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Research article

Fixed/predefined-time control of social network-based delayed interconnected system with discontinuity

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Abstract: This article aims to study the fixed/predefined-time control problem of a social network-based delayed interconnected system with a discontinuous interconnection. First, unlike the fractional power function, hyperbolic tangent function and logistic function, a particular exponential function with an indefinite coefficient is utilized to establish fixed/predefined-time stability criteria for discontinuous delayed differential equations based on delayed differential inclusion. The settling time of fixed-time stability is estimated, and the estimation formula is independent of the system's initial states. The Lyapunov energy function method in this article extends the existing fixed/predefined time stability to the case of indefinite derivatives. Second, taking time-varying coefficients, discontinuity, and the time-delay effect, a social network-based interconnected model is established. With the help of the established fixed/predefined-time theorems and the design of the state feedback control schemes, the fixed/predefined-time stabilization is implemented on the social network-based delayed interconnected system with discontinuity. Finally, a numerical example is given to verify the effectiveness of the fixed/predefined-time control. Numerical simulation shows that, under the state feedback control scheme, the social network-based delayed interconnected system with two subsystems is stabilized in a fixed time at zero within 0.69 seconds. Under the predefined time control scheme, the stabilization at zero for the predefined time can be ensured within any set predefined time $T_{\mathrm{p}} = 0.25$. It is of practical significance to study the social network-based delayed interconnected system and its fixed/predefined-time control for the rapid formation of stable relationships and the implementation of accurate social governance.

Keywords: fixed-time stability; predefined-time stability; social network-based delayed interconnected system; state feedback control scheme

1. Introduction

A social network-based delayed interconnected system (SNDIS) is a social relationship network system in which the social individuals in its subsystems are connected through communication or physical space, and are affected by transmission delays during the connection process [1–3]. A SNDIS focuses on the interaction and connection between people, which will affect people's social behavior. A SNDIS is connected by many nodes (individuals or organizations) to form a social structure. A SNDIS contains various social relationships. Through these social relationships, individuals or organizations from casual acquaintances to closely connected social relationships are interconnected. Social relationships include friendships, classmates, business partners, racial and religious relationships, etc. However, the changing rates of social relations and state variables in the SNDIS will change with the evolution of time, and the transmission of signals between social individuals is time-delayed [4–6]. Therefore, a delayed differential equation (DDE) is often used to establish SNDIS models. The SNDIS evolves from a general interconnected system. As of today, the research on SNDISs and interconnected systems (ISs) has attracted the attention of numerous scholars and achieved many excellent results [7–9]. Given that the communication or physical connections between individuals in society are often intermittent, it is more realistic to utilize discontinuous SNDIS to establish the model. Some of the few existing results have used discontinuous interconnection functions to model ISs [10–12], but the indefinite time-varying coefficients have not been taken into account. When discontinuous interconnections are introduced into a SNDIS for modeling, some complex dynamics may occur, and the stability of the SNDIS may be disrupted. In order to maintain relatively stable social relationships, the discontinuous interconnections and stabilization control of SNDIS should be further studied, which also stimulated the research motivation for this article.

Due to the phenomenon of time-delay in signal transmission between social individuals in a SNDIS, the future states of the system are not only influenced by the current states but also restricted by the past states. At this time, the SNDIS can be described by a DDE [13]. Time-delay can also cause the SNDIS to transition from a stable state to an unstable state. Therefore, how to design appropriate control schemes to achieve stability in SNDISs under time-delay is a topic worthy of research. However, when both time-delay and discontinuity are involved in social network systems, new problems will arise. For example, the existence, uniqueness, and stability of solutions for discontinuous SNDIS are difficult to guarantee. Fortunately, Aubin and Cellina developed the theory of delayed differential inclusion (DDI) to study discontinuous DDEs [14], which can effectively handle the problem of the solutions for SNDIS possessing discontinuity. Specifically, when dealing with the stability and stabilization control problems of a discontinuous SNDIS, the Lyapunov stability method of DDI is a highly effective tool [15–17]. In order to make the states of the social individuals in a discontinuous SNDIS realize harmonious stability as soon as possible within a finite time and achieve the precise control goal, the fixed-time (FxT) and predefined-time (PdT) control of discontinuous SNDISs will be problems that need to be focused on and solved in the next step.

The implementation of FxT/PdT control for discontinuous SNDISs not only requires the design of appropriate control schemes but also largely relies on the theory of fixed-time stability (FxTS) and predefined-time stability (PdTS). FxTS and PdTS are both established on the basis of finite time stability (FnTS), thus inheriting the advantages of a fast response capability and disturbance-rejection seen in FnTS. The FnTS means that the solution trajectories of a system reach a certain point within a finite time (FnT) [10]. FnTS has significant application value in the field of control engineering and has been the subject of extensive research [18–20]. However, a major limitation of FnTS is that its settling time (StT) to reach stability is constrained by the initial states of the system's solution trajectory. Given that the initial states of many systems in reality are unknown, and the StT of FnTS is also unknown, this is not conducive to precise control of the time when the system reaches stability. In 2012, Polyakov further defined the concept of FxTS, where the StT is no longer constrained by any initial values in the system [21]. From then on, the research on FxTS and FxT control of various real-world engineering systems has become an important research hotspot. In [22], the FxTS theorem was established via an implicit Lyapunov function. In [23], FxT stabilization control was implemented for autonomous systems. In [24], FxT stabilization was realized by special functions. In [25–27], some novel FxTS criteria were promoted and applied to neural network models. In [28], FxT fractional-order sliding mode control was realized for pressurized water reactor system. In order to ensure that the convergence time for the system to reach a stable state can be flexibly adjusted according to practical needs, the concept of PdTS was proposed by Jiménez-Rodríguez et al [29]. StT is not subject the constraints of the system's parameters and starting states. The predefined time of PdTS can be set to any numerical value, and it is neither constrained by the initial states nor affected by system's coefficients. Currently, PdTS and PdT control for various continuous and discontinuous systems have received increasing attention and research [29–31]. It should be pointed out that the existing FxTS/PdTS established via the Lyapunov method are mostly based on some common functions (e.g., the power function, exponential function, and arctangent function). In view of the time-varying coefficients and time-delay factors that appear in the discontinuous system, the Lyapunov indefinite derivative method for determining the FxTS/PdTS is an urgently needed research topic. Therefore, some novel and diverse criteria for FxTS/PdTS need to be further established and are beneficial for research into FxT/PdT control of discontinuous SNDIS.

There are three innovative points in this article. (1) Taking time-varying parameters and time-delay effects into account, the FxTS/PdTS criteria are established for discontinuous DDEs, and a StT estimation formula for FxTS is provided. (2) The Lyapunov indefinite derivative method is developed via a particular exponential function possessing an indefinite coefficient. (3) A distinctive state feedback control scheme and PdT control scheme, which include a time-varying control coefficient, are designed for FxT/PdT control implementation of SNDISs. This article has five parts. Section 2 formulates a discontinuous SNDIS model and elaborates on some basic preparatory knowledge. Section 3 presents two crucial FxTS/PdTS theorems and provides their proofs. Subsequently, two state feedback control schemes are designed to implement FxT/PdT stabilization for a SNDIS. Section 4 validates the main results through a numerical example. Section 5 summarizes this article.

Notations: The (non-negative) set of real numbers is represented by $\mathbb{R}$ ($\mathbb{R}_+$) and the set of positive integers is denoted as $\mathbb{N}_+$. Here, $a \vee b$ represents the maximum of a and b for $a, b \in \mathbb{R}$; $\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$ is the scalar product of x and y for $x, y \in \mathbb{R}^n$. Given $x \in \mathbb{R}^n$, the norm $\|x\| = \sqrt{\sum_{i=1}^{n} x_i^2}$, ∂V is the generalized gradient of V, $\overline{\text{co}}$ represents a convex closure, and Ψ^{-1} denotes the inverse function of Ψ.

2. Model formulation and preliminaries

This section formulates a discontinuous SNDIS model. The concepts of FnTS/FxTS/PdTS of DDEs are defined using DDI theory [32]. Finally, the chain rule for calculating derivatives and an inequality are given as lemmas [33–35].

2.1. Dynamic model of the SNDIS

The SNDIS model with N subsystems is formulated as

$$\dot{z}_i(t) = C_i(t)z_i(t) + \sum_{j=1}^{N} g_{ij}(t, z_j(t-\tau(t))) + H_i(t, z_i(t-\tau(t))) + u_i(t), \tag{2.1}$$

where $i = 1, 2, \ldots, N$; $j = 1, 2, \ldots, N$; N represents the number of subsystems; $C_i(t) = \left(c^i_{lm}(t)\right)_{n_i \times n_i}$ denotes a known self-inhibition matrix function which is bounded and continuous; $z_i(t) = (z_{i1}(t), \ldots, z_{in_i}(t))^{\mathrm{T}} \in \mathbb{R}^{n_i}$ denotes the state of an individual member of society in ith subsystem $\mathbf{S}_i$; $H_i(\cdot) = (H_{i1}(\cdot), \ldots, H_{in_i}(\cdot))^{\mathrm{T}}$ is an unknown continuous time-delayed external disturbance which satisfies $H_i(t, 0) = 0, \forall t \in \mathbb{R}$ and $\|H_i(t, z_i(t-\tau(t)))\| \leq D_i$ with a known constant $D_i \geq 0$; $u_i(t) = (u_{i1}(t), \ldots, u_{in_i}(t))^{\mathrm{T}} \in \mathbb{R}^{n_i}$ is the control scheme; $g_{ij}(\cdot) = (g_{ij1}(\cdot), g_{ij2}(\cdot), \ldots, g_{ijn_i}(\cdot))^{\mathrm{T}}$ denotes the interconnection between subsystems $\mathbf{S}_i$ and $\mathbf{S}_j$ if $i \neq j$; and the delay $\tau(t)$ is continuous and $0 \leq \tau(t) \leq \tau$, where the constant $\tau > 0$. The architecture of SNDIS (2.1) is present in Figure 1. All parameters of the SNDIS model (2.1) are shown in Table 1.

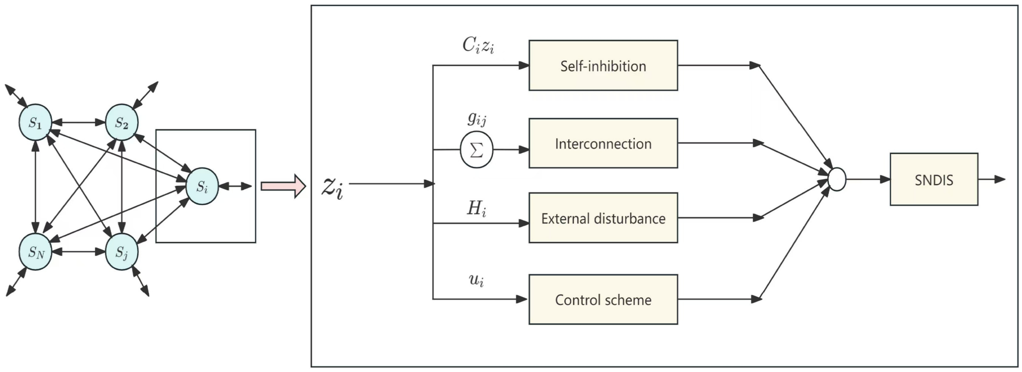

Figure 1. The architecture of the SNDIS (2.1).

Assumption 1. Given any $i, j, n_i \in \mathbb{N}_+$, $g_{ij} : \mathbb{R} \times \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$ possesses Lebesgue measurability and essentially local boundedness. In this situation, $g_{ij}(\cdot)$ is allowed to be continuous.

