac{1}{|Q_k|^{\frac{1}{q}+\gamma}} \left(\int_{Q_k} |T(\chi_{Q_k^{**}})(x)|^q dx \right)^{\frac{1}{q}} \leq C \frac{|Q_k|^{\frac{1}{p}}}{|Q_k|^{\frac{1}{q}+\gamma}}.$$

Observe that $|x_k - y| \sim |x - y|$ when $x \in Q_k$, using the size condition (1) of $K(x,y)$, then

$$\begin{aligned}
T(\chi_{(Q_k^{**})^c})(x) &\leq C \sum_{j=2}^{\infty} \int_{2^j r_k < |x_k - y| < 2^{j+1} r_k} \frac{1}{|x-y|^{2-\gamma}} dy \\
&\leq C \sum_{j=2}^{\infty} \frac{2^j r_k}{(2^j r_k)^{2-\gamma}} \leq C (r_k)^{\gamma}.
\end{aligned}$$

Thus we obtain that the inequality (3.5) holds. Hence, it follows that

$$\frac{1}{|B_k|} \int_{B_k} |I_3| dx = |f_{B_k}| \frac{1}{|B_k|} \int_{B_k} |\widetilde{T}1(x)| dx \leq C |f_{B_k}| \leq C \|f\|_{BMO_{L_\alpha}}.$$

To sum up, we conclude that (A_k) in Lemma 2.3 holds for T with a constant that does not depend on k.

Next we prove that T satisfies the condition (B_k) in Lemma 2.3. Let $B = B(x_0, r_0) \subseteq Q_k^{**}$ where $x_0 \in (0, \infty)$ and $r_0 > 0$. We separate into the two cases: $r_0 \geq \rho_{L_\alpha}(x_0)$ and $r_0 < \rho_{L_\alpha}(x_0)$.

If $r_0 \geq \rho_{L_\alpha}(x_0)$. Notice that $\rho_{L_\alpha}(x_0) \sim \rho_{L_\alpha}(x_k) \sim r_0$. Combining with the computation above, we obtain that T satisfies (A_k),

$$\frac{1}{|B|^{1+\beta+\gamma}} \int_B |Tf(x) - (Tf)_B| dx \leq C \frac{1}{|B|^{1+\beta+\gamma}} \int_B |Tf(x)| dx \leq C\|f\|_{BMO^\beta_{L_\alpha}}.$$

If $r_0 < \rho_{L_\alpha}(x_0)$. By $f = f_1 + f_2 + f_3$ as in (3.2) and the decomposition of Tf in (3.3), we have for $x, z \in B$,

$$\begin{aligned}\frac{1}{|B|^{1+\beta+\gamma}} \int_B |Tf(x) - (Tf)_B| dx \leq & \frac{1}{|B|^{1+\beta+\gamma}} \int_B \frac{1}{|B|} \int_B |Tf_1(x) - Tf_1(z)| dz dx \\ & + \frac{1}{|B|^{1+\beta+\gamma}} \int_B \frac{1}{|B|} \int_B |F(x) - F(z)| dz dx \\ & + \frac{1}{|B|^{1+\beta+\gamma}} \int_B |Tf_3(x) - (Tf_3)_B| dx \\ =: & J_1 + J_2 + J_3.\end{aligned} \tag{3.6}$$

Here we write

$$F(x) = \int_{(B^{**})^c} K(x, y) f_2(y) dy.$$

For the first term J_1, note that T is bounded from L^p into L^q; applying Hölder's inequality, it follows that

$$\begin{aligned} J_1 \leq & \frac{2}{|B|^{1+\beta+\gamma}} \int_B |T_1 f(x)| dx \\ \leq & \frac{C}{|B|^{\frac{1}{q}+\beta+\gamma}} \Big(\int_{B^*} |f(x) - f_B|^q dx \Big)^{\frac{1}{q}} \leq C\|f\|_{BMO^\beta_{L_\alpha}}. \end{aligned} \tag{3.7}$$

For the second term J_2. Let $x, z \in B$, then $|x - z| < r_0$. Note that $|x - y| \sim |x_0 - y|$ when $y \in (B^{**})^c$. Applying the smoothness of the kernel in Definition 1.1, it holds that

$$\begin{aligned} \frac{1}{|B|^{\beta+\gamma}} |F(x) - F(z)| \leq & \frac{C}{|B|^{\beta+\gamma}} \int_{(B^{**})^c} |K(x, y) - K(z, y)| |f(y) - f_B| dy \\ \leq & \frac{C}{|B|^{\beta+\gamma}} \sum_{j=2}^{\infty} \int_{2^j r_0 < |x_0 - y| < 2^{j+1} r_0} \frac{|x - z|^\delta}{|x - y|^{1-\gamma+\delta}} |f(y) - f_B| dy \\ \leq & \frac{C}{|B|^{\beta+\gamma}} \sum_{j=2}^{\infty} \frac{r_0^\delta}{(2^j r_0)^{1-\gamma+\delta}} \int_{|x_0 - y| < 2^{j+1} r_0} |f(y) - f_B| dy \\ \leq & C \sum_{j=2}^{\infty} \frac{(2^j)^{\gamma+\beta-\delta}}{(2^j r_0)^{1+\beta}} \int_{|x_0 - y| < 2^{j+1} r_0} \Big| f(y) - f_{2^{j+1}B} + \sum_{k=0}^{j} (f_{2^{k+1}B} - f_{2^k B}) \Big| dy. \end{aligned}$$

