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https://www.aimspress.com/journal/era

ERA, 33(10): 6012–6035.
DOI: 10.3934/era.2025267
Received: 21 July 2025
Revised: 24 September 2025
Accepted: 09 October 2025
Published: 16 October 2025

Research article

Pell coefficient polynomials for solving linear hyperbolic first-order partial differential equations via the Tau approach

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Abstract: The main aims of this study are the introduction of Pell coefficient polynomials and their numerical treatment of the first-order hyperbolic partial differential equations. Our suggested numerical algorithm will be derived from the utilization of some novel formulas of the Pell coefficient polynomials, along with the application of the spectral tau method. For the proposed expansion, we investigate the convergence and error estimations in detail. The presented numerical results indicate that the suggested numerical method is accurate, converges exponentially, and is computationally efficient.

Keywords: Pell numbers; special polynomials; spectral methods; matrix system; convergence analysis

1. Introduction

Partial differential equations (PDEs) play important roles in modeling and analyzing various physical, biological, and engineering phenomena; see, for example, [1, 2]. These equations often do not admit analytical solutions, making it essential to solve them using various numerical algorithms. The authors of [3] used a quantum variational technique to solve certain nonlinear PDEs. In [4], a deep learning framework was proposed to handle nonlinear PDEs. Moreover, another approach for addressing such equations was developed in [5], where physics-informed neural networks were utilized. Additionally, a method for approximating solutions to a class of PDEs was introduced in [6]. The authors of [7] developed some analytical and numerical solutions for certain nonlinear PDEs. Among the important PDEs are the hyperbolic partial differential equations (HPDEs). Many numerical algorithms were devoted to treating these types of equations. In [8], the authors introduced many-stage optimal stabilized Runge–Kutta methods. A matrix approach was followed using Vieta–Lucas polynomials in [9] to treat some HPDEs. In addition, a collocation algorithm was presented in [10] to handle the HPDEs. The method of lines combined with the Runge–Kutta integration method was employed in [11] for solving systems of certain delay HPDEs. Additionally, the authors of [12] presented an exponential Jacobi spectral method to handle HPDEs.

When dealing with differential equations (DEs) of various types, spectral approaches are very useful tools. These approaches are excellent for solving high-order ordinary DEs, PDEs, and other types of DEs. Their key benefit over the standard numerical approaches is their excellent performance for smooth problems, achieved by exponential or high-order convergence. These techniques provide an accurate approximation with few degrees of freedom by expressing the solution in terms of global basis functions, which are often particular functions or special polynomials. Spectral approaches have found useful applications in many domains, including models of biological systems, fluid dynamics, and quantum physics. One can refer to [13, 14] for some applications of these methods. Collocation, tau, and Galerkin methods are the three most common spectral approaches. One benefit of collocation techniques is that they can handle all DEs governed by any conditions; see, for example, [15–17]. Two sets of basis functions, trial and test, are required to apply both the Galerkin and Tau approaches. In the Galerkin method, these sets should be coincident; see, for example, [18–20], while the tau method has no restrictions in choosing the basis functions; see, for example, [21–23].

The sequences of numbers and polynomials have important roles in the different fields of applied sciences. For example, the sequences of Fibonacci and their generalized sequences are crucial [24]. Many polynomial sequences are beneficial in numerical analysis and, in particular, for treating all types of DEs. For example, the authors of [25] introduced telephone polynomials and utilized them to solve some models numerically. Shifted Lucas polynomials were utilized in [26] together with the collocation method to handle the fractional FitzHugh–Nagumo DEs. Fibonacci polynomials were employed in [27] to treat certain types of variable-order fractional DEs. Some specific Horadam polynomials were used in [28] to solve certain Korteweg–de Vries (KdV)-type equations. Bernstein polynomials were employed in [29] to solve some initial value problems. Multidimensional sinh-Gordon equations were treated in [30] using Lucas polynomials. In addition, modified Lucas polynomials were used in [31] to treat mixed-type fractional functional DEs. Vieta-Lucas polynomials were employed in [32] to handle certain types of fractional optimal control problems.

Among the important sequences that have roles in different disciplines is the Pell sequence. Many contributions were devoted to deriving formulas for the different Pell number sequences; see, for example, [33–35]. In addition, Pell polynomials were used in many articles related to numerical analysis to solve several important models. The authors of [36] followed a numerical approach based on Pell polynomials to solve certain stochastic fractional DEs. The authors of [37] utilized Pell polynomials to address nonlinear variable-order space fractional PDEs. The authors of [38] used the Pell polynomials to handle certain fractional DEs. The authors of [39] developed a Crank–Nicolson spectral Pell matrix algorithm to simulate the Rosenau–Burgers equations numerically.

The main objectives of this article are to introduce Pell coefficient polynomials and utilize them numerically to treat the linear hyperbolic first-order PDEs. More definitely, we can list the goals in the following issues:

- Introducing Pell coefficient polynomials.
- Developing the inversion formula of these polynomials.
- Developing new moments, linearization, and derivative formulas of these polynomials.
- Utilizing the introduced polynomials to handle the linear hyperbolic first-order PDEs.
- Studying the error analysis of the suggested Pell coefficients expansion.
- Evaluating our numerical algorithm through some numerical examples.

Here are the main contributions and novelty of the current paper:

- Introducing new Pell coefficient polynomials: As far as we know, this is the first time that these kinds of polynomials have been defined and studied in the literature. This motivates us to study and utilize these polynomials.
- New theoretical results: Several new formulas concerning the Pell coefficient polynomials are introduced as the foundations to design the proposed numerical algorithm.
- Utilizing the spectral tau method: The suggested Pell coefficient polynomials are used as basis functions to numerically solve linear hyperbolic first-order PDEs.
- Convergence and error analysis: The proposed expansion is thoroughly examined to determine its convergence characteristics and error estimates.
- Numerical performance: The proposed method is tested by presenting some examples supported with comparisons with other schemes in the literature.

The contents of the paper are structured as follows. Section 2 gives some fundamental characteristic properties of the Pell numbers. In addition, we introduce the polynomials, namely, Pell coefficient polynomials. Some new formulas of these formulas are developed in Section 3. Section 4 is devoted to the numerical treatment of the one-dimensional linear HPDEs. Convergence and error analysis are investigated in Section 5. Illustrative examples are presented in Section 6 to ensure the accuracy and applicability of the proposed algorithm. Finally, some conclusions are given in Section 7.

2. Pell coefficient polynomials

In this section, we give an overview of the Pell numbers. In addition, we introduce polynomials whose coefficients are the Pell numbers.

The standard Pell numbers meet the following recursive formula [40]:

$$P_{j+2} = 2\,P_{j+1} + P_j, \quad P_0 = 0,\ P_1 = 1. \tag{2.1}$$

A few Pell numbers are $0, 1, 2, 5, 12, 29, 70, 169$.

They have the following combinatorial formula [41]:

$$P_i = \sum_{m=0}^{\lfloor \frac{i-1}{2} \rfloor} 2^m \binom{i}{2m+1}. \tag{2.2}$$

Now, we define the new Pell coefficient polynomials $\mathrm{P}_i^*(\theta)$ as

$$\mathrm{P}_i^*(\theta) = \sum_{k=0}^{i} P_{k+1}\,\theta^{i-k}, \tag{2.3}$$

where P_k represents the standard Pell numbers.
Formula (2.3) can also be written in the form

$$\mathrm{P}_i^*(\theta) = \sum_{r=0}^{i} P_{i-r+1}\,\theta^{r}. \tag{2.4}$$

From the expression (2.4), it is evident that $\mathsf{P}_i^*(\theta)$ satisfies the following recurrence relation:

$$\mathsf{P}_i^*(\theta) - \theta\,\mathsf{P}_{i-1}^*(\theta) - P_{i+1} = 0. \tag{2.5}$$

Remark 1. *To the best of our knowledge, the introduced Pell coefficient polynomials are new, so their theoretical background is traceless in the literature.*

Remark 2. *The following theorem presents the inversion formula of* $\mathrm{P}_i^*(\theta)$. *This formula, together with the power form representation, will be pivotal in yielding further properties of these polynomials in the next section.*

