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Research article

Some results in generalized topological groups

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Abstract: It is proved in this paper that if G is a $\mathcal{G}$-topological group and $\{K_i : i \in \omega\}$ is a family of generalized open subsets containing the identity element e in G satisfying $K_{i+1}^2 \subset K_i$ and $K_i^{-1} = K_i$ for every $i \in \omega$, then G/H is metrizable where $H = \bigcap_{i\in\omega} K_i$. Let (G, τ) be a $\mathcal{G}$-topological group satisfying for every $e \in U \in \tau$ that there is $e \in O \in \tau$ such that $O^2 \subset U$. Then, we obtain that 1) if H is generalized κ-narrow and $cl_G H = G$, then G is also generalized κ-narrow; 2) AB is generalized κ-narrow in G provided that A and B are generalized κ-narrow in G. Finally, we consider coset spaces. It is shown that 3) for a subgroup H of G, if H and the generalized quotient space G/H have countable generalized pseudocharacter, then G also has countable generalized pseudocharacter; 4) for a subgroup H, if H is the generalized connected component containing e in G, G/H is generalized totally disconnected; 5) if H is a generalized closed subgroup and generalized totally disconnected, then G is also generalized totally disconnected when G/H is generalized totally disconnected.

Keywords: generalized topology groups; coset spaces; total disconnectedness; cardinal invariants; metrizability

1. Introduction

In 2002, Császár first introduced the concept of a generalized topology in [1] which mainly investigated the generalized continuity of mappings and obtained some equivalence conditions for generalized continuity. Recall that a generalized topology on a given set X is a collection $\mathscr{G}$ of subsets of X meeting $\emptyset \in \mathscr{G}$ and $G_i \in \mathscr{G}$ for $i \in I \neq \emptyset$ implies $\bigcup_{i\in I} G_i \in \mathscr{G}$ [1]. Noting that the generalized topology is not closed under finite intersections, if X is a nonempty set and $\mathscr{G}$ is a generalized topology on X, then the pair $(X, \mathscr{G})$ is called a generalized topological space [1]. Afterwards, many topologists devoted themselves to the study of generalized topological spaces and obtained a series of results.

In 2004, Császár introduced separation axioms in generalized topological spaces, similar to those in topological spaces as discussed in [2]. Császár also studied generalized connectedness in [3]. In 2009, Császár [4] extended the product of topological spaces to the product of generalized topological spaces, exploring its properties and applications. Shen [5] further investigated the generalized product of generalized topological spaces based on the literature [4]. Ge and Ying [6] and Sarsak [7] further investigated the generalized separability in generalized topological spaces in 2010 and 2011, respectively.

De Arruda Saraiva in [8] introduced a definition of a generalized quotient topology. Let (X, τ) be a generalized topological space and Y a nonempty set. If the mapping $\pi : X \to Y$ is surjective, then $\tau' = \{B \subset Y : \pi^{-1}(B) \in \tau\}$ is a generalized topology on Y which is called the generalized quotient topology on Y. Call π is the generalized quotient mapping. Some characterizations of the generalized quotient topology are given. For a generalized topological group (G, τ) and a subgroup H, in section 4 we shall discuss some properties on the quotient group G/H which is endowed with the generalized topology with respect to $\pi : G \to G/H$, defined as $\pi(x) = xH$ for each $x \in G$.

A topological group has both an algebraic structure of group and a topological structure [9]. In the past few years, research on topological groups has yielded prolific achievements. Researchers interested in this topic may refer to [10–12]. As a generalization of countable compactness in general topological spaces, the authors in [10] mainly investigated strong q-spaces, strict q-spaces, and point pseudocompact spaces in topological groups. Drawing insights from the concept of topological groups, Hussain et al. [13] and Li and Li [14] introduced generalized topological groups in 2013 and 2014, respectively, and achieved a series of significant achievements. Notably, a generalized topological group has both an algebraic structure of group and a generalized topological structure. In 2013, Hussain et al. [15] introduced the notions of ultra Hausdorffness and ultra $\mathcal{G}$-Hausdorffness and gave the relation between the ultra $\mathcal{G}$-Hausdorffness and $\mathcal{G}$-compactness.

The theory of soft sets is aimed at addressing uncertainty. Soft topology, in turn, emerges from the combination of classical topology and the theory of soft sets. The concept of soft topological groups was proposed by two different scholars in 2014, see [16, 17]. Despite sharing the same name, they have different structures. The authors in [18] introduced the concept of soft rough topological groups which generalized the two different definitions of soft topological groups. Hence, soft rough topological groups are different from generalized topological groups.

It is well known that the following result holds in topological groups.

Theorem 1.1 ([9, Theorem 1.4.30])**.** *Assume that (G, τ) is a topological group. If K is compact in G and F is closed in G, then both KF and FK are closed.*

Recall that a generalized topological space (Y, τ) is said to be *compact* if each cover of Y composed of elements of τ admits a finite subcover [5].

Example 1.2 ([14, Example 3.3])**.** Let R denote the set consisting of all real numbers which is a group with the usual addition. Set $\mathscr{U} = \{[c, d] : -\infty < c < d < +\infty\}$ and $\tau = \{U : U = \bigcup \mathscr{U}' \text{ for some } \mathscr{U}' \subset \mathscr{U}\} \cup \{\emptyset\}$. Then, (R, τ) is a generalized topological group.

Example 1.3. By Example 1.2, (R, τ) is a generalized topological group. Let $K = \{\frac{1}{n} : n \in N_+\} \cup \{0\}$ and $F = \{0\}$. Clearly, K is compact and F is generalized closed. Since $K + F = F + K = K$ and $R \setminus K$ is not generalized open, $K + F$ is not generalized closed.

Example 1.4 ([13, Example 2.5])**.** The set R consisting of all real numbers endowed with the usual addition is a group. Let $\mathscr{V} = \{(-\infty, a), (b, +\infty) : a, b \in R\}$ and $\tau = \{B : B = \bigcup \mathscr{V}' \text{ for some } \mathscr{V}' \subset \mathscr{V}\} \cup \{\emptyset\}$. Then, (R, τ) is a generalized topological group.

Example 1.5. Consider the generalized topological group (R, τ) in Example 1.4. Set $F = \{1, 2, 3\}$ and $P = (0, 1)$. Then, $F + P = (1, 2) \cup (2, 3) \cup (3, 4)$. Obviously, F is compact, P is generalized closed in R, and $G \setminus (F + P) = (-\infty, 1] \cup \{2, 3\} \cup [4, +\infty) \notin \tau$. Hence, $F + P$ is not generalized closed.

