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Research article

Classical and special Rota-Baxter operators on the real Lie algebra $\mathfrak{so}(3)$

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Abstract: This paper presents a classification of real matrix representations of Rota-Baxter operators on the real Lie algebra $\mathfrak{so}(3)$. We obtained all non-isomorphic classical Rota-Baxter operators of weight 0 up to orthogonal similarity, as well as special Rota-Baxter operators (multiplicative and pseudo-Rota-Baxter operators) of weights 0 and 1. For weight 0, we found three canonical symmetric matrix forms that solve the classical Yang-Baxter equation and used them to construct explicit left-symmetric algebra structures. Interestingly, no non-trivial multiplicative Rota–Baxter operators exist on $\mathfrak{so}(3)$ for weights 0 or 1. Pseudo-Rota-Baxter operators exhibit a rich structure: for weight 1, we found two families of symmetric matrices, while for weight 0, there were five families parameterized by continuous parameters $g_{11} \in [0, 1]$ and $g_{22} \in [\frac{1}{2}, 1]$. These findings contribute to the theoretical exploration of Rota-Baxter operators on real orthogonal Lie algebras by presenting explicit 3×3 matrix solutions, which offer valuable tools for their practical implementation in robotic mechanisms.

Keywords: Lie algebra $\mathfrak{so}(3)$; classical Rota-Baxter operators; multiplicative Rota-Baxter operators; pseudo-Rota-Baxter operators; Yang-Baxter equations; left-symmetric algebra structures

1. Introduction

The real Lie algebra $\mathfrak{so}(3)$ has an important role due to its compactness and semi-simplicity, and is crucial in diverse fields from classical mechanics to quantum control theory [1]. Its structural properties continue to inspire new developments in both pure and applied mathematics. In mathematics, $\mathfrak{so}(3)$ is isomorphic to the Lie algebra $(\mathbb{R}^3, \times)$. It is a simple 3-dimensional real Lie algebra. In rigid body physics, $\mathfrak{so}(3)$ is the local linearization of the rotation group, i.e., the special orthogonal group $SO(3)$ at the unit element, describing the instantaneous angular velocity of a rotating body in 3D. For example, in robotic path planning, the Lie algebra $\mathfrak{so}(3)$ describes the state of rotational motion and its transformations, which involves navigating obstacles to reach a target location [2]. Furthermore, the Rota-Baxter operators can be used to conduct research on the kinematics analysis of hybrid mechanisms [3].

The Rota-Baxter operator, an important tool in the theory of algebraic structures, originated from Baxter's research on wave equations in probability theory [4]. In 1969, Rota [5, 6] studied the Rota-Baxter operator from algebraic and combinatorial perspectives and connected it with hypergeometric functions and symmetric functions, gradually making it an important tool in the theory of algebraic structure decomposition and integral equations. In 1972, Cartier [7] derived a more efficient set of identities for Baxter algebras and presented the direct structure of Rota-Baxter algebras. Subsequently, numerous scholars have conducted in-depth research on the extensions and applications of the Rota-Baxter operators. Guo and Keigher [8] studied the relationship between Baxter algebras and shuffle products, and Gao et al. [9] studied differential Rota-Baxter operators. Their core idea was to decompose an algebraic structure into a direct sum of subalgebras via linear operators satisfying specific identities. In recent years, applications of the Rota-Baxter operator in Lie algebras, associative algebras, and mathematical physics have attracted much attention. For example, it plays an important role in the Yang-Baxter equation [10, 11], integrable systems [12, 13], and quantum group theory [14, 15]. Moreover, the Rota-Baxter operator provides a new framework for studying the structure of algebras, and is connected to pre-Lie algebras [16], post-Lie algebras [17, 18], deformation theory [19], associative algebras [20], and matrix algebras [21].

Moreover, it is worth noting that for any nonzero weight λ, a Rota-Baxter operator of weight λ can be reduced to the case of weight 1 by a simple scaling argument. Therefore, the classification of Rota-Baxter operators of weight 1 effectively covers all nonzero weights, while the weight 0 case stands apart due to its distinctive algebraic structure.

It is well known that the complexification of the real Lie algebra $\mathfrak{so}(3)$ is isomorphic to $\mathfrak{sl}(2,\mathbb{C})$, while the real forms $\mathfrak{so}(3)$ and $\mathfrak{sl}(2,\mathbb{R})$ are not isomorphic over $\mathbb{R}$. In this work, we classify, up to orthogonal similarity, all classical Rota-Baxter operators of weight 0 on $\mathfrak{so}(3)$ with respect to the standard basis. Through a systematic analysis of the nonlinear algebraic system determined jointly by the Lie bracket structure of $\mathfrak{so}(3)$ and the corresponding Rota-Baxter identities, we prove that every such operator is represented in the chosen basis by a real symmetric 3×3 matrix. These results complete the classification left open in [10] for the real orthogonal Lie algebra and yield explicit matrix solutions of the classical Yang-Baxter equations that are directly applicable to three-dimensional integrable systems.

To identify Rota-Baxter operators that are also Lie algebra homomorphisms, we impose the homomorphism condition $P([x, y]) = [P(x), P(y)]$, leading to the notion of multiplicative Rota-Baxter operators. In studying Rota-Baxter operators of weight 1 on $\mathfrak{so}(3)$, we introduce a relaxed version of this condition, which gives rise to a class of operators we term pseudo-Rota-Baxter operators (the prefix "pseudo" indicates that these operators are "close to" but not exactly multiplicative). Note that every multiplicative Rota-Baxter operator is automatically pseudo, but the converse does not hold. Interestingly, for weight 0, nontrivial pseudo-Rota-Baxter operators also exist. A noteworthy observation is that all such operators, for both weights 0 and 1, are represented by symmetric matrices.

Motivated by the distinctions between classical Rota-Baxter operators and their multiplicative and pseudo variants, we introduce the term special Rota-Baxter operators as a unifying name for the latter two types (i.e., the multiplicative and pseudo variants). In this paper, we establish their explicit matrix representations with respect to a basis. These explicit classifications advance the theory of Rota-Baxter operators on real orthogonal algebras and provide computational tools for applications in integrable systems and robotic path planning.

This paper is organized as follows. Section 2 provides the necessary preliminaries on the real Lie algebra $\mathfrak{so}(3)$, including its standard basis and bracket relations, and then recalls the definitions of classical, multiplicative, and pseudo-Rota-Baxter operators. Section 3 presents a complete classification (up to orthogonal similarity) of the classical Rota-Baxter operators of weight 0 on $\mathfrak{so}(3)$, giving explicit matrix representations, corresponding solutions of the classical Yang-Baxter equation, and induced left-symmetric algebra structures. Section 4 proves that no nontrivial multiplicative Rota-Baxter operator of weight 1 exists on $\mathfrak{so}(3)$, and classifies all pseudo-Rota-Baxter operators of weight 1 via two families of symmetric matrices. Section 5 addresses the weight 0 case for special Rota-Baxter operators, demonstrating again the absence of nontrivial multiplicative operators, and classifying pseudo-Rota-Baxter operators into five distinct families parameterized by continuous variables.

2. Preliminaries

Throughout this paper, $\mathbb{R}$ and $\mathbb{R}^*$ denote the field of real numbers and the set of non-zero real numbers, respectively. All vector spaces, matrices, and linear transformations considered in this paper are defined over $\mathbb{R}$ and are finite-dimensional.

Definition 2.1. [22] Let F be a field. A Lie algebra over F is an F-vector space $\mathfrak{g}$, together with a bilinear map, the Lie bracket $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, $(x, y) \to [x, y]$, satisfying the following properties:

(1) $[x, x] = 0$ for all $x \in \mathfrak{g}$,
(2) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in \mathfrak{g}$.

The Lie bracket $[x, y]$ is often referred to as the commutator of x and y. Condition (2) is called the Jacobi identity.

For a more detailed discussion of Lie algebras, see [22].

Definition 2.2. [3] Let $\mathfrak{g}$ be a 3-dimensional $\mathbb{R}$-vector space with a basis $\{e_1, e_2, e_3\}$. Define a Lie bracket on $\mathfrak{g}$ as follows:

$$[e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2,$$
$$[e_j, e_i] = -[e_i, e_j], \quad \forall i, j = 1, 2, 3.$$

Then $(\mathfrak{g}, [\cdot, \cdot])$ is a Lie algebra isomorphic to the Lie algebra of the rotation group SO(3), and we denote it by $\mathfrak{so}(3)$.