Theorem 1. θ^i *can be expanded as*

$$\theta^i = \sum_{r=0}^{i} \mu_{r,i}\,\mathrm{P}_r^*(\theta),\ i \geq 0, \tag{2.6}$$

where

$$\mu_{r,i} = \begin{cases} 1, & \text{if } r = i, \\ -2, & \text{if } r = i-1, \\ -1, & \text{if } r = i-2, \\ 0, & \text{if } 0 \leq r \leq i-3. \end{cases} \tag{2.7}$$

Proof. We will prove that the following identity holds:

$$\theta^i = \mathrm{P}_i^*(\theta) - 2\,\mathrm{P}_{i-1}^*(\theta) - \mathrm{P}_{i-2}^*(\theta). \tag{2.8}$$

Assume the following polynomial:

$$\eta_i(\theta) = \mathrm{P}_i^*(\theta) - 2\,\mathrm{P}_{i-1}^*(\theta) - \mathrm{P}_{i-2}^*(\theta),$$

then, it is sufficient to show that

$$\eta_i(\theta) = \theta^i.$$

Based on the representation in (2.3), $\eta_i(\theta)$ may be expressed in the form

$$\eta_i(\theta) = \sum_{r=0}^{i} P_{r+1}\,\theta^{i-r} - 2\sum_{r=0}^{i-1} P_{r+1}\,\theta^{i-r-1} - \sum_{r=0}^{i-2} P_{r+1}\,\theta^{i-r-2}, \tag{2.9}$$

which is equal to

$$\eta_i(\theta) = \sum_{r=0}^{i} P_{r+1}\,\theta^{i-r} - 2\sum_{r=1}^{i} P_r\,\theta^{i-r} - \sum_{r=2}^{i} P_{r-1}\,\theta^{i-r}, \tag{2.10}$$

and accordingly, we have

$$\eta_i(\theta) = \theta^i + \sum_{r=2}^{i} (P_{r+1} - 2\,P_r - P_{r-1})\;\theta^{i-r}, \tag{2.11}$$

and therefore, the recurrence relation (2.1) leads to

$$\eta_i(\theta) = \theta^i. \tag{2.12}$$

This ends the proof.

Remark 3. *Formula* (2.6) *can be written in the following alternative form:*

$$\theta^i = \sum_{r=0}^{i} G_{r,i}\,\mathrm{P}^*_{i-r}(\theta), \tag{2.13}$$

with

$$G_{r,i} = \begin{cases} 1, & \text{if } r = 0,\\ -2, & \text{if } r = 1,\\ -1, & \text{if } r = 2,\\ 0, & \text{if } r \geq 3. \end{cases} \tag{2.14}$$

3. Some further new formulas of the Pell coefficient polynomials

This section is confined to deriving some new formulas for the Pell coefficient polynomials, which will be useful in determining our proposed algorithm.

The following two theorems give the expressions of the moment and linearization formulas of the $\mathrm{P}^*_i(\theta)$.

Theorem 2. *The following moment formula holds for every nonnegative integers r and s:*

$$x^r\,\mathrm{P}^*_s(\theta) = \sum_{m=0}^{s+r} M_{m,s}\,\mathrm{P}^*_{s+r-m}(\theta), \tag{3.1}$$

with

$$M_{m,s} = \sum_{r=0}^{\min(m,s)} P_{r+1}\,G_{m-r,\,s+r-k}, \tag{3.2}$$

and $G_{r,i}$ are given in (2.14)*.*

Proof. The analytic form in (2.3) gives

$$\theta^r\,\mathrm{P}^*_s(\theta) = \sum_{k=0}^{s} P_{k+1}\,\theta^{s-k+r}. \tag{3.3}$$

Applying formula (2.13) leads to the following identity:

$$\theta^r \,\mathrm{P}^*_s(\theta) = \sum_{k=0}^{s} P_{k+1} \sum_{t=0}^{s-k+r} G_{t,s+r-k}\, \mathrm{P}^*_{s+r-k-t}(\theta),$$

which can be rearranged to give

$$\theta^r \,\mathrm{P}^*_s(\theta) = \sum_{m=0}^{s+r} \left(\sum_{r=0}^{\min(m,s)} P_{r+1}\, G_{m-r,\, s+r-k} \right) \mathrm{P}^*_{s+r-m}(\theta). \tag{3.4}$$

This proves Theorem 2.

Theorem 3. *For every two nonnegative integers i and r, the following linearization formula is valid:*

$$\mathrm{P}^*_i(\theta)\, \mathrm{P}^*_r(\theta) = \sum_{m=0}^{r+i} \zeta_{m,i,r}\; \mathrm{P}^*_{r+i-m}(\theta), \tag{3.5}$$

with the linearization coefficients $\zeta_{m,i,r}$ given as

$$\zeta_{m,i,r} = \sum_{s=0}^{\min(i,m)} P_{s+1} \sum_{L=0}^{\min(m-s,r)} P_{L+1}\, G_{m-L-s,r+i-s+L}. \tag{3.6}$$

Proof. Starting with the representation (2.3) leads to

$$\mathrm{P}^*_i(\theta)\, \mathrm{P}^*_r(\theta) = \sum_{s=0}^{i} P_{s+1}\, \theta^{i-s}\, \mathrm{P}^*_r(\theta). \tag{3.7}$$

With the aid of the moment formula (3.4), the last formula turns into

$$\mathrm{P}^*_i(\theta)\, \mathrm{P}^*_r(\theta) = \sum_{s=0}^{i} P_{s+1} \sum_{m=0}^{r+i-s} R_{m,r}\, \mathrm{P}^*_{r+i-m-s}(\theta), \tag{3.8}$$

where

$$R_{m,r} = \sum_{L=0}^{\min(m,r)} P_{L+1} G_{m-L,r+i-s+L}.$$

Formula (3.8) can be written alternatively in the form

$$\mathrm{P}^*_i(\theta)\, \mathrm{P}^*_r(\theta) = \sum_{m=0}^{r+i} \left(\sum_{s=0}^{\min(i,m)} P_{s+1} R_{m-s,r} \right) \mathrm{P}^*_{r+i-m}(\theta), \tag{3.9}$$

and accordingly, Formula (3.5) can be obtained.

Corollary 1. *The following integral formula holds:*

$$\int_0^{\ell} \mathrm{P}^*_i(\theta)\, \mathrm{P}^*_r(\theta)\, d\theta = \sum_{m=0}^{r+i} \zeta_{m,i,r}\; J_{i+r-m}, \tag{3.10}$$

where

$$J_s = \sum_{r=0}^{s} \frac{\ell^{s-r+1}}{s-r+1}\, P_{s+1}, \tag{3.11}$$

and the coefficients $\zeta_{m,i,r}$ are those given in (3.6).

Proof. From the explicit expression (2.3), we can write

$$\int_0^{\ell} \mathrm{P}_r^*(\theta)\, d\theta = \sum_{k=0}^{r} P_{k+1} \int_0^{\ell} \theta^{r-k}\, d\theta, \tag{3.12}$$

and thus, we can write

$$\int_0^{\ell} \mathrm{P}_r^*(\theta)\, d\theta = J_r, \tag{3.13}$$

with

$$J_r = \sum_{k=0}^{r} \frac{P_{k+1}\, \ell^{r-k+1}}{r-k+1}. \tag{3.14}$$

Now, integrating both sides of the linearization formula (3.5), we get

$$\int_0^{\ell} \mathrm{P}_i^*(\theta)\, \mathrm{P}_r^*(\theta)\, d\theta = \sum_{m=0}^{r+i} \zeta_{m,i,r} \int_0^{\ell} \mathrm{P}_{r+i-m}^*(\theta)\, d\theta. \tag{3.15}$$

Using the integral in (3.13), the following integral formula can be obtained:

$$\int_0^{\ell} \mathrm{P}_i^*(\theta)\mathrm{P}_r^*(\theta)\, d\theta = \sum_{m=0}^{r+i} \zeta_{m,i,r}\, J_{r+i-m}. \tag{3.16}$$

This ends the proof.

The first-order derivatives are given in the following theorem in a closed form.