By Examples 1.3 and 1.5, we see some properties of topological groups not valid in generalized topological groups. We mainly discuss a prenorm in a special class of generalized topological groups, the cardinal functions of generalized topological groups, and some basic properties in the quotient spaces of generalized topological groups. Next, we recall some definitions, lemmas, and notations that will be used.

Definition 1.6. [14] Let (Y, τ) be a *generalized topological space* (briefly $\mathcal{G}$-topological space). Each element in τ is called a generalized open subset of Y. The complements are called generalized closed subsets. The family of all generalized open subsets of Y and the family of all generalized closed subsets of Y are denoted by $\tau(Y)$ and $\mathcal{F}(Y)$, respectively.

Definition 1.7. [19] Assume that (Y, τ) is a $\mathcal{G}$-topological space. If $Y \in \tau$, then the generalized topology τ is called strong.

Definition 1.8. [7] Let (Y, τ) be a $\mathcal{G}$-topological space. (Y, τ) is said to be *a generalized T_1 space*, if for any $x, y \in Y$ and $x \neq y$, there exists a generalized open neighborhood U of x such that $y \notin U$ and there exists a generalized open neighborhood V of y such that $x \notin V$.

Definition 1.9. [1] Assume that Y, Z are $\mathcal{G}$-topological spaces and $g : Y \to Z$ is a map.

1) g is said to be *generalized continuous* if for any $V \in \tau(Z)$, then $g^{-1}(V) \in \tau(Y)$;

2) g is said to be *generalized continuous* at a point y in Y if for any generalized open set V containing $g(y)$, there is a generalized open set U containing y such that $g(U) \subset V$;

3) g is said to be generalized *open* (*closed*) if the image of every generalized open set (generalized closed set) in Y is a generalized open set (generalized closed set) of Z;

4) g is called a *generalized homeomorphism* if g is bijective and g, g^{-1} are generalized continuous.

Definition 1.10. [13] Assume that $(G, \cdot)$ is a group and (G, τ) is a $\mathcal{G}$-topological space. We denote two mappings by

$$\mu\colon G \times G \to G \text{ satisfying } \mu(a, b) = ab, \forall a, b \in G$$
$$Inv : G \to G \text{ satisfying } Inv(g) = g^{-1}, \forall g \in G.$$

If μ and Inv are generalized continuous, then $(G, \cdot, \tau)$ is called *a generalized topological group* which is abbreviated as a $\mathcal{G}$-topological group. If μ is generalized continuous, then $(G, \cdot, \tau)$ is said to be a generalized paratopological group which is abbreviated as a $\mathcal{G}$-paratopological group [20].

Lemma 1.11 ([14, Proposition 4.5]). *Suppose (G, τ) is a $\mathcal{G}$-topological group. Let E, F be subsets of G and $x \in G$.*

1) *If $E \in \tau(G)$) (resp. $E \in \mathscr{F}(G)$), then $Ex, xE \in \tau(G)$ (resp. $Ex, xE \in \mathscr{F}(G)$);*

2) *If $E \in \tau(G)$, then $EF, FE \in \tau(G)$.*

Definition 1.12. [1] Let (Y, τ) be a $\mathcal{G}$-topological space, and B a subset of Y, $y \in Y$. If there is $U \in \tau$ such that $y \in U \subseteq B$, then B is said to be *a generalized neighborhood of* y.

Definition 1.13. [14] A $\mathcal{G}$-topological space H is said to be *a generalized homogeneous space* if for any $x, y \in H$ there exists a generalized homeomorphism mapping $g : H \to H$ such that $g(x) = y$.

Assume that $(G, \cdot)$ is a group and that (G, τ) is a $\mathcal{G}$-topological space. For $g \in G$, we denote, respectively, the right action and left action

$$r_a : G \to G \quad by \quad r_g(x) = xg \quad for \quad any \quad x \in G,$$

$$l_a : X \to X \quad by \quad l_g(x) = gx \quad for \quad any \quad x \in G.$$

Lemma 1.14 ([14, Proposition 4.2])**.** *Assume that X is a $\mathcal{G}$-topological group and that $a \in X$. Then, both the right action r_a and left action l_a are generalized homeomorphisms.*

Lemma 1.15 ([13, Theorem 2.13])**.** *Suppose G is a $\mathcal{G}$-topological group with the identity element e. For $g \in G$, the base at g is equal to $\mathscr{B}_g = \{gU : U \in \mathscr{B}_e\}$, where $\mathscr{B}_e$ is a generalized neighborhood base of e.*

2. A special class of $\mathcal{G}$-topological groups

The main goal of this section is to explore a special class of $\mathcal{G}$-topological groups which have a family $\{W_i : i \in \omega\}$ of generalized open neighborhoodsof the identity element e satisfying $W_{i+1}^2 \subset W_i$ and $W_i^{-1} = W_i$ for every $i \in \omega$.

In Example 1.2, if we put $W_i = [-1/i, 1/i]$, then the sequence $\{W_{2^i} : i \in \omega\}$ of generalized open neighborhood of 0 in G satisfies that $W_{2^{i+1}}^2 \subset W_{2^i}$ and $W_{2^i}^{-1} = W_{2^i}$ for every $i \in \omega$.

Consider Example 1.4. For any $c, d \in R$, we have $(-\infty, d/2)^2 = (-\infty, d/2) + (-\infty, d/2) = \{x + y : x, y \in (-\infty, d/2)\} \subset (-\infty, d)$ and $(c/2, +\infty)^2 \subset (c, +\infty)$, but $(-\infty, d)^{-1} = (-d, +\infty) \neq (-\infty, d)$ and $(c, +\infty)^{-1} = (-\infty, -c) \neq (c, +\infty)$.

Let N be a real-valued function on a given group G with the neutral element e. N is a *prenorm* [9] if it satisfies that $N(e) = 0$, $N(gh) \leq N(g) + N(h)$ and $N(g^{-1}) = N(g)$ for all $g, h \in G$.

