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Research article

A basis construction for Free subarrangements of Shi arrangements

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Abstract: D. Suyama and H. Terao established an exact basis construction for the derivation modules of the cone over the Shi arrangements utilizing Bernoulli polynomials. In this paper, building on the above basis, we introduce an explicit basis construction for a class of Free arrangements that lie between the cone of Linial arrangements and Shi arrangements.

Keywords: hyperplane arrangement; Shi arrangement; Free arrangement; Bernoulli polynomial; subarrangement

1. Introduction

Let V be an ℓ-dimensional vector space over a field $\mathbb{K}$ of characteristic 0. An affine arrangement of hyperplanes $\mathcal{A}$ is a finite collection of affine hyperplanes in V. $\mathcal{A}$ is called central if every hyperplane $H \in \mathcal{A}$ goes through the origin. Let S be the algebra of polynomial functions on V, and let $\mathrm{Der}_{\mathbb{K}}(S)$ be the set of $\mathbb{K}-$linear maps $\theta : S \to S$ such that

$$\theta(fg) = f\theta(g) + g\theta(f), \quad f, g \in S.$$

When $\mathcal{A}$ is central, for each $H \in \mathcal{A}$, choose $\alpha_H \in V^*$ with $\ker(\alpha_H) = H$. Define the module of $\mathcal{A}-$derivations by

$$\mathcal{D}(\mathcal{A}) := \{\theta \in \mathrm{Der}_{\mathbb{K}}(S) \,|\, \theta(\alpha_H) \in \alpha_H S \text{ for all } H \in \mathcal{A}\}.$$

An arrangement $\mathcal{A}$ is called a Free arrangement if $\mathcal{D}(\mathcal{A})$ is a free module over S. In this case, $\mathcal{D}(\mathcal{A})$ possesses a basis comprising ℓ homogeneous elements. For an affine arrangement $\mathcal{A}$ in V, $\mathbf{c}\mathcal{A}$ denotes the cone over $\mathcal{A}$, which is a central arrangement in an $(\ell+1)$-dimensional vector space by adding the new coordinate z.

Let $E = \mathbb{R}^{\ell}$ be an ℓ-dimensional Euclidean space with a coordinate system $x_1, \ldots, x_\ell$, and let Φ be a crystallographic irreducible root system. Fix a positive root system $\Phi^+ \subset \Phi$. For each positive root

$\alpha \in \Phi^+$ and $k \in \mathbb{Z}$, we define an affine hyperplane

$$H_{\alpha,k} := \{v \in V \,|(\alpha, v) = k\}.$$

The Coxeter arrangement $\{H_{\alpha,0}|\alpha \in \Phi^+\}$ was the first class of arrangements confirmed to be free, as demonstrated by Saito in [1]. Around 2000, Terao proved the freeness of Coxeter multiarrangements with constant multiplicities in [2].

In the study of the Kazhdan—Lusztig representation theory of the affine Weyl groups, Shi in [3] introduced the Shi arrangement

$$\mathrm{Shi}\,(\ell) := \left\{H_{\alpha,k} \left|\alpha \in \Phi^+, 0 \leq k \leq 1\right.\right\},$$

when the root system is of type $A_{\ell-1}$. The cone of $\mathrm{Shi}\,(\ell)$ was defined by

$$\mathbf{c}\mathrm{Shi}\,(\ell) := \left\{\mathbf{c}H_{\alpha,k} \left|\alpha \in \Phi^+, 0 \leq k \leq 1\right.\right\} \cup \{\{z = 0\}\},$$

where

$$\mathbf{c}H_{\alpha,k} := \{v \in V \,|(\alpha, v) = kz\}.$$

Indeed, there has been extensive research on the Shi arrangements and their relatives [4–6]. In [7], Yoshinaga demonstrated the freeness of the cones over the extended Catalan and Shi arrangements. In [8], Abe and Terao proved that the conings of the $W-$equivariant deformations are Free arrangements under a Shi-Catalan condition. However, there has been limited research on the construction of their bases; no general method for determining the bases has been discovered thus far. In the case of types $A_{\ell-1}$, B_ℓ, C_ℓ, and D_ℓ, explicit bases for the cones over the Shi arrangements were constructed in [9–11]. Additionally, a notable basis for the extended Shi arrangements of type A_2 was determined in [12]. In 2021, Suyama and Yoshinaga constructed the explicit bases for the extended Catalan and Shi arrangements of type $A_{\ell-1}$ using discrete integrals in [13]. Among these works, Suyama and Terao were the first to provide an explicit basis construction for the derivation module of the cone over the Shi arrangement in [9]. The essential ingredient of this recipe is the Bernoulli polynomials. The following are the relevant definitions.