Theorem 4. *The first derivative of* $\mathrm{P}_i^*(\theta)$ *can be expressed as*

$$\frac{d\,\mathrm{P}_i^*(\theta)}{d\theta} = \sum_{m=0}^{i-1} H_{m,i} \mathrm{P}_m^*(\theta), \quad i \geq 1, \tag{3.17}$$

where

$$H_{m,i} = -(m+3)\, P_{i-m-2} - 2(m+2)\, P_{i-m-1} + (m+1)\, P_{i-m}. \tag{3.18}$$

Proof. The power form representation of $\mathrm{P}_i^*(\theta)$ in (2.3) enables one to write

$$\frac{d\,\mathrm{P}_i^*(\theta)}{d\theta} = \sum_{k=0}^{i-1} P_{k+1}\, (i-k)\, \theta^{i-k-1}. \tag{3.19}$$

Inserting the inversion formula (2.8) into the last formula leads to

$$\frac{d\,\mathrm{P}_i^*(\theta)}{d\theta} = \sum_{k=0}^{i-1} P_{k+1}\, (i-k)(\mathrm{P}_{i-k-1}^*(\theta) - 2\mathrm{P}_{i-k-2}^*(\theta) - \mathrm{P}_{i-k-3}^*(\theta)), \tag{3.20}$$

which can be written as

$$\begin{aligned} \frac{d\,\mathrm{P}_i^*(\theta)}{d\theta} =& i\mathrm{P}_{i-1}^*(\theta) - 2\mathrm{P}_{i-2}^*(\theta) \\ &+ \sum_{L=2}^{i-1} ((L-i-2)P_{L-1} + 2(L-i-1)P_L + (i-L)P_{L+1})\, \mathrm{P}_{i-L-1}^*(\theta), \end{aligned} \tag{3.21}$$

and accordingly, we can write

$$\frac{d\,\mathrm{P}_i^*(\theta)}{d\theta} = \sum_{L=0}^{i-1} \left((L-i-2)P_{L-1} + 2(L-i-1)P_L + (i-L)P_{L+1}\right) \mathrm{P}_{i-L-1}^*(\theta). \tag{3.22}$$

An alternative formula for (3.22) is

$$\frac{d\,\mathrm{P}_i^*(\theta)}{d\theta} = \sum_{m=0}^{i-1} \left(-(m+3)\,P_{i-m-2} - 2(m+2)\,P_{i-m-1} + (m+1)\,P_{i-m}\right) \mathrm{P}_m^*(\theta). \tag{3.23}$$

This proves Theorem 4.

Corollary 2. *The following integral formula holds:*

$$\int_0^{\ell} \frac{d\mathrm{P}_i^*(\theta)}{d\theta}\, \mathrm{P}_r^*(\theta)\, d\,\theta = \sum_{m=0}^{i-1} H_{m,i} S_{m,r}, \tag{3.24}$$

where $H_{m,i}$ *are those given in* (3.18)*,* $S_{m,r}$ *are given by*

$$S_{m,r} = \sum_{p=0}^{r+m} \zeta_{p,m,r}\, J_{r+m-p},$$

and the coefficients $\zeta_{p,m,r}$*, and* J_m *are, respectively, given by* (3.6) *and* (3.11).

Proof. Making use of the first-order derivative in (3.17), we get

$$\frac{d\mathrm{P}_i^*(\theta)}{d\theta}\mathrm{P}_r^*(\theta) = \sum_{m=0}^{i-1} H_{m,i}\, \mathrm{P}_m^*(\theta)\, \mathrm{P}_r^*(\theta). \tag{3.25}$$

If we integrate both sides of the last formula, and make use of formula (3.25), then the following formula can be obtained:

$$\int_0^{\ell} \frac{d\mathrm{P}_i^*(\theta)}{d\theta}\mathrm{P}_r^*(\theta)\, d\theta = \sum_{m=0}^{i-1} H_{m,i} S_{m,r}. \tag{3.26}$$

This ends the proof.

4. Approximate solution of the one-dimensional linear HPDEs of first-order

Consider the following one-dimensional linear HPDEs of first-order [42]:

$$\partial_t \mathrm{Z}(\theta,t) - \nu_1\, \partial_\theta \mathrm{Z}(\theta,t) - \nu_2\, \mathrm{Z}(\theta,t) = f(\theta,t), \quad 0<\theta<\ell, \quad 0<t<\tau, \tag{4.1}$$

subject to the following initial and boundary conditions:

$$\mathrm{Z}(\theta,0) = \mathrm{Z}_0(\theta), \quad 0<\theta<\ell, \tag{4.2}$$

$$\mathrm{Z}(0,t) = \mathrm{Z}_1(t), \quad 0<t<\tau, \tag{4.3}$$

where $f(\theta, t)$, $Z_0(\theta)$, and $Z_1(t)$ are given functions, and a, b are positive constants. Moreover, $f(\theta, t) \neq 0$. If we define the following space:

$$\Delta^{\mathcal{N}} = \text{span}\{P^*_i(\theta)\, P^*_j(t) : 0 \leq i, j \leq \mathcal{N}\},$$

then any function $Z^{\mathcal{N}}(\theta, t) \in \Delta^{\mathcal{N}}$ may be assumed to have the following expression:

$$Z^{\mathcal{N}}(\theta, t) = \sum_{i=0}^{\mathcal{N}} \sum_{j=0}^{\mathcal{N}} c_{ij}\, P^*_i(\theta)\, P^*_j(t) = \mathbf{P^*}(\theta)\, \boldsymbol{C}\, \mathbf{P^*}(t)^T, \tag{4.4}$$

where $\mathbf{P^*}(\theta) = [P^*_0(\theta), P^*_1(\theta), \ldots, P^*_{\mathcal{N}}(\theta)]$, $\mathbf{P}(t)^T = [P^*_0(t), P^*_1(t), \ldots, P^*_{\mathcal{N}}(t)]^T$, and $\boldsymbol{C} = (c_{ij})_{0\leq i,j\leq \mathcal{N}}$ is the unknown matrix of dimension $(\mathcal{N} + 1)^2$.
The residual $\mathcal{R}(\theta, t)$ of Eq (4.1) can be expressed as

$$\mathcal{R}(\theta, t) = \partial_t Z^{\mathcal{N}}(\theta, t) - a\, \partial_\theta Z^{\mathcal{N}}(\theta, t) - b\, Z^{\mathcal{N}}(\theta, t) - f(\theta, t). \tag{4.5}$$

Now, the application of the Tau method leads to

$$(\mathcal{R}(\theta, t)\, ,\, P^*_r(\theta)\, P^*_s(t)) = 0, \quad 0 \leq r,\ s \leq \mathcal{N} - 1. \tag{4.6}$$

Now, if we let

$$\boldsymbol{F} = (f_{r,s})_{\mathcal{N}\times\mathcal{N}}, \quad f_{rs} = (f(\theta, t)\, ,\, P^*_r(\theta)\, P^*_s(t)), \tag{4.7}$$

$$\boldsymbol{\mathcal{G}} = (g_{i,r})_{(\mathcal{N}+1)\times\mathcal{N}}, \quad g_{i,r} = (P^*_i(\theta)\, ,\, P^*_r(\theta)), \tag{4.8}$$

$$\boldsymbol{\mathcal{H}} = (h_{ir})_{(\mathcal{N}+1)\times\mathcal{N}}, \quad h_{ir} = \left(\frac{d\, P^*_i(\theta)}{d\,\theta}\, ,\, P^*_r(\theta)\right), \tag{4.9}$$

then Eq (4.6) can be rewritten as

$$\sum_{i=0}^{\mathcal{N}} \sum_{j=0}^{\mathcal{N}} c_{ij}\, g_{i,r}\, h_{j,s} - \nu_1 \sum_{i=0}^{\mathcal{N}} \sum_{j=0}^{\mathcal{N}} c_{ij}\, h_{i,r}\, g_{j,s} - \nu_2 \sum_{i=0}^{\mathcal{N}} \sum_{j=0}^{\mathcal{N}} c_{ij}\, g_{i,r}\, g_{j,s} = f_{r,s}, \quad 0 \leq r,\ s \leq \mathcal{N} - 1, \tag{4.10}$$

or in matrix form as

$$\boldsymbol{\mathcal{G}}^T \boldsymbol{C}\, \boldsymbol{\mathcal{H}} - \nu_1\, \boldsymbol{\mathcal{H}}^T \boldsymbol{C}\, \boldsymbol{\mathcal{G}} - \nu_2\, \boldsymbol{\mathcal{G}}^T \boldsymbol{C}\, \boldsymbol{\mathcal{G}} = \boldsymbol{F}. \tag{4.11}$$

In addition, the conditions (4.2) and (4.3) lead to

$$\sum_{i=0}^{\mathcal{N}} \sum_{j=0}^{\mathcal{N}} c_{ij}\, g_{i,r}\, P^*_j(0) = (Z_0(\theta)\, ,\, P^*_r(\theta)), \quad 0 \leq r \leq \mathcal{N}, \tag{4.12}$$

$$\sum_{i=0}^{\mathcal{N}} \sum_{j=0}^{\mathcal{N}} c_{ij}\, g_{j,s}\, P^*_i(0) = (Z_1(t)\, ,\, P^*_s(t)), \quad 0 \leq s \leq \mathcal{N} - 1. \tag{4.13}$$

An appropriate method may now be employed to solve the resultant algebraic system of equations of order $(\mathcal{N} + 1)^2$, including Eqs (4.11)–(4.13).