Assumption 2. $\forall i, j \in \mathbb{N}_+$, $\forall t \in \mathbb{R}$, and $0 \in \overline{\mathrm{co}}[g_{ij}(t, 0)]$. Moreover, it can seek out bounded Lebesgue summable functions $\alpha_{ij}(t) \geq 0$ and $\beta_{ij}(t) \geq 0$, such that (s.t.)

$$\|\gamma_{ij}(t)\| \leq \alpha_{ij}(t)\|z_j\| + \beta_{ij}(t),$$

for almost everywhere (a.e.) $t \in \mathbb{R}$ and $\gamma_{ij}(t) \in \overline{\mathrm{co}}[g_{ij}(t, z_j)]$. Let us set $\alpha^{\sup}_{ij} = \sup_{t\in\mathbb{R}} \left\{\alpha_{ij}(t)\right\}$ and $\beta^{\sup}_{ij} = \sup_{t\in\mathbb{R}} \left\{\beta_{ij}(t)\right\}$.

Table 1. Definitions of parameters in the SNDIS model (2.1).

Parameter	Meaning
$z_i(t)$	State of an individual member of society in the ith subsystem
$C_i(t)$	Known self-inhibition matrix function
g_{ij}	Interconnection between subsystems $\mathbf{S}_i$ and $\mathbf{S}_j$
H_i	External disturbance
$u_i(t)$	Control scheme
$\tau(t)$	Time-delay
i	Sorting of states or subsystems
j	Sorting of states or subsystems
N	Number of subsystems
t	Time
n_i	Number of states in the ith subsystem

For a constant $\tau > 0$, let $\mathbb{C}_\tau = \mathbb{C}([-\tau, 0], \mathbb{R}^n)$ be Banach space which is composed of all continuous functions $\varphi : [-\tau, 0] \to \mathbb{R}^n$ and it possesses a norm $\|\varphi\|_{\mathbb{C}_\tau} = \sup_{s\in[-\tau,0]} \|\varphi(s)\|$. If $t_0 \in \mathbb{R}_+$, and for $\mathcal{L} \in (0, +\infty]$, $z(t) : [t_0-\tau, \mathcal{L}) \to \mathbb{R}^n$ is continuous, then $z_t \in \mathbb{C}_\tau$, in which $z_t(\sigma) = z(t+\sigma)$ for $\sigma \in [-\tau, 0]$ and $t_0 \le t < \mathcal{L}$. In order to study the FxT/PdT stabilization control of the SNDIS model (2.1), let us consider a DDE

$$\dot{z} = f(t, z_t), \tag{2.2}$$

where $f : \mathbb{R} \times \mathbb{C}_\tau \to \mathbb{R}^n$ is discontinuous and has Lebesgue measurability and essentially local boundedness. Moreover, $z_{t_0} \in \mathbb{C}_\tau$ is the initial state at the initial time $t_0 \ge 0$. The SNDIS model (2.1) can be regarded as a special case of the DDE (2.2).

Definition 1 ([14, 17]). $z(t)$ is called a Filippov solution of the DDE (2.2) for $t_0 \le t < \mathcal{L}$ if it is absolutely continuous and

$$\dot{z} \in F(t, z_t) \overset{\text{def}}{=} \bigcap_{\delta>0} \bigcap_{\mu(\Theta)=0} \overline{\text{co}}[f(t, B(z_t, \delta) \setminus \Theta)], \tag{2.3}$$

where $B(z_t, \delta) = \{z_t^* \in \mathbb{C}_\tau \big| \|z_t^* - z_t\|_{\mathbb{C}_\tau} < \delta\}$ and $\mu(\Theta)$ is the Lebesgue measure.

This article requires $0 \in F(t, 0)$ to hold for all $t \in \mathbb{R}$.

In accordance with Definition 1, if $z_i(t)$ is a Filippov solution of the SNDIS (2.1) with the initial condition $(t_0, z_{it_0}) \in \mathbb{R}_+ \times \mathbb{C}([-\tau, 0], \mathbb{R}^{n_i})$, then it satisfies the DDI

$$\dot{z}_i(t) \in C_i(t)z_i(t) + \sum_{j=1}^{N} \overline{\text{co}}[g_{ij}(t, z_j(t - \tau(t)))] + H_i(t, z_i(t - \tau(t))) + u_i(t) = F(t, z_t). \tag{2.4}$$

According to the measurable selection theorem (see [36]), we can find measurable functions $\gamma_{ij}(t - \tau(t)) \in \overline{\text{co}}[g_{ij}(t, z_j(t - \tau(t)))]$ s.t.

$$\dot{z}_i(t) = C_i(t)z_i(t) + \sum_{j=1}^{N} \gamma_{ij}(t - \tau(t)) + H_i(t, z_i(t - \tau(t))) + u_i(t). \tag{2.5}$$

Remark 1. In Assumption 1, the Lebesgue measurability and essentially local boundedness are used for the interconnection function g_{ij}, since the SNDIS model (2.1) is a special case of the DDE (2.2). Due to the essentially local boundedness of $f(t, z_t)$, the set-valued function $F(t, z_t)$ is upper semi-continuous (USC) with nonempty, compact, and convex values and is locally bounded. Thus, for any initial state $z_{t_0} \in \mathbb{C}_\tau$ and initial time $t_0 \geq 0$, the local existence of a Filippov solution can be guaranteed for the SNDIS model (2.1) under Assumption 1. Assumption 2 is a growth condition with the bounded Lebesgue summable functions $\alpha_{ij}(t)$ and $\beta_{ij}(t)$. Under Assumption 2, the extension of the Filippov solution can be guaranteed for the SNDIS model (2.1). For more related knowledge, please refer to [14, 17, 37]. Taking [37] as an example, under the growth condition, the the maximal existing interval of each Filippov solution is extended to $[0, +\infty)$. To sum up, the role of Assumption 1 is to ensure the local existence of the Filippov solution for the SNDIS model (2.1) under the framework of differential inclusion. The role of Assumption 2 is to ensure the global existence of the Filippov solution for the SNDIS model (2.1) (i.e., the extension of the Filippov solution). Moreover, another role of Assumption 2 is to produce an effect on the FxT stabilization condition through inequality scaling in the proof process of Theorem 3.

2.2. Preliminaries

Definition 2 ([17]). The DDE (2.2) is called stable at zero if $\forall\varepsilon > 0$, $\forall t_0 \geq 0$, and we can find $\delta = \delta(\varepsilon, t_0) > 0$ s.t. $\|z(t_0, z_{t_0})(t)\| < \varepsilon$ for every $z_{t_0} \in B(0, \delta) = \{z_{t_0} \in \mathbb{C}_\tau \big| \|z_{t_0}\|_{\mathbb{C}_\tau} < \delta\}$ and $t \geq t_0$.

Definition 3 ([17]). The DDE (2.2) is called uniformly stable at zero if $\forall\varepsilon > 0$, $\forall t_0 \geq 0$, and we can find $\delta = \delta(\varepsilon) > 0$ which is not constrained by t_0, s.t. $\|z(t_0, z_{t_0})(t)\| < \varepsilon$ for every $z_{t_0} \in B(0, \delta)$ and $t \geq t_0$.

Definition 4 ([27]). The DDE (2.2) is called FnT attractive at zero if we can find $0 \leq T(t_0, z_{t_0}) < +\infty$ s.t. $\lim_{t\to T(t_0, z_{t_0})} z(t_0, z_{t_0})(t) = 0$, and $z(t_0, z_{t_0})(t) = 0$ for all $t \geq T(t_0, z_{t_0})$. Here, $T(t_0, z_{t_0})$ is named the settling time (StT).

Definition 5 ([27]). The DDE (2.2) is called FxT (uniformly) stable at zero if it is (uniformly) stable and FnT-attractive at zero, and the StT $T(t_0, z_{t_0})$ is bounded concerning z_{t_0}; i.e., we can find a constant $T^{\max} > 0$ s.t. $T(t_0, z_{t_0}) \leq t_0 + T^{\max}$, for all $z_{t_0} \in \mathbb{C}_\tau$.

Definition 6 ([29]). Given a predefined time $T_{\rm p} > 0$ which is completely unrestricted by the system's parameters and initial states, if $\lim_{t\to T_{\rm p}} z(t_0, z_{t_0})(t) = 0$ and $z(t_0, z_{t_0})(t) = 0$ for every $t \geq t_0 + T_{\rm p}$, then the DDE (2.2) is named PdT-attractive at zero.

Definition 7 ([29]). The DDE (2.2) is called PdTS at zero if it is stable and PdT-attractive at zero.

Definition 8 ([17]). A strictly increasing continuous function $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a K^∞-function (marked as $\Psi \in K^\infty$), if $\lim_{s\to+\infty} \Psi(s) = +\infty$ and $\Psi(0) = 0$.

Definition 9 ([33]). We say that $V : \mathbb{R}^n \to \mathbb{R}$ is C-regular if and only if

1) V is regular;
2) $V(0) = 0$ and $V(z) > 0$ for $z \neq 0$;
3) $V(z) \to +\infty$ as $\|z\| \to +\infty$.

Lemma 1 ([33,34]). If $V : \mathbb{R}^n \to \mathbb{R}$ is C-regular and $z(t) : [t_0, +\infty) \to \mathbb{R}^n$ is absolutely continuous, then $z(t)$ and $V(z(t)) : [t_0, +\infty) \to \mathbb{R}$ are differentiable for a.e. $t \geq t_0$, and

$$\frac{\mathrm{d}V(z(t))}{\mathrm{d}t} = \langle \mathscr{C}(t), \dot{z}(t) \rangle, \forall \mathscr{C}(t) \in \partial V(z(t)). \tag{2.6}$$

Lemma 2 ([35]). Let $Y_1, Y_2, \ldots, Y_n$ be non-negative numbers and $b > a > 0$, then

$$n^{\frac{b-a}{ab}} \left(\sum_{i=1}^{n} Y_i^b \right)^{1/b} \geq \left(\sum_{i=1}^{n} Y_i^a \right)^{1/a} \geq \left(\sum_{i=1}^{n} Y_i^b \right)^{1/b}. \tag{2.7}$$

3. Main results

In this section, two novel FxTS/PdTS criteria will be developed for the DDE (2.2). Then, the established FxTS/PdTS criteria are applied to the implementation of FxT/PdT control for the SNDIS (2.1). For convenience, let us assume $z(t) = z(t_0, z_{t_0})(t)$.