In the subsequent sections, the basis $\{e_1, e_2, e_3\}$ of $\mathfrak{so}(3)$ is called the standard basis. The matrix representations of all linear transformations are presented with respect to the standard basis.

Definition 2.3. [16] Let $(\mathfrak{g}, [\cdot, \cdot])$ be a Lie algebra and $\lambda \in \mathbb{R}$. A linear map $P : \mathfrak{g} \to \mathfrak{g}$ is called a *classical Rota-Baxter operator of weight* λ if

$$[P(x), P(y)] = P([P(x), y] + [x, P(y)] + \lambda[x, y]), \quad \forall x, y \in \mathfrak{g}. \tag{2.1}$$

If P additionally satisfies the homomorphism condition $[P(x), P(y)] = P([x, y])$ for all $x, y \in \mathfrak{g}$, then it is called a *multiplicative Rota-Baxter operator of weight* λ.

It is obvious that the zero mapping, which is called the trivial Rota-Baxter operator, includes both classical Rota-Baxter operators and multiplicative Rota-Baxter operators.

Definition 2.4. [16] A Rota-Baxter operator of weight 0 on a Lie algebra $\mathfrak{g}$ is called a solution to the classical Yang-Baxter equation of $\mathfrak{g}$.

Recall that a *left-symmetric algebra* (or *pre-Lie algebra*) is a vector space A equipped with a bilinear product $\star : A \times A \to A$ satisfying the following identity:

$$(x \star y) \star z - x \star (y \star z) = (y \star x) \star z - y \star (x \star z), \quad \forall x, y, z \in A.$$

Such algebras naturally arise in the study of Rota-Baxter operators and the classical Yang-Baxter equation on Lie algebras. For a detailed discussion, we refer the reader to [16] and the references therein.

Lemma 2.1. *[16] Let $\mathfrak{g}$ be a Lie algebra, and let R be a solution to the classical Yang-Baxter equation on $\mathfrak{g}$. Define a new operation on $\mathfrak{g}$:*

$$x \star y = [R(x), y], \quad \forall x, y \in \mathfrak{g}.$$

Then $(\mathfrak{g}, \star)$ forms a left-symmetric algebra.

Definition 2.5. Let $(\mathfrak{g}, [\cdot, \cdot])$ be a Lie algebra and $\lambda \in \mathbb{R}$. A linear map $H : \mathfrak{g} \to \mathfrak{g}$ is called a *pseudo-Rota-Baxter operator of weight* λ if

$$H([x, y]) = H([H(x), y] + [x, H(y)] + \lambda[x, y]), \quad \forall x, y \in \mathfrak{g}.$$

3. Classical Rota-Baxter operators of weight 0

In this section, we classify classical Rota-Baxter operators of weight 0 on $\mathfrak{so}(3)$ by solving the operator equation derived from its Lie bracket structure. Let P be classical Rota-Baxter operator of weight 0 on $\mathfrak{so}(3)$ with the matrix representation:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad \forall a_{ij} \in \mathbb{R}, i, j = 1, 2, 3.$$

Theorem 3.1. *Up to orthogonal similarity, all matrix representations of classical Rota-Baxter operators of weight 0 on $\mathfrak{so}(3)$ with respect to the standard basis are as follows:*

$$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} \frac{a_{12}^2}{a_{22}} & a_{12} & 0 \\ a_{12} & a_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} \frac{a_{13}^2}{a_{33}} & \frac{a_{13}a_{23}}{a_{33}} & a_{13} \\ \frac{a_{13}a_{23}}{a_{33}} & \frac{a_{23}^2}{a_{33}} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix},$$

where the parameters are real numbers with $a_{22}, a_{33} \neq 0$.

Proof. Combining the definition of $\mathfrak{so}(3)$ and the matrix representation of the classical Rota-Baxter operator of weight 0, we know that P satisfies the equation

$$[P(e_i), P(e_j)] = P([P(e_i), e_j] + [e_i, P(e_j)]),\ i, j = 1, 2, 3.$$

Comparing the coefficients on both sides of the above equation, we obtain

$$a_{11}(a_{31} - a_{13}) - a_{22}(a_{13} + a_{31}) + a_{32}(a_{12} + a_{21}) = 0, \tag{3.1}$$
$$-a_{11}(a_{12} - a_{21}) + a_{22}(a_{32} - a_{23}) + a_{31}(a_{12} + a_{21}) = 0, \tag{3.2}$$
$$a_{11}(a_{12} - a_{21}) + a_{33}(a_{12} + a_{21}) - a_{23}(a_{13} + a_{31}) = 0, \tag{3.3}$$
$$a_{11}(a_{23} + a_{32}) + a_{33}(a_{32} - a_{23}) - a_{21}(a_{13} + a_{31}) = 0, \tag{3.4}$$
$$a_{22}(a_{12} - a_{21}) - a_{33}(a_{12} + a_{21}) + a_{13}(a_{23} + a_{32}) = 0, \tag{3.5}$$
$$-a_{22}(a_{13} + a_{31}) + a_{33}(a_{13} - a_{31}) + a_{12}(a_{23} + a_{32}) = 0, \tag{3.6}$$
$$a_{11}a_{22} - a_{11}a_{33} - a_{22}a_{33} - a_{12}a_{21} + a_{31}^2 + a_{32}^2 = 0, \tag{3.7}$$
$$a_{11}a_{22} - a_{11}a_{33} + a_{22}a_{33} + a_{13}a_{31} - a_{21}^2 - a_{23}^2 = 0, \tag{3.8}$$
$$a_{11}a_{22} + a_{11}a_{33} - a_{22}a_{33} + a_{23}a_{32} - a_{12}^2 - a_{13}^2 = 0. \tag{3.9}$$

The strategy of the proof is to analyze the system (3.1)–(3.9) by cases, depending on how many of the diagonal elements a_{11}, a_{22}, a_{33} are zero. There are four distinct cases: all three are zero, exactly two are zero, exactly one is zero, and none are zero. The case with exactly two zero further splits into three subcases and the case with exactly one zeros also splits into three subcases, resulting in a total of $1 + 3 + 3 + 1 = 8$ cases to consider.

Case (1): If $a_{11} = a_{22} = a_{33} = 0$, then Eqs (3.1)–(3.9) can be simplified as follows:

$$a_{32}(a_{12} + a_{21}) = 0, \tag{3.10}$$
$$a_{31}(a_{12} + a_{21}) = 0, \tag{3.11}$$
$$a_{23}(a_{13} + a_{31}) = 0, \tag{3.12}$$
$$a_{21}(a_{13} + a_{31}) = 0, \tag{3.13}$$
$$a_{13}(a_{23} + a_{32}) = 0, \tag{3.14}$$
$$a_{12}(a_{23} + a_{32}) = 0, \tag{3.15}$$
$$a_{12}a_{21} - a_{31}^2 - a_{32}^2 = 0, \tag{3.16}$$
$$a_{13}a_{31} - a_{21}^2 - a_{23}^2 = 0, \tag{3.17}$$
$$a_{23}a_{32} - a_{12}^2 - a_{13}^2 = 0. \tag{3.18}$$

Then, we will discuss whether $a_{12} + a_{21}$ is equal to zero.

The condition $a_{12} + a_{21} = 0$ implies $a_{12} = -a_{21}$. Equation (3.16) implies $-a_{12}^2 - a_{31}^2 - a_{32}^2 = 0$. In $\mathbb{R}$, the only solution is $a_{12} = a_{31} = a_{32} = 0$. Equations (3.17) and (3.18) likewise imply $a_{13} = a_{23} = 0$. Therefore, in this case, P is the trivial operator.

Conversely, if $a_{12} + a_{21} \neq 0$, then (3.10) and (3.11) force $a_{32} = a_{31} = 0$. Substituting this into (3.17) and (3.18) yields $a_{21}^2 + a_{23}^2 = a_{12}^2 + a_{13}^2 = 0$, implying $a_{21} = a_{23} = a_{12} = a_{13} = 0$. This contradicts the assumption that $a_{12} + a_{21} \neq 0$. Hence, this case is impossible.