Lemma 2.1 ([9, Proposition 3.3.1])**.** *Let N be a prenorm on a given group G. Then, $|N(g) - N(h)| \leq N(g^{-1}h)$ holds for all $g, h \in G$.*

Theorem 2.2. *Let N be a prenorm on a $\mathcal{G}$-topological group G. N is a generalized continuous map iff for any $\varepsilon > 0$ there is a generalized neighborhood V containing e satisfying $N(g) < \varepsilon$ for arbitrary $g \in V$.*

Proof. If N is generalized continuous, then for each positive number ε there exists a generalized neighborhood V of e such that $N(V) \subset (0 - \varepsilon, 0 + \varepsilon)$.

Next, we prove the sufficiency. For an arbitrary point a in G and arbitrary $\varepsilon > 0$, there is a generalized neighborhood V of e such that $N(x) < \varepsilon$ for arbitrary $x \in V$. $\forall y \in aV$, we have $a^{-1}y \in V$, and hence $N(a^{-1}y) < \varepsilon$. Since $|N(a) - N(y)| \leq N(a^{-1}y) < \varepsilon$, N is generalized continuous. □

Lemma 2.3 ([9, Lemma 3.3.6])**.** *Let G be a group. For a bounded real-valued function ψ, the formula $N_\psi(x) = sup\{|\psi(yx) - \psi(y)| : y \in G\}$ define a function N_ψ on G for each $x \in G$. Then ψ is a a prenorm on G.*

Theorem 2.4. *Let G be a $\mathcal{G}$-topological group and $\{W_i : i \in \omega\}$ a sequence of generalized open neighborhoods of the identity element e in G satisfying $W_{i+1}^2 \subset W_i$ and $W_i^{-1} = W_i$ for every $i \in \omega$. Then, there is a prenorm N defined on G such that the following condition holds:*

$$\{a \in G : N(a) < \frac{1}{2^i}\} \subset W_i \subset \{a \in G : N(a) \leq \frac{2}{2^i}\}.$$

Consequently, the prenorm N is generalized continuous.

Proof. Let $V(1) = W_0$, take $i \in \omega$, and assume that generalized open neighborhoods $V\left(\frac{j}{2^i}\right)$ of e are defined for every $j = 1, 2, \cdots, 2^i$. Let $V\left(\frac{1}{2^{i+1}}\right) = W_{i+1}, V\left(\frac{2j}{2^{i+1}}\right) = V\left(\frac{j}{2^i}\right)$ for any $j = 1, \cdots, 2^i$, and let $V(\frac{2j+1}{2^{i+1}}) = V(\frac{j}{2^i}) \cdot W_{i+1} = V(\frac{j}{2^i}) \cdot V(\frac{1}{2^{i+1}})$ for $j = 1, 2, \cdots, 2^i - 1$. We have defined generalized open neighborhoods $V(t)$ of e for every positive dyadic rational number $t \leq 1$. If $j > 2^i$, then we set $V(\frac{j}{2^i}) = G$.

Claim: $V\left(\frac{j}{2^i}\right) \cdot V\left(\frac{1}{2^i}\right) \subset V\left(\frac{j+1}{2^i}\right)$ for any integers $j > 0, i \geq 0$.

Indeed, we need only verify the case when $i < 2^j$. If $i = 1$, then $j = 1$ and $V\left(\frac{1}{2}\right) \cdot V\left(\frac{1}{2}\right) = W_1^2 \subset W_0 = V(1)$, which implies that the claim holds for $i = 1$. Assume that the claim holds for some i. Next, we shall verify it for $i + 1$.

If $j = 2c, c \in \omega$, then

$$V\left(\frac{j+1}{2^{i+1}}\right) = V(\frac{2c+1}{2^{i+1}}) = V(\frac{2c}{2^{i+1}}) \cdot V(\frac{1}{2^{i+1}}) = V(\frac{j}{2^{i+1}}) \cdot V(\frac{1}{2^{i+1}}).$$

If $j = 2c + 1 < 2^{i+1}, c \in \omega$, then

$$\begin{aligned} V(\frac{j}{2^{i+1}}) \cdot V(\frac{1}{2^{i+1}}) &= V(\frac{2c+1}{2^{i+1}}) \cdot W_{i+1} \\ &= V(\frac{c}{2^i}) \cdot W_{i+1} \cdot W_{i+1}. \\ &\subset V(\frac{c}{2^i}) \cdot W_i \\ &= V(\frac{c}{2^i}) \cdot V(\frac{1}{2^i}). \end{aligned}$$

However, $V(\frac{c}{2^i}) \cdot V(\frac{1}{2^i}) \subset V(\frac{c+1}{2^i}) = V(\frac{j+1}{2^{i+1}})$ hold based on the inductive hypothesis. Hence, the claim holds for any integers $j > 0, i \geq 0$.

Define a function $g : G \to R$ by $g(x) = inf\{t > 0 : x \in V(t)\}$ for every $x \in G$. Since $x \in V_2 = G$ for every $x \in G$, the function g is well-defined. By the claim, it can be inferred that if $0 < t < s$, where t and s are positive dyadic rational numbers, then $V(t) \subset V(s)$. Next, we consider s and t exclusively as positive dyadic rational numbers. Therefore, we can conclude that if $g(x) < t$, then $x \in V(t)$.

Then, g is a non-negative function that is bounded above by 2. According to Lemma 2.3, the function N defined as

$$N(x) = \sup_{b \in G} |g(bx) - g(b)|$$

for every $x \in G$, is a prenorm on G.

It is clear that $g(e) = 0$. If $N(a) < \frac{1}{2^i}$, for some $a \in G$, then

$$g(a) = |g(ea) - g(e)| \leq N(a) < \frac{1}{2^i},$$

which indicates that $a \in V\left(\frac{1}{2^i}\right) = W_i$. Hence, we have proved that

$$\{a \in G : N(a) < \frac{1}{2^i}\} \subset W_i.$$

Next, we shall check that $W_i \subset \{a \in G : N(a) \leq \frac{2}{2^i}\}$, which indicates the generalized continuity of N. We need only verify that for any $w \in V\left(\frac{1}{2^i}\right) = W_i$, we have $|g(bw) - g(b)| \leq \frac{2}{2^i}$ for each $b \in G$.