3.1. FxTS/PdTS criteria for DDE

Theorem 1. For DDE (2.2), assume that there can seek out a C-regular and locally Lipschitz continuous (LLc) function $V : \mathbb{R} \times \mathbb{C}_\tau \to \mathbb{R}_+$ satisfying $V(t, 0) = 0$ for all $t \in \mathbb{R}$. If

(A1) $\Psi_1 \in K^\infty$ exists s.t. $\Psi_1(\|\varphi(0)\|) \leq V(t, \varphi)$ holds for each $(t, \varphi) \in \mathbb{R} \times \mathbb{C}_\tau$;
(A2) The constants $b > 0$ and $0 < q < 1$ exist s.t. for a.e. $t \geq t_0$ and $\forall \varphi \in \mathbb{C}_\tau$, the derivative of V along the solution trajectories of the DDE (2.2) can be calculated as

$$\frac{\mathrm{d}V(t, \varphi)}{\mathrm{d}t}\Big|_{(2.2)} \leq (a(t) - b) \exp\left(V^q(t, \varphi)\right) V^{1-q}(t, \varphi); \tag{3.1}$$

(A3) The continuous function $a(t)$ satisfies $\int_{t_0}^{t} a(s)\mathrm{d}s \leq 0$ for each $t \geq t_0$ and $\int_0^{+\infty} a^+(s)\mathrm{d}s \overset{\text{def}}{=} a^{\mathbf{M}} < +\infty$, where $a^+(s) = 0 \vee a(s)$,

then the DDE (2.2) is FxTS at zero. Moreover, the estimation of StT is $T(t_0, z_{t_0}) \leq t_0 + T^{\max}$, where

$$T^{\max} = \frac{1 + qa^{\mathbf{M}}}{qb}. \tag{3.2}$$

Proof. The proof includes two steps.
Step 1. The DDE (2.2) is stable at zero.
Due to the arbitrariness of $\varphi \in \mathbb{C}_\tau$, it can take

$$\varphi(\sigma) = z_t(\sigma), \text{ for } \sigma \in [-\tau, 0]. \tag{3.3}$$

It is inferred from (3.1) and (3.3) that

$$\frac{\mathrm{d}V(t, z_t)}{\mathrm{d}t} \leq (a(t) - b) \exp\left(V^q(t, z_t)\right) V^{1-q}(t, z_t). \tag{3.4}$$

Set $W(V) = 1 - \exp(-V^q)$, and thus $\mathrm{d}W = q\exp(-V^q)V^{q-1}\mathrm{d}V$. Thus, the inequality (3.4) can be rewritten as

$$\frac{\mathrm{d}W(V(t,z_t))}{\mathrm{d}t} \le qa(t) - qb. \tag{3.5}$$

By integrating (3.5) from t_0 to t, we obtain

$$\int_{t_0}^{t} \frac{\mathrm{d}W(V(t,z_t))}{\mathrm{d}t}\mathrm{d}t \le \int_{t_0}^{t} (qa(s) - qb)\mathrm{d}s,$$

which yields

$$W(V(t,z_t)) - W(V(t_0,z_{t_0})) \le \int_{t_0}^{t} (qa(s) - qb)\mathrm{d}s. \tag{3.6}$$

Since $qb > 0$, from (3.6), we can derive

$$W(V(t,z_t)) - W(V(t_0,z_{t_0})) \le q\int_{t_0}^{t} a(s)\mathrm{d}s. \tag{3.7}$$

Recalling the condition $\int_{t_0}^{t} a(s)\mathrm{d}s \le 0$ for each $t \ge t_0$ in Hypothesis (A3), (3.7) implies that

$$W(V(t,z_t)) \le W(V(t_0,z_{t_0})). \tag{3.8}$$

Because $V(t_0,z_{t_0})$ is continuous at z_{t_0} and $V(t_0,0) = 0$, then $\forall\varepsilon > 0$ and $\forall t_0 \ge 0$, we can find $\delta = \delta(\varepsilon,t_0) > 0$ s.t. for any $z_{t_0} \in B(0,\delta) = \{z_{t_0} \in \mathbb{C}_\tau \big| \|z_{t_0}\|_{\mathbb{C}_\tau} < \delta\}$, it has $0 < W(\Psi_1(\varepsilon)) < 1$ and

$$V(t_0,z_{t_0}) < \Psi_1(\varepsilon). \tag{3.9}$$

Therefore, for all $t \ge t_0$, it follows from (3.8) and (3.9) that

$$\begin{aligned} 1 - \exp(-V^q(t,z_t)) = W(V(t,z_t)) &\le W(V(t_0,z_{t_0})) \\ &< W(\Psi_1(\varepsilon)) = 1 - \exp(-\Psi_1^q(\varepsilon)), \end{aligned}$$

which implies

$$V(t,z_t) < \Psi_1(\varepsilon), \text{ for all } t \ge t_0. \tag{3.10}$$

Recalling Hypothesis (A1), we can deduce from (3.10) that

$$\|z(t)\| = \|z_t(0)\| \le \Psi_1^{-1}(V(t,z_t)) < \Psi_1^{-1}(\Psi_1(\varepsilon)) = \varepsilon. \tag{3.11}$$

This means that the DDE (2.2) possesses stability at zero.

Step 2. The DDE (2.2) is FnT-attractive at zero and the StT $T(t_0,z_{t_0})$ is bounded in terms of z_{t_0}.

By Hypothesis (A3), we deduce from (3.6) that

$$W(V(t,z_t)) - W(V(t_0,z_{t_0})) \le \int_{t_0}^{t} (qa^+(s) - qb)\mathrm{d}s$$

$$\leq q\int_0^{+\infty} a^+(s)\mathrm{d}s - \int_{t_0}^{t} qb\mathrm{d}s = qa^{\mathbf{M}} - qb(t-t_0). \tag{3.12}$$

That is,

$$W(V(t,z_t)) \leq W(V(t_0,z_{t_0})) + qa^{\mathbf{M}} - qb(t-t_0). \tag{3.13}$$

Notice that $W(V(t,z_t)) \geq 0$ because $V(t,z_t) \geq 0$. In (3.13), let

$$W(V(t_0,z_{t_0})) + qa^{\mathbf{M}} - qb(t-t_0) \leq 0, \tag{3.14}$$

and thus $W(V(t,z_t)) = 0$ holds for all

$$\begin{aligned} t &\geq t_0 + \frac{W(V(t_0,z_{t_0})) + qa^{\mathbf{M}}}{qb} \\ &= t_0 + \frac{1-\exp(-V^q(t_0,z_{t_0})) + qa^{\mathbf{M}}}{qb} \triangleq T(t_0,z_{t_0}). \end{aligned} \tag{3.15}$$

Therefore, $V(t,z_t) = W^{-1}(0) = 0$ for all $t \geq T(t_0,z_{t_0})$. This implies that $z(t) = z_t(0) = 0$ for all $t \geq T(t_0,z_{t_0})$. Since $0 < \exp(-V^q(t_0,z_{t_0})) \leq 1$, the StT $T(t_0,z_{t_0}) \geq 0$ and it is bounded in terms of z_{t_0}, that is,

$$T(t_0,z_{t_0}) = t_0 + \frac{1-\exp(-V^q(t_0,z_{t_0})) + qa^{\mathbf{M}}}{qb} \leq t_0 + \frac{1+qa^{\mathbf{M}}}{qb} \triangleq t_0 + T^{\max}. \tag{3.16}$$

In a word, $z(t) = 0$ for all $t \geq t_0 + T^{\max}$.

Remark 2. In particular, if $a(t) = 0$, then $\int_{t_0}^{t} a(s)\mathrm{d}s = 0$ for all $t \geq t_0$ and $a^{\mathbf{M}} = \int_0^{+\infty} a^+(s)\mathrm{d}s = 0 < +\infty$. Thus, the StT in Theorem 1 can be estimated by $T(t_0,z_{t_0}) \leq t_0 + T_1^{\max}$, where

$$T_1^{\max} = \frac{1}{qb}. \tag{3.17}$$

Remark 3. In Theorem 1, if it is further supposed that

(A4) $V(t,\varphi) \leq \Psi_2(\|\varphi\|_{\mathbb{C}_\tau})$ holds for all $(t,\varphi) \in \mathbb{R}\times\mathbb{C}_\tau$ and $\Psi_2 \in K^\infty$,

then DDE (2.2) is FxT uniformly stable at zero. In fact, the FnT attractiveness of DDE (2.2) at zero has been proved in Theorem 1. Regarding the uniform stability, it can be ensured by using Hypothesis (A4). Actually, using the hypothesis that $V(t,\varphi) \leq \Psi_2(\|\varphi\|_{\mathbb{C}_\tau})$ for all $(t,\varphi) \in \mathbb{R}\times\mathbb{C}_\tau$ in (A4), this leads to

$$V(t_0,z_{t_0}) \leq \Psi_2(\|z_{t_0}\|_{\mathbb{C}_\tau})$$

Combining the inequality above and (3.8), it can be deduced that

$$\begin{aligned} 1-\exp(-V^q(t,z_t)) &= W(V(t,z_t)) \leq W(V(t_0,z_{t_0})) \\ &\leq W(\Psi_2(\|z_{t_0}\|_{\mathbb{C}_\tau})) = 1-\exp(-\Psi_2^q(\|z_{t_0}\|_{\mathbb{C}_\tau})), \end{aligned}$$

which yields

$$V(t,z_t) \leq \Psi_2(\|z_{t_0}\|_{\mathbb{C}_\tau}).$$

By Hypothesis (A1), it follows from the inequality above that

$$\|z(t)\| = \|z_t(0)\| \le \Psi_1^{-1}(V(t,z_t)) \le \Psi_1^{-1}(\Psi_2(\|z_{t_0}\|_{\mathbb{C}_\tau})).$$

When $\forall\varepsilon > 0$, we may choose $\delta = \Psi_2^{-1}(\Psi_1(\varepsilon))$, which is not constrained by t_0. Therefore, for every $z_{t_0} \in B(0,\delta) = \{z_{t_0} \in \mathbb{C}_\tau \big| \|z_{t_0}\|_{\mathbb{C}_\tau} < \delta\}$ and $t \ge t_0$, we have

$$\|z(t)\| \le \Psi_1^{-1}(\Psi_2(\|z_{t_0}\|_{\mathbb{C}_\tau})) < \Psi_1^{-1}(\Psi_2(\Psi_2^{-1}(\Psi_1(\varepsilon))) = \varepsilon.$$

This means that the DDE (2.2) is uniformly stable at zero.

Similar to Theorem 1, it can imply the following result.

Theorem 2. For the DDE (2.2), assume that there is a C-regular and LLc function $V : \mathbb{R} \times \mathbb{C}_\tau \to \mathbb{R}_+$ satisfying $V(t,0) = 0$ for all $t \in \mathbb{R}$. For given predefined time $T_{\mathrm{p}} > 0$, if Hypotheses (A1) and (A3) hold, and for a.e. $t \ge t_0$ and $\forall\varphi \in \mathbb{C}_\tau$, the derivative of V along the solution trajectories of the DDE (2.2) can be calculated as

$$\frac{\mathrm{d}V(t,\varphi)}{\mathrm{d}t}\Big|_{(2.2)} \le \left(a(t) - \frac{bT^{\max}}{T_{\mathrm{p}}}\right)\exp\left(V^q(t,\varphi)\right) V^{1-q}(t,\varphi), \tag{3.18}$$

where the constants $b > 0$ and $0 < q < 1$, and $T^{\max}$ has been given in (3.2), then the DDE (2.2) is PdTS at zero.