Case (2): If $a_{11} = a_{22} = 0$ and $a_{33} \neq 0$, then Eqs (3.1)–(3.9) can be simplified as follows:

$$a_{32}(a_{12} + a_{21}) = 0, \tag{3.19}$$
$$a_{31}(a_{12} + a_{21}) = 0, \tag{3.20}$$
$$a_{33}(a_{12} + a_{21}) = a_{23}(a_{13} + a_{31}), \tag{3.21}$$
$$a_{33}(a_{32} - a_{23}) = a_{21}(a_{13} + a_{31}), \tag{3.22}$$
$$a_{33}(a_{12} + a_{21}) = a_{13}(a_{23} + a_{32}), \tag{3.23}$$
$$a_{33}(a_{12} + a_{21}) = -a_{12}(a_{23} + a_{32}), \tag{3.24}$$
$$a_{31}^2 + a_{32}^2 = a_{12}a_{21}, \tag{3.25}$$
$$a_{21}^2 + a_{23}^2 = a_{13}a_{31}, \tag{3.26}$$
$$a_{12}^2 + a_{23}^2 = a_{23}a_{32}. \tag{3.27}$$

Similarly, we discuss whether $a_{12} + a_{21}$ is equal to zero. In the case $a_{12} + a_{21} = 0$ (i.e., $a_{12} = -a_{21}$), substituting into Eq (3.25) gives $-a_{12}^2 - a_{31}^2 - a_{32}^2 = 0$. This implies $a_{12} = a_{31} = a_{32} = 0$. It then follows from Eqs (3.26) and (3.27) that $a_{23} = a_{13} = 0$. The resulting matrix is

$$P_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_{33} \end{pmatrix}, \text{ with } a_{33} \neq 0.$$

Conversely, if $a_{12} + a_{21} \neq 0$, then Eqs (3.19) and (3.20) show that $a_{32} = a_{31} = 0$. It follows from Eqs (3.26) and (3.27) that $a_{21} = a_{12} = 0$, which contradicts the assumption. Therefore, this subcase leads to a contradiction.

Case (3): If $a_{11} = a_{33} = 0$ and $a_{22} \neq 0$, then we have an argument analogous to **Case (2)**, and the resulting matrix is

$$P_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ with } a_{22} \neq 0.$$

Case (4): If $a_{22} = a_{33} = 0$ and $a_{11} \neq 0$, then we have an argument analogous to **Case (2)**, and the resulting matrix is

$$P_3 = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ with } a_{11} \neq 0.$$

The matrices P_1, P_2, and P_3 have the same eigenvalues $\{0, 0, \lambda\}$, where λ is a nonzero real parameter. Since they are also symmetric, these three matrix forms are orthogonal similar.

Case (5): If $a_{11} = 0$ and $a_{22}a_{33} \neq 0$, then Eqs (3.1)–(3.9) can be simplified as follows:

$$a_{22}(a_{13} + a_{31}) - a_{32}(a_{12} + a_{21}) = 0, \tag{3.28}$$
$$a_{22}(a_{32} - a_{23}) + a_{31}(a_{12} + a_{21}) = 0, \tag{3.29}$$
$$a_{33}(a_{12} + a_{21}) - a_{23}(a_{13} + a_{31}) = 0, \tag{3.30}$$
$$a_{33}(a_{32} - a_{23}) - a_{21}(a_{13} + a_{31}) = 0, \tag{3.31}$$
$$a_{22}(a_{12} - a_{21}) - a_{33}(a_{12} + a_{21}) - a_{13}(a_{23} + a_{32}) = 0, \tag{3.32}$$

$$a_{22}(a_{13} + a_{31}) - a_{33}(a_{13} - a_{31}) - a_{12}(a_{23} + a_{32}) = 0, \tag{3.33}$$
$$a_{22}a_{33} + a_{12}a_{21} - a_{31}^2 - a_{32}^2 = 0, \tag{3.34}$$
$$a_{22}a_{33} + a_{13}a_{31} - a_{21}^2 - a_{23}^2 = 0, \tag{3.35}$$
$$a_{22}a_{33} - a_{23}a_{32} + a_{12}^2 + a_{13}^2 = 0. \tag{3.36}$$

Still we apply the methods in **Case (1)** to discuss whether $a_{12} + a_{21}$ is zero.

Assume $a_{12} + a_{21} = 0$, and then Eq (3.28) implies $a_{13} + a_{31} = 0$. It follows from (3.29) or (3.31) that $a_{23} = a_{32}$. Substituting these results ($a_{12} = -a_{21}$, $a_{13} = -a_{31}$, and $a_{23} = a_{32}$) into Eqs (3.34), (3.35), and (3.36) gives:

$$a_{22}a_{33} = a_{12}^2 + a_{13}^2 + a_{23}^2,$$
$$a_{22}a_{33} = a_{23}^2.$$

Simplifying these two equations, we find that $a_{12}^2 + a_{13}^2 = 0$. Since $a_{12}, a_{13} \in \mathbb{R}$, this forces $a_{12} = a_{13} = 0$. The second equation now becomes $a_{22}a_{33} = a_{23}^2$. Therefore, $a_{23} \neq 0$ and $a_{22} = \frac{a_{23}^2}{a_{33}}$. Thus, the matrix in this case is:

$$P_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{a_{23}^2}{a_{33}} & a_{23} \\ 0 & a_{23} & a_{33} \end{pmatrix}.$$

Conversely, assume $a_{12} + a_{21} \neq 0$. If $a_{13} + a_{31} = 0$, then Eq (3.30) becomes $a_{33}(a_{12} + a_{21}) = 0$. This contradicts the given conditions $a_{33} \neq 0$ and $a_{12} + a_{21} \neq 0$. Hence, $a_{13} + a_{31} \neq 0$.

Now, from Eqs (3.28) and (3.30), we obtain:

$$a_{22}(a_{13} + a_{31}) = a_{32}(a_{12} + a_{21}),$$
$$a_{33}(a_{12} + a_{21}) = a_{23}(a_{13} + a_{31}).$$

Multiplying both sides of the first equation by the corresponding sides of the second equation gives:

$$a_{22}a_{33}(a_{13} + a_{31})(a_{12} + a_{21}) = a_{23}a_{32}(a_{13} + a_{31})(a_{12} + a_{21}).$$

Since $a_{12} + a_{21} \neq 0$ and $a_{13} + a_{31} \neq 0$, we have $(a_{12} + a_{21})(a_{13} + a_{31}) \neq 0$. Thus, from the above equation, we obtain $a_{22}a_{33} = a_{23}a_{32}$. Substituting this into Eq (3.36) yields $a_{12}^2 + a_{13}^2 = 0$. Since $a_{12}, a_{13} \in \mathbb{R}$, then $a_{12} = a_{13} = 0$.

Substitute these results into Eqs (3.32) and (3.33), respectively, and simplifying the resulting equations, we obtain:

$$(a_{22} + a_{33})a_{21} = 0,$$
$$(a_{22} + a_{33})a_{31} = 0.$$

Since $a_{22}a_{33} \neq 0$, we must have $a_{22} + a_{33} \neq 0$. Indeed, if $a_{22} + a_{33} = 0$, then Eq (3.34) would imply $a_{22} = 0$ or $a_{33} = 0$, which is false. Consequently, we must have $a_{21} = 0$ and $a_{31} = 0$. This means that $a_{12} + a_{21} = a_{13} + a_{31} = 0$, which contradicts our initial assumptions that $a_{12} + a_{21} \neq 0$ and $a_{13} + a_{31} \neq 0$. This contradiction shows that $a_{12} + a_{21} \neq 0$ is impossible.

Case (6): If $a_{22} = 0$ and $a_{11}a_{22} \neq 0$, then we have an argument analogous to **Case (5)**, and the resulting matrix is

$$P_5 = \begin{pmatrix} \frac{a_{12}^2}{a_{22}} & a_{12} & 0 \\ a_{12} & a_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Case (7): If $a_{33} = 0$ and $a_{11}a_{22} \neq 0$, then we have an argument analogous to **Case (5)**, and the resulting matrix is

$$P_6 = \begin{pmatrix} \frac{a_{13}^2}{a_{33}} & 0 & a_{13} \\ 0 & 0 & 0 \\ a_{13} & 0 & a_{33} \end{pmatrix}.$$

Furthermore, one can verify that the three matrices P_4, P_5, P_6 are orthogonal similar.