Let $B_k\,(x)$ denote the k-th Bernoulli polynomial. Let $B_k\,(0) = B_k$ denote the k-th Bernoulli number. For $(s, t) \in (\mathbb{Z}_{\geq 0})^2$, Suyama in [9] defined the polynomial

$$B_{s,t}\,(x) := \sum_{i=0}^{s} \frac{1}{t+i+1} \begin{pmatrix} s \\ i \end{pmatrix} \{B_{t+i+1}\,(x) - B_{t+i+1}\}.$$

For example,

$$B_{1,1}\,(x) = B_{0,1}\,(x) + B_{0,2}\,(x) = \frac{1}{3}\left(x^3 - x\right).$$

Note that $B_{s,t}\,(x)$ is a polynomial of degree $s + t + 1$. The homogenization of $B_{s,t}\,(x)$ is defined by

$$\overline{B}_{s,t}\,(x, z) := z^{s+t+1} B_{s,t}\left(\frac{x}{z}\right).$$

Next, we define the arrangement for $p \in \mathbb{Z}^+$,

$$\begin{aligned}\mathcal{A}[p,\ell] &:= \{\{x_1 - x_n = 0\} \mid 2 \le n \le p \le \ell\} \cup \{\{x_m - x_\ell = 0\} \mid 2 \le m \le \ell - 1\} \\ &\cup \{\{x_m - x_n - z = 0\} \mid 1 \le m < n \le \ell\} \cup \{z = 0\}.\end{aligned}$$

Then $\mathcal{A}[p,\ell]$ is an arrangement between the Linial arrangement and the Shi arrangement. According to [6]-Theorem 3, we can get the arrangement $\mathcal{A}[p,\ell]$ is an central arrangement and free with exponents $\left(0, 1, (\ell-1)^{\ell-p}, \ell^{p-1}\right)$.

In this paper, we will provide an explicit construction of a basis for $\mathcal{D}(\mathcal{A}[p,\ell])$, $1 \le p \le \ell - 1$. The main results of this paper are as follows.

Theorem 1.1. *Define homogeneous derivations*

$$\begin{aligned}\eta_1 &:= \partial_1 + \partial_2 + \cdots + \partial_\ell \in \mathcal{D}(\mathcal{A}[1,\ell]), \\ \eta_2 &:= x_1\partial_1 + x_2\partial_2 + \cdots + x_\ell\partial_\ell + z\partial_z \in \mathcal{D}(\mathcal{A}[1,\ell]), \\ \varphi_1^{(1)} &:= \prod_{s=2}^{\ell} (x_1 - x_s - z)\,\partial_1 \in \mathcal{D}(\mathcal{A}[1,\ell]), \\ \varphi_j^{(1)} &:= \left(x_j - x_{j+1} - z\right) \quad \sum_{i=1}^{\ell} \sum_{\substack{0 \le k_1 \le j-2 \\ 0 \le k_2 \le \ell-j-1}} (-1)^{k_1+k_2} I_{[2,j-1]}^{j-k_1-2} I_{[j+2,\ell]}^{\ell-j-k_2-1} \overline{B}_{k_1,k_2}(y_i, z)\,\partial_i,\end{aligned}$$

for $2 \le j \le \ell - 1$, where

$$y_i = \begin{cases} x_1 - z, & i = 1, \\ x_i, & 2 \le i \le \ell, \end{cases}$$

and $\partial_i\,(1 \le i \le \ell)$ and ∂_z denote $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial z}$ respectively. $I_{[u,v]}^{k}$ denotes the elementary symmetric function in the variables $\{x_u, x_{u+1}, \ldots, x_v\}$ of degree k for $1 \le u \le v \le \ell$.

Then, the derivations $\eta_1, \eta_2, \varphi_1^{(1)}, \ldots, \varphi_{\ell-1}^{(1)}$ form a basis for $\mathcal{D}(\mathcal{A}[1,\ell])$.

Theorem 1.2. *For $2 \le p \le \ell - 1$, $1 \le j \le \ell - 1$, define the homogeneous derivations*

$$\varphi_j^{(p)} := \begin{cases} (x_1 - x_2)\,\varphi_1^{(1)} + (-1)^\ell (x_1 - x_2 - z) \sum_{m=3}^{p} (m-2)\,\varphi_m^{(1)}, & j = 1, \\ (-1)^\ell \varphi_1^{(1)} + \sum_{m=2}^{p} (m-1)\,\varphi_m^{(1)}, & j = 2, \\ \left(x_1 - x_j\right)\varphi_j^{(1)} - \left(x_j - x_{j+1} - z\right) \sum_{m=j+1}^{p} \varphi_m^{(1)}, & 3 \le j \le p-1, \\ \left(x_1 - x_j\right)\varphi_j^{(1)}, & j = p \ge 3, \\ \varphi_j^{(1)}, & p+1 \le j \le \ell - 1. \end{cases}$$

Then, the derivations $\eta_1, \eta_2, \varphi_1^{(p)}, \ldots, \varphi_{\ell-1}^{(p)}$ form a basis for $\mathcal{D}(\mathcal{A}[p,\ell])$.

2. The proof of main results

In order to prove Theorem 1.1, we require the following basis as defined by Suyama and Terao in [9].