Proof. Similar to the proof of Theorem 1, the proof includes two steps.
Step 1 (Stability): Because $\frac{qbT^{\max}}{T_{\mathrm{p}}} > 0$, similar to (3.6) and (3.7), we can get

$$\begin{aligned} W(V(t,z_t)) - W(V(t_0,z_{t_0})) &\le \int_{t_0}^{t} (qa(s) - \frac{qbT^{\max}}{T_{\mathrm{p}}})\mathrm{d}s \\ &\le q\int_{t_0}^{t} a(s)\mathrm{d}s. \end{aligned}$$

Thus, the stability of the DDE (2.2) at zero can be guaranteed.
Step 2 (PdT attractiveness): Similar to (3.12), by Hypothesis (A3), we deduce that

$$\begin{aligned} W(V(t,z_t)) - W(V(t_0,z_{t_0})) &\le \int_{t_0}^{t} (qa^+(s) - \frac{qbT^{\max}}{T_{\mathrm{p}}})\mathrm{d}s \\ &\le q\int_{0}^{+\infty} a^+(s)\mathrm{d}s - \int_{t_0}^{t} \frac{qbT^{\max}}{T_{\mathrm{p}}}\mathrm{d}s = qa^{\mathbf{M}} - \frac{qbT^{\max}}{T_{\mathrm{p}}}(t-t_0). \end{aligned}$$

That is,

$$W(V(t,z_t)) \le W(V(t_0,z_{t_0})) + qa^{\mathbf{M}} - \frac{qbT^{\max}}{T_{\mathrm{p}}}(t-t_0).$$

Notice that $W(V(t,z_t)) \ge 0$ because of $V(t,z_t) \ge 0$. In the inequality above, let

$$W(V(t_0,z_{t_0})) + qa^{\mathbf{M}} - \frac{qbT^{\max}}{T_{\mathrm{p}}}(t-t_0) \le 0,$$

then $W(V(t, z_t)) = 0$ holds for all

$$
\begin{aligned}
t &\geq t_0 + \frac{W(V(t_0, z_{t_0})) + qa^{\mathbf{M}}}{qbT^{\max}} \cdot T_{\mathrm{p}} \\
&= t_0 + \frac{1 - \exp(-V^q(t_0, z_{t_0})) + qa^{\mathbf{M}}}{qbT^{\max}} \cdot T_{\mathrm{p}} \triangleq T(t_0, z_{t_0}).
\end{aligned}
$$

Therefore, $V(t, z_t) = W^{-1}(0) = 0$ for all $t \geq T(t_0, z_{t_0})$. This implies $z(t) = z_t(0) = 0$ for all $t \geq T(t_0, z_{t_0})$. Since $0 < \exp(-V^q(t_0, z_{t_0})) \leq 1$

$$
T(t_0, z_{t_0}) = t_0 + \frac{1 - \exp(-V^q(t_0, z_{t_0})) + qa^{\mathbf{M}}}{qbT^{\max}} \cdot T_{\mathrm{p}} \leq t_0 + \frac{1 + qa^{\mathbf{M}}}{qbT^{\max}} \cdot T_{\mathrm{p}} = t_0 + T_{\mathrm{p}}.
$$

In a word, $z(t) = 0$ for all $t \geq t_0 + T_{\mathrm{p}}$. That is, the DDE (2.2) is PdT-attractive at zero.

Remark 4. Because the value of function $a(t)$ is indefinite and can be taken as positive, negative, or zero, the derivative of V no longer needs to be negative/seminegative definite. Therefore, the FxTS/PdTS criteria in Theorem 1 and Theorem 2 have a wider range of applicability. On the other hand, in the previous FxTS/PdTS criteria, the derivative of the V function was implemented through power functions [29–31], the hyperbolic tangent function, and the logistic function [38], while in the FxTS/PdTS criteria of the earlier article, the derivative of the V function is implemented through a special exponential function. This also indicates that the FxTS/PdTS results of the previous article have been improved.

Remark 5. The StT estimation formula in Theorem 1 is different from the previous estimation formulas (see [21–23]) and very simple. This is very advantageous for designing appropriate control schemes when implementing FxT control of SNDISs. In Theorem 2, the predefined time to achieve the PdTS is not restricted by the system's parameters and initial states, which allows the system's states to quickly stabilize at the desired point, and the predefined time T_{p} can also be flexibly set according to the actual needs. Furthermore, the FxTS/PdTS criteria proposed in this article are applicable to non autonomous DDEs with discontinuities and can be further applied to realization of FxT/PdT control for discontinuous SNDISs.

Remark 6. Compared with infinite-time stability (i.e. asymptotic/exponential stability [15–17]), the most significant advantage of fixed/predefined-time stability is that their convergence speeds are accelerated. On the other hand, the advantage of fixed/predefined-time stability over infinite-time stability is that StT is not limited by system's initial states, and the T_p of the predetermined-time stability can be arbitrarily preset, while the convergence time for infinite-time stability cannot be preset.

Remark 7. Given the frequent occurrence of of stochastic phenomena in the real world, the fixed/predefined-time stability criteria derived in this article are expected to be further extended to stochastic discontinuous systems. This is theoretically possible. However, there are still some realistic difficulties that need to be overcome. For example, the fixed/predefined-time stability criteria in this article are applicable to discontinuous non stochastic differential equations based on differential inclusion. If the fixed/predefined-time stability criteria are extended to stochastic discontinuous systems, what is the best way to use stochastic differential inclusion to handle the solutions of stochastic discontinuous systems? This motivates us to further investigate the fixed/predefined-time stability problem of stochastic discontinuous systems in the future.

Remark 8. In [39], fixed/predefined-time control was realized for discontinuous systems based on an intermittent control strategy. In [40], fixed/predefined-time synchronization was implemented for neutral-type competitive networks possessing discontinuities. In [41], the predefined-time control for switching systems was executed. In [42], the fractional-order sliding mode coordinated control strategy was developed for ensuring the PdTS of a nuclear steam supply system. However, the focus of this paper is on the implementation of fixed/predefined-time control of SNDISs, which presents a novel research approach to address the control issues related to social models.

3.2. FxT stabilization control of SNDISs

Given the matrices

$$C_i(t) = \left(c^i_{lm}(t)\right)_{n_i\times n_i} = \begin{pmatrix} c^i_{11}(t) & c^i_{12}(t) & \cdots & c^i_{1n_i}(t) \\ c^i_{21}(t) & c^i_{22}(t) & \cdots & c^i_{2n_i}(t) \\ \vdots & \vdots & \ddots & \vdots \\ c^i_{n_i1}(t) & c^i_{n_i2}(t) & \cdots & c^i_{n_in_i}(t) \end{pmatrix},$$

where $i = 1, 2, \ldots, N$, we note that

$$\mathscr{I}^{\sup}_i = \sup_{t\in\mathbb{R}}\left\{\max_{1\le l,m\le n_i}\{|c^i_{lm}(t)|\}\right\}.$$

We design a state-feedback control scheme as follows:

$$\begin{aligned} u_i(t) = &-\lambda_i z_i(t) - \vartheta_i \mathbf{Sig}(z_i(t))\mathbf{E}_i - \eta_i \mathbf{Sig}(z_i(t))\mathbf{E}_i \sum_{j=1}^{N} \|z_j(t-\tau(t))\| \\ &+ \frac{1}{2}\left(\mathscr{M} a^+(t) - a^-(t)\right)\mathbf{Sig}(z_i(t))|z_i(t)|^{1-q}\cdot \exp\left(\left(\sum_{i=1}^{N} z_i^{\mathrm{T}}(t)z_i(t)\right)^{\frac{q}{2}}\right) \\ &- \ell_i \mathbf{Sig}(z_i(t))|z_i(t)|^{1-q}\cdot \exp\left(\left(\sum_{i=1}^{N} z_i^{\mathrm{T}}(t)z_i(t)\right)^{\frac{q}{2}}\right), \end{aligned} \tag{3.19}$$

where $i = 1, 2, \ldots, N$, with the control gains $\lambda_i \ge 0$, $\vartheta_i \ge 0$, $\eta_i \ge 0$, $\ell_i > 0$ and $0 < q < 1$. $\mathbf{Sig}(z_i(t)) = \mathrm{diag}(\mathrm{sign}(z_{i1}(t)), \ldots, \mathrm{sign}(z_{in_i}(t)))$. $\mathbf{E}_i = (\underbrace{1, \ldots, 1}_{n_i})^{\mathrm{T}}$. $|z_i(t)|^{1-q} = (|z_{i1}(t)|^{1-q}, \ldots, |z_{in_i}(t)|^{1-q})^{\mathrm{T}}$. The time-varying control gains $a^+(t) = 0 \vee a(t)$ and $a^-(t) = 0 \vee [-a(t)]$, where the function $a(t)$ is continuous and satisfies Hypothesis (A3). The constant $\mathscr{M}$ is given as

$$\mathscr{M} = \frac{1}{\max\limits_{1\le i\le N}\left\{n_i^{\frac{q}{2}}\right\}\cdot N^{\frac{q}{2}}}. \tag{3.20}$$

The control block diagram of the state feedback control scheme (3.19) for FxT stabilization is shown in Figure 2.

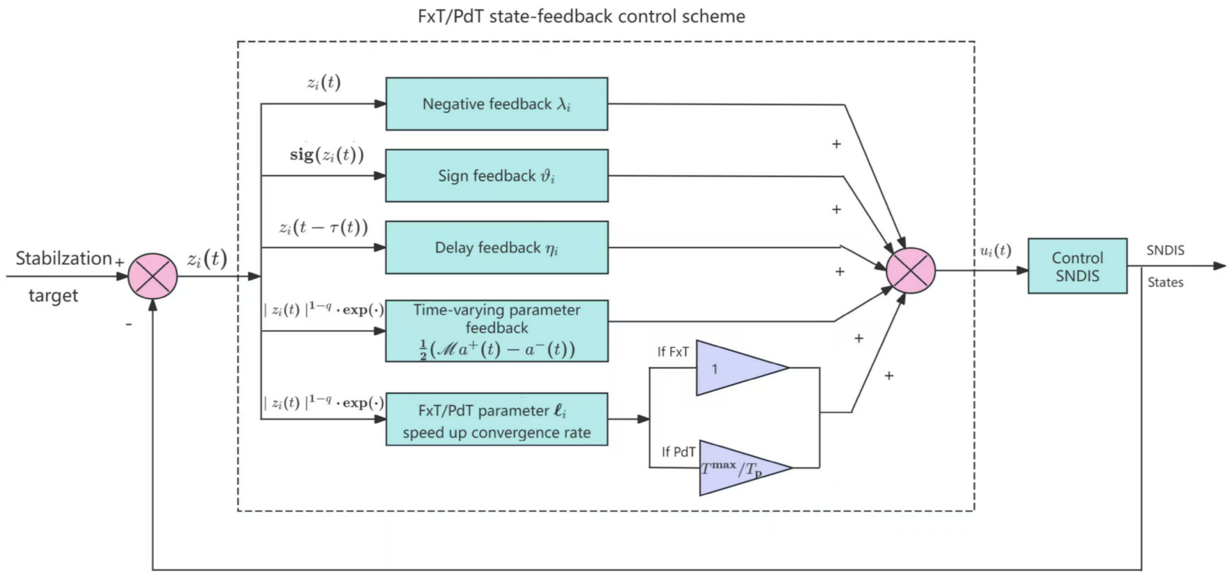

Figure 2. The control block diagram of state-feedback control schemes for FxT/PdT stabilization.