Moreover, we have

$$
\begin{aligned}
\frac{1}{|B|^{\beta+\gamma}}|F(x)-F(z)| \leq & C\sum_{j=2}^{\infty}(2^j)^{\gamma+\beta-\delta}\Big(\frac{1}{(2^j r_0)^{1+\beta}}\int_{|x_0-y|<2^{j+1}r_0}|f(y)-f_{2^{j+1}B}|dy \\
& + \sum_{k=0}^{j}\frac{1}{(2^{k+1}r_0)^{1+\beta}}\int_{2^{k+1}r_0}|f(y)-f_{2^{k+1}B}|dy\Big) \\
\leq & C\sum_{j=2}^{\infty}(2^j)^{\gamma+\beta-\delta}\Big(\|f\|_{BMO^{\beta}_{L_\alpha}}+(j+1)\|f\|_{BMO^{\beta}_{L_\alpha}}\Big) \\
\leq & C\|f\|_{BMO^{\beta}_{L_\alpha}}.
\end{aligned}
$$

Hence, we obtain that $J_2 \leq C\|f\|_{BMO^{\beta}_{L_\alpha}}$. Now we turn to estimate J_3. Thanks to Lemma 2.4 (1) and the hypothesis condition of $T1$, we get

$$
\begin{aligned}
J_3 &\leq \frac{|f_B|}{|B|^{1+\beta+\gamma}}\int_B |T1(x)-(T1)_B|dx \\
&\leq C\|f\|_{BMO^{\beta}_{L_\alpha}}\Big(\frac{\rho_{L_\alpha}(x)}{r}\Big)^{\beta}\frac{1}{|B|^{1+\gamma}}\int_B|T1(x)-(T1)_B|dx \\
&\leq C\|f\|_{BMO^{\beta}_{L_\alpha}}.
\end{aligned}
$$

Combining with all the computation above, we obtain that the condition in Lemma 2.3 holds. Hence T is bounded from $BMO^{\beta}_{L_\alpha}((0,\infty))$ into $BMO^{\beta+\gamma}_{L_\alpha}((0,\infty))$.

Now we start to show the opposite statement. Suppose that the operator T is bounded from $BMO^{\beta}_{L_\alpha}((0,\infty))$ into $BMO^{\beta+\gamma}_{L_\alpha}((0,\infty))$. Let $B = B(x_0, r_0)$ where $x_0 \in (0,\infty)$ and $0 < r_0 < \rho_{L_\alpha}(x_0)$. We write the function $\widetilde{f}(x) \equiv f(x, x_0, r_0)$ defined in Lemma 2.6, by using (3.2), and it follows that

$$
\widetilde{f} = (\widetilde{f}-(\widetilde{f})_B)\chi_{(B^{**})} + (\widetilde{f}-(\widetilde{f})_B)\chi_{(B^{**})^c} + (\widetilde{f})_B =: \widetilde{f_1} + \widetilde{f_2} + \widetilde{f_B}.
$$

Then $\widetilde{f_B}T1(y) = T\widetilde{f}(y) - T\widetilde{f_1}(y) - T\widetilde{f_2}(y)$. Hence we have

$$
\begin{aligned}
\widetilde{f_B}\frac{1}{|B|^{1+\beta+\gamma}}\int_B|T1(y)-(T1)_B|dy \leq & \frac{1}{|B|^{1+\beta+\gamma}}\int_B|T\widetilde{f}(y)-(T\widetilde{f})_B|dy \\
& + \frac{1}{|B|^{1+\beta+\gamma}}\int_B|T\widetilde{f_1}(y)-(T\widetilde{f_1})_B|dy \\
& + \frac{1}{|B|^{1+\beta+\gamma}}\int_B|T\widetilde{f_2}(y)-(T\widetilde{f_2})_B|dy \\
=: & H_1 + H_2 + H_3.
\end{aligned}
$$

We can check that each of $H_i (i = 1, 2, 3)$ above is controlled by $\|\widetilde{f}\|_{BMO^{\beta}_{L_\alpha}} \leq C$, where C is independent of x_0 and r_0. Indeed, for the first term H_1, it holds that H_1 is controlled by $\|\widetilde{f}\|_{BMO^{\beta}_{L_\alpha}} \leq C$ because T is bounded from $BMO^{\beta}_{L_\alpha}((0,\infty))$ into $BMO^{\beta+\gamma}_{L_\alpha}((0,\infty))$. The second term H_2 follows by Hölder's inequality and $L^p \to L^q$ boundedness of T. The last term H_3 is done as J_2 in (3.6). Note that $\widetilde{f_B} \sim \Big(\rho_{L_\alpha}(x_0)^{\beta} - {r_0}^{\beta}\Big)$, we obtain that

$$
\Big(\frac{\rho_{L_\alpha}(x_0)}{r_0}\Big)^{\beta}\frac{1}{|B|^{1+\gamma}}\int_B|T1(y)-(T1)_B|dy \leq C.
$$

Hence the proof of this theorem is completed. □

Proof of Theorem 1.3. The proof of this theorem is similar to Theorem 1.2, putting $\beta = 0$ everywhere, and expecting just two differences. The first one is the estimation of the term J_3; here, we need to apply Lemma 2.4. The second difference is the proof of the opposite statement, where we need to consider the function $h(x_0, r_0)(x)$ of Lemma 2.6. □

Proof of Corollary 1. Suppose that g is a measurable function on $(0, \infty)$ satisfying the conditions (1) and (2) of Corollary 1. From the proof of Theorem 1.2, we know that g is a pointwise multiplier on $BMO^{\beta}_{L_\alpha}((0, \infty))$. In particular, the kernel of operator T is zero.