Remark 4. *The elements $g_{i,r}$ and $h_{i,r}$ in* (4.10) *are given in Corollaries 1 and 2, where*

$$\begin{aligned} &(a) \quad g_{i,r} = \int_0^{\ell} P^*_i(\theta)\, P^*_r(\theta)\, d\theta. \\ &(b) \quad h_{i,r} = \int_0^{\ell} \frac{d\, P^*_i(\theta)}{d\,\theta}\, P^*_r(\theta)\, d\theta. \end{aligned} \tag{4.14}$$

5. Convergence and error analysis

This section is interested in analyzing in detail the expansion in terms of $P_i^*(\theta)$. Thus, some important lemmas and theorems are presented and proved.

Lemma 1. *Let $\theta \in [0, \ell]$, $\quad \ell > 0$. This inequality holds:*

$$|P_i^*(\theta)| \leq (\ell\, \varepsilon)^{i+2}, \qquad \forall\, i \geq 0, \tag{5.1}$$

where $\varepsilon = \sqrt{2} + 1$.

Proof. The application of Eq (2.4) enables us to write

$$|P_i^*(\theta)| = \sum_{r=0}^{i} |\lambda_{i-r}|\, |\theta^r|. \tag{5.2}$$

Now, λ_{i-r} can be summed as

$$\lambda_{i-r} = \frac{\left(\sqrt{2}+1\right)^{i-r+1}}{2\sqrt{2}}. \tag{5.3}$$

Therefore, $|P_i^*(\theta)|$ is given by

$$|P_i^*(\theta)| = \sum_{r=0}^{i} \left| \frac{\left(\sqrt{2}+1\right)^{i-r+1}}{2\sqrt{2}} \right| |\theta^r|. \tag{5.4}$$

The previous equation can be summed and written after using $|\theta^r| \leq \ell^r$ as

$$|P_i^*(\theta)| = \frac{\left(\sqrt{2}+2\right)\left(\left(\sqrt{2}+1\right)^{i+1} - \ell^{i+1}\right)}{4\left(-\ell + \sqrt{2}+1\right)}. \tag{5.5}$$

At the end, using the following estimation:

$$\frac{\left(\sqrt{2}+2\right)\left(\left(\sqrt{2}+1\right)^{i+1} - \ell^{i+1}\right)}{4\left(-\ell + \sqrt{2}+1\right)} \leq (\ell\, \varepsilon)^{i+2}, \qquad \forall\, i \geq 0, \tag{5.6}$$

where $\varepsilon = \sqrt{2} + 1$, we get the desired result.

Lemma 2. *Let $g(\theta)$ be an infinitely differentiable function at the origin that may be expressed as*

$$g(\theta) = \sum_{n=0}^{\infty} \sum_{s=n}^{\infty} \frac{g^s(0)\, \mu_{n,s}}{s!}\, P_n^*(\theta), \tag{5.7}$$

where $\mu_{n,s}$ is defined in Eq (2.7).

Proof. First, write $g(\theta)$ as

$$g(\theta) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!}\,\theta^n. \tag{5.8}$$

Thanks to the inversion formula (2.6), the previous expansion turns into

$$g(\theta) = \sum_{n=0}^{\infty}\sum_{r=0}^{n} \frac{g^{(n)}(0)\,\mu_{r,n}}{n!}\,\mathrm{P}^*_r(\theta), \tag{5.9}$$

which can be transformed again into the following form, based on some algebraic computations:

$$g(\theta) = \sum_{n=0}^{\infty}\sum_{s=n}^{\infty} \frac{g^{s}(0)\,\mu_{n,s}}{s!}\,\mathrm{P}^*_n(\theta). \tag{5.10}$$

This ends the proof.

Theorem 5. *If* $g(\theta)$ *is defined on* $[0,\ell]$, *and* $|g^{(i)}(0)| \le \lambda^i,\ i>0$, *where* $\lambda>0$, *and* $g(\theta) = \sum_{i=0}^{\infty} \hat{g}_i\,\mathrm{P}^*_i(\theta)$, *then we get*

$$|\hat{g}_i| \le \frac{2\,e^{\lambda}\lambda^i}{i!}, \quad \forall i>0. \tag{5.11}$$

Furthermore, the series is absolutely convergent.

Proof. Using Lemma 2, we can write

$$|\hat{g}_i| = \left|\sum_{s=i}^{\infty} \frac{f^s(0)\,\mu_{i,s}}{s!}\right| = \sum_{s=i}^{\infty} \frac{|f^s(0)|\,\left|\mu_{i,s}\right|}{s!}. \tag{5.12}$$

The application of the assumption $|g^{(i)}(0)| \le \lambda^i, i>0$ enables us to write

$$|\hat{g}_i| \le \sum_{s=i}^{\infty} \frac{2\,\lambda^s}{s!} = \frac{2\,e^{\lambda}((i-1)!-\Gamma(i,\lambda))}{(i-1)!}, \tag{5.13}$$

where $\Gamma(i,\lambda)$ is the incomplete gamma function.
If we make use of the following inequality:

$$\frac{2\,e^{\lambda}((i-1)!-\Gamma(i,\lambda))}{(i-1)!} \le \frac{2\,e^{\lambda}\,\lambda^i}{i!}, \quad \forall\ i>0, \tag{5.14}$$

then the inequality in (5.13) leads to

$$|\hat{g}_i| \le \frac{2\,e^{\lambda}\lambda^i}{i!}. \tag{5.15}$$

To prove the second part of this theorem, using inequalities (5.1) and (5.11), we can write

$$\begin{aligned}\left|\sum_{i=0}^{\infty} \hat{g}_i\,\mathrm{P}_i(\theta)\right| &= \sum_{i=0}^{\infty} |\hat{g}_i|\,\left|\mathrm{P}^*_i(\theta)\right| \\ &\le \sum_{i=0}^{\infty} \frac{2\,e^{\lambda}\lambda^i\,(\ell\,\varepsilon)^{i+2}}{i!} \\ &= 2\,\varepsilon^2\,\ell^2\,e^{\lambda\,(1+\varepsilon\,\ell)},\end{aligned} \tag{5.16}$$

so the series converges absolutely.

Theorem 6. *Let* $\mathrm{Z}^{\mathcal{N}}(\theta,t) = f_1(\theta)\, f_2(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_{ij}\, \mathrm{P}_i^*(\theta)\, \mathrm{P}_j^*(t)$*, with* $|f_1^{(i)}(0)| \le Q_1^i$ *and* $|f_2^{(i)}(0)| \le Q_2^i$*, where* Q_1 *and* Q_2 *are positive constants. One has*

$$|c_{ij}| \le \frac{4\, e^{Q_1+Q_2} Q_1^i\, Q_2^j}{i!\, j!}. \tag{5.17}$$

Moreover, the series converges absolutely.