Theorem 3. Under Assumptions 1 and 2 and the state-feedback control scheme (3.19), if the function $a(t)$ satisfies Hypothesis (A3) and

$$\min_{1\le i\le N}\{\lambda_i\} \ge \max_{1\le i\le N}\left\{n_i\mathscr{I}_i^{\sup}\right\}, \tag{3.21}$$

$$\min_{1\le i\le N}\{\vartheta_i\} \ge \max_{1\le i\le N}\left\{\sum_{j=1}^{N}\beta_{ij}^{\sup}\right\} + \max_{1\le i\le N}\{D_i\}, \tag{3.22}$$

$$\min_{1\le i\le N}\{\eta_i\} \ge \max_{1\le i,j\le N}\left\{\alpha_{ij}^{\sup}\right\}, \tag{3.23}$$

then the SNDIS (2.1) can be FxT stabilized at zero, the StT is estimated by $T(t_0, z_{t_0}) \le t_0 + \widetilde{T}^{\max}$, where

$$\widetilde{T}^{\max} = \frac{2 + qa^{\mathbf{M}}}{2q\cdot\min_{1\le i\le N}\{\ell_i\}}. \tag{3.24}$$

Proof. Select a Lyapunov function

$$V(t, z_t) = \sum_{i=1}^{N} z_i^{\mathrm{T}}(t)z_i(t). \tag{3.25}$$

Differentiating (3.25) via Lemma 1, from (2.5), we find that

$$\begin{aligned}\frac{\mathrm{d}V(t,z_t)}{\mathrm{d}t} &= 2\sum_{i=1}^{N} z_i^{\mathrm{T}}(t)\dot{z}_i(t)\\ &= 2\sum_{i=1}^{N} z_i^{\mathrm{T}}(t)\left[C_i(t)z_i(t) + \sum_{j=1}^{N}\gamma_{ij}(t-\tau(t)) + H_i(t,z_i(t-\tau(t))) + u_i(t)\right].\end{aligned} \tag{3.26}$$

Substituting (3.19) into (3.26), it implies that

$$\begin{aligned}\frac{\mathrm{d}V(t,z_t)}{\mathrm{d}t} = {} & 2\sum_{i=1}^{N} z_i^{\mathrm{T}}(t)C_i(t)z_i(t) + 2\sum_{i=1}^{N} z_i^{\mathrm{T}}(t)\sum_{j=1}^{N}\gamma_{ij}(t-\tau(t)) + 2\sum_{i=1}^{N} z_i^{\mathrm{T}}(t)H_i(t,z_i(t-\tau(t)))\\ & - 2\sum_{i=1}^{N} z_i^{\mathrm{T}}(t)\lambda_i z_i(t) - 2\sum_{i=1}^{N} z_i^{\mathrm{T}}(t)\vartheta_i\mathbf{Sig}(z_i(t))\mathbf{E}_i\\ & - 2\sum_{i=1}^{N} z_i^{\mathrm{T}}(t)\eta_i\mathbf{Sig}(z_i(t))\mathbf{E}_i\sum_{j=1}^{N}\|z_j(t-\tau(t))\|\\ & + \sum_{i=1}^{N} z_i^{\mathrm{T}}(t)\mathscr{M}a^{+}(t)\mathbf{Sig}(z_i(t))|z_i(t)|^{1-q}\cdot\exp\left(\left(\sum_{i=1}^{N} z_i^{\mathrm{T}}(t)z_i(t)\right)^{\frac{q}{2}}\right)\\ & - \sum_{i=1}^{N} z_i^{\mathrm{T}}(t)a^{-}(t)\mathbf{Sig}(z_i(t))|z_i(t)|^{1-q}\cdot\exp\left(\left(\sum_{i=1}^{N} z_i^{\mathrm{T}}(t)z_i(t)\right)^{\frac{q}{2}}\right)\\ & - 2\sum_{i=1}^{N} z_i^{\mathrm{T}}(t)\ell_i\mathbf{Sig}(z_i(t))|z_i(t)|^{1-q}\cdot\exp\left(\left(\sum_{i=1}^{N} z_i^{\mathrm{T}}(t)z_i(t)\right)^{\frac{q}{2}}\right).\end{aligned} \tag{3.27}$$

We deduce that

$$\begin{aligned}z_i^{\mathrm{T}}(t)C_i(t)z_i(t) &= \sum_{l=1}^{n_i}\sum_{m=1}^{n_i} z_{il}(t)c_{lm}^{i}(t)z_{im}(t)\\ &\le \sum_{l=1}^{n_i}\sum_{m=1}^{n_i}|z_{il}(t)||c_{lm}^{i}(t)||z_{im}(t)|\\ &\le \max_{1\le l,m\le n_i}\{|c_{lm}^{i}(t)|\}\sum_{l=1}^{n_i}\sum_{m=1}^{n_i}|z_{il}(t)||z_{im}(t)|\\ &\le n_i\mathscr{I}_i^{\sup}\sum_{l=1}^{n_i} z_{il}^{2}(t) = n_i\mathscr{I}_i^{\sup}\|z_i(t)\|^2 = n_i\mathscr{I}_i^{\sup}z_i^{\mathrm{T}}(t)z_i(t).\end{aligned} \tag{3.28}$$

It can be inferred from (3.28) that

$$\sum_{i=1}^{N} z_i^{\mathrm{T}}(t)C_i(t)z_i(t) \le \sum_{i=1}^{N} n_i\mathscr{I}_i^{\sup}z_i^{\mathrm{T}}(t)z_i(t)$$

$$
\begin{aligned}
&\leq \max_{1\leq i\leq N}\left\{n_i\mathscr{I}_i^{\sup}\right\}\sum_{i=1}^{N} z_i^{\mathrm{T}}(t)z_i(t)\\
&= \max_{1\leq i\leq N}\left\{n_i\mathscr{I}_i^{\sup}\right\}V(t,z_t). \qquad (3.29)
\end{aligned}
$$

Under Assumption 2, using the Cauchy Schwarz inequality and Lemma 2, we can see that

$$
\begin{aligned}
\sum_{i=1}^{N} z_i^{\mathrm{T}}(t)\sum_{j=1}^{N}\gamma_{ij}(t-\tau(t)) &\leq \sum_{i=1}^{N}\|z_i(t)\|\cdot\sum_{j=1}^{N}\|\gamma_{ij}(t-\tau(t))\|\\
&\leq \sum_{i=1}^{N}\|z_i(t)\|\cdot\left[\sum_{j=1}^{N}(\alpha_{ij}(t)\|z_j(t-\tau(t))\|+\beta_{ij}(t))\right]\\
&\leq \max_{1\leq i,j\leq N}\left\{\alpha_{ij}^{\sup}\right\}\cdot\sum_{i=1}^{N}\sum_{j=1}^{N}\|z_i(t)\|\|z_j(t-\tau(t))\|\\
&\quad+\max_{1\leq i\leq N}\left\{\sum_{j=1}^{N}\beta_{ij}^{\sup}\right\}\cdot\sum_{i=1}^{N}\|z_i(t)\|. \qquad (3.30)
\end{aligned}
$$

Recalling the hypothesis $\|H_i(t,z_i(t-\tau(t)))\|\leq D_i$, we have

$$
\begin{aligned}
\sum_{i=1}^{N} z_i^{\mathrm{T}}(t)H_i(t,z_i(t-\tau(t))) &\leq \sum_{i=1}^{N}\|z_i(t)\|\|H_i(t,z_i(t-\tau(t)))\|\\
&\leq \sum_{i=1}^{N} D_i\|z_i(t)\|\\
&\leq \max_{1\leq i\leq N}\{D_i\}\cdot\sum_{i=1}^{N}\|z_i(t)\|. \qquad (3.31)
\end{aligned}
$$

It is obvious that

$$
\sum_{i=1}^{N} z_i^{\mathrm{T}}(t)\lambda_i z_i(t)\geq \min_{1\leq i\leq N}\{\lambda_i\}\cdot\sum_{i=1}^{N} z_i^{\mathrm{T}}(t)z_i(t) = \min_{1\leq i\leq N}\{\lambda_i\}V(t,z_t). \qquad (3.32)
$$

Applying Lemma 2, we have

$$
\begin{aligned}
z_i^{\mathrm{T}}(t)\vartheta_i\mathbf{Sig}(z_i(t))\mathbf{E}_i &= \vartheta_i\sum_{l=1}^{n_i}|z_{il}(t)|\\
&\geq \vartheta_i\left(\sum_{l=1}^{n_i}|z_{il}(t)|^2\right)^{1/2} = \vartheta_i\|z_i(t)\|, \qquad (3.33)
\end{aligned}
$$

which implies

$$
\sum_{i=1}^{N} z_i^{\mathrm{T}}(t)\vartheta_i\mathbf{Sig}(z_i(t))\mathbf{E}_i\geq \min_{1\leq i\leq N}\{\vartheta_i\}\cdot\sum_{i=1}^{N}\|z_i(t)\|. \qquad (3.34)
$$

Similar to (3.34), we can deduce that

$$\sum_{i=1}^{N} z_i^{\mathrm{T}}(t)\eta_i \mathbf{Sig}(z_i(t))\mathbf{E}_i \sum_{j=1}^{N} \|z_j(t-\tau(t))\| \geq \min_{1\leq i\leq N}\{\eta_i\} \sum_{i=1}^{N} \|z_i(t)\| \sum_{j=1}^{N} \|z_j(t-\tau(t))\|$$
$$= \min_{1\leq i\leq N}\{\eta_i\} \cdot \sum_{i=1}^{N}\sum_{j=1}^{N} \|z_i(t)\|\|z_j(t-\tau(t))\|. \tag{3.35}$$

Because $0 < q < 1$ and $2 > 2-q > 0$, utilizing Lemma 2, we have

$$n_i^{\frac{q}{2(2-q)}} \left(\sum_{l=1}^{n_i} |z_{il}(t)|^2\right)^{1/2} \geq \left(\sum_{l=1}^{n_i} |z_{il}(t)|^{2-q}\right)^{1/(2-q)}, \tag{3.36}$$

which yields

$$\sum_{l=1}^{n_i} |z_{il}(t)|^{2-q} \leq n_i^{\frac{q}{2}} \left(\sum_{l=1}^{n_i} |z_{il}(t)|^2\right)^{(2-q)/2} = n_i^{\frac{q}{2}} \|z_i(t)\|^{2-q}. \tag{3.37}$$

Recalling $a^+(t) = 0 \vee a(t) \geq 0$, by using (3.37), we have

$$z_i^{\mathrm{T}}(t)\mathscr{M} a^+(t)\mathbf{Sig}(z_i(t))|z_i(t)|^{1-q} = \mathscr{M} a^+(t) \sum_{l=1}^{n_i} |z_{il}(t)|^{2-q}$$
$$\leq \mathscr{M} a^+(t) n_i^{\frac{q}{2}} \|z_i(t)\|^{2-q}. \tag{3.38}$$

We can infer from (3.38) that

$$\sum_{i=1}^{N} z_i^{\mathrm{T}}(t)\mathscr{M} a^+(t)\mathbf{Sig}(z_i(t))|z_i(t)|^{1-q} \leq \mathscr{M} a^+(t) \max_{1\leq i\leq N}\left\{n_i^{\frac{q}{2}}\right\} \sum_{i=1}^{N} \|z_i(t)\|^{2-q}$$
$$= \mathscr{M} a^+(t) \max_{1\leq i\leq N}\left\{n_i^{\frac{q}{2}}\right\} \sum_{i=1}^{N} \left(z_i^{\mathrm{T}}(t) z_i(t)\right)^{\frac{2-q}{2}}. \tag{3.39}$$

Because $1 > (2-q)/2 > 0$, using Lemma 2, we have

$$N^{\frac{1-\frac{2-q}{2}}{\frac{2-q}{2}\cdot 1}} \sum_{i=1}^{N} Y_i \geq \left(\sum_{i=1}^{N} Y_i^{\frac{2-q}{2}}\right)^{1/(\frac{2-q}{2})} \tag{3.40}$$

holds for non-negative numbers $Y_1, Y_2, \ldots, Y_N$.

