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Research article

Liouville-type theorems for positive solutions to $\Delta_{p_1,\cdots,p_r} v + f(v) = 0$ in $\mathbb{R}^m$

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Abstract: In this paper, we employed the Bernstein method to prove some Liouville-type theorems of the equation $\Delta_{p_1,\cdots,p_r} v + f(v) = 0$. Here, $\Delta_{p_1,\cdots,p_r} v := \operatorname{div}(\sum_{i=1}^r |\nabla v|^{p_i-2}\nabla v)$. This could be regarded as a natural generalization of the p-Laplacian and the (p,q)-Laplacian. As applications, we derived Liouville-type theorems of positive solutions to some generalized static Fisher-KPP equation, Allen-Cahn equation, static Newell-Whitehead equation, and Lichnerowicz equation.

Keywords: Liouville-type theorems; $(p_1,\cdots,p_r)$-Laplacian; nonlinear elliptic equations

1. Introduction

The p-Laplace equation serves as a fundamental model for nonlinear diffusion (e.g., in porous medium flow) and image processing (e.g., in TV denoising models). And, the Liouville theorem offers a benchmark for the qualitative analysis of solutions. As a nonlinear elliptic equation, the p-Laplace equation is studied in the context of Liouville-type theorems to classify bounded solutions (or those satisfying certain growth conditions) defined on the whole space or non-compact manifolds. The central question is: Are all such solutions constant? The classical result establishes that when $p = 2$ (the Laplace equation), any bounded harmonic function on the entire space must be constant (see [1]). For $p \neq 2$, the conclusion depends on both p and the spatial dimension n (see [2,3]).

In this paper, we consider the following elliptic equation:

$$\Delta_{p_1,\cdots,p_r} v + f(v) = 0 \qquad \text{in} \quad \mathbb{R}^m, \tag{1.1}$$

where $m \geq 1$ and the semilinear term f is sufficiently smooth and may be positive or negative. Here, $\Delta_{p_1,\cdots,p_r} v := \operatorname{div}(\sum_{i=1}^r |\nabla v|^{p_i-2}\nabla v)$, where $r \in \mathbb{N}^+$, $p_i > 1$ for any $1 \leq i \leq r$, and $1 < p_1 < \cdots < p_r$. When $r = 1$, the $(p_1,\cdots,p_r)$-Laplacian reduces to the classical p-Laplacian (see [4,5]). When $r = 2$, the

$(p_1, \cdots, p_r)$-Laplacian reduces to the classical (p, q)-Laplacian (see [6–8]). Under certain conditions on f and p_i, we prove some Liouville-type theorems for solutions of (1.1) in $\mathbb{R}^m$.

It is well known that Liouville-type theorems of nonlinear elliptic equations are very important in the study of nonlinear PDEs (see [9–11]). Furthermore, the research of Liouville-type theorems has also been widely pursued (see [12–14]). With the aim to investigate the problem (1.1), we mainly focus on four papers.

Gidas and Spruck [15] established Liouville-type theorems of positive solutions to the Lane-Emden equation

$$\Delta v + v^{\alpha} = 0 \tag{1.2}$$

in $\mathbb{R}^m$ when $m > 2$ and $1 \leq \alpha < \frac{m+2}{m-2}$. This finding extends the classical Liouville theorem on harmonic functions in Euclidean space.

Serrin and Zou [16] established the Liouville-type theorems of positive solutions in $\mathbb{R}^m$ for the equation

$$\Delta_p v + f(v) = 0 \tag{1.3}$$

under the conditions $1 < p < m$, $f \geq 0$ is subcritical, and there is $q > p$ such that $f(t) \geq t^{q-1}$ for t large. Recall that $f \geq 0$ is subcritical if there is $0 < \beta < \frac{mp}{m-p} - 1$ such that

$$f'(t) \leq \beta \frac{f(t)}{t}, \quad \forall \ t > 0. \tag{1.4}$$

This might be considered an extension of the previously mentioned study by Gidas and Spruck [15].

McCoy [17] investigated the equation

$$\Delta v + f(v) = 0, \tag{1.5}$$

in $\mathbb{R}^m$ $(m \geq 2)$ and proved that if the inequality

$$f'(t) \leq \frac{m+1}{m-1} \frac{f(t)}{t}, \quad \forall \ t > 0.$$

holds, then any positive solution to (1.5) is constant.