Lemma 2.1. [[9], Theorem 3.5] *The arrangement* $\mathbf{c}\mathrm{Shi}(\ell)$ *is free with the exponents* $\left(0,1,\ell^{\ell-1}\right)$*. The homogeneous derivations* η_1,η_2*, and*

$$\psi_j^{(\ell)} := \left(x_j - x_{j+1} - z\right)\sum_{i=1}^{\ell}\sum_{\substack{0\le k_1\le j-1\\0\le k_2\le \ell-j-1}}(-1)^{k_1+k_2} I_{[1,j-1]}^{j-k_1-1} I_{[j+2,\ell]}^{\ell-j-k_2-1}\overline{B}_{k_1,k_2}(x_i,z)\,\partial_i,$$

form a basis for $\mathcal{D}(\mathbf{c}\mathrm{Shi}(\ell))$*, where* $1\le j\le \ell-1$.

Suyama and Terao reached the above conclusion using Saito's criterion, which is a crucial theorem for determining the basis of a free arrangement.

Lemma 2.2. [[14], Saito's criterion] *Let* $\mathcal{A}$ *be a central arrangement, and let* $Q(\mathcal{A})$ *be the defining polynomial of* $\mathcal{A}$*. Given* $\theta_1,\ldots,\theta_\ell\in\mathcal{D}(\mathcal{A})$*, the following two conditions are equivalent:*

1) $\det \mathrm{M}(\theta_1,\ldots,\theta_\ell)\doteq Q(\mathcal{A})$,

2) $\theta_1,\ldots,\theta_\ell$ *form a basis for* $\mathcal{D}(\mathcal{A})$ *over* S,

where $\mathrm{M}(\theta_1,\ldots,\theta_\ell)=\left(\theta_j(x_i)\right)_{\ell\times\ell}$ *denotes the coefficient matrix, and* $A\doteq B$ *means that* $A=cB$, $c\in\mathbb{K}\setminus\{0\}$.

Let $\alpha_\ell=(1,\ldots,1)^T$ and $\beta_\ell=(x_1,\ldots,x_\ell)^T$ be the $\ell\times 1$ column vectors, and define $\psi_{i,j}^{(\ell)}:=\psi_j^{(\ell)}(x_i)$ for $1\le i\le\ell$, $1\le j\le\ell-1$. Suyuma in [9] proved the equality

$$\begin{aligned}&\det\mathrm{M}\left(\eta_1,\eta_2,\psi_1^{(\ell)},\ldots,\psi_{\ell-1}^{(\ell)}\right)\\&=\det\begin{pmatrix}\alpha_\ell & \beta_\ell & \left(\psi_{i,j}^{(\ell)}\right)_{\ell\times(\ell-1)}\\ 0 & z & 0_{1\times(\ell-1)}\end{pmatrix}_{(\ell+1)\times(\ell+1)}\\&\doteq z\prod_{1\le m<n\le\ell}(x_m-x_n)(x_m-x_n-z),\end{aligned}$$

then we obtain

$$\det\left(\alpha_\ell\ \left(\psi_{i,j}^{(\ell)}\right)_{\ell\times(\ell-1)}\right)\doteq\prod_{1\le m<n\le\ell}(x_m-x_n)(x_m-x_n-z). \tag{2.1}$$

Proof of Theorem 1.1. Write $\varphi_{i,j}^{(1)}:=\varphi_j^{(1)}(x_i)$ for $1\le i\le\ell$, $1\le j\le\ell-1$, from the definitions of $\varphi_j^{(1)}$ and $\psi_j^{(\ell)}$, we can get

$$\varphi_{i,j}^{(1)}=\psi_{i-1,j-1}^{(\ell-1)}\Big|_{(x_1,\ldots,x_{\ell-1})\to(x_2,\ldots,x_\ell)} \tag{2.2}$$

for $2\le i\le\ell$, $2\le j\le\ell-1$. Consequently, for $2\le m<n\le\ell$, it follows that $\varphi_j^{(1)}(x_m-x_n)$ is divisible by x_m-x_n, and $\varphi_j^{(1)}(x_m-x_n-z)$ is divisible by x_m-x_n-z. Let the congruence notation $\overset{(n,k)}{\equiv}$ in the subsequent calculation denote modulo the ideal (x_1-x_n-kz). For $2\le n\le\ell$, we derive

$$\begin{aligned}&\varphi_j^{(1)}(x_1-x_n-z)\\&=\left(x_j-x_{j+1}-z\right)\sum_{\substack{0\le k_1\le j-2\\0\le k_2\le\ell-j-1}}(-1)^{k_1+k_2}I_{[2,j-1]}^{j-k_1-2}I_{[j+2,\ell]}^{\ell-j-k_2-1}\left[\overline{B}_{k_1,k_2}(x_1-z,z)-\overline{B}_{k_1,k_2}(x_n,z)\right]\\&\overset{(n,1)}{\equiv}0,\end{aligned}$$

which implies that $\varphi_j^{(1)}(x_1 - x_n - z)$ is divisible by $x_1 - x_n - z$. Thus $\varphi_j^{(1)} \in \mathcal{D}(\mathcal{A}[1,\ell])$ for $2 \le j \le \ell - 1$. Therefore, we have $\eta_1, \eta_2, \varphi_1^{(1)}, \ldots, \varphi_{\ell-1}^{(1)} \in \mathcal{D}(\mathcal{A}[1,\ell])$. Additionally, we obtain