The inequality (3.40) can be rewritten as

$$\sum_{i=1}^{N} Y_i^{\frac{2-q}{2}} \leq N^{\frac{q}{2}} \left(\sum_{i=1}^{N} Y_i\right)^{\frac{2-q}{2}}. \tag{3.41}$$

Taking $Y_i = z_i^{\mathrm{T}}(t)z_i(t)$ in (3.41), we get

$$\sum_{i=1}^{N}\left(z_i^{\mathrm{T}}(t)z_i(t)\right)^{\frac{2-q}{2}} \le N^{\frac{q}{2}}\left(\sum_{i=1}^{N} z_i^{\mathrm{T}}(t)z_i(t)\right)^{\frac{2-q}{2}} = N^{\frac{q}{2}}V^{1-\frac{q}{2}}(t,z_t). \tag{3.42}$$

Combining (3.20), from (3.39) and (3.42), we can derive that

$$\begin{aligned}\sum_{i=1}^{N} z_i^{\mathrm{T}}(t)\mathscr{M}a^{+}(t)\mathbf{Sig}(z_i(t))|z_i(t)|^{1-q} &\le \mathscr{M}a^{+}(t)\max_{1\le i\le N}\left\{n_i^{\frac{q}{2}}\right\}\cdot N^{\frac{q}{2}}V^{1-\frac{q}{2}}(t,z_t)\\ &= a^{+}(t)V^{1-\frac{q}{2}}(t,z_t).\end{aligned} \tag{3.43}$$

Because $2 > 2-q > 0$, by applying Lemma 2, we have

$$\left(\sum_{l=1}^{n_i}|z_{il}(t)|^{2-q}\right)^{1/(2-q)} \ge \left(\sum_{l=1}^{n_i}|z_{il}(t)|^{2}\right)^{1/2}, \tag{3.44}$$

which yields

$$\sum_{l=1}^{n_i}|z_{il}(t)|^{2-q} \ge \left(\sum_{l=1}^{n_i}|z_{il}(t)|^{2}\right)^{(2-q)/2} = \|z_i(t)\|^{2-q}. \tag{3.45}$$

Since $a^{-}(t) = 0 \vee [-a(t)] \ge 0$, by using (3.45), we obtain

$$z_i^{\mathrm{T}}(t)a^{-}(t)\mathbf{Sig}(z_i(t))|z_i(t)|^{1-q} = a^{-}(t)\sum_{l=1}^{n_i}|z_{il}(t)|^{2-q} \ge a^{-}(t)\|z_i(t)\|^{2-q}. \tag{3.46}$$

From (3.46), we can deduce that

$$\begin{aligned}\sum_{i=1}^{N} z_i^{\mathrm{T}}(t)a^{-}(t)\mathbf{Sig}(z_i(t))|z_i(t)|^{1-q} &\ge a^{-}(t)\sum_{i=1}^{N}\|z_i(t)\|^{2-q}\\ &= a^{-}(t)\sum_{i=1}^{N}\left(z_i^{\mathrm{T}}(t)z_i(t)\right)^{\frac{2-q}{2}}.\end{aligned} \tag{3.47}$$

Since $1 > (2-q)/2 > 0$, using Lemma 2, we get

$$\left(\sum_{i=1}^{N} Y_i^{\frac{2-q}{2}}\right)^{1/(\frac{2-q}{2})} \ge \sum_{i=1}^{N} Y_i \tag{3.48}$$

holds for the given non-negative numbers $Y_1, Y_2, \ldots, Y_N$.

The inequality (3.48) can be rewritten as

$$\sum_{i=1}^{N} Y_i^{\frac{2-q}{2}} \ge \left(\sum_{i=1}^{N} Y_i\right)^{\frac{2-q}{2}}. \tag{3.49}$$

Taking $Y_i = z_i^{\mathrm{T}}(t)z_i(t)$ in (3.49), we have

$$\sum_{i=1}^{N}\left(z_i^{\mathrm{T}}(t)z_i(t)\right)^{\frac{2-q}{2}} \geq \left(\sum_{i=1}^{N} z_i^{\mathrm{T}}(t)z_i(t)\right)^{\frac{2-q}{2}} = V^{1-\frac{q}{2}}(t, z_t). \tag{3.50}$$

We deduce from (3.47) and (3.50) that

$$\sum_{i=1}^{N} z_i^{\mathrm{T}}(t)a^-(t)\mathbf{Sig}(z_i(t))|z_i(t)|^{1-q} \geq a^-(t)V^{1-\frac{q}{2}}(t, z_t). \tag{3.51}$$

Similar to (3.47), it follows that

$$\begin{aligned}\sum_{i=1}^{N} z_i^{\mathrm{T}}(t)\ell_i\mathbf{Sig}(z_i(t))|z_i(t)|^{1-q} &\geq \min_{1\leq i\leq N}\{\ell_i\}\cdot\sum_{i=1}^{N} z_i^{\mathrm{T}}(t)\mathbf{Sig}(z_i(t))|z_i(t)|^{1-q} \\ &\geq \min_{1\leq i\leq N}\{\ell_i\}\sum_{i=1}^{N}\|z_i(t)\|^{2-q} \\ &= \min_{1\leq i\leq N}\{\ell_i\}\sum_{i=1}^{N}\left(z_i^{\mathrm{T}}(t)z_i(t)\right)^{\frac{2-q}{2}}. \end{aligned}\tag{3.52}$$

It is inferred from (3.50) and (3.52) that

$$\sum_{i=1}^{N} z_i^{\mathrm{T}}(t)\ell_i\mathbf{Sig}(z_i(t))|z_i(t)|^{1-q} \geq \min_{1\leq i\leq N}\{\ell_i\}\, V^{1-\frac{q}{2}}(t, z_t). \tag{3.53}$$

Utilizing the conditions (3.21)–(3.23), combined with (3.29)–(3.32), (3.34), (3.35), (3.43), (3.51) and (3.53), from (3.27), we can find that

$$\begin{aligned}\frac{\mathrm{d}V(t,z_t)}{\mathrm{d}t} &\leq -2\left(\min_{1\leq i\leq N}\{\lambda_i\} - \max_{1\leq i\leq N}\left\{n_i\mathscr{I}_i^{\sup}\right\}\right)V(t,z_t) \\ &\quad - 2\left(\min_{1\leq i\leq N}\{\vartheta_i\} - \max_{1\leq i\leq N}\left\{\sum_{j=1}^{N}\beta_{ij}^{\sup}\right\} - \max_{1\leq i\leq N}\{D_i\}\right)\sum_{i=1}^{N}\|z_i(t)\| \\ &\quad - 2\left(\min_{1\leq i\leq N}\{\eta_i\} - \max_{1\leq i,j\leq N}\left\{\alpha_{ij}^{\sup}\right\}\right)\sum_{i=1}^{N}\sum_{j=1}^{N}\|z_i(t)\|\|z_j(t-\tau(t))\| \\ &\quad + \left(a^+(t) - a^-(t)\right)V^{1-\frac{q}{2}}(t,z_t)\cdot\exp\left(V^{\frac{q}{2}}(t,z_t)\right) - bV^{1-\frac{q}{2}}(t,z_t)\cdot\exp\left(V^{\frac{q}{2}}(t,z_t)\right) \\ &\leq \left(a^+(t) - a^-(t)\right)V^{1-\frac{q}{2}}(t,z_t)\cdot\exp\left(V^{\frac{q}{2}}(t,z_t)\right) - bV^{1-\frac{q}{2}}(t,z_t)\cdot\exp\left(V^{\frac{q}{2}}(t,z_t)\right), \end{aligned}\tag{3.54}$$

where $0 < q/2 < 1$ and the constant $b = 2\min_{1\leq i\leq N}\{\ell_i\} > 0$.

Because $a(t) = a^+(t) - a^-(t)$, (3.54) implies that

$$\begin{aligned}\frac{\mathrm{d}V(t,z_t)}{\mathrm{d}t} &\leq a(t)V^{1-\frac{q}{2}}(t,z_t)\cdot\exp\left(V^{\frac{q}{2}}(t,z_t)\right) - bV^{1-\frac{q}{2}}(t,z_t)\cdot\exp\left(V^{\frac{q}{2}}(t,z_t)\right) \\ &= (a(t)-b)\exp\left(V^{\frac{q}{2}}(t,z_t)\right)\cdot V^{1-\frac{q}{2}}(t,z_t). \end{aligned}\tag{3.55}$$

According to Theorem 1, the SNDIS (2.1) is FxT-stabilized at zero under the state feedback control scheme (3.19). Moreover, the StT is estimated by $T(t_0, z_{t_0}) \leq t_0 + \widetilde{T}^{\max}$, where $\widetilde{T}^{\max}$ has already been given in (3.24).

3.3. PdT stabilization control of SNDISs

Design a PdT state feedback control scheme as

$$\begin{aligned}u_i(t) =& -\lambda_i z_i(t) - \vartheta_i \mathbf{Sig}(z_i(t))\mathbf{E}_i - \eta_i \mathbf{Sig}(z_i(t))\mathbf{E}_i \sum_{j=1}^{N} \|z_j(t-\tau(t))\| \\ &+ \frac{1}{2}\left(\mathscr{M} a^+(t) - a^-(t)\right) \mathbf{Sig}(z_i(t))|z_i(t)|^{1-q} \cdot \exp\left(\left(\sum_{i=1}^{N} z_i^{\mathrm{T}}(t) z_i(t)\right)^{\frac{q}{2}}\right) \\ &- \frac{\widetilde{T}^{\max}}{T_{\mathrm{p}}} \ell_i \mathbf{Sig}(z_i(t))|z_i(t)|^{1-q} \cdot \exp\left(\left(\sum_{i=1}^{N} z_i^{\mathrm{T}}(t) z_i(t)\right)^{\frac{q}{2}}\right),\end{aligned} \tag{3.56}$$

where $i = 1, 2, \ldots, N$, $T_{\mathrm{p}} > 0$ represents a predefined time, $\widetilde{T}^{\max}$ has been given by (3.24), and the remaining parameters are the same as in (3.19). The control block diagram of the state feedback control scheme (3.56) for PdT stabilization is presented in Figure 2.

Theorem 4. Under Assumptions 1 and 2 and the PdT state feedback control scheme (3.56), if the function $a(t)$ satisfies Hypothesis (A3) and the conditions (3.21)–(3.23) hold, then the SNDIS (2.1) can be PdT-stabilized at zero within a predefined-time T_{p}.

Proof. We still select (3.25) as the Lyapunov function. Similar to the proof of Theorem 3, we obtain

$$\frac{\mathrm{d}V(t, z_t)}{\mathrm{d}t} \leq \left(a(t) - \frac{b\widetilde{T}^{\max}}{T_{\mathrm{p}}}\right) \exp\left(V^{\frac{q}{2}}(t, z_t)\right) \cdot V^{1-\frac{q}{2}}(t, z_t). \tag{3.57}$$

where $0 < q/2 < 1$ and $b = 2 \min\limits_{1 \leq i \leq N} \{\ell_i\} > 0$. According to Theorem 2, the SNDIS (2.1) is PdT-stabilized at zero in a predefined-time T_{p} under the PdT state feedback control scheme (3.56).

Remark 9. Unlike the fixed/predefined-time stabilization in [11, 43, 44], the advantages of fixed/predefined-time stabilization in Theorems 3 and 4 are reflected in the fact that it can deal with the discontinuous SNDIS with indefinite time-varying coefficients. Moreover, the fixed/predefined-time stabilization in Theorems 3 and 4 offers a faster convergence rate and more flexible convergence time.

4. Simulation results

In this section, a numerical example is provided to verify the FxT- and PdT-stabilization of SNDISs.

Lemma 3. Suppose that $a(t) = -\frac{1}{5}(\sin 3t + \cos 3t)e^{-3t}$. Then the function $a(t)$ is indefinite and satisfies $\int_0^t a(s)\mathrm{d}s \leq 0$ for all $t \geq 0$ and $a^{\mathbf{M}} = \int_0^{+\infty} a^+(s)\mathrm{d}s < +\infty$.