Proof. The application of Lemma 2, along with the assumption $\mathrm{Z}^{\mathcal{N}}(\theta,t) = f_1(\theta)\, f_2(t)$, enables us to write

$$c_{ij} = \sum_{p=i}^{\infty}\sum_{q=j}^{\infty} \frac{f_1^p(0)\, f_2^q(0)\, \mu_{j,q}\, \mu_{i,p}}{p!\, q!}. \tag{5.18}$$

If we make use of the assumption $|f_1^{(i)}(0)| \le Q_1^i$ and $|f_2^{(i)}(0)| \le Q_2^i$, then we can write

$$|c_{ij}| \le \sum_{p=i}^{\infty} \frac{Q_1^p\, \mu_{i,p}}{p!} \times \sum_{q=j}^{\infty} \frac{Q_2^q\, \mu_{j,q}}{q!}. \tag{5.19}$$

Ultimately, by replicating the methods analogous to the proof of Theorem 5, we obtain

$$|c_{ij}| \le \frac{4\, e^{Q_1+Q_2} Q_1^i\, Q_2^j}{i!\, j!}. \tag{5.20}$$

Theorem 7. *The following upper estimation holds if* $\mathrm{Z}^{\mathcal{N}}(\theta,t)$ *satisfies the assumptions of Theorem 6.*

$$|\mathrm{Z}(\theta,t) - \mathrm{Z}^{\mathcal{N}}(\theta,t)| < \frac{4\, \ell^2\, \tau^2\, \varepsilon^{\mathcal{N}+4}\, e^{Q_1(1+\varepsilon\ell)}\, e^{Q_2(1+\varepsilon\tau)} \left[(\ell\, Q_1)^{\mathcal{N}} + (\tau\, Q_2)^{\mathcal{N}}\right]}{\mathcal{N}!}. \tag{5.21}$$

Proof. The application of definitions $\mathrm{Z}(\theta,t)$ and $\mathrm{Z}^{\mathcal{N}}(\theta,t)$ enables us to write

$$\begin{aligned} |\mathrm{Z}(\theta,t) - \mathrm{Z}^{\mathcal{N}}(\theta,t)| &= \left| \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_{ij}\, \mathrm{P}_i^*(\theta)\, \mathrm{P}_j^*(t) - \sum_{i=0}^{\mathcal{N}}\sum_{j=0}^{\mathcal{N}} c_{ij}\, \mathrm{P}_i^*(\theta)\, \mathrm{P}_j^*(t) \right| \\ &\le \left| \sum_{i=0}^{\mathcal{N}}\sum_{j=\mathcal{N}+1}^{\infty} c_{ij}\, \mathrm{P}_i^*(\theta)\, \mathrm{P}_j^*(t) \right| + \left| \sum_{i=\mathcal{N}+1}^{\infty}\sum_{j=0}^{\infty} c_{ij}\, \mathrm{P}_i^*(\theta)\, \mathrm{P}_j^*(t) \right|. \end{aligned} \tag{5.22}$$

With the help of Theorem 6 along with Lemma 1, the following estimations may be obtained.

$$\begin{aligned}
\sum_{i=0}^{\mathcal{N}} \frac{2\,e^{Q_1} Q_1^i\,(\ell\,\varepsilon)^{i+2}}{i!} &= \frac{2\,\varepsilon^2 \ell^2 e^{\lambda\varepsilon\ell+Q_1}\Gamma(N+1, Q_1\varepsilon\ell)}{\mathcal{N}!} < 2\,\varepsilon^2\,\ell^2\,e^{Q_1\,(1+\varepsilon\,\ell)},\\
\sum_{j=\mathcal{N}+1}^{\infty} \frac{2\,e^{Q_2} Q_2^j\,(\tau\,\varepsilon)^{j+2}}{j!} &= \frac{2(\mathcal{N}+1)Q_2^{\mathcal{N}} e^{Q_2\varepsilon\tau+Q_2}(\varepsilon\tau)^{\mathcal{N}+2}(Q_2\varepsilon\tau)^{-\mathcal{N}}(\mathcal{N}!-\Gamma(\mathcal{N}+1, Q_2\varepsilon\tau))}{(\mathcal{N}+1)!}\\
&< \frac{2Q_2^{\mathcal{N}} e^{Q_2(1+\varepsilon\tau)}(\varepsilon\tau)^{\mathcal{N}+2}}{\mathcal{N}!},\\
\sum_{i=\mathcal{N}+1}^{\infty} \frac{2\,e^{Q_1} Q_1^i\,(\ell\,\varepsilon)^{i+2}}{i!} &= \frac{2(\mathcal{N}+1)Q_1^{\mathcal{N}} e^{Q_1\varepsilon\ell+Q_1}(\varepsilon\ell)^{\mathcal{N}+2}(Q_1\varepsilon\ell)^{-\mathcal{N}}(\mathcal{N}!-\Gamma(\mathcal{N}+1, Q_1\varepsilon\ell))}{(\mathcal{N}+1)!}\\
&< \frac{2Q_1^{\mathcal{N}} e^{Q_1(1+\varepsilon\ell)}(\varepsilon\ell)^{\mathcal{N}+2}}{\mathcal{N}!},\\
\sum_{j=0}^{\infty} \frac{2\,e^{Q_2} Q_2^j\,(\tau\,\varepsilon)^{j+2}}{j!} &= 2\,\varepsilon^2\,\tau^2\,e^{Q_2\,(1+\varepsilon\,\tau)}.
\end{aligned} \tag{5.23}$$

Using the estimations in (5.23), we get the following estimation:

$$|\mathrm{Z}(\theta,t) - \mathrm{Z}^{\mathcal{N}}(\theta,t)| < \frac{4\,\ell^2\,\tau^2\,\varepsilon^{\mathcal{N}+4}\,e^{Q_1(1+\varepsilon\ell)}\,e^{Q_2(1+\varepsilon\tau)}\left[(\ell\,Q_1)^{\mathcal{N}} + (\tau\,Q_2)^{\mathcal{N}}\right]}{\mathcal{N}!}, \tag{5.24}$$

which is the desired result.

6. Some illustrative examples

This section presents three illustrative examples to ensure the proposed algorithm's applicability and accuracy.

Test Problem 1. *[43, 44]. Consider the following equation:*

$$\partial_t \mathrm{Z}(\theta,t) + \partial_\theta \mathrm{Z}(\theta,t) + \mathrm{Z}(\theta,t) = (\theta-t)^2, \quad 0<\theta,\ t<1, \tag{6.1}$$

with the following conditions:

$$\mathrm{Z}(\theta,0) = \theta^2, \quad \mathrm{Z}(0,t) = t^2, \quad 0<\theta, t<1, \tag{6.2}$$

whose exact solution is given as: $\mathrm{Z}(\theta,t) = (\theta-t)^2$.
For $\mathcal{N}=2$*, applying Theorem 4 yields*

$$\begin{aligned}
\mathcal{G} &= \begin{pmatrix} 1 & \frac{5}{2} \\ \frac{5}{2} & \frac{19}{3} \\ \frac{19}{3} & \frac{193}{12} \end{pmatrix},\\
\mathcal{H} &= \begin{pmatrix} 0 & 0 \\ 1 & \frac{5}{2} \\ 3 & \frac{23}{3} \end{pmatrix}.
\end{aligned} \tag{6.3}$$

Therefore, Eq (4.11) *can be rewritten as*

$$\begin{pmatrix} 1 & \frac{5}{2} & \frac{19}{3} \\ \frac{5}{2} & \frac{19}{3} & \frac{193}{12} \end{pmatrix}\begin{pmatrix} c_{00} & c_{01} & c_{02} \\ c_{10} & c_{11} & c_{12} \\ c_{20} & c_{21} & c_{22} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & \frac{5}{2} \\ 3 & \frac{23}{3} \end{pmatrix}$$
$$+\begin{pmatrix} 0 & 1 & 3 \\ 0 & \frac{5}{2} & \frac{23}{3} \end{pmatrix}\begin{pmatrix} c_{00} & c_{01} & c_{02} \\ c_{10} & c_{11} & c_{12} \\ c_{20} & c_{21} & c_{22} \end{pmatrix}\begin{pmatrix} 1 & \frac{5}{2} \\ \frac{5}{2} & \frac{19}{3} \\ \frac{19}{3} & \frac{193}{12} \end{pmatrix} \tag{6.4}$$
$$+\begin{pmatrix} 1 & \frac{5}{2} & \frac{19}{3} \\ \frac{5}{2} & \frac{19}{3} & \frac{193}{12} \end{pmatrix}\begin{pmatrix} c_{00} & c_{01} & c_{02} \\ c_{10} & c_{11} & c_{12} \\ c_{20} & c_{21} & c_{22} \end{pmatrix}\begin{pmatrix} 1 & \frac{5}{2} \\ \frac{5}{2} & \frac{19}{3} \\ \frac{19}{3} & \frac{193}{12} \end{pmatrix} = \begin{pmatrix} \frac{1}{6} & \frac{5}{12} \\ \frac{5}{12} & \frac{37}{36} \end{pmatrix}.$$