Case (8): If a_{11}, a_{22}, a_{33} are all non-zero, then a calculation shows that any non-symmetric solutions are complex. Therefore, we have $a_{ij} = a_{ji}$ for $i \neq j$. Substituting these symmetry conditions into Eqs (3.1)–(3.9) yields the following equations:

$$\begin{aligned}
a_{12}a_{13} &= a_{11}a_{23}, \\
a_{12}a_{23} &= a_{22}a_{13}, \\
a_{13}a_{23} &= a_{33}a_{12}, \\
a_{22}a_{33} &= a_{23}^2, \\
a_{11}a_{22} &= a_{12}^2, \\
a_{11}a_{33} &= a_{13}^2.
\end{aligned}$$

Solving the above equations, the matrix in this case is given by

$$P_7 = \begin{pmatrix} \frac{a_{13}^2}{a_{33}} & \frac{a_{13}a_{23}}{a_{33}} & a_{13} \\ \frac{a_{13}a_{23}}{a_{33}} & \frac{a_{23}^2}{a_{33}} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}, \quad \forall a_{13}, a_{23}, a_{33} \in \mathbb{R}^*.$$

In conclusion, up to isomorphism, there are three non-trivial matrix forms of classical Rota-Baxter operators of weight 0 on $\mathfrak{so}(3)$ with respect to the standard basis. □

Corollary 3.2. *Solutions of the classical Yang-Baxter equation in* $\mathfrak{so}(3)$ *are as follows:*

(1) $R_1 = ae_{11}$,
(2) $R_2 = \frac{a^2}{b}e_{11} + a(e_{12} + e_{21}) + be_{22}$,
(3) $R_3 = \frac{a^2}{c}e_{11} + \frac{b^2}{c}e_{22} + \frac{ab}{c}(e_{12} + e_{21}) + a(e_{13} + e_{31}) + b(e_{23} + e_{32})$,

where $a, b, c \in \mathbb{R}^*$ *and* e_{ij} *denotes the* 3×3 *matrix unit with* 1 *in the* (i, j)*-th entry and* 0 *elsewhere.*

The left-symmetric algebra structures on $\mathfrak{so}(3)$ are given by Lemma 2.1 for the solutions in Corollary 3.2.

Corollary 3.3. *The left-symmetric algebra structures on the Lie algebra* $\mathfrak{so}(3)$ *are as follows:*

(1) $e_1 \star e_2 = ae_3,\ e_1 \star e_3 = -ae_2$,
(2) $e_1 \star e_1 = -ae_3,\ e_1 \star e_2 = \frac{a^2}{b}e_3,\ e_1 \star e_3 = ae_1 - \frac{a^2}{b}e_2$,
$e_2 \star e_1 = -be_3,\ e_2 \star e_2 = ae_3,\ e_2 \star e_3 = be_1 - ae_2$,
(3) $e_1 \star e_1 = ae_2 - \frac{ab}{c}e_3,\ e_1 \star e_2 = -ae_1 + \frac{a^2}{c}e_3,\ e_1 \star e_3 = \frac{ab}{c}e_1 - \frac{a^2}{c}e_2$,
$e_2 \star e_1 = be_2 - \frac{b^2}{c}e_2,\ e_2 \star e_2 = -be_1 + \frac{ab}{c}e_3,\ e_2 \star e_3 = \frac{b^2}{c}e_1 - \frac{ab}{c}e_2$,
$e_3 \star e_1 = ce_2 - be_3,\ e_3 \star e_2 = -ce_1 + ae_3,\ e_3 \star e_3 = be_1 - ae_2$.

4. Special Rota–Baxter operators of weight 1

According to [23], the only Rota-Baxter operators of weight 1 on a compact simple Lie algebra are 0 and $-id$. Since the real Lie algebra $\mathfrak{so}(3)$ is compact and simple, we have the following result.

Theorem 4.1. *All classical Rota-Baxter operators of weight* 1 *on* $\mathfrak{so}(3)$ *are* 0 *and* $-id$.

The aim of this section is to classify two specific types of Rota-Baxter operators of weight 1 on $\mathfrak{so}(3)$: the multiplicative Rota-Baxter operators and the pseudo-Rota-Baxter operators. Let $\bar{P}$ and H denote a multiplicative and a pseudo-Rota-Baxter operator with real matrix representations $\left(b_{ij}\right)$ and $\left(c_{ij}\right)$ for all $i, j = 1, 2, 3$, respectively.

Theorem 4.2. *There is no non-trivial multiplicative Rota-Baxter operator of weight* 1 *on* $\mathfrak{so}(3)$.

Proof. By combining the equations in Definition 2.3, we obtain $\bar{P}([\bar{P}(x), y] + [x, \bar{P}(y)]) = 0$. This implies $[\bar{P}(x), y] + [x, \bar{P}(y)] \in \ker \bar{P}$. Since the Lie algebra $\mathfrak{so}(3)$ is simple and has no non-trivial ideals, it follows from $\ker \bar{P} \lhd \mathfrak{so}(3)$ that either $\ker \bar{P} = \{0\}$ or $\ker \bar{P} = \mathfrak{so}(3)$.

We now analyze $[\bar{P}(x), y] + [x, \bar{P}(y)] = 0$ by substituting the matrix representation of $\bar{P}$. Comparing the coefficients yields the following system of equations:

$$\begin{aligned}
&b_{12} = -b_{21} = 0,\\
&b_{13} = -b_{31} = 0,\\
&b_{23} = -b_{32} = 0,\\
&b_{11} + b_{22} = 0,\\
&b_{22} + b_{33} = 0,\\
&b_{11} + b_{33} = 0.
\end{aligned}$$

The only solution of this system is $b_{ij} = 0,\ i, j = 1, 2, 3$. This implies that $\bar{P}$ must be the zero map. Consequently, its kernel is $\ker \bar{P} = \mathfrak{so}(3)$. Therefore, the only multiplicative Rota-Baxter operator of weight 1 on $\mathfrak{so}(3)$ is the zero map. □

Theorem 4.3. *Up to orthogonal similarity, all matrix representations of the pseudo-Rota-Baxter operator of weight* 1 *on* $\mathfrak{so}(3)$ *with respect to the standard basis are as follows:*

$$\begin{pmatrix} c_1 & \sqrt{c_1c_2} & \sqrt{c_1c_3} \\ \sqrt{c_1c_2} & c_2 & \sqrt{c_2c_3} \\ \sqrt{c_1c_3} & \sqrt{c_2c_3} & c_3 \end{pmatrix}, \quad \begin{pmatrix} c_1 & -\sqrt{c_1c_2} & -\sqrt{c_1c_3} \\ -\sqrt{c_1c_2} & c_2 & -\sqrt{c_2c_3} \\ -\sqrt{c_1c_3} & -\sqrt{c_2c_3} & c_3 \end{pmatrix},$$

where the parameters $c_i,\ i = 1, 2, 3$, *are real numbers and share the same sign.*

Proof. The proof proceeds by a systematic case analysis based on the diagonal entries of the matrix representation of H. We begin by substituting the matrix form of H into the defining equation of the pseudo-Rota-Baxter operator of weight 1:

$$H([e_i, e_j]) = H([H(e_i), e_j] + [e_i, H(e_j)] + [e_i, e_j]),\ i, j = 1, 2, 3.$$

Comparing coefficients yields the following system of equations:

$$(c_{11}+c_{22})c_{13} = c_{11}c_{31} + c_{12}c_{32}, \tag{4.1}$$
$$(c_{11}+c_{22})c_{23} = c_{31}c_{21} + c_{32}c_{22}, \tag{4.2}$$
$$(c_{11}+c_{22})c_{33} = c_{31}^2 + c_{32}^2, \tag{4.3}$$
$$(c_{22}+c_{33})c_{11} = c_{12}^2 + c_{13}^2, \tag{4.4}$$
$$(c_{22}+c_{33})c_{21} = c_{12}c_{22} + c_{13}c_{23}, \tag{4.5}$$
$$(c_{22}+c_{33})c_{31} = c_{12}c_{32} + c_{13}c_{33}, \tag{4.6}$$
$$(c_{11}+c_{33})c_{12} = c_{11}c_{21} + c_{23}c_{13}, \tag{4.7}$$
$$(c_{11}+c_{33})c_{22} = c_{21}^2 + c_{23}^2, \tag{4.8}$$
$$(c_{11}+c_{33})c_{32} = c_{21}c_{31} + c_{23}c_{33}. \tag{4.9}$$

Now, we consider cases depending on whether the diagonal elements c_{11}, c_{22}, c_{33} are zero or not.

Case 1: $c_{11} = 0$.

From Eq (4.4), we have $c_{12}^2 + c_{13}^2 = 0$, which implies $c_{12} = c_{13} = 0$. Substituting into Eqs (4.1)–(4.9) yields the reduced system:

$$c_{22}(c_{23} - c_{32}) = c_{31}c_{21}, \tag{4.10}$$
$$c_{22}c_{33} = c_{31}^2 + c_{32}^2, \tag{4.11}$$
$$(c_{22}+c_{33})c_{21} = 0, \tag{4.12}$$
$$(c_{22}+c_{33})c_{31} = 0, \tag{4.13}$$
$$c_{22}c_{33} = c_{21}^2 + c_{23}^2, \tag{4.14}$$
$$c_{33}(c_{32} - c_{23}) = c_{21}c_{31}. \tag{4.15}$$

We now consider subcases based on c_{22}.