$$
\begin{aligned}
&\det \mathrm{M}\left(\eta_1, \eta_2, \varphi_1^{(1)}, \ldots, \varphi_{\ell-1}^{(1)}\right) \\
&= (-1)^{\ell+1} z \det \begin{pmatrix} 1 & \prod\limits_{s=2}^{\ell} (x_1 - x_s - z) & \varphi_{1,2}^{(1)} & \cdots & \varphi_{1,\ell-1}^{(1)} \\ 1 & 0 & \varphi_{2,2}^{(1)} & \cdots & \varphi_{2,\ell-1}^{(1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \varphi_{\ell,2}^{(1)} & \cdots & \varphi_{\ell,\ell-1}^{(1)} \end{pmatrix}_{\ell \times \ell} \\
&= (-1)^{\ell} z \prod_{s=2}^{\ell} (x_1 - x_s - z) \det \begin{pmatrix} 1 & \varphi_{2,2}^{(1)} & \cdots & \varphi_{2,\ell-1}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \varphi_{\ell,2}^{(1)} & \cdots & \varphi_{\ell,\ell-1}^{(1)} \end{pmatrix}_{(\ell-1)\times(\ell-1)} \\
&= (-1)^{\ell} z \prod_{s=2}^{\ell} (x_1 - x_s - z) \det \begin{pmatrix} \alpha_{\ell-1} & \left(\varphi_{i,j}^{(1)}\right)_{(2 \le i \le \ell, 2 \le j \le \ell-1)} \end{pmatrix}_{(\ell-1)\times(\ell-1)}.
\end{aligned}
$$

According to the equalities (2.1) and (2.2), we have

$$
\det \begin{pmatrix} \alpha_{\ell-1} & \left(\varphi_{i,j}^{(1)}\right)_{(2 \le i \le \ell, 2 \le j \le \ell-1)} \end{pmatrix} \doteq \prod_{2 \le m < n \le \ell} (x_m - x_n)(x_m - x_n - z).
$$

Hence, we obtain

$$
\begin{aligned}
&\det \mathrm{M}\left(\eta_1, \eta_2, \varphi_1^{(1)}, \ldots, \varphi_{\ell-1}^{(1)}\right) \\
&\doteq z \prod_{s=2}^{\ell} (x_1 - x_s - z) \prod_{2 \le m < n \le \ell} (x_m - x_n)(x_m - x_n - z) \\
&= z \prod_{2 \le m < n \le \ell} (x_m - x_n) \prod_{1 \le m < n \le \ell} (x_m - x_n - z) \\
&= Q(\mathcal{A}[1,\ell]).
\end{aligned}
$$

By applying Lemma 2.2, we conclude that the derivations $\eta_1, \eta_2, \varphi_1^{(1)}, \ldots, \varphi_{\ell-1}^{(1)}$ form a basis for $\mathcal{D}(\mathcal{A}[1,\ell])$.

Lemma 2.3. *For* $2 \le n \le \ell$, $2 \le j \le \ell - 1$, *we have*

$$
\varphi_j^{(1)}(x_1 - x_n) \overset{(n,0)}{\equiv} (-1)^{\ell} z \left(x_j - x_{j+1} - z\right) \prod_{s=2}^{j-1} (x_n - x_s) \prod_{s=j+2}^{\ell} (x_n - x_s - z). \tag{2.3}
$$

Proof. We have the following congruence relation of polynomials modulo the ideal $(x_1 - x_n)$.

$$
\begin{aligned}
&\varphi_j^{(1)}(x_1 - x_n)\\
&= \left(x_j - x_{j+1} - z\right) \sum_{\substack{0\le k_1\le j-2\\ 0\le k_2\le \ell-j-1}} (-1)^{k_1+k_2} I_{[2,j-1]}^{j-k_1-2} I_{[j+2,\ell]}^{\ell-j-k_2-1}\left[\overline{B}_{k_1,k_2}(x_1 - z, z) - \overline{B}_{k_1,k_2}(x_n, z)\right]\\
&\overset{(n,0)}{\equiv} \left(x_j - x_{j+1} - z\right) \sum_{\substack{0\le k_1\le j-2\\ 0\le k_2\le \ell-j-1}} (-1)^{k_1+k_2+1} I_{[2,j-1]}^{j-k_1-2} I_{[j+2,\ell]}^{\ell-j-k_2-1}\left[\overline{B}_{k_1,k_2}(x_n, z) - \overline{B}_{k_1,k_2}(x_n - z, z)\right]\\
&= \left(x_j - x_{j+1} - z\right) \sum_{\substack{0\le k_1\le j-2\\ 0\le k_2\le \ell-j-1}} (-1)^{k_1+k_2+1} I_{[2,j-1]}^{j-k_1-2} I_{[j+2,\ell]}^{\ell-j-k_2-1} z^{k_1+k_2+1}\left(\frac{x_n}{z}\right)^{k_1}\left(\frac{x_n - z}{z}\right)^{k_2}\\
&= (-z)\left(x_j - x_{j+1} - z\right)\sum_{k_1=0}^{j-2} I_{[2,j-1]}^{j-k_1-2}(-x_n)^{k_1}\sum_{k_2=0}^{\ell-j-1} I_{[j+2,\ell]}^{\ell-j-k_2-1}[-(x_n - z)]^{k_2}\\
&= (-1)^{\ell} z\left(x_j - x_{j+1} - z\right)\prod_{s=2}^{j-1}(x_n - x_s)\prod_{s=j+2}^{\ell}(x_n - x_s - z).
\end{aligned}
$$