Now we suppose that g satisfies the above condition. Let the function $f(r, x_0)(x)$ of Lemma 2.6, where $x, x_0 \in (0, \infty)$ and $0 < r_0 < \rho_{L_\alpha}(x_0)$, belong to $BMO^{\beta}_{L_\alpha}((0, \infty))$, then $\rho_{L_\alpha}(x_0)^\beta \leq C_\beta(\rho_{L_\alpha}(x_0)^\beta - r_0^\beta)$. Applying Lemma 2.4 ($h = hg$), it follows that

$$\begin{aligned}\left(\frac{\rho_{L_\alpha}(x_0)}{r}\right)^\beta \frac{1}{|B|}\int_B |g(x)|dx &= C_\beta \frac{\rho_{L_\alpha}(x_0)^\beta - r_0^\beta}{|B|^{1+\beta}}\int_B |h(x)g(x)|dx \\ &\leq \frac{C_\beta}{|B|^{1+\beta}}\int_B |(hg)(x) - (hg)_B|dx + \frac{C_\beta}{|B|^\beta}(hg)_B \\ &\leq C_\beta ||hg||_{BMO^{\beta}_{L_\alpha}} \\ &\leq C_\beta ||h||_{BMO^{\beta}_{L_\alpha}}.\end{aligned}$$

Hence $|g|_B \leq C$, which does not depend on B.

On the other hand, if $x_0 \in (0, \infty)$ and $0 < r_0 < \rho_{L_\alpha}(x_0)$, by the boundedness in $BMO^{\beta}_{L_\alpha}((0, \infty))$ of T_g, we obtain that

$$\begin{aligned}\left(\frac{\rho_{L_\alpha(x_0)}}{r_0}\right)^\beta \frac{1}{|B|}\int_B |g(x) - g_B|dx \leq& C_\beta \frac{(\rho_{L_\alpha}(x_0))^\beta - r_0^\beta}{|B|^{1+\beta}}\int_B |g(x) - g_B|dx \\ \leq& \frac{C_\beta}{|B|^{1+\beta}}\int_B |g(x)h(x, r, x_0) - (gh(x, r, x_0))_B|dx \\ \leq& C_\beta ||gh(\cdot, r, x_0)||_{BMO^{\beta}_{L_\alpha}} \\ \leq& C_\beta ||h(\cdot, r, x_0)||_{BMO^{\beta}_{L_\alpha}},\end{aligned}$$

where C_β is independent of B. □

4. Proof of Theorem 1.4

In this section, we establish the boundedness on $BMO^{\beta}_{L_\alpha}((0, \infty))$ $(0 \leq \beta \leq 1)$ of the maximal operators for semigroups related to the Laguerre differential operator L_α by using the $T1$ criterion.

4.1. Maximal operators for the heat-diffusion semigroup e^{-tL_α}.

In this subsection, we establish the boundedness on $BMO^{\beta}_{L_\alpha}((0, \infty))$ $(0 \leq \beta \leq 1)$ of the maximal operator for heat semigroup related to the Laguerre operator. In order to prove $BMO^{\beta}_{L_\alpha}$-boundedness of the maximal operator $W_*^{L_\alpha} f(x) := \sup_{t>0} |W_t^{L_\alpha} f(x)|$, then it is enough to prove that the vector-valued operator $\Lambda(f) := \{W_t^{L_\alpha} f\}_{t>0}$ is bounded from $BMO^\beta_{L_\alpha}((0,\infty))$ into $BMO^\beta_{L_\alpha}((0,\infty);E)$. By the spectral theorem, it is easy to check that the vector-valued operator Λ is bounded from $L^2((0,\infty))$ into $L^2((0,\infty);E)$. Hence, we only need to estimate the vector-valued kernel $\|W_t^\alpha(x,y)\|_E$; see Proposition 4.1.

Now we introduce the properties of the heat-diffusion semigroup generated by L_α, $\alpha > -\frac{1}{2}$. Let $\{W_t^{L_\alpha}\}_{t\geq 0}$ be the heat-diffusion semigroup generated by L_α. For $f \in L^2((0,\infty))$, we have

$$W_t^{L_\alpha} f(x) \equiv e^{tL_\alpha} f(x) = \int_0^\infty W_t^\alpha(x,y) f(y) dy,\ x \in (0,\infty), t > 0, \tag{4.1}$$

where the kernel

$$W_t^\alpha(x,y) = \left(\frac{2e^{-t}}{1-e^{-2t}}\right)^{\frac{1}{2}} \left(\frac{2xye^{-t}}{1-e^{-2t}}\right)^{\frac{1}{2}} I_\alpha\left(\frac{2xye^{-t}}{1-e^{-2t}}\right) e^{-\frac{1}{2}\frac{1+e^{-2t}}{1-e^{-2t}}(x^2+y^2)}, \tag{4.2}$$

I_α is the modified Bessel function of the first kind and order α, see, e.g., [20].