(a) $c_{22} = 0$.

Then Eqs (4.11) and (4.14) imply $c_{31}^2 + c_{32}^2 = c_{21}^2 + c_{23}^2 = 0$, so $c_{31} = c_{32} = c_{21} = c_{23} = 0$. Thus, we obtain the matrix

$$H_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c_{33} \end{pmatrix}, \forall c_{33} \in \mathbb{R}.$$

(b) $c_{22} \neq 0$.

We further consider c_{33}.

(b1) $c_{33} = 0$.

Equations (4.11) and (4.14) give $c_{31} = c_{32} = c_{21} = c_{23} = 0$. Thus,

$$H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & c_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \forall c_{22} \in \mathbb{R}.$$

(b2) $c_{33} \neq 0$.

Then $c_{22}+c_{33} \neq 0$, so Eqs (4.12) and (4.13) imply $c_{21} = c_{31} = 0$. Substituting into Eqs (4.10), (4.11), and (4.14) yields $c_{23} = c_{32}$ and $c_{23}^2 = c_{22}c_{33}$. Therefore, for $c_{22}, c_{33} \in \mathbb{R}^*$ with $c_{22}c_{33} > 0$, we have

$$H_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & c_{22} & \sqrt{c_{22}c_{33}} \\ 0 & \sqrt{c_{22}c_{33}} & c_{33} \end{pmatrix},$$

$$H_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & c_{22} & -\sqrt{c_{22}c_{33}} \\ 0 & -\sqrt{c_{22}c_{33}} & c_{33} \end{pmatrix}.$$

Case 2: $c_{11} \neq 0$.

(c) $c_{22} = 0$.

Equation (4.8) gives $c_{21} = c_{23} = 0$. The system (4.1)–(4.9) reduces to:

$$c_{11}(c_{13} - c_{31}) = c_{12}c_{32}, \tag{4.16}$$

$$c_{11}c_{33} = c_{31}^2 + c_{32}^2, \tag{4.17}$$

$$c_{11}c_{33} = c_{12}^2 + c_{13}^2, \tag{4.18}$$

$$c_{33}(c_{13} - c_{31}) = c_{12}c_{32}, \tag{4.19}$$

$$(c_{11} + c_{33})c_{12} = 0, \tag{4.20}$$

$$(c_{11} + c_{33})c_{32} = 0. \tag{4.21}$$

(c1) $c_{33} = 0$.

Equations (4.17) and (4.18) imply $c_{31} = c_{32} = c_{12} = c_{13} = 0$. Thus,

$$H_5 = \begin{pmatrix} c_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \forall c_{11} \in \mathbb{R}^*.$$

(c2) $c_{33} \neq 0$.

Since $c_{33} \neq 0$, Eqs (4.20)–(4.21) give $c_{12} = c_{32} = 0$. Then Eqs (4.16) and (4.17) imply $c_{13} = c_{31}$ and $c_{13}^2 = c_{11}c_{33}$. Thus, for $c_{11}, c_{33} \in \mathbb{R}^*$, with $c_{11}c_{33} > 0$, we have

$$H_6 = \begin{pmatrix} c_{11} & 0 & \sqrt{c_{11}c_{33}} \\ 0 & 0 & 0 \\ \sqrt{c_{11}c_{33}} & 0 & c_{33} \end{pmatrix},$$

$$H_7 = \begin{pmatrix} c_{11} & 0 & -\sqrt{c_{11}c_{33}} \\ 0 & 0 & 0 \\ -\sqrt{c_{11}c_{33}} & 0 & c_{33} \end{pmatrix}.$$

(d) $c_{22} \neq 0$.

(d1) $c_{33} = 0$.

Equation (4.3) gives $c_{31} = c_{32} = 0$. The system reduces to:

$$(c_{11} + c_{22})c_{13} = 0, \tag{4.22}$$

$$(c_{11} + c_{22})c_{23} = 0, \tag{4.23}$$
$$c_{11}c_{22} = c_{12}^2 + c_{13}^2, \tag{4.24}$$
$$c_{22}(c_{12} - c_{21}) = c_{13}c_{23}, \tag{4.25}$$
$$c_{11}(c_{12} - c_{21}) = c_{13}c_{23}, \tag{4.26}$$
$$c_{11}c_{22} = c_{21}^2 + c_{23}^2. \tag{4.27}$$

Since $c_{11} + c_{22} \neq 0$, Eqs (4.22) and (4.23) imply $c_{13} = c_{23} = 0$. Then Eqs (4.24) and (4.25) give $c_{12} = c_{21}$ and $c_{12}^2 = c_{11}c_{22}$. Thus, for $c_{11}, c_{22} \in \mathbb{R}^*$, with $c_{11}c_{22} > 0$, we have

$$H_8 = \begin{pmatrix} c_{11} & \sqrt{c_{11}c_{22}} & 0 \\ \sqrt{c_{11}c_{22}} & c_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$H_9 = \begin{pmatrix} c_{11} & -\sqrt{c_{11}c_{22}} & 0 \\ -\sqrt{c_{11}c_{22}} & c_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(d2) $c_{33} \neq 0$.

We first show that the matrix H is symmetric. From Eqs (4.1) and (4.6), we derive:

$$(c_{11} + c_{22})c_{13} - (c_{22} + c_{33})c_{31} = c_{11}c_{31} + c_{12}c_{32} - c_{12}c_{32} - c_{13}c_{33},$$

which simplifies to

$$(c_{11} + c_{22} + c_{33})c_{13} = (c_{11} + c_{22} + c_{33})c_{31}.$$

Since $c_{11} + c_{22} + c_{33} \neq 0$, we have $c_{13} = c_{31}$. Similarly, we obtain $c_{23} = c_{32}, c_{12} = c_{21}$.

Substituting these into Eqs (4.3), (4.4), and (4.8) yields:

$$(c_{11} + c_{22})c_{33} = c_{13}^2 + c_{23}^2,$$
$$(c_{22} + c_{33})c_{11} = c_{12}^2 + c_{13}^2,$$
$$(c_{11} + c_{33})c_{22} = c_{12}^2 + c_{23}^2.$$

Solving this system, we find:

$$c_{13}^2 = c_{11}c_{33}, c_{12}^2 = c_{11}c_{22}, c_{23}^2 = c_{22}c_{33}.$$

Since $c_{11}, c_{22}, c_{33} \neq 0$, the parameters c_{12}, c_{13}, c_{23} are also all non-zero, and c_{11}, c_{22}, c_{33} must have the same sign. Thus, we obtain eight matrices as follows:

$$H_{10} = \begin{pmatrix} c_{11} & \sqrt{c_{11}c_{22}} & \sqrt{c_{11}c_{33}} \\ \sqrt{c_{11}c_{22}} & c_{22} & \sqrt{c_{22}c_{33}} \\ \sqrt{c_{11}c_{33}} & \sqrt{c_{22}c_{33}} & c_{33} \end{pmatrix},$$

$$H_{11} = \begin{pmatrix} c_{11} & -\sqrt{c_{11}c_{22}} & \sqrt{c_{11}c_{33}} \\ -\sqrt{c_{11}c_{22}} & c_{22} & \sqrt{c_{22}c_{33}} \\ \sqrt{c_{11}c_{33}} & \sqrt{c_{22}c_{33}} & c_{33} \end{pmatrix},$$

$$H_{12}=\begin{pmatrix} c_{11} & \sqrt{c_{11}c_{22}} & -\sqrt{c_{11}c_{33}} \\ \sqrt{c_{11}c_{22}} & c_{22} & \sqrt{c_{22}c_{33}} \\ -\sqrt{c_{11}c_{33}} & \sqrt{c_{22}c_{33}} & c_{33} \end{pmatrix},$$

$$H_{13}=\begin{pmatrix} c_{11} & \sqrt{c_{11}c_{22}} & \sqrt{c_{11}c_{33}} \\ \sqrt{c_{11}c_{22}} & c_{22} & -\sqrt{c_{22}c_{33}} \\ \sqrt{c_{11}c_{33}} & -\sqrt{c_{22}c_{33}} & c_{33} \end{pmatrix},$$

$$H_{14}=\begin{pmatrix} c_{11} & \sqrt{c_{11}c_{22}} & -\sqrt{c_{11}c_{33}} \\ \sqrt{c_{11}c_{22}} & c_{22} & -\sqrt{c_{22}c_{33}} \\ -\sqrt{c_{11}c_{33}} & -\sqrt{c_{22}c_{33}} & c_{33} \end{pmatrix},$$