Remark 2.4. *In equality* (2.3)*, we observe that* $\prod_{s=2}^{j-1}(x_n - x_s) = 0$ *for* $2 \le n \le j-1$. *This implies that* $\varphi_j^{(1)}(x_1 - x_n)$ *is divisible by* $x_1 - x_n$ *for* $3 \le j \le \ell - 1$ *and* $2 \le n \le j - 1$.

In the following, we will prove that the derivations $\varphi_1^{(p)}, \dots, \varphi_{\ell-1}^{(p)}$ belong to the module $\mathcal{D}(\mathcal{A}[p,\ell])$ for $2 \le p \le \ell - 1$. It suffices to prove $\varphi_j^{(p)}(x_1 - x_n)$ is divisible by $x_1 - x_n$ for $2 \le n \le p$, $1 \le j \le \ell - 1$.

For convenience, we define the notations for $f, g, h \in \mathbb{Z}^+$,

$$
\mathrm{A}_f^{[g,h]} := \prod_{s=g}^{h}\left(x_f - x_s\right), \quad \mathrm{B}_f^{[g,h]} := \prod_{s=g}^{h}\left(x_f - x_s - z\right).
$$

If $g > h$, we agree that $\mathrm{A}_f^{[g,h]} = \mathrm{B}_f^{[g,h]} = 1$.

Lemma 2.5. *For any* $u \ge 4$, $v \ge 2$, *and* $w \ge j \ge 3$, *we have the following three equalities.*

$$
\begin{aligned}
\mathrm{B}_{u+1}^{[3,u]} &= (x_{u+1} - x_u - z)\left[x_{u+1} - x_{u-1} - (u-3)z\right]\mathrm{A}_{u+1}^{[3,u-2]}\\
&\quad + \sum_{m=3}^{u-2}(m-2)(x_m - x_{m+1} - z)\,\mathrm{A}_{u+1}^{[3,m-1]}\mathrm{B}_{u+1}^{[m+2,u+1]}.
\end{aligned} \tag{2.4}
$$

$$
\mathrm{B}_{v+1}^{[2,v]} = v\mathrm{A}_{v+1}^{[2,v]} + \sum_{m=2}^{v}(m-1)(x_m - x_{m+1} - z)\,\mathrm{A}_{v+1}^{[2,m-1]}\mathrm{B}_{v+1}^{[m+2,v+1]}. \tag{2.5}
$$

$$
\mathrm{A}_{w+1}^{[2,j]}\mathrm{B}_{w+1}^{[j+2,w+1]} = \mathrm{A}_{w+1}^{[2,w]} + \sum_{m=j+1}^{w}(x_m - x_{m+1} - z)\mathrm{A}_{w+1}^{[2,m-1]}\mathrm{B}_{w+1}^{[m+2,w+1]}. \tag{2.6}
$$

Proof. We will only prove equality (2.4) by induction on u. The proofs of equality (2.5) and (2.6) are similar. For $u = 4$, $\mathrm{B}_5^{[3,4]} = (x_5-x_3-z)(x_5-x_4-z)$, the equality holds. Assume that for $u = k$, the equality holds. Then, we replace x_{k+1} with x_{k+2}, and multiply both sides of the equality by $x_{k+2} - x_{k+1} - z$ to get

$$\mathrm{B}_{k+2}^{[3,k+1]} = (x_{k+2} - x_{k+1} - z)\,[x_{k+2} - x_k - (k-2)\,z]\,\mathrm{A}_{k+2}^{[3,k-1]} \sum_{m=3}^{k-1} (m-2)\,(x_m - x_{m+1} - z)\,\mathrm{A}_{k+2}^{[3,m-1]}\mathrm{B}_{k+2}^{[m+2,k+2]}.$$

We have completed the induction.