In order to estimate the heat kernel $W_t^\alpha(x,y)$ conveniently, we introduce some properties of the Bessel function I_α (see [5]):

$$I_\alpha(z) \sim z^\alpha \quad z \to 0, \tag{4.3}$$

$$z^{\frac{1}{2}} I_\alpha(z) = \frac{1}{\sqrt{2\pi}} e^z (1 + O(\frac{1}{z})) \quad z \to \infty, \tag{4.4}$$

$$\frac{d}{dz}\left(z^{-\alpha} I_\alpha(z)\right) = z^{-\alpha} I_{\alpha+1}(z) \quad z \in (0,\infty). \tag{4.5}$$

Let $r = e^{-2t}$. Based on the discussion of $W_t^\alpha(x,y)$ in [20], the heat kernel can be decomposed as

$$W_t^\alpha(x,y) = H(r,x,y)\Phi(r,x,y)\Psi_\alpha(r,x,y) \tag{4.6}$$

where

$$H(r,x,y) = e^{-t(2\alpha+1)} \frac{(1+r)^{1/2}}{(1-r)^{1/2}} e^{-\frac{1}{2}\frac{1+r}{1-r}|x-y|^2},$$

$$\Phi(r,x,y) = \frac{\sqrt{2}}{(1+r)^{1/2} r^{(2\alpha+1)/4}} e^{-\frac{1-r}{(1+\sqrt{r})^2} xy},$$

$$\Psi_\alpha(r,x,y) = \left(\frac{2r^{1/2}xy}{(1-r)}\right)^{1/2} e^{-\frac{2r^{1/2}xy}{(1-r)}} I_\alpha\left(\frac{2r^{1/2}xy}{(1-r)}\right).$$

If $x,y \in (0,\infty)$ and $t > 1$, notice that $r < e^{-2}$, and from (4.3) and (4.6), it follows that

$$\begin{aligned} W_t^\alpha(x,y) &\leq Ce^{-ct} e^{-c\frac{|x-y|^2}{t}} \left(\frac{2xye^{-t}}{1-e^{-2t}}\right)^{\frac{1}{2}+\alpha} e^{-\frac{2xye^{-t}}{1-e^{-2t}}} \\ &\leq Ce^{-ct} e^{-c|x-y|^2}. \end{aligned}$$

If $x,y \in (0,\infty)$ and $0 < t < 1$, applying (4.4) and (4.6), we have

$$\begin{aligned} W_t^\alpha(x,y) &\leq Ct^{-\frac{1}{2}} e^{-c\frac{|x-y|^2}{t}} e^{-txy} \frac{1}{\sqrt{2\pi}} e^{\frac{2xye^{-t}}{1-e^{-2t}}} e^{-\frac{2xye^{-t}}{1-e^{-2t}}} (1 + O(\frac{1}{z})) \\ &\leq Ct^{-\frac{1}{2}} e^{-c\frac{|x-y|^2}{t}} e^{-txy}. \end{aligned}$$

Hence, we have for $x, y \in (0,\infty)$ and $t > 0$,

$$W_t^\alpha(x,y) \le Ct^{-\frac{1}{2}}e^{-c\frac{|x-y|^2}{t}}e^{-txy}\chi_{(0,1]}(t) + Ce^{-ct}e^{-c|x-y|^2}\chi_{(1,\infty)}(t). \tag{4.7}$$

We write $\widetilde{W}_t^\alpha(x,y) = \frac{1}{\sqrt{4\pi t}}e^{-\frac{|x-y|^2}{4t}}$, then $\widetilde{W}_t^\alpha f(x) = \int_0^\infty \widetilde{W}_t^\alpha(x,y)f(y)dy$.

Lemma 4.1. *There exists a constant $C > 0$ such that*

$$|W_t^\alpha(x,y) - \widetilde{W}_t^\alpha(x,y)| \le C\Big(\frac{\sqrt{t}}{\rho_{L_\alpha}(x)}\Big)^2 t^{-\frac{1}{2}}e^{-c\frac{|x-y|^2}{t}}.$$

Proof. Using definition of $\rho_{L_\alpha}(x)$ and (4.7),

$$\begin{aligned}|W_t^\alpha(x,y) - \widetilde{W}_t^\alpha(x,y)| \le& C|t^{-\frac{1}{2}}e^{-c\frac{|x-y|^2}{t}}e^{-txy}\chi_{(0,1]}(t) - \frac{1}{\sqrt{4\pi t}}e^{-c\frac{|x-y|^2}{4t}}| \\ &+ C|e^{-ct}e^{-c|x-y|^2}\chi_{(1,\infty)}(t) - \frac{1}{\sqrt{4\pi t}}e^{-c\frac{|x-y|^2}{4t}}| \\ \le& C(|e^{-txy}\chi_{(0,1]}(t) - 1| + |\sqrt{t}e^{-ct}\chi_{(1,\infty)}(t) - 1|)t^{-\frac{1}{2}}e^{-c\frac{|x-y|^2}{t}} \\ \le& C(|txy\chi_{(0,1]}(t)| + |t\chi_{(1,\infty)}(t)|)t^{-\frac{1}{2}}e^{-c\frac{|x-y|^2}{t}} \\ \le& C\Big(\frac{\sqrt{t}}{\rho_{L_\alpha}(x)}\Big)^2 t^{-\frac{1}{2}}e^{-c\frac{|x-y|^2}{t}}.\end{aligned}$$

The proof of this lemma is completed. □

Combining the above Lemma and its proof, we can get the following lemma.