Moreover, conditions (4.12) *and* (4.13)*, after putting* $Z_0(\theta) = \theta^2$ *and* $Z_1(t) = t^2$ *enable us to write*

$$\begin{pmatrix} c_{00} + 2c_{01} + 5c_{02} + \frac{5}{2}c_{10} + 5c_{11} + \frac{25}{2}c_{12} + \frac{19}{3}c_{20} + \frac{38}{3}c_{21} + \frac{95}{3}c_{22} \\ \frac{5}{2}c_{00} + 5c_{01} + \frac{25}{2}c_{02} + \frac{19}{3}c_{10} + \frac{38}{3}c_{11} + \frac{95}{3}c_{12} + \frac{193}{12}c_{20} + \frac{193}{6}c_{21} + \frac{965}{12}c_{22} \\ \frac{19}{3}c_{00} + \frac{38}{3}c_{01} + \frac{95}{3}c_{02} + \frac{193}{12}c_{10} + \frac{193}{6}c_{11} + \frac{965}{12}c_{12} + \frac{613}{15}c_{20} + \frac{1226}{15}c_{21} + \frac{613}{3}c_{22} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{11}{12} \\ \frac{71}{30} \end{pmatrix}, \tag{6.5}$$

and

$$\begin{pmatrix} c_{00} + \frac{5}{2}c_{01} + \frac{19}{3}c_{02} + 2c_{10} + 5c_{11} + \frac{38}{3}c_{12} + 5c_{20} + \frac{25}{2}c_{21} + \frac{95}{3}c_{22} \\ \frac{5}{2}c_{00} + \frac{19}{3}c_{01} + \frac{193}{12}c_{02} + 5c_{10} + \frac{38}{3}c_{11} + \frac{193}{6}c_{12} + \frac{25}{2}c_{20} + \frac{95}{3}c_{21} + \frac{965}{12}c_{22} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{11}{12} \end{pmatrix}. \tag{6.6}$$

Now, we get a system of nine equations from (6.4)*,* (6.5)*, and* (6.6)*, and this system can be solved exactly to give:*

$$c_{00} \to -10, c_{01} \to 2, c_{02} \to 1, c_{10} \to 2, c_{11} \to -2, c_{12} \to 0, c_{20} \to 1, c_{21} \to 0, c_{22} \to 0,$$

and, therefore: $Z_2(\theta, t) = (\theta - t)^2$, *which is the exact solution.*

Test Problem 2. *[44] Consider the following equation:*

$$\partial_t Z(\theta, t) + \partial_\theta Z(\theta, t) + Z(\theta, t) = \cos(\theta + t) - 2\sin(\theta + t), \quad 0 < \theta,\ t < 1, \tag{6.7}$$

with the following conditions:

$$Z(\theta, 0) = \cos(\theta), \quad Z(0, t) = \cos(t), \quad 0 < \theta, t < 1, \tag{6.8}$$

whose exact solution is given as: $Z(\theta, t) = \cos(\theta + t)$.

In Table 1, we give a comparison of absolute errors (AEs) between our method and the method in [44] at various θ *and t. Table 2 shows the maximum absolute errors (MAEs),* L_∞*-errors, and* L_2*-errors at different values of* $\mathcal{N}$*. Figure 1 shows the AEs (left) and approximate solution (right) at* $\mathcal{N} = 6$*. Figure 2 shows the AEs at* $\theta = t$ *and different values of* $\mathcal{N}$*. Finally, Figure 3 shows the stability* $|Z_{\mathcal{N}+1}(\theta, 0.3) - Z_{\mathcal{N}}(\theta, 0.3)|$ *at different values of* $\mathcal{N}$*.*

Table 1. Comparison of the AEs for Example 2.

(θ, t)	Legendre wavelet at $M = 6$ [44]	Chebyshev wavelet at $M = 6$ [44]	Present method at $\mathcal{N} = 6$
(0.1,0.1)	9.12×10^{-5}	4.53×10^{-4}	1.74541×10^{-7}
(0.2,0.2)	3.36×10^{-5}	2.07×10^{-4}	1.80389×10^{-7}
(0.3,0.3)	1.86×10^{-5}	6.92×10^{-5}	2.69712×10^{-7}
(0.4,0.4)	2.42×10^{-5}	1.39×10^{-4}	2.95508×10^{-7}
(0.5,0.5)	8.30×10^{-5}	2.64×10^{-4}	2.69978×10^{-7}
(0.6,0.6)	1.38×10^{-4}	7.62×10^{-5}	2.67529×10^{-7}
(0.7,0.7)	9.90×10^{-5}	2.10×10^{-5}	3.3939×10^{-7}
(0.8,0.8)	1.20×10^{-6}	2.01×10^{-5}	4.73848×10^{-7}
(0.9,0.9)	9.00×10^{-5}	3.00×10^{-4}	5.27481×10^{-7}

Table 2. Errors of Example 2.

$\mathcal{N}$	2	3	4	5	6
MAEs	1.78021×10^{-2}	6.68349×10^{-4}	4.37397×10^{-5}	2.65074×10^{-6}	6.09552×10^{-7}
L_∞-errors	1.80157×10^{-2}	6.98176×10^{-4}	4.39709×10^{-5}	1.67709×10^{-6}	6.09552×10^{-7}
L_2-errors	7.18447×10^{-3}	3.14754×10^{-4}	1.89696×10^{-5}	7.28894×10^{-7}	3.05884×10^{-7}

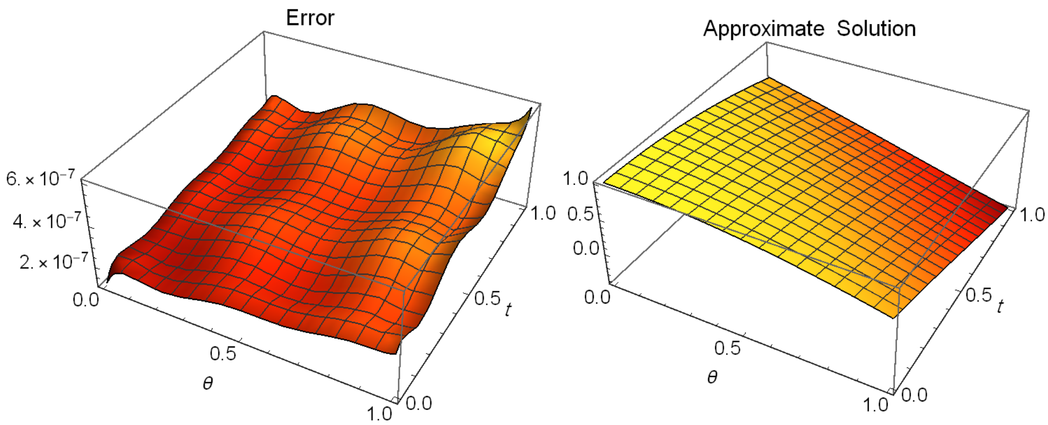

Figure 1. The AEs (left) and approximate solution (right) of Example 2 at $\mathcal{N} = 6$.

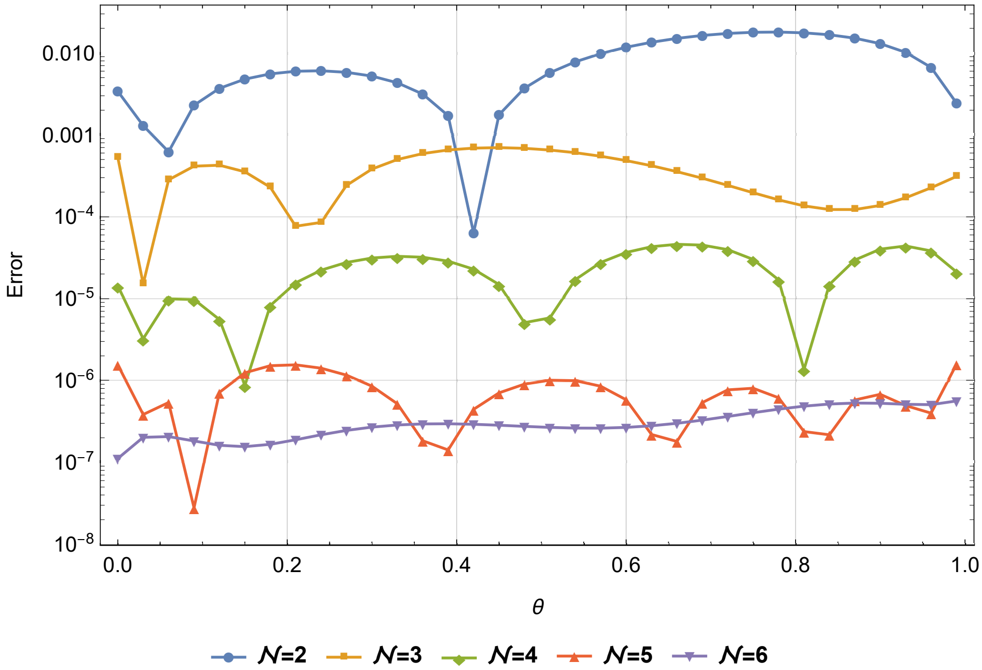

Figure 2. The AEs of Example 2 at $\theta = t$ and different values of $\mathcal{N}$.