Indeed, for each $b \in G$ there exists $j \in \omega \setminus \{0\}$ such that $\frac{(j-1)}{2^i} \leq g(b) < \frac{j}{2^i}$. Then, $b \in V\left(\frac{j}{2^i}\right)$. Since $w \in W_i = V\left(\frac{1}{2^i}\right)$ and $w^{-1} \in V\left(\frac{1}{2^i}\right)$, it follows that bw and bw^{-1} are in $V\left(\frac{j}{2^i}\right) \cdot V\left(\frac{1}{2^i}\right)$. It follows from the claim that $V\left(\frac{j}{2^i}\right) \cdot V\left(\frac{1}{2^i}\right) \subset V\left(\frac{j+1}{2^i}\right)$. Therefore, $g(bw) \leq \frac{j+1}{2^i}$ and $g\left(bw^{-1}\right) \leq \frac{j+1}{2^i}$. It follows from the above equation and the inequality $\frac{j-1}{2^i} \leq g(b)$ that $g(bw) - g(b) \leq \frac{2}{2^i}$ and $g\left(bw^{-1}\right) - g(b) \leq \frac{2}{2^i}$. By the the arbitrariness of b in G, we get $g(b) - g(bw) = g(bww^{-1}) - g(bw) \leq \frac{2}{2^i}$. Together with the previous inequality, we have $|g(bw) - g(b)| \leq \frac{2}{2^i}$ for every $b \in G$. By the definition of N, we know $N(w) \leq \frac{2}{2^i}$, i.e., $W_i \subset \{a \in G : N(a) \leq \frac{2}{2^i}\}$, which completes our proof. □

Theorem 2.5. *Suppose that G is a $\mathcal{G}$-topological group and $\{K_i : i \in \omega\}$ is a sequence consisting of generalized open neighborhoods of the identity element e in G satisfying $K_{i+1}^2 \subset K_i$ and $K_i^{-1} = K_i$ for every $i \in \omega$. Set $H = \bigcap_{i\in\omega} K_i$. Then, G/H is metrizable.*

Proof. Since $K_{i+1}^2 \subset K_i$ and $K_i^{-1} = K_i$ for every $i \in \omega$, we conclude that $\overline{K_{i+1}} \subset K_i$. In fact, if $x \in \overline{K_{i+1}}$, then $xK_{i+1} \cap K_{i+1} \neq \emptyset$ and hence $x \in K_{i+1}K_{i+1}^{-1} \subset K_i$, which implies that $\overline{K_{i+1}} \subset K_i$. By $\overline{K_{i+1}} \subset K_i \subset \overline{K_i}$ for each $i \in \omega$, we know that $H = \bigcap_{i\in\omega} K_i = \bigcap_{i\in\omega} \overline{K_i}$ is closed in G. By induction on K_i, we know that $\bigcap_{i\in\omega} K_i$ is also a subgroup of G.

By Theorem 2.4, there is a generalized continuous prenorm N on G.

Claim: $\{a \in G : N(a) = 0\} = H = \bigcap_{i\in\omega} K_i$.

In fact, if $N(a) = 0$, then $a \in B_N\left(\frac{1}{2^i}\right) = \left\{a \in G : N(a) < \frac{1}{2^i}\right\} \subset K_i$ for every $i \in \omega$. Hence, $a \in H$. If $a \in H$, then $a \in K_i$ for any $i \in \omega$. By Theorem 2.4, there is a prenorm N on G such that $B_N\left(\frac{1}{2^i}\right) = \left\{a \in G : N(a) < \frac{1}{2^i}\right\} \subset K_i \subset \{a \in G : N(a) \leq \frac{2}{2^i}\}$, which implies that $N(a) = 0$.

Let $\rho_N(x, z) = N\left(xz^{-1}\right)$ for any $x, z \in G$. Clearly, for any a belonging to G, $\rho_N(a, a) = N(e) = 0$. For any a, b belonging to G, we have $\rho_N(a, b) = N\left(ab^{-1}\right) = N\left(\left(ab^{-1}\right)^{-1}\right) = N\left(ba^{-1}\right) = \rho_N(b, a)$. For any a, b, c belonging to G, we have $\rho_N(a, c) = N\left(ac^{-1}\right) = N\left(ab^{-1}bc^{-1}\right) \leq N\left(ab^{-1}\right) + N\left(bc^{-1}\right) = \rho_N(a, b) + \rho_N(b, c)$.

Let $d(aH, bH) = N\left(ab^{-1}\right)$ for any $a, b \in G$. Next, we prove that d is a well-defined function from $G/H \times G/H$ to the real number set.

It is clear that if $aH = a'H$, then $a^{-1}a' \in H$, and hence

$$d(a'H, bH) = N\left(a'b^{-1}\right) = N\left(a'a^{-1}ab^{-1}\right) \leq N(a'a^{-1}) + N(ab^{-1}) = N(ab^{-1}) = d(aH, bH).$$

Similarly, we can check that $d(aH, bH) \leq d(a'H, bH)$, so $d(aH, bH) = d(a'H, bH)$. Further, if $bH = b'H$, then $d(a'H, b'H) = d(aH, b'H) = N(ab'^{-1}) = N(b'a^{-1}) = d(b'H, aH) = d(bH, aH) = d(aH, bH)$.

Next, we can check that $d(aH, bH) = 0$ if and only if $aH = bH$. It follows from $d(aH, bH) = N\left(ab^{-1}\right)$ and $N\left(ab^{-1}\right) = N(ba^{-1})$ that $d(aH, bH) = d(bH, aH)$. For any $a, b, c \in G$, $d(aH, cH) = N\left(ac^{-1}\right) = N\left(ab^{-1}bc^{-1}\right) \leq N(ab^{-1}) + N(bc^{-1}) = d(aH, bH) + d(bH, cH)$. Thus, d is a metric on G/H. □

3. Generalized ω-narrow properties

In this section, the notations ω and κ denote the first infinite ordinal number and an infinite cardinal number. Let H be a $\mathcal{G}$-topological group. If for any generalized open set U containing e, there is $K \subset G$ with $|K| \leq \kappa$ such that $KU = G$, then H is called κ-narrow where κ is an infinite cardinal.

Proposition 3.1. Let (H, τ) be a $\mathcal{G}$-topological group. If H is generalized ω-narrow, then τ is strong.

Proof. Because H is generalized ω-narrow, there is a countable subset A of G such that $AU = H$ for any generalized open neighborhood U containing e in H. Hence, τ is strong by Definition 1.7. □

Proposition 3.2. Let $(H, \cdot)$ be a group and (H, τ) a $\mathcal{G}$-topological space. If Inv is generalized continuous, then the following conditions are equivalent.

1) H is generalized κ-narrow.

2) For any generalized open subset O containing e in H, there is $C \subset H$ such that $|C| \leq \kappa$ and $H = OC$.