Lemma 4.2. *Suppose that $0 < \delta < 2$, $x, y \in (0,\infty)$ and $t > 0$. If $|y-z| < \rho_{L_\alpha}(y)$ and $|y-z| < \frac{1}{4}|x-y|$, then*

$$|W_t^\alpha(x,y) - \widetilde{W}_t^\alpha(x,y) - (W_t^\alpha(x,z) - \widetilde{W}_t^\alpha(x,z))| \le C\Big(\frac{|y-z|}{\rho_{L_\alpha}(x)}\Big)^\delta t^{-\frac{1}{2}}e^{-c\frac{|x-y|^2}{t}}.$$

Lemma 4.3. *Let $0 < \delta < 2$. If $|y-z| < \sqrt{t}$, then there exists a constant $C > 0$ such that*

$$|W_t^\alpha(x,y) - W_t^\alpha(x,z)| \le C\Big(\frac{|y-z|}{\sqrt{t}}\Big)^\delta t^{-\frac{1}{2}}e^{-c\frac{|x-y|^2}{t}}.$$

Proof. If $|y-z| < \sqrt{t}$ and $\frac{1}{4}|x-y| \ge |y-z|$, by applying Lemma 4.2, it follows that

$$\begin{aligned}|W_t^\alpha(x,y) - W_t^\alpha(x,z)| \le& C\Big(\frac{|y-z|}{\sqrt{t}}\Big)^\delta\Big(1 + \frac{\sqrt{t}}{\rho_{L_\alpha}(x)}\Big)^\delta t^{-\frac{1}{2}}e^{-c\frac{|x-y|^2}{t}} \\ \le& C\Big(\frac{|y-z|}{\sqrt{t}}\Big)^\delta t^{-\frac{1}{2}}e^{-c\frac{|x-y|^2}{t}}.\end{aligned}$$

If $\frac{1}{4}|x-y| \le |y-z| < \sqrt{t}$, we can obtain that the conclusion above holds by the semigroup property. □

The semigroup $\{W_t^{L_\alpha}\}_{t\geq 0}$ is contractive in $L^p(0,\infty)$ for $(1 \leq p \leq \infty$, and selfadjoint in $L^2((0,\infty))$ but it is not Markovian. On the other hand, for every $f \in L^p((0,\infty))$, $1 \leq p \leq \infty$, $\lim_{t\to 0^+} W_t^\alpha f(x) = f(x)$ in $L^p((0,\infty))$, and a. e. $x \in (0,\infty)$. Suppose that $f \in BMO_{L_\alpha}^\beta((0,\infty))$, the integral

$$W_t^{L_\alpha} f(x) \equiv \int_0^\infty W_t^\alpha(x,y) f(y) dy$$

is absolutely convergent, for every $t \in (0,\infty)$ and $x \in (0,\infty)$.

For enough good function f, we define the maximal operators $W_*^{L_\alpha}$ associated with heat semigroup by

$$W_*^{L_\alpha} f(x) := \sup_{t>0} |W_t^{L_\alpha} f(x)|.$$

Obviously, we have $W_*^{L_\alpha} f(x) = \|W_t^{L_\alpha} f\|_E$, where $E = L^\infty((0,\infty), dx)$. Applying Theorem 1.2 and Remark 1.2, in order to prove that the maximal operator $W_*^{L_\alpha}$ is bounded from $BMO_{L_\alpha}^\beta((0,\infty))$ into itself, it is enough to prove that the vector-valued operator

$$\Lambda(f) := \{W_t^{L_\alpha} f\}_{t>0}$$

is bounded from $BMO_{L_\alpha}^\beta((0,\infty))$ into $BMO_{L_\alpha}^\beta((0,\infty); E)$, where the space $BMO_{L_\alpha}^\beta((0,\infty); E)$ is defined in the obvious way by replacing the absolute values $|\cdot|$ by norms $\|\cdot\|_E$. By the spectral theorem, it is easy to check that the vector-valued operator Λ is bounded from $L^2((0,\infty))$ into $L^2((0,\infty); E)$, see, e.g., [25]. Hence, we need to estimate the vector-valued kernel $\|W_t^\alpha(x,y)\|_E$. For vector-vector kernel $W_t^\alpha(x,y)$, we have the following estimates.

Proposition 4.1. *There exists a constant C such that for $x, y, z \in (0,\infty)$,*

(i) $\|W_t^\alpha(x,y)\|_E \leq \frac{C}{|x-y|} e^{-c|x-y|^2}$, *$x \neq y$;*

(ii) $\|W_t^\alpha(x,y) - W_t^\alpha(x,z)\|_E + \|W_t^\alpha(y,x) - W_t^\alpha(z,x)\|_E \leq C_\delta \frac{|y-z|^\delta}{|x-y|^{1+\delta}}$, *$x \neq y$ and $|x-y| > 2|y-z|$, for all* $0 < \delta < 1$*;*

(iii) For all $B = B(x,r)$ with $0 < r < \rho_{L_\alpha}(x)$, then

$$\log\left(\frac{\rho_{L_\alpha}(x)}{r}\right)\frac{1}{|B|}\int_B \|W_t^\alpha 1(y) - (W_t^\alpha 1)_B\|_E dy \leq C.$$

In particular, suppose that $\beta < min\{1,\delta\}$, then

$$\left(\frac{\rho_{L_\alpha}(x)}{r}\right)^\beta \frac{1}{|B|}\int_B \|W_t^\alpha 1(y) - (W_t^\alpha 1)_B\|_E dy \leq C.$$