Cuccu et al. [18] studied equation (1.3) in $\mathbb{R}^m$ $(m \geq 2)$, where the function f may change sign. They proved that when

$$f'(t) \leq (p-1) \frac{m+1}{m-1} \frac{f(t)}{t}, \quad \forall \ t > 0, \tag{1.6}$$

holds, then any positive solution to (1.3) is constant.

In this paper, we study some Liouville-type theorems of (1.1) in $\mathbb{R}^m$ $(m \geq 2)$. Our aim is to extend some work of [16–18] to the general $(p_1, \cdots, p_r)$-Laplacian.

Our main theorem is stated as follows

Theorem 1.1. *Assume that p_i satisfies*

$$1 < p_1 < \cdots < p_r < \min\left\{\frac{5}{2}, \frac{m}{m-1}(p_1 - 1)^2 - (p_1 - 2)(p_1 - 3) + 1\right\} \tag{1.7}$$

or

$$\max\left\{\frac{5}{2}, 1 + \sqrt{\frac{m-1}{m}\left((p_r - 1) + (p_r - 2)(p_r - 3)\right)}\right\} < p_1 < \cdots < p_r. \tag{1.8}$$

If one of the following conditions holds,

(i) f *is nonnegative and*

$$f'(t) \leq (p_1 - 1)\frac{m+1}{m-1}\frac{f(t)}{t}, \tag{1.9}$$

or

(ii) f *is nonpositive and*

$$f'(t) \leq (p_r - 1)\frac{m+1}{m-1}\frac{f(t)}{t}, \tag{1.10}$$

then any positive solution to (1.1) *in* $\mathbb{R}^m$ *is constant.*

Remark 1.1. 1) In [16], f is required to be non-negative. In our theorem, we do not require this constraint and f can be negative. Furthermore, we extend the p-Laplacian operator to a more general $(p_1, \cdots, p_r)$-Laplacian operator.

2) When $r = 1$ and $p_1 = 2$, the Liouville-type theorem established by McCoy [17] can be recovered by the above Theorem 1.1.

3) When $r = 1$ and $p_1 = p$, Cuccu-Mohammed-Porru's Liouville-type theorems (see [18]) can be recovered by the above Theorem 1.1.

Corollary 1.2. *Let v be a positive solution of*

$$\Delta_{p_1,\cdots,p_r} v - \sum_{i=1}^{k} a_i v^{\alpha_i} = 0, \tag{1.11}$$

where $a_i > 0$. Suppose p_i satisfy conditions (1.7) *or* (1.8). *If* $\min\limits_{1\leq i\leq k}\{\alpha_i\} \geq (p_r - 1)\frac{m+1}{m-1}$, *then v is constant.*

Corollary 1.3. *Let v be a positive solution of*

$$\Delta_{p_1,\cdots,p_r} v + \sum_{i=1}^{k} a_i v^{\alpha_i} = 0, \tag{1.12}$$

where $a_i > 0$. Suppose p_i satisfy conditions (1.7) *or* (1.8). *If* $\max\limits_{1\leq i\leq k}\{\alpha_i\} \leq (p_1 - 1)\frac{m+1}{m-1}$, *then v is constant.*

Corollary 1.4. *Let v be a positive solution of*

$$\Delta_{p_1,\cdots,p_r} v + \sum_{i=1}^{k} a_i v^{\alpha_i} - \sum_{j=1}^{l} b_j v^{\beta_j} = 0, \tag{1.13}$$

where $a_i > 0$, $b_j > 0$, and α_i and β_j are constants. Suppose p_i satisfy conditions (1.7) *or* (1.8). *If one of the following conditions holds,*

(i) $\sum_{i=1}^{k} a_i v^{\alpha_i} - \sum_{j=1}^{l} b_j v^{\beta_j} \geq 0$ *and* $\max\limits_{1\leq i\leq k}\{\alpha_i\} \leq (p_1 - 1)\frac{m+1}{m-1} \leq \min\limits_{1\leq j\leq l}\{\beta_j\}$, *or*

(ii) $\sum_{i=1}^{k} a_i v^{\alpha_i} - \sum_{j=1}^{l} b_j v^{\beta_j} \leq 0$ *and* $\max\limits_{1\leq i\leq k}\{\alpha_i\} \leq (p_r - 1)\frac{m+1}{m-1} \leq \min\limits_{1\leq j\leq l}\{\beta_j\}$,

then v is constant.

Remark 1.2. Theorem 1.1 can apply to the following cases.