3) For any generalized open subset O containing e in H, there is a set D contained in H satisfying $|C| \leq \kappa$ and $H = OD = DO$.

Proof. It is easy to see that 3) ⇒ 1) and 3) ⇒ 2) hold.

Next, we check that 1) ⇒ 2). It follows from Inv is generalized continuous that for each generalized open subset O containing e, there is a generalized open subset U of G such that $U^{-1} \subset O$. By 1), there exists $B \subset H$ with $|B| \leq \kappa$ such that $H = BU$. Set $C = B^{-1}$. Then, $|C| \leq \kappa$ and $H = H^{-1} = U^{-1}B^{-1} \subset OC$, which implies that $H = OC$. By a similar proof, we can show that 2) ⇒ 1).

Next, we verify that 2) ⇒ 3). If 2) holds, then there are subsets A and B with $|A| \leq \kappa$ and $|B| \leq \kappa$ such that $G = AO$ and $G = OB$. Let $D = A \cup B$. Then, $H = DO = OD$ and $|D| \leq \kappa$. □

Theorem 3.3. *Let G, H be $\mathcal{G}$-topological groups and Φ a generalized continuous homomorphism from G onto H. If G is generalized κ-narrow, then H is generalized κ-narrow.*

Proof. Let e_H and e_G be the neutral elements of the group H and G, respectively. It follows from Φ being generalized continuous that $\Phi^{-1}(W)$ is a generalized open subset of G for any generalized open neighborhood W of e_H in H. By the fact that G is generalized κ-narrow, one can obtain a set $K \subset G$ with $|K| \leq \kappa$ such that $\Phi^{-1}(W)K = G$. It follows that $\Phi(\Phi^{-1}(W)K) = \Phi(G) = H$. Since Φ is homomorphic and onto, $\Phi(\Phi^{-1}(W)) \cdot \Phi(K) = W\Phi(K) = H$. It remains to note that $|\Phi(K)| \leq \kappa$ by $|K| \leq \kappa$, which completes the proof. □

Definition 3.4. [4] Let $\{(G_\alpha, \tau_\alpha) : \alpha \in \Lambda\}$ be a family of $\mathcal{G}$-topological spaces. Set $\mathscr{B} = \{\prod_{\alpha\in\Lambda} M_\alpha : M_\alpha \in \tau_\alpha$, and $M_\alpha = \bigcup \tau_\alpha$ except for finitely many $\alpha \in \Lambda\}$. Call $\mathscr{B}$ a basis of some generalized topology τ on $G = \prod_{\alpha\in\Lambda} G_\alpha$ and τ a product of generalized topologies τ_α.

Lemma 3.5 ([20, Theorem 8.3]). *Assume that $\{(X_\alpha, \cdot_\alpha, \tau_\alpha) : \alpha \in \Lambda\}$ is a family of $\mathcal{G}$-paratopological groups, and, for every $\alpha \in \Lambda$, the set $\cup\tau_\alpha$ satisfies the condition that $a \cdot_\alpha b \in \cup\tau_\alpha$ for any $a, b \in \cup\tau_\alpha$. Then, $G = \prod_{\alpha\in\Lambda} G_\alpha$ endowed with the product of generalized topologies τ_α and the product operation defined coordinatewise is a $\mathcal{G}$-paratopological group. In particular, the product of a family of strong $\mathcal{G}$-paratopological groups is a $\mathcal{G}$-paratopological group.*

Theorem 3.6. *Let $\{(G_\alpha, \tau_\alpha) : \alpha \in \varDelta\}$ be a family of generalized κ-narrow $\mathcal{G}$-topological groups. If τ_α is strong for every $\alpha \in \varDelta$, then $G = \prod_{\alpha\in\varDelta} G_\alpha$ is a generalized κ-narrow $\mathcal{G}$-topological group.*

Proof. For any generalized open neighborhood $U = \prod_{\alpha\in\varDelta} U_\alpha$ where $U_\alpha \in \tau_\alpha$ and $U_\alpha = G_\alpha$ except for finitely many α, which is denoted by $\alpha_i, 1 \leq i \leq n$ for some $n \in N_+$. Since for every $1 \leq i \leq n$, there exists $A_{\alpha_i} \subset G$ with $|A_{\alpha_i}| \leq \kappa$ such that $U_{\alpha_i}A_{\alpha_i} = A_{\alpha_i}U_{\alpha_i} = G_{\alpha_i}$, for any $\alpha \in \varDelta \setminus \{\alpha_i : 1 \leq i \leq n\}$, set $A_\alpha = \{e\}$. Then, $|\prod_{\alpha\in\varDelta} A_\alpha| \leq \kappa$ and $(\prod_{\alpha\in\varDelta} A_\alpha)U = U(\prod_{\alpha\in\varDelta} A_\alpha) = \prod_{\alpha\in\varDelta} G_\alpha$.

For any $g = (g_\alpha)_{\alpha\in\varDelta} \in G$ and every basic generalized open neighborhood $W = \prod_{\alpha\in\varDelta} W_\alpha$ of g^{-1} in G, where $W_\alpha \in \tau_\alpha$ and $W_\alpha = G_\alpha$ except for finitely many α, which is denoted by $\alpha_i, 1 \leq i \leq n$ for some $n \in N_+$. There are generalized open sets $O_{\alpha_i} \in \tau_{\alpha_i}$ containing x_{α_i} such that $(O_{\alpha_i})^{-1} \subset W_{\alpha_i}$ for $1 \leq i \leq n$. For any $\alpha \in \varDelta \setminus \{\alpha_i : 1 \leq i \leq n\}$, set $O_\alpha = G_\alpha$. Then, $O = \prod_{\alpha\in\varDelta} O_\alpha$ is a generalized open subset of G containing x and $O^{-1} \subset W$. Combining this with Lemma 3.5, we know G is a $\mathcal{G}$-topological group. We have proved that G is a generalized κ-narrow $\mathcal{G}$-topological group. □

Assume that (G, τ) is a $\mathcal{G}$-topological group and κ is an infinite cardinal.We define the generalized Lindelöf number $gl(G)$ of G as the least cardinal κ such that any generalized open covering $\mathscr{P}$ of G contains a subcover $\mathscr{P}'$ with $|\mathscr{P}'| \leq \kappa$.

Proposition 3.7. Suppose that (G, τ) is a $\mathcal{G}$-topological group. If $gl(G) = \kappa$, then G is generalized κ-narrow.