Proof. Clearly, $a(\frac{\pi}{6}) = -\frac{\pi}{5}e^{-\frac{\pi}{2}} < 0$ and $a(\frac{\pi}{3}) = \frac{1}{5}e^{-\pi} > 0$, this means that $a(t)$ is indefinite. On the other hand

$$\int_0^t a(s)\mathrm{d}s = -\int_0^t \frac{1}{5}(\sin 3s + \cos 3s)e^{-3s}\mathrm{d}s = \frac{2}{15}\left(\frac{\cos 3t}{e^{3t}} - 1\right) - \int_0^t a(s)\mathrm{d}s. \tag{4.1}$$

From (4.1), we find that

$$\int_0^t a(s)\mathrm{d}s = \frac{1}{15}\left(\frac{\cos 3t}{e^{3t}} - 1\right) \le 0 \text{ for all } t \ge 0. \tag{4.2}$$

Moreover

$$\begin{aligned} a^{\mathbf{M}} = \int_0^{+\infty} a^+(s)\mathrm{d}s \le \int_0^{+\infty} |a(s)|\mathrm{d}s = \int_0^{+\infty} \frac{|\sin 3s + \cos 3s|}{5e^{3s}}\mathrm{d}s \\ \le \int_0^{+\infty} \frac{2}{5e^{3s}}\mathrm{d}s = \frac{2}{15} < +\infty. \end{aligned} \tag{4.3}$$

Example 1. Consider the SNDIS (2.1) with two subsystems, where $N = 2, n_i = 2$ and $i = 1, 2$. The delay is $\tau(t) = 1$. The disturbance is $H_i(t, z_i(t-\tau(t))) = (H_{i1}(z_{i1}(t-\tau(t))), H_{i2}(t, z_{i2}(t-\tau(t))))^{\mathrm{T}}$, where $H_{i1}(z_{i1}(t-\tau(t))) = \frac{2\sqrt{2}\sin(z_{i1}(t-\tau(t)))}{20}$ and $H_{i2}(z_{i2}(t-\tau(t))) = \frac{2\sqrt{2}\sin(z_{i2}(t-\tau(t)))}{20}$. Obviously, $H_i(t, 0) = 0, \forall t \in \mathbb{R}$, and $\|H_i(t, z_i(t-\tau(t)))\| \le D_i = 0.2$. All parameter values of the SNDIS (2.1) are presented in Table 2. The matrices are given as

$$C_1(t) = \begin{pmatrix} -3.2 - 0.2\cos t & 1.5 - 0.2\cos t \\ 2.8 + 0.1\sin t & -2.5 + 0.1\sin t \end{pmatrix},$$

$$C_2(t) = \begin{pmatrix} -3.1 & 1.1 - 0.3\sin t \\ 2.7 - 0.2\cos t & -2.3 - 0.4\cos t \end{pmatrix},$$

Table 2. Simulation parameters in Example 1.

System/Scheme	Parameters
SNDIS (2.1)	$N = 2, n_i = 2, i = 1, 2, \tau(t) = 1,\ j = 1, 2$
	$H_{i1}(z_{i1}(t-\tau(t))) = \frac{2\sqrt{2}\sin(z_{i1}(t-\tau(t)))}{20}$
	$H_{i2}(z_{i2}(t-\tau(t))) = \frac{2\sqrt{2}\sin(z_{i2}(t-\tau(t)))}{20}$
	$c^1_{11} = -3.2 - 0.2\cos t,\ c^1_{12} = 1.5 - 0.2\cos t$
	$c^1_{21} = 2.8 + 0.1\sin t,\ c^1_{22} = -2.5 + 0.1\sin t$
	$c^2_{11} = -3.1,\ c^2_{12} = 1.1 - 0.3\sin t$
	$c^2_{21} = 2.7 - 0.2\cos t,\ c^2_{22} = -2.3 - 0.4\cos t$
Control scheme (3.19)	$N = 2, n_i = 2, i = 1, 2, \tau(t) = 1,\ j = 1, 2$
	$\lambda_1 = \lambda_2 = 7,\ \vartheta_1 = \vartheta_2 = 1.5$
	$\eta_1 = \eta_2 = 1,\ \ell_1 = \ell_2 = 3,\ q = \frac{1}{2}$
	$a(t) = -\frac{1}{5}(\sin 3t + \cos 3t)e^{-3t},\ \mathscr{M} = \sqrt{2}/2$
PdT control scheme (3.56)	$N = 2, n_i = 2, i = 1, 2, \tau(t) = 1,\ j = 1, 2$
	$\lambda_1 = \lambda_2 = 6.9,\ \vartheta_1 = \vartheta_2 = 1.6$
	$\eta_1 = \eta_2 = 0.9,\ \ell_1 = \ell_2 = 2.5,\ q = \frac{1}{2}$
	$a(t) = -\frac{1}{5}(\sin 3t + \cos 3t)e^{-3t}$
	$\mathscr{M} = \sqrt{2}/2,\ T_{\mathrm{p}} = 0.25$

The interconnection function $g_{ij}(t, z_j) = (g_{ij1}(t, z_{j1}), g_{ij2}(t, z_{j2}), \ldots, g_{ijn_i}(t, z_{jn_i}))^{\mathrm{T}}$ is given as

$$g_{ijk}(t,\theta) = \begin{cases} (0.5 - 0.2\cos t)(\tanh(\theta) - 0.6), & \theta \le 0, \\ (0.5 - 0.2\cos t)(\tanh(\theta) + 0.2), & 0 < \theta \le 1.5, \\ (0.5 - 0.2\cos t)(\tanh(\theta) - 0.5), & 1.5 < \theta \le 3, \\ (0.5 - 0.2\cos t)(\tanh(\theta) + 0.3), & \theta > 3, \end{cases}$$

for $\theta = z_{jk}$ and $k = 1, 2, \ldots, n_i$. It is obvious that $g_{ijk}(t, \theta)$ is discontinuous at $\theta = 0$, $\theta = 1.5$, and $\theta = 3$ (see Figure 3). It is easy to verify that $|g_{ijk}(t, \theta)| \le (0.5 - 0.2\cos t)|\theta| + 0.6$. Thus, $g_{ij}(\cdot)$ satisfies Assumptions 1 and 2 with $\alpha_{ij}(t) = 0.5 - 0.2\cos t$ and $\beta_{ij}(t) = 0.6$.

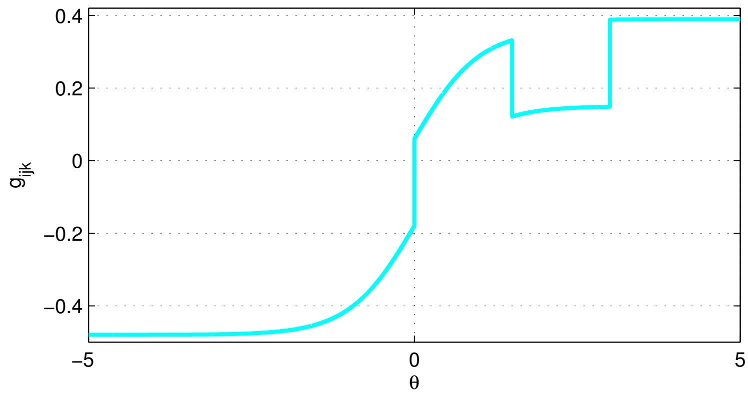

Figure 3. The interconnection $g_{ijk}(t, \theta)$ when $t = 0$.

If there is no control scheme in the system, the states of the SNDIS (2.1) is present in Figure 4 with the initial time $t_0 = 0$ and the initial states $z_{1t_0} = z_1(0) = (1.5, -1.5)^{\mathrm{T}}$ and $z_{2t_0} = z_2(0) = (-2.5, 2.5)^{\mathrm{T}}$. As shown in Figure 4, the SNDIS (2.1) is not stabilized at zero.

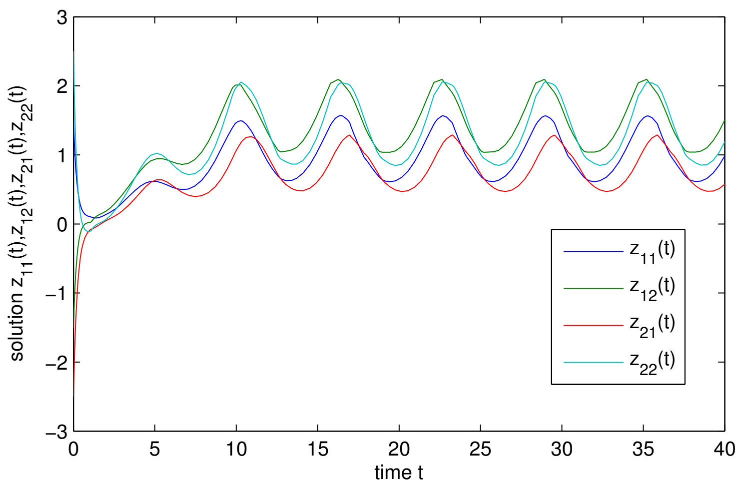

Figure 4. The states of the SNDIS (2.1) without a control scheme in Example 1.

Select the control scheme (3.19) with $\lambda_1 = \lambda_2 = 7$, $\vartheta_1 = \vartheta_2 = 1.5$, $\eta_1 = \eta_2 = 1$, $\ell_1 = \ell_2 = 3$, $q = \frac{1}{2}$

and

$$\mathscr{M} = \frac{1}{\max\limits_{1\le i\le N}\left\{n_i^{\frac{q}{2}}\right\}\cdot N^{\frac{q}{2}}} = \frac{\sqrt{2}}{2}.$$

Let us take $a(t) = -\frac{1}{5}(\sin 3t+\cos 3t)e^{-3t}$, using Lemma 3, it can verify that $a(t)$ satisfies Hypothesis (A3) with $a^{\mathbf{M}} \le \frac{2}{15}$. All parameter values of the control scheme (3.19) are present in Table 2. By calculation, we obtain

$$7 = \min_{1\le i\le N}\{\lambda_i\} \ge \max_{1\le i\le N}\left\{n_i\mathscr{I}_i^{\sup}\right\} = 6.8,$$

$$1.5 = \min_{1\le i\le N}\{\vartheta_i\} \ge \max_{1\le i\le N}\left\{\sum_{j=1}^{N}\beta_{ij}^{\sup}\right\} + \max_{1\le i\le N}\{D_i\} = 1.4,$$

$$1 = \min_{1\le i\le N}\{\eta_i\} \ge \max_{1\le i,j\le N}\left\{\alpha_{ij}^{\sup}\right\} = 0.7.$$

Therefore, all conditions of Theorem 3 hold and it shows that SNDIS (2.1) can be FxT-stabilized at zero via a control scheme (3.19), which is shown in Figures 5 and 6. Furthermore, the estimation of StT is $T(t_0, z_{t_0}) \le t_0 + \widetilde{T}^{\max}$, where $\widetilde{T}^{\max} \le 0.69$.

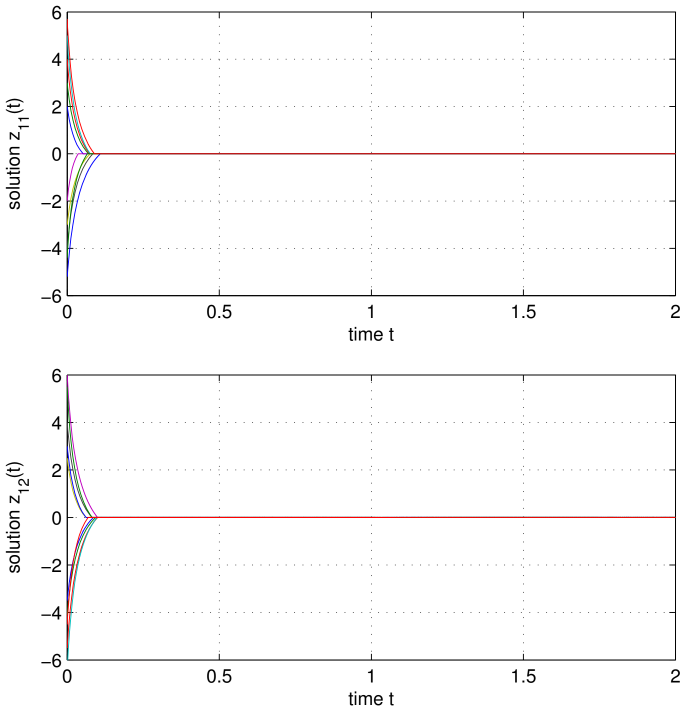

Figure 5. The states $z_{11}(t)$ and $z_{12}(t)$ in the subsystem $\mathbf{S}_1$ of the SNDIS (2.1) under the control scheme (3.19).