Proposition 2.6. *The derivation $\varphi_1^{(p)}$ belongs to the module $\mathcal{D}(\mathcal{A}[p,\ell])$ for $2 \le p \le \ell - 1$.*

Proof. We will establish this by induction on p. From Lemma 2.3, for $2 \le n \le p$, we have

$$\begin{aligned}
&\varphi_1^{(p)}(x_1 - x_n)\\
&= (x_1 - x_2)\,\varphi_1^{(1)}(x_1 - x_n) + (-1)^{\ell}(x_1 - x_2 - z)\sum_{m=3}^{p}(m-2)\,\varphi_m^{(1)}(x_1 - x_n)\\
&\overset{(n,0)}{\equiv} (x_n - x_2)\,\mathrm{B}_n^{[2,\ell]} + z\,(x_n - x_2 - z)\sum_{m=3}^{p}(m-2)\,(x_m - x_{m+1} - z)\,\mathrm{A}_n^{[2,m-1]}\mathrm{B}_n^{[m+2,\ell]}.
\end{aligned}$$

1) For $p = 2$, it is obvious that $\varphi_1^{(2)} = (x_1 - x_2)\,\varphi_1^{(1)}$ is divisible by $x_1 - x_2$.

2) For $p = 3$, we get

$$\begin{aligned}
&\varphi_1^{(3)}(x_1 - x_n)\\
&\overset{(n,0)}{\equiv} (x_n - x_2)\,\mathrm{B}_n^{[2,\ell]} + z\,(x_n - x_2 - z)\,(x_3 - x_4 - z)\,(x_n - x_2)\,\mathrm{B}_n^{[5,\ell]}\\
&= (x_n - x_2)\,(x_n - x_2 - z)\,\mathrm{B}_n^{[5,\ell]}\,[(x_n - x_3 - z)\,(x_n - x_4 - z) + z\,(x_3 - x_4 - z)]\,.
\end{aligned}$$

If $n = 2, 3$, we have $\varphi_1^{(3)}(x_1 - x_n) \overset{(n,0)}{\equiv} 0$; that is, $\varphi_1^{(3)}(x_1 - x_n)$ is divisible by $x_1 - x_n$ for $n = 2, 3$. Therefore, $\varphi_1^{(3)} \in \mathcal{D}(\mathcal{A}[3,\ell])$.

3) For any $3 \le k \le \ell - 2$, assume that $\varphi_1^{(k)} \in \mathcal{D}(\mathcal{A}[k,\ell])$, which implies that $\varphi_1^{(k)}(x_1 - x_n)$ is divisible by $x_1 - x_n$ for $2 \le n \le k$.

For $p = k + 1$, we can see

$$\varphi_1^{(k+1)} = \varphi_1^{(k)} + (-1)^{\ell}(k-1)(x_1 - x_2 - z)\varphi_{k+1}^{(1)}.$$

According to the induction hypothesis and Remark 2.4, it suffices to prove that $\varphi_1^{(k+1)}(x_1 - x_{k+1})$ is divisible by $x_1 - x_{k+1}$. By using the equality (2.4), we have

$$\begin{aligned}
&\varphi_1^{(k+1)}(x_1 - x_{k+1})\\
&\overset{(k+1,0)}{\equiv} (x_{k+1} - x_2)\,\mathrm{B}_{k+1}^{[2,\ell]} + z\,(x_{k+1} - x_2 - z)\sum_{m=3}^{k+1}(m-2)\,(x_m - x_{m+1} - z)\,\mathrm{A}_{k+1}^{[2,m-1]}\mathrm{B}_{k+1}^{[m+2,\ell]}\\
&= 0.
\end{aligned}$$

Therefore, $\varphi_1^{(k+1)}(x_1 - x_{k+1})$ is divisible by $x_1 - x_{k+1}$, and we have $\varphi_1^{(k+1)} \in \mathcal{D}(\mathcal{A}[k+1,\ell])$.

Hence, we may conclude that for any $2 \le p \le \ell - 1$, $\varphi_1^{(p)} \in \mathcal{D}(\mathcal{A}[p,\ell])$.

Proposition 2.7. *The derivation $\varphi_2^{(p)}$ belongs to the module $\mathcal{D}(\mathcal{A}[p,\ell])$ for $2 \le p \le \ell - 1$.*

Proof. From Lemma 2.3, we can get for $2 \le n \le p$,

$$\varphi_2^{(p)}(x_1 - x_n) \overset{(n,0)}{\equiv} (-1)^\ell \Big[\mathrm{B}_n^{[2,\ell]} + z \sum_{m=2}^{p} (m-1)(x_m - x_{m+1} - z)\, \mathrm{A}_n^{[2,m-1]} \mathrm{B}_n^{[m+2,\ell]} \Big].$$

For $p = 2, 3$, this conclusion is straightforward to verify.

For any $3 \le k \le \ell - 2$, assume that $\varphi_2^{(k)} \in \mathcal{D}(\mathcal{A}[k,\ell])$. For $p = k+1$, we have

$$\varphi_2^{(k+1)} = \varphi_2^{(k)} + k\varphi_{k+1}^{(1)}.$$

By using the equality (2.5), we have

$$\begin{aligned}
&\varphi_2^{(k+1)}(x_1 - x_{k+1}) \\
&\overset{(k+1,0)}{\equiv} (-1)^\ell \mathrm{B}_{k+1}^{[2,\ell]} + (-1)^\ell z \sum_{m=2}^{k+1} (m-1)(x_m - x_{m+1} - z)\, \mathrm{A}_{k+1}^{[2,m-1]} \mathrm{B}_{k+1}^{[m+2,\ell]} \\
&= 0.
\end{aligned}$$

Therefore, $\varphi_2^{(k+1)}(x_1 - x_{k+1})$ is divisible by $x_1 - x_{k+1}$. According to the induction hypothesis and Remark 2.4, we have $\varphi_2^{(k+1)} \in \mathcal{D}(\mathcal{A}[k+1,\ell])$.