Proof. Let U be any generalized open neighborhood of identity element e in G. Then $\{xU : x \in G\}$ is a generalized open covering of G. It follows from $gl(G) \leq \kappa$ that there exists a subset K of G such that $|K| \leq \kappa$ and the family $\{xU : x \in K\}$ covers G, which implies $KU = G$. Then, we have that G is generalized κ-narrow. □

A subset A of a $\mathcal{G}$-topological group G is called generalized κ-narrow if for every generalized open subset U of G containing the identity e, there exists a set $K \subset G$ and $|K| \leq \kappa$ such that $A \subset KU \cap UK$ where κ is an infinite cardinal.

Theorem 3.8. *Let (G, τ) be a $\mathcal{G}$-topological group, and A and B generalized κ-narrow subsets of G. Then, A^{-1} is generalized κ-narrow in G. Further, if for any $U \in \tau$ satisfying $e \in U$, there is $V \in \tau$ containing e such that $V^2 \subset U$, then AB is also generalized κ-narrow in G.*

Proof. Let U be any generalized open subset of G containing the identity e. Then there exist a generalized neighborhood O of e and $F \subset G$ with $|F| \leq \kappa$ such that $O^{-1} \subset U$ *and* $A \subset FO \cap OF$. Let $K = F^{-1}$. Then we have that $A^{-1} \subset (FO)^{-1} \cap (OF)^{-1} = O^{-1}K \cap KO^{-1} \subset UK \cap KU$. This implies that A^{-1} is generalized κ-narrow in G.

Next, we prove that AB is generalized κ-narrow in G. Indeed, there exists a generalized neighborhood V of e in G such that $V^2 \subset U$. There exists a subset L of G such that $|L| \leq \kappa$ and $B \subset LV$. For any $y \in L$, there exists a generalized open neighborhood W_y of e such that $y^{-1}W_y y \subset V$.

Since A is generalized κ-narrow in G, there exists a subset $K_y \subset G$ such that $\left|K_y\right| \leq \kappa$ and $A \subset K_y W_y$. Set $K = \bigcup_{y \in L} K_y$ and $M = KL$. Obviously, $|M| \leq \kappa$. For any $a \in A$ and $b \in B$, there are $y \in L$ and $x \in K_y$ such that $b \in yV$ and $a \in xW_y$. Then, we have $ab \in xW_y yV = xy\left(y^{-1}W_y y\right)V \subset xyVV \subset xyU \subset MU$, which implies $AB \subset MU$. Similarly, there exists $Q \subset G$ such that $|Q| \leq \kappa$ and $AB \subset UQ$. Set $H = M \cup Q$. Clearly, we have that $|H| \leq \tau$ and $AB \subset HU \cap UH$, which completes our proof. □

Let X be a subset of a group G. If G is the smallest group containing X, then X is said to algebraically generate G denoted by $G = \langle X \rangle$.

Theorem 3.9. *Let X be a generalized κ-narrow subset of a $\mathcal{G}$-topological group (G, τ) and $G = \langle X \rangle$. If for any $U \in \tau$ satisfying $e \in U$, there exists $V \in \tau$ containing e such that $V^2 \subset U$, then G is generalized κ-narrow.*

Proof. Let $A_0 = X \cup X^{-1}$, $A_1 = A_0^2$, and $A_k = A_{k-1}^2$ for any $k \geq 2$. Since X is generalized κ-narrow, A_k is generalized κ-narrow for each $k \in \omega$ by Theorem 3.8. It follows from $G = \langle X \rangle = \bigcup_{k \in \omega} A_k$ that for each generalized open set U of e in G, there are $F_k \subset G$ with $|F_k| \leq \kappa$ such that $G \subset (\bigcup_{k \in \omega} F_k)U$. Since $|\bigcup_{k \in \omega} A_k| \leq \kappa$, G is generalized κ-narrow. □

Let (X, τ) be a generalized topological group and $F \subset X$. We denote by $Int_X F$ the generalized inter of F, and by $cl_X F$ the generalized closure of F. The generalized inter of F is defined by the union of all generalized open subsets contained in F. The generalized closure of F is defined by the intersection of all generalized closed subsets containing F.

Lemma 3.10 ([13, Theorem 2.29]). *Assume that X is a $\mathcal{G}$-topological group and $\mathscr{F}$ is a generalized neighborhood base of e in X, $A \subset X$. Then, $cl_X A = \bigcap_{F \in \mathscr{F}} FA = \bigcap_{F \in \mathscr{F}} AF$.*

Theorem 3.11. *Let G be a $\mathcal{G}$-topological group such that for every $e \in W \in \tau$, there exists $e \in O \in \tau$ such that $O^2 \subset W$. If H is generalized κ-narrow and $cl_G H = G$, then G is also generalized κ-narrow.*

Proof. For any generalized open subset W containing e in G, there is a generalized open set O containing e such that $O^2 \subset W$. It follows from H being generalized κ-narrow that there exists $K \subset G$ with $|K| \leq \kappa$ such that $H \subset OK$. By Lemma 3.10, $G \subset OH \subset OOK \subset WK$, that is, G is generalized κ-narrow. □

4. Some properties in the generalized quotient spaces

Suppose that H is a subgroup of a $\mathcal{G}$-topological group (G, τ). We endow the set $G/H = \{aH : a \in G\}$ with the generalized quotient topology with respect to the generalized quotient map π defined by $\pi(x) = xH$ for every $x \in G$, that is, $\tau(G/H) = \{B \subset G/H : \pi^{-1}(B) \in \tau\}$.

Theorem 4.1. *Assume that (G, τ) is a $\mathcal{G}$-topological group and H is a generalized closed subgroup of G. Then, the family $\{\pi(xV) : V \in \tau, e \in V\}$ is a basis of xH in the space G/H for each $x \in G$ and the generalized quotient mapping π is generalized open. Further, G/H is generalized homogeneous and generalized T_1.*

Proof. It follows from $\pi^{-1}(\pi(xV)) = xVH$ and Lemma 1.11 that $\pi(xV)$ is a generalized open subset of G/H for any $e \in V \in \tau$. Next fix every generalized open set P of xH in G/H. Then, $x \in \pi^{-1}(P)$ and there exists $e \in V \in \tau$ such that $xV \subset \pi^{-1}(P)$. Then, $\pi(xV) \subset \pi(\pi^{-1}(P)) = P$, and hence $\{\pi(xV) : e \in V \in \tau\}$ is a basis of xH in G/H and π is generalized open.