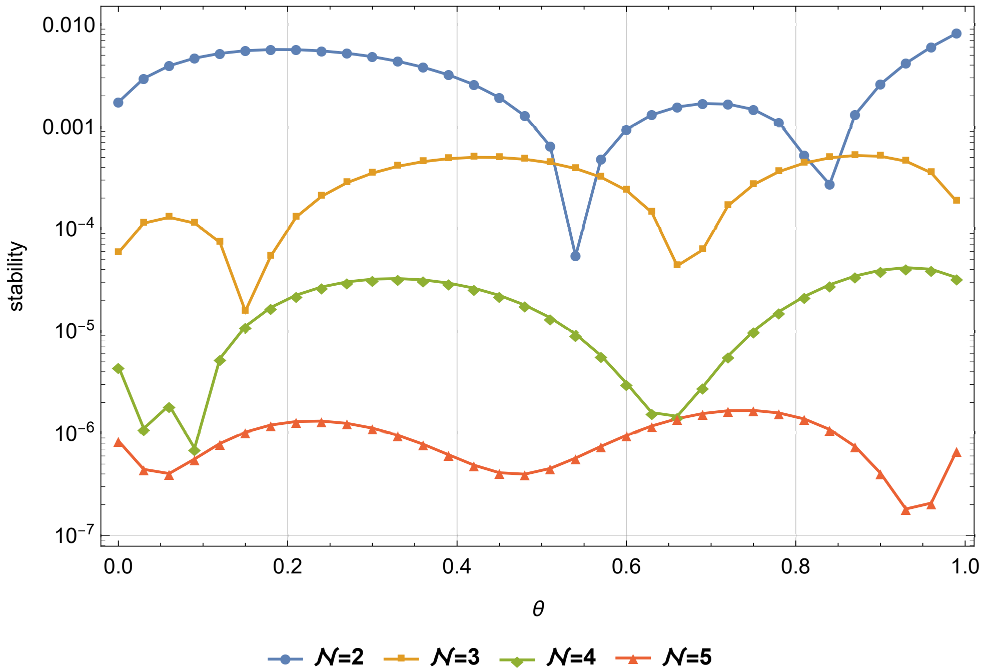

Figure 3. Stability $|Z_{\mathcal{N}+1}(\theta, 0.3) - Z_{\mathcal{N}}(\theta, 0.3)|$ of Example 2.

Test Problem 3. *[45] Consider the following equation:*

$$\partial_t Z(\theta, t) + \partial_\theta Z(\theta, t) + Z(\theta, t) = -\sqrt{2}\, e^{-\sqrt{2}\, t - \theta}, \quad 0 < \theta,\ t < 1, \tag{6.9}$$

with the following conditions:

$$Z(\theta,0)=e^{-\theta},\quad Z(0,t)=e^{-\sqrt{2}t},\quad 0<\theta,t<1, \tag{6.10}$$

whose exact solution is given as: $Z(\theta,t)=e^{-(\sqrt{2}t+\theta)}$.

In Table 3, we give a comparison of AEs between our method and the method in [45] at different values of θ *when* $t=0.1$. *Also, Table 4 presents a comparison of AEs between our method and the method in [45] at different values of* θ *when* $t=0.5$. *Table 5 shows the MAEs,* L_∞*-errors, and* L_2*-errors at different values of* $\mathcal{N}$. *Figure 4 shows the AEs (left) and approximate solution (right) at* $\mathcal{N}=6$. *Figure 5 shows the AEs at different values of* $\mathcal{N}$ *and* θ *when* $t=0.1$. *Finally, Figure 6 shows the AEs at different values of* $\mathcal{N}$ *and* θ *when* $t=0.9$.

Table 3. Comparison of the AEs for Example 3 at $t=0.1$.

θ	Method in [45] at $M=N=16$	Present method at $\mathcal{N}=6$
0.1	2.84×10^{-7}	2.7611×10^{-7}
0.2	8.79×10^{-6}	4.1743×10^{-7}
0.3	1.20×10^{-5}	4.40967×10^{-7}
0.4	1.12×10^{-5}	3.70131×10^{-7}
0.5	7.95×10^{-6}	3.0252×10^{-7}
0.6	3.29×10^{-6}	3.26524×10^{-7}
0.7	1.78×10^{-6}	4.40008×10^{-7}
0.8	6.53×10^{-6}	5.26572×10^{-7}
0.9	1.04×10^{-5}	4.39639×10^{-7}

Table 4. Comparison of the AEs for Example 3 at $t=0.5$.

θ	Method in [45] at $M=N=16$	Present method at $\mathcal{N}=6$
0.1	8.95×10^{-6}	2.82859×10^{-7}
0.2	4.10×10^{-6}	3.28433×10^{-7}
0.3	1.39×10^{-5}	5.28007×10^{-7}
0.4	2.07×10^{-5}	7.55095×10^{-7}
0.5	2.47×10^{-5}	8.30558×10^{-7}
0.6	2.62×10^{-5}	6.76593×10^{-7}
0.7	2.55×10^{-5}	3.66795×10^{-7}
0.8	2.31×10^{-5}	1.03841×10^{-7}
0.9	1.93×10^{-5}	1.53321×10^{-7}

Table 5. Errors of Example 3.

$\mathcal{N}$	2	3	4	5	6
MAEs	1.05726×10^{-2}	9.47791×10^{-4}	6.56318×10^{-5}	3.56142×10^{-6}	1.39114×10^{-6}
L_∞-errors	1.13462×10^{-2}	9.94362×10^{-4}	4.70445×10^{-5}	3.71657×10^{-6}	1.63616×10^{-6}
L_2-errors	6.10254×10^{-3}	5.10838×10^{-4}	3.37253×10^{-5}	1.87435×10^{-6}	5.93552×10^{-7}

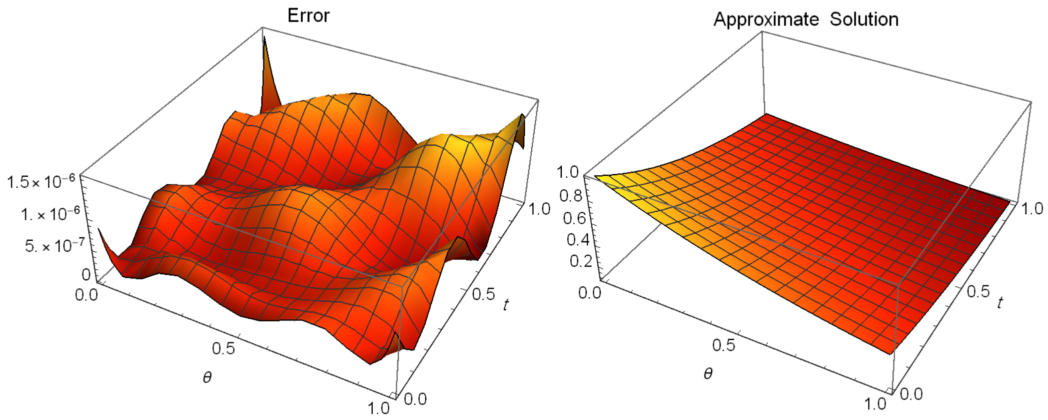

Figure 4. AEs (left) and approximate solution (right) of Example 3 at $\mathcal{N} = 6$.