$$H_{15}=\begin{pmatrix} c_{11} & -\sqrt{c_{11}c_{22}} & \sqrt{c_{11}c_{33}} \\ -\sqrt{c_{11}c_{22}} & c_{22} & -\sqrt{c_{22}c_{33}} \\ \sqrt{c_{11}c_{33}} & -\sqrt{c_{22}c_{33}} & c_{33} \end{pmatrix},$$

$$H_{16}=\begin{pmatrix} c_{11} & -\sqrt{c_{11}c_{22}} & -\sqrt{c_{11}c_{33}} \\ -\sqrt{c_{11}c_{22}} & c_{22} & \sqrt{c_{22}c_{33}} \\ -\sqrt{c_{11}c_{33}} & \sqrt{c_{22}c_{33}} & c_{33} \end{pmatrix},$$

$$H_{17}=\begin{pmatrix} c_{11} & -\sqrt{c_{11}c_{22}} & -\sqrt{c_{11}c_{33}} \\ -\sqrt{c_{11}c_{22}} & c_{22} & -\sqrt{c_{22}c_{33}} \\ -\sqrt{c_{11}c_{33}} & -\sqrt{c_{22}c_{33}} & c_{33} \end{pmatrix}.$$

The matrices H_1 to H_9 are special cases of H_{10} to H_{17} when some parameters are zero. Moreover, the matrices within the sets $\{H_{10}, H_{14}, H_{15}, H_{16}\}$ and $\{H_{11}, H_{12}, H_{13}, H_{17}\}$ are orthogonally similar, as they share the same eigenvalues. However, the two sets are not similar to each other.

Therefore, up to orthogonal similarity, the pseudo-Rota-Baxter operators of weight 1 on $\mathfrak{so}(3)$ are represented by the following two families of symmetric matrices:

$$\begin{pmatrix} c_1 & \sqrt{c_1c_2} & \sqrt{c_1c_3} \\ \sqrt{c_1c_2} & c_2 & \sqrt{c_2c_3} \\ \sqrt{c_1c_3} & \sqrt{c_2c_3} & c_3 \end{pmatrix}, \quad \begin{pmatrix} c_1 & -\sqrt{c_1c_2} & -\sqrt{c_1c_3} \\ -\sqrt{c_1c_2} & c_2 & -\sqrt{c_2c_3} \\ -\sqrt{c_1c_3} & -\sqrt{c_2c_3} & c_3 \end{pmatrix},$$

where c_1, c_2, c_3 are not zero and share the same sign. □

5. Special Rota-Baxter operators of weight 0

The classification of special Rota-Baxter operators on $\mathfrak{so}(3)$ is completed in this section with an analysis of the weight 0 case. We determine all multiplicative and pseudo-Rota-Baxter operators of weight 0, paralleling the results for weight 1.

Theorem 5.1. *There exists no non-trivial multiplicative Rota-Baxter operator of weight* 0 *on* $\mathfrak{so}(3)$.

Proof. By Theorem 3.1, up to orthogonal similarity, any classical Rota-Baxter operator P of weight 0 on $\mathfrak{so}(3)$ is represented by a matrix of one of the following three forms with respect to the standard basis:

$$(I)\quad \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (II)\quad \begin{pmatrix} \frac{a_{12}^2}{a_{22}} & a_{12} & 0 \\ a_{12} & a_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (III)\quad \begin{pmatrix} \frac{a_{13}^2}{a_{33}} & \frac{a_{13}a_{23}}{a_{33}} & a_{13} \\ \frac{a_{13}a_{23}}{a_{33}} & \frac{a_{23}^2}{a_{33}} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}.$$

where $a_{ij} \in \mathbb{R}$ with $a_{22}, a_{33} \neq 0$ in forms (II) and (III).

A multiplicative Rota-Baxter operator must additionally satisfy the homomorphism condition. We now verify this condition for each case.

Case I: The operator acts as

$$P(e_1) = a_{11}e_1,\ P(e_2) = P(e_3) = 0.$$

Consider the elements e_2 and e_3. We have:

$$[P(e_2), P(e_3)] = 0,\ P([e_2, e_3]) = a_{11}e_1.$$

The homomorphism condition requires $a_{11} = 0$. Thus, the only multiplicative operator in this case is zero.

Case II: The operator acts as:

$$P(e_1) = \frac{a_{13}^2}{a_{33}}e_1 + a_{12}e_2,\ P(e_2) = a_{12}e_1 + a_{22}e_2,\ P(e_3) = 0.$$

Consider the elements e_1 and e_3. We have:

$$[P(e_1), P(e_3)] = 0,\ P([e_1, e_3]) = -a_{12}e_1 - a_{22}e_2.$$

The homomorphism condition implies $-a_{21}e_1 - a_{22}e_2 = 0$, forcing $a_{21} = a_{22} = 0$. However, this contradicts the requirement that $a_{22} \neq 0$ for a non-trivial form (II) operator. The only possibility is thus the zero operator.

Case III: The operator acts as:

$$\begin{aligned}
P(e_1) &= \frac{a_{13}^2}{a_{33}}e_1 + \frac{a_{13}a_{23}}{a_{33}}e_2 + a_{13}e_3,\\
P(e_2) &= \frac{a_{13}a_{23}}{a_{33}}e_1 + \frac{a_{23}^2}{a_{33}}e_2 + a_{23}e_3,\\
P(e_3) &= a_{13}e_1 + a_{23}e_2 + a_{33}e_3.
\end{aligned}$$

Consider the elements e_1 and e_3. We compute:

$$[P(e_1), P(e_3)] = 0,\ P([e_1, e_3]) = P(-e_2) = -\frac{a_{13}a_{23}}{a_{33}}e_1 - \frac{a_{23}^2}{a_{33}}e_2 - a_{23}e_3.$$

The homomorphism condition requires:

$$-\frac{a_{13}a_{23}}{a_{33}}e_1 - \frac{a_{23}^2}{a_{33}}e_2 - a_{23}e_3 = 0,$$

which implies $a_{23} = 0$. Substituting this into the matrix form (III), we obtain a simplified operator. Now consider the elements e_2 and e_3:

$$[P(e_2), P(e_3)] = 0,\ P([e_2, e_3]) = P(e_1) = \frac{a_{13}^2}{a_{33}}e_1 + a_{13}e_2.$$

The homomorphism condition gives $\frac{a_{13}^2}{a_{33}}e_1 + a_{13}e_2 = 0$, forcing $a_{13} = 0$. Finally, with $a_{13} = a_{23} = 0$, the operator becomes $P(e_3) = a_{33}e_3$ and $P(e_1) = P(e_2) = 0$. Testing again with e_1 and e_2:

$$[P(e_1), P(e_2)] = 0,\ P([e_1, e_2]) = P(e_3) = a_{33}e_3,$$

which requires $a_{33} = 0$. Thus, the only multiplicative operator in this case is also zero.

Since in all three cases the homomorphism condition forces the operator to be zero, we conclude that there exists no non-trivial multiplicative Rota-Baxter operator of weight 0 on $\mathfrak{so}(3)$. $\square$

Theorem 5.2. *Up to orthogonal similarity, all real matrix representations of the pseudo-Rota-Baxter operator of weight* 0 *on* $\mathfrak{so}(3)$ *with respect to the standard basis are as follows:*

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}, \quad \begin{pmatrix} g_{11} & 0 & \sqrt{g_{11}(1-g_{11})} \\ 0 & 1 & 0 \\ \sqrt{g_{11}(1-g_{11})} & 0 & 1-g_{11} \end{pmatrix},$$
$$\begin{pmatrix} g_{22} & g_{22}-1 & \sqrt{-1+3g_{22}-2g_{22}^2} \\ g_{22}-1 & g_{22} & \sqrt{-1+3g_{22}-2g_{22}^2} \\ \sqrt{-1+3g_{22}-2g_{22}^2} & \sqrt{-1+3g_{22}-2g_{22}^2} & 2-2g_{22} \end{pmatrix},$$
$$\begin{pmatrix} g_{22} & 1-g_{22} & \sqrt{-1+3g_{22}-2g_{22}^2} \\ 1-g_{22} & g_{22} & \sqrt{-1+3g_{22}-2g_{22}^2} \\ \sqrt{-1+3g_{22}-2g_{22}^2} & \sqrt{-1+3g_{22}-2g_{22}^2} & 2-2g_{22} \end{pmatrix},$$

where $g_{11} \in [0, 1]$ *and* $g_{22} \in [\frac{1}{2}, 1]$.