Proposition 2.8. *The derivation $\varphi_j^{(p,\ell)}$ belongs to the module $\mathcal{D}(\mathcal{A}[p,\ell])$ for $3 \le j \le \ell - 1$ and $4 \le p \le \ell - 1$.*

Proof. First, from Lemma 2.3, for $3 \le j \le p-1$ and $2 \le n \le p$, we can get

$$\varphi_j^{(p)}(x_1 - x_n) \overset{(n,0)}{\equiv} (-1)^\ell z \big(x_j - x_{j+1} - z\big) \left[\mathrm{A}_n^{[2,j]} \mathrm{B}_n^{[j+2,\ell]} - \sum_{m=j+1}^{p} (x_m - x_{m+1} - z)\, \mathrm{A}_n^{[2,m-1]} \mathrm{B}_n^{[m+2,\ell]} \right].$$

1) For $p = 4$, it is obvious that $\varphi_j^{(4)} \in \mathcal{D}(\mathcal{A}[4,\ell])$.

2) For any $4 \le k \le \ell - 2$, assume that $\varphi_j^{(k)} \in \mathcal{D}(\mathcal{A}[k,\ell])$, which implies that $\varphi_j^{(k)}(x_1 - x_n)$ is divisible by $x_1 - x_n$ for $2 \le n \le k$ and $3 \le j \le k-1$.

For $p = k+1$, we can see

$$\varphi_j^{(k+1)} = \varphi_j^{(k)} - \big(x_j - x_{j+1} - z\big)\varphi_{k+1}^{(1)}.$$

By using the equality (2.6), we have

$$\begin{aligned}
&\varphi_j^{(k+1)}(x_1 - x_{k+1}) \\
&\overset{(k+1,0)}{\equiv} (-1)^\ell z \big(x_j - x_{j+1} - z\big) \left[\mathrm{A}_{k+1}^{[2,j]} \mathrm{B}_{k+1}^{[j+2,\ell]} - \sum_{m=j+1}^{k+1} (x_m - x_{m+1} - z)\, \mathrm{A}_{k+1}^{[2,m-1]} \mathrm{B}_{k+1}^{[m+2,\ell]} \right] \\
&= 0.
\end{aligned}$$

Therefore, $\varphi_j^{(k+1)}(x_1 - x_{k+1})$ is divisible by $x_1 - x_{k+1}$. According to the induction hypothesis and Remark 2.4, we have $\varphi_j^{(k+1)} \in \mathcal{D}(\mathcal{A}[k+1,\ell])$.

For $p \le j \le \ell - 1$, it is evident that $\varphi_j^{(p)} \in \mathcal{D}(\mathcal{A}[p,\ell])$.

Proof of Theorem 1.2. According to Propositions 2.6, 2.7, and 2.8, it suffices to prove $\det \mathrm{M}\left(\eta_1, \eta_2, \varphi_1^{(p)}, \ldots, \varphi_{\ell-1}^{(p)}\right) \doteq Q(\mathcal{A}[p,\ell])$ for $2 \le p \le \ell - 1$. Let

$$\gamma_1 = \left(x_1 - x_2, 0, (-1)^\ell (x_1 - x_2 - z), \ldots, (-1)^\ell (p-2)(x_1 - x_2 - z)\right)^T$$

and

$$\gamma_2 = \left((-1)^\ell, 1, 2, \ldots, p-1\right)^T$$

be the $p \times 1$ column vectors, and define a matrix

$$M_{p\times(p-2)} := \begin{pmatrix} 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \\ x_1 - x_3 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ -(x_3 - x_4 - z) & \cdots & x_1 - x_{p-1} & 0 \\ -(x_3 - x_4 - z) & \cdots & -\left(x_{p-1} - x_p - z\right) & x_1 - x_p \end{pmatrix}.$$

Write $\widetilde{M}_{p\times p} := \left(\gamma_1, \gamma_2, M_{p\times(p-2)}\right)$, then

$$\det \widetilde{M}_{p\times p} = \prod_{s=2}^{p} (x_1 - x_s).$$

Thus, we obtain the following equality:

$$\begin{aligned} &\left(\eta_1, \eta_2, \varphi_1^{(p)}, \ldots, \varphi_{\ell-1}^{(p)}\right)_{(\ell+1)\times(\ell+1)} \\ &= \left(\eta_1, \eta_2, \varphi_1^{(1)}, \ldots, \varphi_{\ell-1}^{(1)}\right) \begin{pmatrix} E_2 & 0_{2\times p} & 0_{2\times(\ell-p-1)} \\ 0_{p\times 2} & \widetilde{M}_{p\times p} & 0_{p\times(\ell-p-1)} \\ 0_{(\ell-p-1)\times 2} & 0_{(\ell-p-1)\times p} & E_{\ell-p-1} \end{pmatrix}. \end{aligned}$$