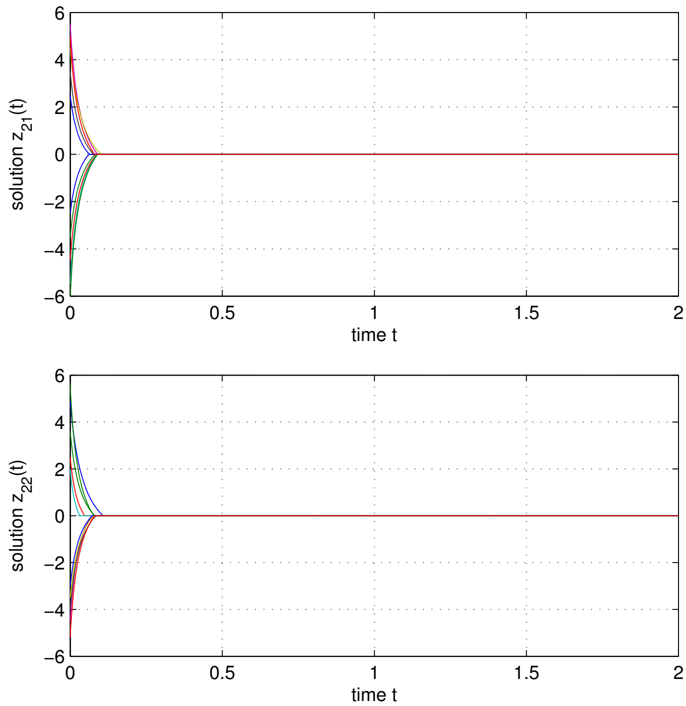

Figure 6. The states $z_{21}(t)$ and $z_{22}(t)$ in the subsystem $\mathbf{S}_2$ of the SNDIS (2.1) under the control scheme (3.19).

If we select the PdT control scheme (3.56) with $\lambda_1 = \lambda_2 = 6.9$, $\vartheta_1 = \vartheta_2 = 1.6$, $\eta_1 = \eta_2 = 0.9$, $\ell_1 = \ell_2 = 2.5$, $q = \frac{1}{2}$, and $\mathscr{M} = \frac{\sqrt{2}}{2}$. The PdT is taken as $T_{\mathrm{p}} = 0.25$. All parameter values of the control scheme (3.56) are presented in Table 2. By calculation, we have

$$6.9 = \min_{1\le i\le N} \{\lambda_i\} \ge \max_{1\le i\le N} \left\{n_i \mathscr{I}_i^{\sup}\right\} = 6.8,$$

$$1.6 = \min_{1\le i\le N} \{\vartheta_i\} \ge \max_{1\le i\le N} \left\{\sum_{j=1}^{N} \beta_{ij}^{\sup}\right\} + \max_{1\le i\le N} \{D_i\} = 1.4,$$

$$0.9 = \min_{1\le i\le N} \{\eta_i\} \ge \max_{1\le i,j\le N} \left\{\alpha_{ij}^{\sup}\right\} = 0.7.$$

Let us take $a(t) = -\frac{1}{5}(\sin 3t + \cos 3t)e^{-3t}$, which satisfies Hypothesis (A3). Therefore, all conditions of Theorem 4 hold, and it shows that the SNDIS (2.1) can be PdT-stabilized at zero via the control scheme (3.56), which is presented in Figures 7 and 8.

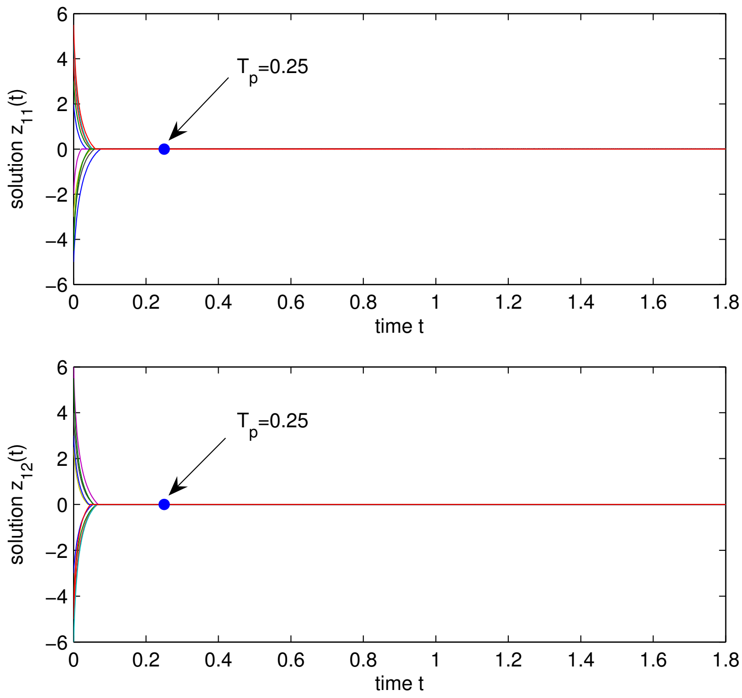

Figure 7. The states $z_{11}(t)$ and $z_{12}(t)$ in the subsystem $\mathbf{S}_1$ of the SNDIS (2.1) under the control scheme (3.56).

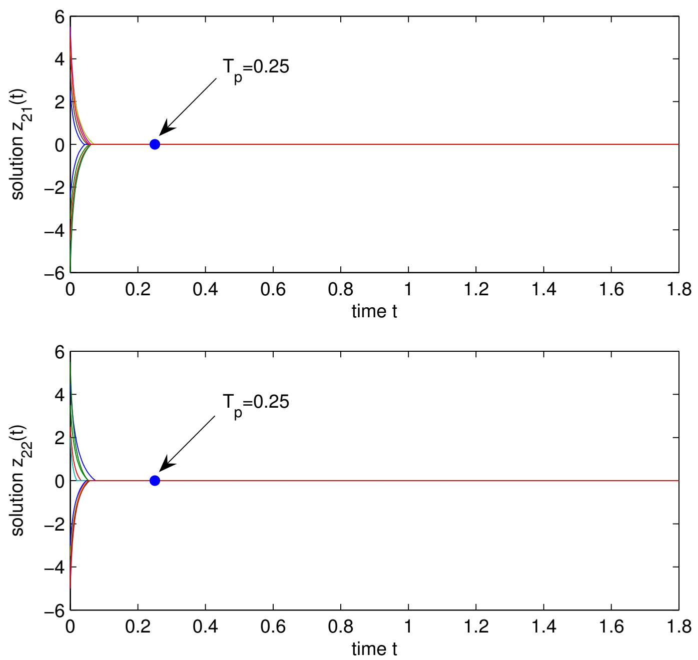

Figure 8. The states $z_{21}(t)$ and $z_{22}(t)$ in the subsystem $\mathbf{S}_2$ of the SNDIS (2.1) under the control scheme (3.56).

Remark 10. In [11], the FxT stabilization of ISs was realized by using an event-triggered control strategy and Lyapunov method with power functions, where the StT estimations for two subsystems

$\mathbf{S}_1$ and $\mathbf{S}_2$ are $T_1^* = 8.0458$ and $T_2^* = 3.4200$. In [43], the FxT stabilization of an interconnected system was achieved via an event-triggered mechanism, where the StT estimation is 5.672. However, our FxT stabilization of the SNDIS is implemented by a state feedback control scheme and the Lyapunov method with a particular exponential function, where the StT estimation does not exceed 0.69. This means that the convergence time of FxT stabilization in this paper is faster. In addition, unlike the event-triggered predefined time output feedback control design in [44], our PdT stabilization is based on PdT state feedback control scheme with indefinite time-varying parameters, where the PdT is arbitrarily set to $T_{\mathrm{p}} = 0.25$. Because $0 < q < 1$, it is clear that $\lim_{z_i \to 0} u_i(t) = 0$. That is, as the system converges really fast, the limit of the control input is zero (see Figure 9). On the other hand, because the control scheme contains a linear negative feedback term, a sign feedback term, a delay feedback term, a time-varying parameter feedback term, and a accelerating convergence rate term, one limitation of using a high-magnitude input is the increased economic cost of the control scheme. Although a larger amplitude of input is used in the control scheme, FxT/PdT stabilization control can be realized effectively. Moreover, the design of the time-varying parameter feedback term improves the flexibility of the control scheme. Due to the design of the exponential function term $\exp(\cdot)$, the upper bound estimation $\widetilde{T}^{\max}$ of the settling time is relatively simple and distinctive.

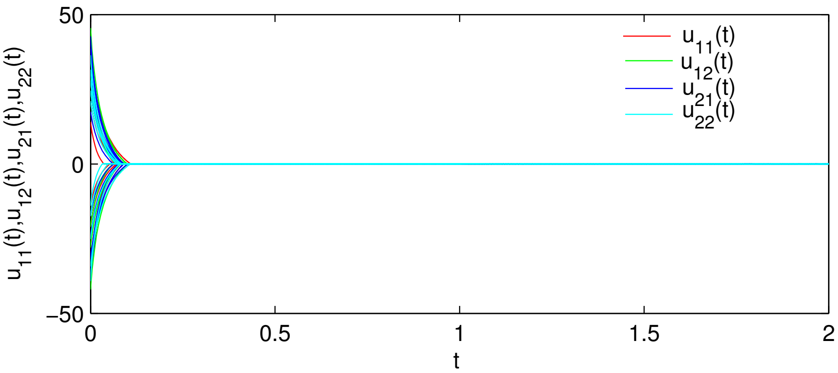

Figure 9. Evolution of the control input (3.19).

5. Conclusions

In this article, with the help of DDI theory and the Lyapunov energy function method, two FxTS/PdTS theorems applicable to discontinuous DDEs have been derived through a special exponential function with an indefinite time-varying coefficient. Unlike existing StT estimations for FxTS, this paper has presents a relatively simple StT estimation, whose upper bound is not restricted by any initial states of the system. By considering the jump discontinuity of the communication between social individuals, the time-delay effect, the time-varying coefficients, and the influence of disturbance factors, this article has established a SNDIS model possessing discontinuous interconnections. By cleverly designing switching state feedback control scheme and a PdT state feedback control scheme possessing an indefinite control gain, FxT/PdT stabilization control of the discontinuous SNDIS has been ultimately achieved. FxT/PdT control of SNDISs has important guiding significance for quickly building stable social relationships and achieving precise control objectives. Further research will focus on the FxT/PdT control problem of pulse and random social network-based ISs. The multistability problem of discontinuous SNDISs is also an interesting research topic, and some of the research methods involved are given in [45] and [46]. In addition, the diffusive equation and its FxTS/PdTS are also interesting and worthy of study, and some related work can be referred to in [47–49].

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

Zuowei Cai is a guest editor for Electronic Research Archive and was not involved in the editorial review or the decision to publish this article. The author declares there is no conflict of interest.

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