Proof. Let H be a pseudo-Rota-Baxter operator of weight 0 with matrix representation $\mathbf{H} = (g_{ij})$ with respect to the standard basis. The defining equation

$$H([e_i, e_j]) = H([H(e_i), e_j] + [e_i, H(e_j)] + [e_i, e_j]),\ i, j = 1, 2, 3,$$

yields the following system upon comparing coefficients:

$$(g_{11} + g_{22} - 1)g_{13} = g_{11}g_{31} + g_{12}g_{32}, \tag{5.1}$$
$$(g_{11} + g_{22} - 1)g_{23} = g_{21}g_{31} + g_{22}g_{32}, \tag{5.2}$$
$$(g_{11} + g_{22} - 1)g_{33} = g_{31}^2 + g_{32}^2, \tag{5.3}$$
$$(g_{11} + g_{33} - 1)g_{12} = g_{11}g_{21} + g_{13}g_{23}, \tag{5.4}$$
$$(g_{11} + g_{33} - 1)g_{22} = g_{21}^2 + g_{23}^2, \tag{5.5}$$
$$(g_{11} + g_{33} - 1)g_{32} = g_{31}g_{21} + g_{33}g_{23}, \tag{5.6}$$
$$(g_{22} + g_{33} - 1)g_{11} = g_{12}^2 + g_{13}^2, \tag{5.7}$$
$$(g_{22} + g_{33} - 1)g_{21} = g_{12}g_{22} + g_{13}g_{23}, \tag{5.8}$$
$$(g_{22} + g_{33} - 1)g_{31} = g_{12}g_{32} + g_{13}g_{23}. \tag{5.9}$$

We proceed by case analysis based on the values of the diagonal entries g_{11}, g_{22}, g_{33}.

Case 1: All diagonal entries vanish.

Equations (5.3), (5.5), and (5.7) imply $g_{31}^2 + g_{32}^2 = g_{21}^2 + g_{23}^2 = g_{12}^2 + g_{13}^2 = 0$, forcing all off-diagonal entries to vanish. Thus H is zero.

Case 2: Exactly two diagonal entries vanish.

Subcase 2.1: $g_{11} = g_{22} = 0$, $g_{33} \neq 0$.

From Eqs (5.5) and (5.7), we obtain $g_{21} = g_{23} = g_{12} = g_{13} = 0$. The remaining equations reduce to:

$$\begin{aligned} &g_{31}^2 + g_{32}^2 = -g_{33}, \\ &(g_{33} - 1)g_{32} = 0, \\ &(g_{33} - 1)g_{31} = 0. \end{aligned}$$

If $g_{33} = 1$, then $g_{31}^2 + g_{32}^2 = -1$, which is impossible over $\mathbb{R}$. Thus $g_{33} \neq 1$, forcing $g_{31} = g_{32} = 0$ and then $g_{33} = 0$, which is a contradiction. Hence no solution exists. The other subcases with exactly two zero diagonal entries similarly yield only the trivial solution.

Case 3: Exactly one diagonal entry vanishes.

Subcase 3.1: $g_{11} = 0$ and $g_{22}, g_{33} \neq 0$.

Equation (5.1) gives $g_{12}^2 + g_{13}^2 = 0$. The system reduces to:

$$(g_{22} - 1)g_{23} = g_{21}g_{31} + g_{22}g_{32}, \tag{5.10}$$
$$(g_{22} - 1)g_{33} = g_{31}^2 + g_{32}^2, \tag{5.11}$$
$$(g_{33} - 1)g_{22} = g_{21}^2 + g_{23}^2, \tag{5.12}$$
$$(g_{33} - 1)g_{32} = g_{21}g_{31} + g_{22}g_{33}, \tag{5.13}$$
$$(g_{22} + g_{33} - 1)g_{21} = 0, \tag{5.14}$$
$$(g_{22} + g_{33} - 1)g_{31} = 0. \tag{5.15}$$

From Eqs (5.10) and (5.13), we derive $(g_{22} + g_{33} - 1)g_{23} = (g_{22} + g_{33} - 1)g_{32}$. If $g_{22} + g_{33} = 1$, then Eq (5.11) gives $g_{22}+g_{33} = 1$, forcing $g_{31}^2+g_{32}^2+g_{33}^2 = 0$, which is a contradiction. Thus $g_{23} = g_{32}$. Then Eqs (5.14) and (5.15) give $g_{21} = g_{31} = 0$, and Eq (5.10) yields $g_{23} = 0$. Finally, Eqs (5.11) and (5.12) imply $g_{22} = g_{33} = 1$, giving the matrix

$$H_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The other subcases with exactly one zero diagonal entry similarly yield the matrices

$$H_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad H_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Case 4: All diagonal entries are non-zero.

We now analyze the case $g_{11}, g_{22}, g_{33} \neq 0$. The analysis proceeds by considering the vanishing of the expressions $g_{11} + g_{22} - 1$, $g_{11} + g_{33} - 1$, and $g_{22} + g_{33} - 1$.

Subcase 4.1: $g_{11} + g_{22} - 1 = 0$.

From Eq (5.3), we obtain $g_{31}^2 + g_{32}^2 = 0$, which implies $g_{31} = g_{32} = 0$. Substituting into Eqs (5.6) and (5.9) yields $g_{13} = g_{23} = 0$. The system then reduces to:

$$(g_{11} + g_{33} - 1)g_{12} = g_{21}g_{11}, \tag{5.16}$$
$$(g_{11} + g_{33} - 1)g_{22} = g_{21}^2, \tag{5.17}$$
$$(g_{22} + g_{33} - 1)g_{11} = g_{12}^2, \tag{5.18}$$
$$(g_{22} + g_{33} - 1)g_{21} = g_{12}g_{22}. \tag{5.19}$$

From Eqs (5.16) and (5.19), we derive $g_{33}(g_{12} - g_{21}) = 0$. Since $g_{33} \neq 0$, we obtain $g_{12} = g_{21}$. Equations (5.17) and (5.18) imply $(g_{33} - 1)g_{22} = (g_{33} - 1)g_{11}$.

- If $g_{33} = 1$, then Eq (5.17) gives $g_{12}^2 = g_{21}^2 = g_{11}g_{22} = g_{22}(1 - g_{22})$. For $g_{22} \in [0, 1]$, we obtain the matrices:

$$H_4 = \begin{pmatrix} 1 - g_{22} & \sqrt{g_{22}(1 - g_{22})} & 0 \\ \sqrt{g_{22}(1 - g_{22})} & g_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$H_5 = \begin{pmatrix} 1 - g_{22} & -\sqrt{g_{22}(1 - g_{22})} & 0 \\ -\sqrt{g_{22}(1 - g_{22})} & g_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

- If $g_{33} \neq 1$, then it follows from $g_{11} + g_{22} = 1$ that $g_{11} = g_{22} = \frac{1}{2}$. Equations (5.16) to (5.19) then force $g_{12} = g_{21} = 0$ and $g_{33} = \frac{1}{2}$, yielding:

$$H_6 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

Subcase 4.2: $g_{11} + g_{22} - 1 \neq 0$.

We further analyze based on the other linear combinations:

- If $g_{11} + g_{33} - 1 = 0$, then a similar analysis yields:

$$H_7 = \begin{pmatrix} g_{11} & 0 & \sqrt{g_{11}(1 - g_{11})} \\ 0 & 1 & 0 \\ \sqrt{g_{11}(1 - g_{11})} & 0 & 1 - g_{11} \end{pmatrix},$$
$$H_8 = \begin{pmatrix} g_{11} & 0 & -\sqrt{g_{11}(1 - g_{11})} \\ 0 & 1 & 0 \\ -\sqrt{g_{11}(1 - g_{11})} & 0 & 1 - g_{11} \end{pmatrix},$$

  for $g_{11} \in [0, 1]$, plus the exceptional case H_6.
- If $g_{22} + g_{33} - 1 = 0$, then we obtain:

$$H_9 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & g_{22} & \sqrt{g_{22}(1 - g_{22})} \\ 0 & \sqrt{g_{22}(1 - g_{22})} & 1 - g_{22} \end{pmatrix},$$

$$H_{10} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & g_{22} & -\sqrt{g_{22}(1-g_{22})} \\ 0 & -\sqrt{g_{22}(1-g_{22})} & 1-g_{22} \end{pmatrix}, \text{where } g_{22} \in [0,1],$$

for $g_{22} \in [0, 1]$, plus the exceptional case H_6.