Proof. Let us start to prove (i). For every $x, y, t \in (0,\infty)$, by applying (4.7), we have

$$\begin{aligned} W_t^\alpha(x,y) &\leq C t^{-\frac{1}{2}} e^{-c\frac{|x-y|^2}{t}} e^{-txy} \chi_{(0,1]}(t) + C e^{-ct} e^{-c|x-y|^2} \chi_{(1,\infty)}(t) \\ &\leq C e^{-c|x-y|^2} \chi_{(0,1]}(t) + C e^{-c|x-y|^2} \chi_{(1,\infty)}(t) \\ &\leq C e^{-c|x-y|^2}. \end{aligned}$$

Therefore, it is easy to deduce the desired conclusion.

(ii) The simple fact that $|x-y| \sim |x-z|$ when $|x-y| > 2|y-z|$. For all $0<\delta<1$, if $|y-z| \le \sqrt{t}$, from Lemma 4.3, it follows that

$$\begin{aligned}|W_t^\alpha(x,y) - W_t^\alpha(x,z)| \le & C\Big(\frac{|y-z|}{\sqrt{t}}\Big)^\delta t^{-\frac{1}{2}} e^{-c\frac{|x-y|^2}{t}} \\ \le & C\frac{|y-z|^\delta}{|x-y|^{1+\delta}}\Big(\frac{|x-y|}{\sqrt{t}}\Big)^{1+\delta} e^{-c\frac{|x-y|^2}{t}} \\ \le & C\frac{|y-z|^\delta}{|x-y|^{1+\delta}}.\end{aligned}$$

If $|y-z| \le \sqrt{t}$, from (4.7), it follows that

$$\begin{aligned}|W_t^\alpha(x,y)| \le & Ct^{-\frac{1}{2}} e^{-c\frac{|x-y|^2}{t}} e^{-txy}\chi_{(0,1]}(t) + Ce^{-ct}e^{-c|x-y|^2}\chi_{(1,\infty)}(t) \\ \le & C\Big(\frac{|y-z|}{\sqrt{t}}\Big)^\delta t^{-\frac{1}{2}} e^{-c\frac{|x-y|^2}{t}} \\ \le & C\frac{|y-z|^\delta}{|x-y|^{1+\delta}}.\end{aligned}$$

By the same argument, we can be obtained $|W_t^\alpha(x,z)| \le C\frac{|y-z|^\delta}{|x-y|^{1+\delta}}$. Combining with the computation above, and noting that the symmetry of the kernel $W_t^\alpha(x,y) = W_t^\alpha(y,x)$, we obtain the desired estimates.

Next, we turn to prove (iii). Let $B = B(x,r)$ with $0 < r \le \rho_{L_\alpha}(x)$. We have $\rho_{L_\alpha}(y) \sim \rho_{L_\alpha}(z) \sim \rho_{L_\alpha}(x)$ when $y, z \in B$. Notice that $W_t^\alpha 1(x) \equiv 1$, by Lemma 4.1, it holds that

$$\begin{aligned}|W_t^\alpha 1(y) - W_t^\alpha 1(z)| \le & |W_t^\alpha 1(y) - \widetilde{W}_t^\alpha 1(y)| + |(W_t^\alpha 1(z) - \widetilde{W}_t^\alpha 1(z))| \\ \le & \int_0^\infty \Big(\frac{\sqrt{t}}{\rho_{L_\alpha}(y)}\Big)^2 t^{-\frac{1}{2}} e^{-c\frac{|y-u|^2}{t}} + \Big(\frac{\sqrt{t}}{\rho_{L_\alpha}(z)}\Big)^2 t^{-\frac{1}{2}} e^{-c\frac{|z-u|^2}{t}} du \\ \le & \Big(\frac{\sqrt{t}}{\rho_{L_\alpha}(x)}\Big)^2 \int_0^\infty t^{-\frac{1}{2}} e^{-c\frac{|y-u|^2}{t}} + t^{-\frac{1}{2}} e^{-c\frac{|z-u|^2}{t}} du \\ = & C\Big(\frac{\sqrt{t}}{\rho_{L_\alpha}(x)}\Big)^2.\end{aligned}$$

Hence, when $\sqrt{t} < 2r$, we have

$$|W_t^\alpha 1(y) - W_t^\alpha 1(z)| \le C\Big(\frac{r}{\rho_{L_\alpha}(x)}\Big)^2. \tag{4.8}$$

If $\sqrt{t} > 2r$ and $\sqrt{t} > \rho_{L_\alpha}(x)$, then $\sqrt{t} > 2r > |y-z|$. Applying Lemma 4.3, we get

$$\begin{aligned}|W_t^\alpha 1(y) - W_t^\alpha 1(z)| \le & \int_0^\infty |W_t^\alpha(y,u) - W_t^\alpha(z,u)| du \\ \le & C\Big(\frac{|y-z|}{\sqrt{t}}\Big)^\delta \le C\Big(\frac{r}{\sqrt{t}}\Big)^\delta \\ \le & C\Big(\frac{r}{\rho_{L_\alpha}(x)}\Big)^\delta, 0<\delta<1.\end{aligned} \tag{4.9}$$