Next, we prove that G/H is a generalized homogeneous. For each $a \in G$, we define $\psi_a : G/H \to G/H$ as $\psi_a(xH) = axH$ for each $x \in G$. It is not difficult to check that the mapping ψ_a is well defined. In order to show that ψ_a is a generalized homeomorphism, it follows from that ψ_a is a bijection that it is only required to show that ψ_a is generalized continuous and generalized open for each $a \in G$. Indeed, for each $x \in G$ and every generalized open set V containing the identity e in G, $\psi_a(\pi(xV)) = \psi_a(xVH) = axVH = \pi(axW)$, which indicates that ψ_a is generalized homeomorphism. Taking any $xH, yH \in G/H$, we have $\psi_a(xH) = yH$ where $a = yx^{-1}$.

Since H is a generalized closed subset of G, G/H is a generalized T_1 space. □

Corollary 4.2 ([14, Theorem 4.3]). *Every $\mathcal{G}$-topological group is a generalized homogeneous space.*

Definition 4.3. Assume that (X, τ) is a generalized T_1 topological space and $x \in X$. The cardinal number

$$\psi(X, x) = \min\{|\mathscr{F}| : \mathscr{F} \subseteq \tau, \bigcap_{F \in \mathscr{F}} F = \{x\}\}$$

is called the generalized pseudocharacter of X at x. The generalized pseudocharacter of X is defined by $\psi(X) = sup_{x \in X}\psi(X, x)$.

Theorem 4.4. *Let H be a subgroup of a $\mathcal{G}$-topological group G. If H and G/H have countable generalized pseudocharacter, then G also has countable generalized pseudocharacter.*

Proof. Suppose that $\pi : G \to G/H$ is the generalized quotient map. Since G/H has countable generalized pseudocharacter, one can take a family γ consisting of countably many generalized open sets such that $\bigcap \gamma = \{\pi(e)\}$. Similarly, by the fact that H has countable generalized pseudocharacter, there is a countable family λ consisting of generalized open sets in G such that $H \cap (\bigcap \lambda) = \{e\}$. Let $\mathscr{P} = \lambda \cup \left\{\pi^{-1}(U) : U \in \gamma\right\}$. Then, $\mathscr{P}$ is countable, and every element in $\mathscr{P}$ is a generalized open subset of G containing e. Next, we shall verify that $\bigcap \mathscr{P} = \{e\}$.

Indeed, for each $a \in G$ and $a \neq e$, we have $a \in H$ or $a \notin H$. If $a \in H$, then $a \notin \bigcap \lambda$, and hence there is $V \in \lambda$ such that $a \notin V$, which implies that $a \notin \bigcap \mathscr{P}$. If $a \notin H$, then $\pi(a) \neq \pi(e)$. There exists $U \in \gamma$ such that $\pi(a) \notin U$, and, consequently, we have $a \notin \pi^{-1}(U) \in \mathscr{P}$. Thus, $a \notin \bigcap \mathscr{P}$. We are done. □

Definition 4.5. [3] Assume that X is a $\mathcal{G}$-topological space and $E, F \subset X$. E and F are called separated in X, if $E \cap cl_X(F) = F \cap cl_X(E) = \varnothing$.

Definition 4.6. [3] Let Y be a $\mathcal{G}$-topological space and $K \subset Y$.

1) Y is said to be a generalized connected space, if there exist two separated sets $E, F \subset Y$ such that $Y = E \cup F$, and then $E = \emptyset$ or $F = \emptyset$.

2) K is said to be a generalized connected subset of Y, if the subspace K is a generalized connected space.

Lemma 4.7 ([13, Theorem 3.3]). *Suppose that $\Phi : G \to H$ is a generalized continuous mapping between $\mathcal{G}$-topological spaces. If G is generalized connected, then $\Phi(G)$ is also generalized connected.*

Definition 4.8. [13] Let G be a $\mathcal{G}$-topological space, with $F \subseteq G$. For $x \in F$, the set $F_x = \bigcup_{x \in S \subset F} S$, where S is generalized connected in F, is called the generalized connected component of F containing x.

Definition 4.9. Let G be a $\mathcal{G}$-topological space. If every generalized connected component of G consists of one point, then G is said to be generalized totally disconnected.

The next proposition is clear.

Proposition 4.10. Let G be a $\mathcal{G}$-topological space. For any $g \in G$, $gG_e = G_e g$ is a generalized connected component of g in G.

Lemma 4.11 ([13, Theorem 3.8]). *Assume that G is a $\mathcal{G}$-topological space and H is a generalized connected component of e in G. Then, H is a generalized closed normal subgroup of G.*

Lemma 4.12 ([3, Lemma 1.3]). *Suppose that X is a $\mathcal{G}$-topological space and E, F are separated in X. If K is generalized connected and $K \subset E \cup F$, then either $K \subset E$ or $K \subset F$.*

Theorem 4.13. *Let (G, τ) be a $\mathcal{G}$-topological group, and H be a subgroup and the generalized connected component of e in G. Then, G/H is generalized totally disconnected.*

Proof. Let $\pi : G \to G/H$ be the generalized quotient map from G onto G/H. By Lemma 4.11, H is a generalized closed normal subgroup of G. Let K be the generalized connected component of $\pi(e)$ in the quotient group G/H . To show that G/H is generalized totally disconnected, it is sufficient to show that K is a single point set, that is, $K = \{\pi(e)\}$.