Subcase 4.3: $g_{11} + g_{22} - 1$, $g_{11} + g_{33} - 1$, and $g_{22} + g_{33} - 1$ are all non-zero.
The symmetry of the matrix can be readily established. From Eqs (5.4) and (5.8), we obtain

$$(g_{11} + g_{22} + g_{33} - 1)(g_{12} - g_{21}) = 0.$$

If $g_{11} + g_{22} + g_{33} = 0$, then Eq (5.3) gives $g_{31}^2 + g_{32}^2 + g_{33}^2 = 0$, forcing $g_{31} = g_{32} = g_{33} = 0$, which is a contradiction. Thus $g_{12} = g_{21}$. Similarly, $g_{13} = g_{31}$ and $g_{23} = g_{32}$.

The system then reduces to:

$$(g_{22} - 1)g_{13} = g_{12}g_{23}, \tag{5.20}$$
$$(g_{11} - 1)g_{23} = g_{13}g_{21}, \tag{5.21}$$
$$(g_{33} - 1)g_{12} = g_{13}g_{23}, \tag{5.22}$$
$$(g_{11} + g_{22} - 1)g_{33} = g_{13}^2 + g_{23}^2, \tag{5.23}$$
$$(g_{11} + g_{33} - 1)g_{22} = g_{12}^2 + g_{23}^2, \tag{5.24}$$
$$(g_{22} + g_{33} - 1)g_{11} = g_{12}^2 + g_{13}^2. \tag{5.25}$$

Suppose $g_{11} = 1$, and then Eqs (5.23) and (5.24) imply $g_{22}g_{33} = g_{13}^2 + g_{23}^2$ and $g_{22}g_{33} = g_{12}^2 + g_{23}^2$. Since $g_{22}, g_{33} \neq 0$, we have $g_{12}^2 = g_{13}^2$. From Eq (5.21), we obtain $g_{12}g_{23} = 0$. Thus $g_{12} = g_{13} = 0$. Substituting into Eq (5.25) yields $(g_{22} + g_{33} - 1)g_{11} = 0$, which is a contradiction. Thus, $g_{11} \neq 1$. Similarly, $g_{22} \neq 1$ and $g_{33} \neq 1$.

By using similar assumptions, we prove that g_{12}, g_{13}, g_{23} are all non-zero. Equation (5.20) may be transformed to $g_{13} = \frac{g_{12}g_{23}}{g_{22}-1}$. Equations (5.21) and (5.22) yield:

$$g_{12}^2 = (g_{11} - 1)(g_{22} - 1), \tag{5.26}$$
$$g_{23}^2 = (g_{22} - 1)(g_{33} - 1). \tag{5.27}$$

Similarly, substituting $g_{13} = \frac{g_{12}g_{23}}{g_{22}-1}$ into Eq (5.21), we obtain:

$$g_{13}^2 = (g_{11} - 1)(g_{33} - 1).$$

It follows from Eqs (5.24), (5.26), and (5.27) that

$$g_{11} + g_{22} + g_{33} = 2.$$

Thus,

$$g_{11} = g_{22},\ g_{23}^2 = g_{13}^2.$$

Then

$$g_{12}^2 = g_{21}^2 = (g_{22} - 1)^2,$$

$$g_{23}^2 = g_{13}^2 = g_{32}^2 = -1 + 3g_{22} - 2g_{22}^2.$$

For real solutions, we require $-1 + 3g_{22} - 2g_{22}^2 \geqslant 0$, which holds for $g_{22} \in [\frac{1}{2}, 1]$. This yields the final family of matrices:

$$H_{11} = \begin{pmatrix} g_{22} & g_{22} - 1 & \sqrt{-1 + 3g_{22} - 2g_{22}^2} \\ g_{22} - 1 & g_{22} & \sqrt{-1 + 3g_{22} - 2g_{22}^2} \\ \sqrt{-1 + 3g_{22} - 2g_{22}^2} & \sqrt{-1 + 3g_{22} - 2g_{22}^2} & 2 - 2g_{22} \end{pmatrix},$$

$$H_{12} = \begin{pmatrix} g_{22} & g_{22} - 1 & \sqrt{-1 + 3g_{22} - 2g_{22}^2} \\ g_{22} - 1 & g_{22} & -\sqrt{-1 + 3g_{22} - 2g_{22}^2} \\ \sqrt{-1 + 3g_{22} - 2g_{22}^2} & -\sqrt{-1 + 3g_{22} - 2g_{22}^2} & 2 - 2g_{22} \end{pmatrix},$$

$$H_{13} = \begin{pmatrix} g_{22} & g_{22} - 1 & -\sqrt{-1 + 3g_{22} - 2g_{22}^2} \\ g_{22} - 1 & g_{22} & \sqrt{-1 + 3g_{22} - 2g_{22}^2} \\ -\sqrt{-1 + 3g_{22} - 2g_{22}^2} & \sqrt{-1 + 3g_{22} - 2g_{22}^2} & 2 - 2g_{22} \end{pmatrix},$$

$$H_{14} = \begin{pmatrix} g_{22} & g_{22} - 1 & -\sqrt{-1 + 3g_{22} - 2g_{22}^2} \\ g_{22} - 1 & g_{22} & -\sqrt{-1 + 3g_{22} - 2g_{22}^2} \\ -\sqrt{-1 + 3g_{22} - 2g_{22}^2} & -\sqrt{-1 + 3g_{22} - 2g_{22}^2} & 2 - 2g_{22} \end{pmatrix},$$

$$H_{15} = \begin{pmatrix} g_{22} & 1 - g_{22} & \sqrt{-1 + 3g_{22} - 2g_{22}^2} \\ 1 - g_{22} & g_{22} & \sqrt{-1 + 3g_{22} - 2g_{22}^2} \\ \sqrt{-1 + 3g_{22} - 2g_{22}^2} & \sqrt{-1 + 3g_{22} - 2g_{22}^2} & 2 - 2g_{22} \end{pmatrix},$$

$$H_{16} = \begin{pmatrix} g_{22} & 1 - g_{22} & \sqrt{-1 + 3g_{22} - 2g_{22}^2} \\ 1 - g_{22} & g_{22} & -\sqrt{-1 + 3g_{22} - 2g_{22}^2} \\ \sqrt{-1 + 3g_{22} - 2g_{22}^2} & -\sqrt{-1 + 3g_{22} - 2g_{22}^2} & 2 - 2g_{22} \end{pmatrix},$$

$$H_{17} = \begin{pmatrix} g_{22} & 1 - g_{22} & -\sqrt{-1 + 3g_{22} - 2g_{22}^2} \\ 1 - g_{22} & g_{22} & \sqrt{-1 + 3g_{22} - 2g_{22}^2} \\ -\sqrt{-1 + 3g_{22} - 2g_{22}^2} & \sqrt{-1 + 3g_{22} - 2g_{22}^2} & 2 - 2g_{22} \end{pmatrix},$$

$$H_{18} = \begin{pmatrix} g_{22} & 1 - g_{22} & -\sqrt{-1 + 3g_{22} - 2g_{22}^2} \\ 1 - g_{22} & g_{22} & -\sqrt{-1 + 3g_{22} - 2g_{22}^2} \\ -\sqrt{-1 + 3g_{22} - 2g_{22}^2} & -\sqrt{-1 + 3g_{22} - 2g_{22}^2} & 2 - 2g_{22} \end{pmatrix}.$$

The above real symmetric matrices H_{11} to H_{18} can be classified into the following two similarity classes represented by:

$$\begin{pmatrix} g_{22} & g_{22}-1 & \sqrt{-1+3g_{22}-2g_{22}^2} \\ g_{22}-1 & g_{22} & \sqrt{-1+3g_{22}-2g_{22}^2} \\ \sqrt{-1+3g_{22}-2g_{22}^2} & \sqrt{-1+3g_{22}-2g_{22}^2} & 2-2g_{22} \end{pmatrix},$$
$$\begin{pmatrix} g_{22} & 1-g_{22} & \sqrt{-1+3g_{22}-2g_{22}^2} \\ 1-g_{22} & g_{22} & \sqrt{-1+3g_{22}-2g_{22}^2} \\ \sqrt{-1+3g_{22}-2g_{22}^2} & \sqrt{-1+3g_{22}-2g_{22}^2} & 2-2g_{22} \end{pmatrix},$$

where $g_{11} \in [0,1]$ and $g_{22} \in [\frac{1}{2}, 1]$.

By computing eigenvalues, we verify that the matrices within each of the five families are orthogonally similar, while matrices from different families are not. This completes the classification. □

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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