If $2r < \sqrt{t} < \rho_{L_\alpha}(x)$, we obtain that

$$\begin{aligned}|W_t^\alpha 1(y) - W_t^\alpha 1(z)| =&|W_t^\alpha 1(y) - \widetilde{W}_t^\alpha 1(y) - (W_t^\alpha 1(z) - \widetilde{W}_t^\alpha 1(z))|\\ =&\Big|\Big(\int_{|u-y|>c\rho_{L_\alpha}(x)} + \int_{4|y-z|<|u-y|<C\rho_{L_\alpha}(y)} + \int_{|u-y|<4|y-z|}\Big)\\ &W_t^\alpha(y,u) - \widetilde{W}_t^\alpha(y,u) - (W_t^\alpha(z,u) - \widetilde{W}_t^\alpha(z,u))\Big|\\ =:&|L_1 + L_2 + L_3|.\end{aligned}$$

For the first term L_1, we use the smoothness proved in Part (ii) of this proposition. Note that the same smoothness estimate is valid for the classical heat kernel. So we get

$$|L_1| \le \int_{|u-y|>c\rho_{L_\alpha}(x)} \frac{|y-z|^\delta}{|u-y|^{1+\delta}} du \le C\Big(\frac{r}{\rho_{L_\alpha}(x)}\Big)^\delta.$$

For L_2, note that $\rho_{L_\alpha}(u) \sim \rho_{L_\alpha}(y)$ when $|u-y| < c\rho_{L_\alpha}(y)$, applying Lemma 4.2, we obtain that

$$|L_2| \le C|y-z|^\delta \int_{4|y-z|<|u-y|<C\rho_{L_\alpha}(y)} \frac{1}{(\rho_{L_\alpha}(u))^\delta} t^{-\frac{1}{2}} e^{-c\frac{|u-y|^2}{t}} du \le C\Big(\frac{r}{\rho_{L_\alpha}(x)}\Big)^\delta.$$

For the last term L_3, observe that $\sqrt{t} < \rho_{L_\alpha}(x)$, from Lemma 4.1, it holds that

$$\begin{aligned}|L_3| \le& C\Big(\frac{\sqrt{t}}{\rho_{L_\alpha}(x)}\Big)^2\Big(\int_{|u-y|<4|y-z|} t^{-\frac{1}{2}} e^{-c\frac{|u-y|^2}{t}} du + \int_{|u-z|<4|y-z|} t^{-\frac{1}{2}} e^{-c\frac{|u-z|^2}{t}} du\Big)\\ \le& C\Big(\frac{\sqrt{t}}{\rho_{L_\alpha}(x)}\Big)^2 \int_{|\xi|\le\frac{4|y-z|}{\sqrt{t}}} e^{-c|\xi|^2} d\xi\\ \le& C\Big(\frac{\sqrt{t}}{\rho_{L_\alpha}(x)}\Big)^2 \frac{|y-z|}{\sqrt{t}}\\ \le& C\frac{r}{\rho_{L_\alpha}(x)}.\end{aligned}$$

Hence, for $2r < \sqrt{t} < \rho_{L_\alpha}(x)$, we obtain that

$$|W_t^\alpha 1(y) - W_t^\alpha 1(z)| \le C\Big(\frac{r}{\rho_{L_\alpha}(x)}\Big)^\delta. \tag{4.10}$$

Combining (4.8)–(4.10), we have

$$\|W_t^\alpha 1(y) - W_t^\alpha 1(z)\|_E \le C\Big(\frac{r}{\rho_{L_\alpha}(x)}\Big)^\delta. \tag{4.11}$$

Therefore, we have

$$\begin{aligned}&\log\Big(\frac{\rho_{L_\alpha}(x)}{r}\Big)\frac{1}{|B|}\int_B \|W_t^\alpha 1(y) - (W_t^\alpha 1)_B\|_E dy\\ \le& \log\Big(\frac{\rho_{L_\alpha}(x)}{r}\Big)\frac{1}{|B|}\int_B \frac{1}{|B|}\int_B \|W_t^\alpha 1(y) - (W_t^\alpha 1)_B\|_E dz dy\\ \le& C\Big(\frac{r}{\rho_{L_\alpha}(x)}\Big)^\delta \log\Big(\frac{\rho_{L_\alpha}(x)}{r}\Big) \le C.\end{aligned}$$

The first part of (iii) has been attained. For the second estimate, by (4.11), we have

$$\left(\frac{\rho_{L_\alpha}(x)}{r}\right)^{\beta}\frac{1}{|B|}\int_B \|W_t^\alpha 1(y)-(W_t^\alpha 1)_B\|_E dy \le C\left(\frac{r}{\rho_{L_\alpha}(x)}\right)^{\delta-\beta}\le C,$$

as soon as $\delta-\beta\ge 0$, which can be guaranteed if $\beta\le\min\{1,\delta\}$. □

Proof of Theorem 1.4 for the heat semigroup case. Note that $W_*^{L_\alpha}f(x)=\|W_t^{L_\alpha}f\|_E$, and the operator Λ above is bounded from $L^2((0,\infty))$ into $L^2((0,\infty);E)$; applying Theorem 1.2, Remark 1.2, and Proposition 4.1, we obtain that the maximal operator $W_*^{L_\alpha}$ is bounded from $BMO^{\beta}_{L_\alpha}((0,\infty))$ into itself. □