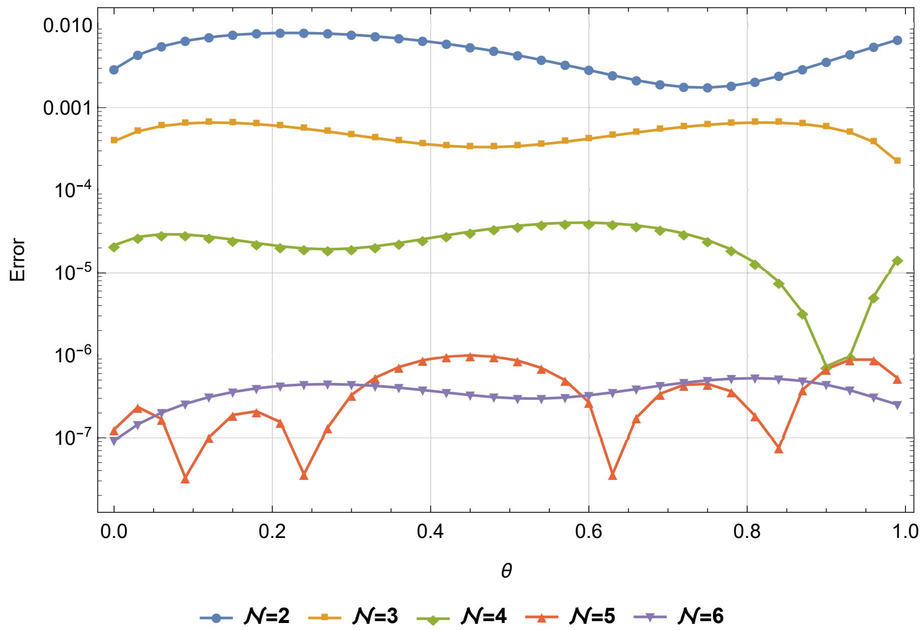

Figure 5. The AEs of Example 3 at $t = 0.1$ and different values of $\mathcal{N}$.

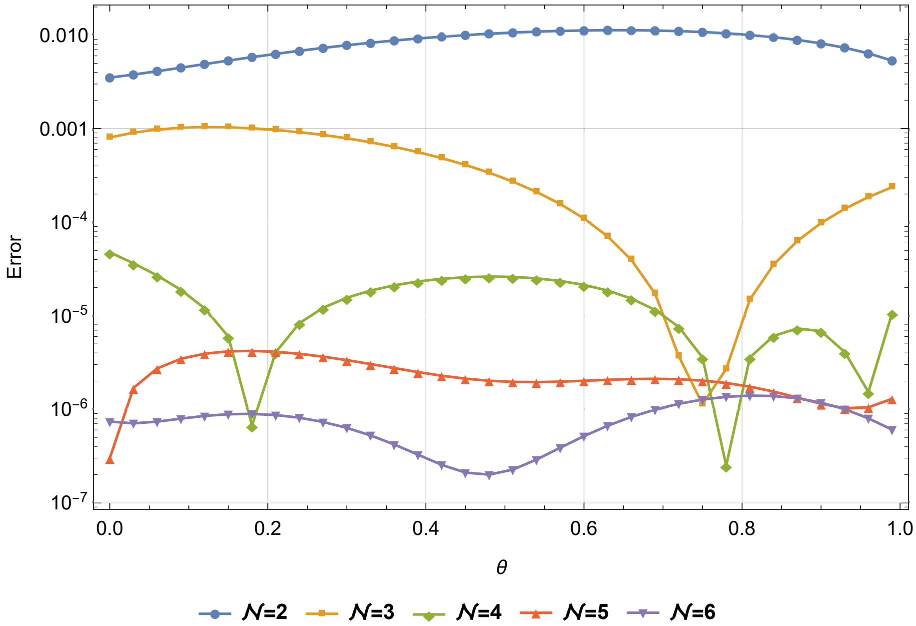

Figure 6. The AEs of Example 3 at $t = 0.9$ and different values of $\mathcal{N}$.

Test Problem 4. *Consider the following equation:*

$$\partial_t Z(\theta, t) + \partial_\theta Z(\theta, t) + Z(\theta, t) = f(\theta, t), \quad 0 < \theta,\ t < 1, \tag{6.11}$$

with the following conditions:

$$Z(\theta, 0) = |\theta|, \quad Z(0, t) = |t|, \quad 0 < \theta, t < 1, \tag{6.12}$$

where $f(\theta, t)$ is chosen such that the exact solution of this problem is $Z(\theta, t) = |\theta - t|$.
Figure 7 shows the AEs (left) and approximate solution (right) at $\mathcal{N} = 4$. Table 6 shows the L_∞-errors and L_2-errors at different values of t when $\mathcal{N} = 4$.

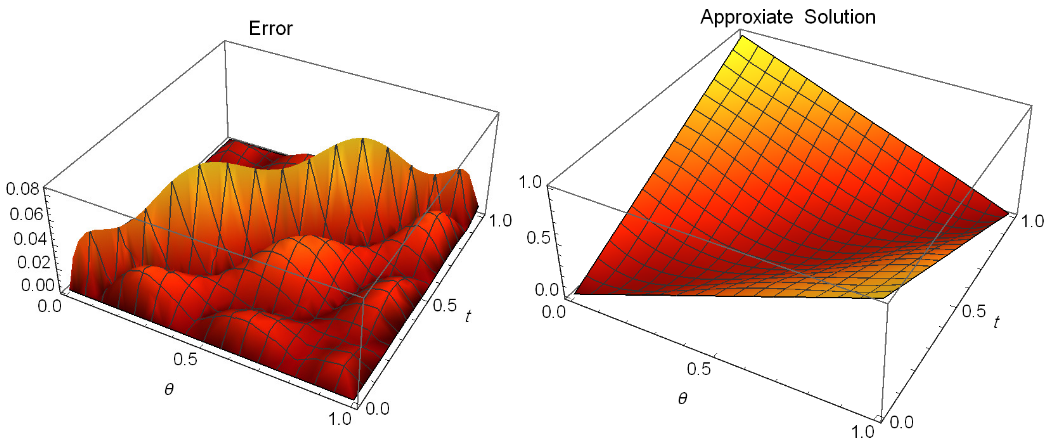

Figure 7. The AEs (left) and approximate solution (right) of Example 4 at $\mathcal{N} = 4$.

Table 6. Errors of Example 4 at $\mathcal{N} = 4$.

t	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
L_∞-errors	2.7×10^{-2}	2.4×10^{-2}	3.0×10^{-2}	3.6×10^{-2}	2.4×10^{-2}	3.6×10^{-2}	2.9×10^{-2}	2.5×10^{-2}	2.8×10^{-2}
L_2-errors	1.6×10^{-2}	1.5×10^{-2}	2.6×10^{-2}	2.5×10^{-2}	2.0×10^{-2}	2.5×10^{-2}	2.6×10^{-2}	1.5×10^{-2}	1.6×10^{-2}

Remark 5. *According to the error analysis section, the solutions need to be smooth enough. For a non-smooth solution, it is better to be constructed fractional-order Pell coefficient polynomials as basis functions.*

7. Concluding remarks

This work was confined to introducing new polynomials whose coefficients are the celebrated Pell numbers. New formulas regarding these polynomials, such as moment, linearization, and derivative formulas of these polynomials, have been found and used to design a numerical algorithm to treat the linear hyperbolic first-order partial differential equations. The spectral tau method was utilized for such a purpose. The numerical results demonstrated the high accuracy of our proposed numerical algorithm. The use of these polynomials in numerical analysis is novel to the best of our knowledge. In future work, we aim to introduce other polynomials that generalize our introduced polynomials. In addition, we believe that these polynomials can be utilized to solve other types of differential equations.

Author contributions

Waleed Mohamed Abd-Elhameed: Conceptualization, Methodology, Validation, Formal analysis, Investigation, Writing—original draft preparation, Writing—review and editing, Supervision; Mohamed A. Abdelkawy: Methodology, Validation, Investigation, Funding acquisition; Omar Mazen Alqubori: Methodology, Validation, Investigation; Ahmed Gamal Atta: Conceptualization, Methodology, Software, Validation, Formal analysis, Investigation, Writing—original draft preparation, Writing—review and editing. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Funding

This work was supported and funded by the Deanship of Scientific Research at Imam Mohammad Ibn Saud Islamic University (IMSIU) (grant number IMSIU-DDRSP2502).

Conflict of interest

The authors declare there is no conflict of interest.

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