Hence,

$$\begin{aligned} &\det \mathrm{M}\left(\eta_1, \eta_2, \varphi_1^{(p)}, \ldots, \varphi_{\ell-1}^{(p)}\right) \\ &= \det \mathrm{M}\left(\eta_1, \eta_2, \varphi_1^{(1)}, \ldots, \varphi_{\ell-1}^{(1)}\right) \det \widetilde{M}_{p\times p} \\ &\doteq z \prod_{s=2}^{\ell} (x_1 - x_s - z) \prod_{2\le m<n\le\ell} (x_m - x_n) \prod_{2\le m<n\le\ell} (x_m - x_n - z) \prod_{s=2}^{p} (x_1 - x_s) \\ &= z \prod_{2\le m<n\le\ell} (x_m - x_n) \prod_{1\le m<n\le\ell} (x_m - x_n - z) \prod_{s=2}^{p} (x_1 - x_s) \\ &\doteq Q(\mathcal{A}[p,\ell]). \end{aligned}$$

We complete the proof.

3. Example

In this section, according to the derivations defined in Theorem 1.3, we will present the hyperplanes contained in the arrangement $\mathcal{A}[3,4]$ and the specific expressions for the basis of its derivation module.

Example 3.1. *This arrangement* $\mathcal{A}[3,4]$ *contains the following 12 hyperplanes:*

$$
\begin{aligned}
&x_1 - x_2 - z = 0,\ x_1 - x_3 - z = 0,\ x_1 - x_4 - z = 0,\\
&x_2 - x_3 - z = 0,\ x_2 - x_4 - z = 0,\ x_3 - x_4 - z = 0,\\
&x_1 - x_2 = 0,\ x_1 - x_3 = 0,\ x_2 - x_3 = 0,\ x_2 - x_4 = 0,\\
&x_3 - x_4 = 0,\ z = 0.
\end{aligned}
$$

According to the basis expression form in Theorem 1.3, substituting the parameters $\ell = 4$ *and* $p = 3$*, we can obtain the following five derivations:* $\eta_1, \eta_2, \varphi_1^{(3)}, \varphi_2^{(3)}$ *and* $\varphi_3^{(3)}$*, which form the basis for* $\mathcal{D}(\mathcal{A}[3,4])$*:*

$$
\begin{aligned}
\eta_1 &= \partial_1 + \partial_2 + \partial_3 + \partial_4,\\
\eta_2 &= x_1\partial_1 + x_2\partial_2 + x_3\partial_3 + x_4\partial_4 + z\partial_z,\\
\varphi_1^{(3)} &= (x_1 - x_2 - z)\left[(x_1 - x_2)(x_1 - x_3 - z)(x_1 - x_4 - z) + \frac{1}{2}(x_1 - z)(x_3 - x_4 - z)(2x_2 - x_1)\right]\partial_1\\
&\quad + \frac{1}{2}x_2(x_2 - z)(x_1 - x_2 - z)(x_3 - x_4 - z)\partial_2 + \frac{1}{2}x_3(x_1 - x_2 - z)(x_3 - x_4 - z)(2x_2 - x_3 - z)\partial_3\\
&\quad + \frac{1}{2}x_4(x_1 - x_2 - z)(x_3 - x_4 - z)(2x_2 - x_4 - z)\partial_4,\\
\varphi_2^{(3)} &= \left\{\prod_{i=2}^{4}(x_1 - x_i - z) + (x_1 - z)\left[\frac{1}{2}(x_2 - x_3 - z)(2x_4 - x_1 + 2z) + (x_3 - x_4 - z)(2x_2 - x_1)\right]\right\}\partial_1\\
&\quad + \left[\frac{1}{2}x_2(x_2 - x_3 - z)(2x_4 - x_2 + z) + x_2(x_2 - z)(x_3 - x_4 - z)\right]\partial_2\\
&\quad + \left[\frac{1}{2}x_3(x_2 - x_3 - z)(2x_4 - x_3 + z) + x_3(x_3 - x_4 - z)(2x_2 - x_3 - z)\right]\partial_3\\
&\quad + \left[\frac{1}{2}x_4(x_4 + z)(x_2 - x_3 - z) + x_4(x_3 - x_4 - z)(2x_2 - x_4 - z)\right]\partial_4,\\
\varphi_3^{(3)} &= \frac{1}{2}(x_1 - x_3)(x_1 - z)(2x_2 - x_1)(x_3 - x_4 - z)\partial_1 + \frac{1}{2}x_2(x_1 - x_3)(x_2 - z)(x_3 - x_4 - z)\partial_2\\
&\quad + \frac{1}{2}x_3(x_1 - x_3)(x_3 - x_4 - z)(2x_2 - x_3 - z)\partial_3 + \frac{1}{2}x_4(x_1 - x_3)(x_3 - x_4 - z)(2x_2 - x_4 - z)\partial_4.
\end{aligned}
$$

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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