Suppose $\pi^{-1}(K) = F_1 \cup F_2$, where F_1, F_2 are disjoint generalized open sets of G. Clearly, F_1 and F_2 are separated in $\pi^{-1}(K)$. For any $a \in \pi^{-1}(K)$, aH is g-connected by Lemma 4.7. It follows from $a \in \pi^{-1}(K)$ that $aH \subset \pi^{-1}(K)$. By Lemma 4.12, we have $aH \subset F_1$ or $aH \subset F_2$. Thus, F_1 and F_2 are union of cosets of H. Next, we prove that $F_1 = \emptyset$ or $F_2 = \emptyset$. If not, then F_1 and F_2 are non-empty generalized open sets, so we get that $\pi(F_1)$ and $\pi(F_2)$ are also non-empty generalized open sets. We get that $K = \pi\left(\pi^{-1}(K)\right) = \pi(F_1) \cup \pi(F_2)$ and $\pi(F_1) \cap \pi(F_2) = \varnothing$ since F_1 and F_2 are union of cosets of H. This contradicts with the fact that K is generalized connected. Thus, $F_1 = \varnothing$ or $F_2 = \varnothing$. Since $\pi^{-1}(K)$ is a generalized connected set containing the unit element e in G, then $\pi^{-1}(K) \subset H$. Combining this with $H \subset \pi^{-1}(K)$, we get that $\pi^{-1}(K) = H$ i.e. $K = \{H\}$. □

Theorem 4.14. *Let (G, τ) be a $\mathcal{G}$-topological group and H a generalized closed subgroup of G. If H and G/H are generalized totally disconnected, then G is generalized totally disconnected.*

Proof. Suppose that P is a generalized connected subset of G and π is a generalized quotient map from G to G/H. It follows from Lemma 4.7 that $\pi(P)$ is a generalized connected subset of G/H. $\pi(P)$ is a single point set, since G/H is generalized totally disconnected. Then there exists $x \in G$ such that $P \subset xH$. It follows from xH is generalized totally disconnected that P is a single point set which implies that G is generalized totally disconnected. □

5. Conclusions

In this paper, some properties of generalized topological groups are studied, which are divided into four sections. Section 1 lists the relevant definitions and conclusions of generalized topological groups. Section 2 discusses a special class of generalized topological groups. Section 3 considers the generalized κ-narrow properties of generalized topological groups, and, lastly, some properties in the quotient spaces of generalized topological groups are obtained. Topological groups have rich topological properties and algebraic properties. In the future, we intend to focus on the following two aspects.

Seek which results in topological groups still hold in generalized topological groups. For example, each T_1 topological group is regular. Assume that Y is a $\mathcal{G}$-topological space. Recall that Y is called *generalized regular* [13], if for each $x \in Y$ and any generalized closed set M such that $x \notin M$, there are two generalized open sets U and V such that $x \in U, M \subset V$ and $U \cap V = \emptyset$. It was shown in [13] that if a $\mathcal{G}$-topological group H has a base at identity e_H consisting of a symmetric neighbourhood, then it is generalized regular. However, not each generalized topological group has a base at identity element consisting of a symmetric neighbourhood. It is natural to ask the following question.

Question 5.1. Let H be a generalized topological group. If H is generalized T_1, is it generalized regular?

On the other hand, find examples to illustrate the differences between topological groups and generalized topological groups. For example, we will consider the following question and we hope to give a negative answer by an example.

Question 5.2. Let H be a generalized topological group. Is the generalized character of H equal to the generalized pseudocharacter of H?

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

References

1. Á. Császár, Generalized topology, generized continuity, *Acta Math. Hung.*, **96** (2002), 351–357. https://doi.org/10.1023/A:1019713018007
2. Á. Császár, Separation axioms for generalized topologies, *Acta Math. Hung.*, **104** (2004), 63–69. https://doi.org/10.1023/b:amhu.0000034362.97008.c6
3. Á. Császár, γ-connected sets, *Acta Math. Hung.*, **101** (2003), 273–279. https://doi.org/10.1023/B:AMHU.0000004939.57085.9e
4. Á. Császár, Product of generalized topologies, *Acta Math. Hung.*, **123** (2009), 127–132. https://doi.org/10.1007/s10474-008-8074-x

5. R. Shen, Remarks on products of generalized topologies, *Acta Math. Hung.*, **124** (2009), 363–369. https://doi.org/10.1007/s10474-009-8207-x

6. G. E. Ge, G. E. Ying, μ-Separations in generalized topological spaces, *Appl. Math. J. Chin. Univ.*, **25** (2010), 243–252. https://doi.org/10.1007/s11766-010-2274-1

7. M. S. Sarsak, Weak separation axioms in generalized topological spaces, *Acta Math. Hung.*, **131** (2011), 110–121. https://doi.org/10.1007/s10474-010-0017-7

8. L. E. De Arruda Saraiva, Generalized quotient topologies, *Acta Math. Hung.*, **132** (2011), 168–173. https://doi.org/10.1007/s10474-010-0047-1

9. A. Arhangel'skii, M. Tkachenko, *Topological Groups and Related Structures, An Introduction to Topological Algebra*, 1^{st} edition, Atlantis Press, 2008. https://doi.org/10.2991/978-94-91216-35-0

10. S. Lin, L. Xie, D. Chen, Some generalized countably compact properties in topological groups, *Topol. Appl.*, **339** (2023), 108705. https://doi.org/10.1016/j.topol.2023.108705

11. A. V. Arkhangel'skii, Topological homogeneity. Topological groups and their continuous images, *Russ. Math. Surv.*, **42** (1987), 83–131. https://doi.org/10.1070/rm1987v042n02abeh001333

12. S. Bagchi, The sequential and contractible topological embeddings of functional groups, *Symmetry*, **12** (2020), 789. https://doi.org/10.3390/sym12050789

13. M. Hussain, M. U. D. Khan, C. Ozel, On generalized topological groups, *Filomat*, **27** (2013), 567–575. https://doi.org/10.2298/FIL1304567H

14. Z. Li, S. Li, Introduction to generalized topological groups, *Publ. Math. Debrecen*, **85** (2014), 1–13. https://doi.org/10.5486/pmd.2014.5597

15. M. Hussain, M. U. D. Khan, C. Ozel, More on generalized topological groups, *Creative Math. Inf.*, **22** (2013), 47–51. https://doi.org/10.37193/CMI.2013.01.07

16. T. Hida, Soft topological group, *Ann. Fuzzy Math. Inf.*, **8** (2014), 1001–1025.

17. T. Shah, S. Shaheen, Soft topological groups and rings, *Ann. Fuzzy Math. Inf.*, **7** (2014), 725–743.

18. O. Tantawy, S. A. Kandil, A. ElShamy, Generalization of soft topological groups, *J. Egypt. Math. Soc.*, **28** (2020), 13. https://doi.org/10.1186/s42787-020-00075-7

19. Á. Császár, Extremally disconnected generalized topologies, *Ann. Univ. Sci. Budap. Rolando Eötvös Nominatae Sect. Math.*, **47** (2004), 91–96.

20. P. Ye, C. Cai, Research on some properties of generalized topological groups, *J. Nanning Normal Univ. (Nat. Sci. Ed.)*, **38** (2021), 27–36. https://doi.org/10.16601/j.cnki.issn2096-7330.2021.02.004

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