4.2. Maximal operators for Poisson semigroup $e^{-t\sqrt{L_\alpha}}$.

In this subsection, we will establish the boundedness on $BMO^{\beta}_{L_\alpha}((0,\infty))(0\le\beta\le 1)$ of the maximal operator for Poisson semigroup related to the Laguerre operator. Applying Bochner's subordination formula, the Poisson semigroup $P_t^{L_\alpha}\equiv e^{-t\sqrt{L_\alpha}}$ associated with the Laguerre differential operator L_α is given by

$$P_t^{L_\alpha}f(x)=e^{-t\sqrt{L_\alpha}}f(x)=\frac{t}{2\sqrt{\pi}}\int_0^\infty e^{-\frac{t^2}{4s}}W_s^{L_\alpha}f(x)\frac{ds}{s^{3/2}},\ x\in(0,\infty),\ t>0. \tag{4.12}$$

Thanks to (4.1), for enough good function f, we obtain that

$$\begin{aligned}P_t^{L_\alpha}f(x)&=\int_0^\infty\frac{t}{2\sqrt{\pi}}\int_0^\infty e^{-\frac{t^2}{4s}}W_s^{\alpha}(x,y)\frac{ds}{s^{3/2}}f(y)dy\\&=\int_0^\infty P_t^\alpha(x,y)f(y)dy,\end{aligned}\tag{4.13}$$

where the Poisson kernel

$$P_t^\alpha(x,y)=\frac{t}{2\sqrt{\pi}}\int_0^\infty e^{-\frac{t^2}{4s}}W_s^{\alpha}(x,y)\frac{ds}{s^{3/2}},\ x\in(0,\infty),\ t>0.$$

To get the boundedness of the maximal operator

$$P_*^{L_\alpha}f(x):=\sup_{t>0}|P_t^{L_\alpha}f(x)|=\|P_t^{L_\alpha}f\|_{L^\infty((0,\infty),dx)}\tag{4.14}$$

in $BMO^{\beta}_{L_\alpha}((0,\infty))$, we proceed using a vector-valued approach and the boundedness of maximal heat semigroup $W_*^{L_\alpha}$. The following proposition is completely analogous to Proposition 4.1.

Proposition 4.2. *Let $E=L^\infty((0,\infty),dx)$. Then, there exists a constant C such that for $x,y,z\in(0,\infty)$,*

(i) $\|P_t^\alpha(x,y)\|_E\le\frac{C}{|x-y|}e^{-|x-y|^2}$, *$x\ne y$;*

(ii) $\|P_t^\alpha(x,y)-P_t^\alpha(x,z)\|_E+\|P_t^\alpha(y,x)-P_t^\alpha(z,x)\|_E\le C_\delta\frac{|y-z|^\delta}{|x-y|^{1+\delta}}$, *$x\ne y$ and $|x-y|>2|y-z|$, for $0<\delta<1$.*

(iii) for all $B=B(x,r)$ with $0<r<\rho_{L_\alpha}(x)$, then

$$\log\left(\frac{\rho_{L_\alpha}(x)}{r}\right)\frac{1}{|B|}\int_B\|P_t^\alpha 1(y)-(P_t^\alpha 1)_B\|_E dy\le C.$$

In particular, if $\beta<min\{1,\delta\}$, then

$$\left(\frac{\rho_{L_\alpha}(x)}{r}\right)^{\beta}\frac{1}{|B|}\int_B\|P_t^\alpha 1(y)-(P_t^\alpha 1)_B\|_E dy\le C.$$

Proof. The estimate of $W_t^\alpha(x,y)$ is transferred to $P_t^\alpha(x,y)$ by the formula (4.11) and the proposition 4.1. It is easy to check that (i) and (ii) hold. We just sketch the proof of (iii). For any $y,z\in B=B(x,r)$, $x\in(0,\infty)$ and $r\le\rho_{L_\alpha}(x)$, by Minkowski's integral, we get

$$\begin{aligned}&\|P_t^\alpha 1(y)-(P_t^\alpha 1)_B\|_E\\ =&\frac{1}{B}\int_B\|P_t^\alpha 1(y)-P_t^\alpha 1(z)\|_E dz\\ \le& C\frac{1}{B}\int_B\frac{1}{\Gamma(\frac{1}{2})}\int_0^\infty e^{-u}\|W_u 1(y)-W_u 1(z)\|_E\frac{du}{u^{1/2}}dz\\ \le& C\Big(\frac{r}{\rho_{L_\alpha}(x)}\Big)^\delta.\end{aligned}$$

Hence, we obtain that

$$\log\Big(\frac{\rho_{L_\alpha}(x)}{r}\Big)\frac{1}{|B|}\int_B\|P_t^\alpha 1(y)-(P_t^\alpha 1)_B\|_E dy\le C$$

and

$$\Big(\frac{\rho_{L_\alpha}(x)}{r}\Big)^\beta\frac{1}{|B|}\int_B\|P_t^\alpha 1(y)-(P_t^\alpha 1)_B\|_E dy\le C.$$

The proof of this proposition is completed. □

Proof of Theorem 1.4 for the Poisson semigroup case. Note that $P_*^{L_\alpha}f(x)=\|P_t^{L_\alpha}f\|_E$; applying Proposition 4.2, by the same argument with the proof of Theorem 1.4 of the Poisson semigroup case, we immediately obtain that the maximal operator $P_*^{L_\alpha}$ is bounded from $BMO_{L_\alpha}^\beta((0,\infty))$ into itself. □

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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