(i) When $k = l = 1$, $a_1 = b_1 > 0$, $\alpha_1 = 1, \beta_1 = 2$, and $p_1 = 2$ ($r = 1$), (1.13) becomes the static Fisher-KPP equation

$$\Delta v + a_1 v - a_1 v^2 = 0. \tag{1.14}$$

If one of the following conditions holds,
(•) $a_1 v - a_1 v^2 \geq 0$ and $m \geq 3$, or
(•) $a_1 v - a_1 v^2 \leq 0$ and $m \geq 3$,
then v is constant.

(ii) When $k = l = 1$, $a_1 = b_1 = \alpha_1 = 1, \beta_1 = 3$, and $p_1 = 2$ ($r = 1$), (1.13) becomes the Allen-Cahn equation

$$\Delta v + v - v^3 = 0. \tag{1.15}$$

If one of the following conditions holds,
(•) $v - v^3 \geq 0$ and $m \geq 2$, or
(•) $v - v^3 \leq 0$ and $m \geq 2$,
then v is constant.

(iii) When $k = l = 1$, $a_1 > 0$, $b_1 > 0$, $\alpha_1 = 1, \beta_1 = 3$, and $p_1 = 2$ ($r = 1$), (1.13) becomes the static Newell-Whitehead equation

$$\Delta v + a_1 v - b_1 v^3 = 0. \tag{1.16}$$

If one of the following conditions holds,
(•) $a_1 v - b_1 v^3 \geq 0$ and $m \geq 2$, or
(•) $a_1 v - b_1 v^3 \leq 0$ and $m \geq 2$,
then v is constant.

(iv) When $k = 2$, $l = 1$, $a_1 > 0$, $a_2 > 0$, $b_1 > 0$, $\alpha_1 = 1$, $\alpha_2 < 1, \beta_1 > 1$, and $p_1 = 2$ ($r = 1$), (1.13) becomes the Lichnerowicz equation

$$\Delta v + a_1 v + a_2 v^{\alpha_2} - b_1 v^{\beta_1} = 0. \tag{1.17}$$

If one of the following conditions holds,
(•) $a_1 v + a_2 v^{\alpha_2} - b_1 v^{\beta_1} \geq 0$ and $m \geq \frac{1+\beta_1}{\beta_1 - 1}$, or
(•) $a_1 v + a_2 v^{\alpha_2} - b_1 v^{\beta_1} \leq 0$ and $m \geq \frac{1+\beta_1}{\beta_1 - 1}$,
then v is constant.

The remainder of this paper is structured as follows. In Section 2, we establish some important lemmas. We employ the Bernstein method to prove Theorem 1.1 in Section 3. The proofs of Corollaries 1.2–1.4 are provided in Section 4.

2. Preliminary

To prove the main theorem, we require the following lemmas.

Lemma 2.1. *Suppose p_i satisfy conditions* (1.7) *or* (1.8). *Then, we have*

$$\sup_{t>0} \frac{\sum_{i=1}^{r} (p_i - 2)(p_i - 3)\, t^{p_i - 2}}{\sum_{i=1}^{r} t^{p_i - 2}} < \left(\frac{m}{m-1}\right)(p_1 - 1)^2 - (p_r - 1).$$

Proof. We note that

$$\frac{\sum_{i=1}^{r}(p_i-2)(p_i-3)t^{p_i-2}}{\sum_{i=1}^{r}t^{p_i-2}} \leq \max\{(p_i-2)(p_i-3), i=1,\cdots,r\}. \tag{2.1}$$

Then, we have

$$\begin{aligned}
&\frac{m}{m-1}(p_1-1)^2-(p_r-1)-\frac{\sum_{i=1}^{r}(p_i-2)(p_i-3)t^{p_i-2}}{\sum_{i=1}^{r}t^{p_i-2}}\\
\geq&\left(\frac{m}{m-1}\right)(p_1-1)^2-(p_r-1)-\max\{(p_i-2)(p_i-3), i=1,\cdots,r\}\\
>&0.
\end{aligned} \tag{2.2}$$

Through calculation, we obtain

$$\begin{aligned}
&(p_r-2)(p_r-3)\\
=&(p_r-p_1+p_1-2)(p_r-p_1+p_1-3)\\
=&(p_r-p_1)^2+(p_r-p_1)(2p_1-5)+(p_1-2)(p_1-3)\\
=&(p_r-p_1)(p_1+p_r-5)+(p_1-2)(p_1-3)\\
\geq&(p_r-p_1)(2p_1-5)+(p_1-2)(p_1-3).
\end{aligned} \tag{2.3}$$

Therefore, we can see that when $p_r > ... > p_1 \geq \frac{5}{2}$, we have

$$\max\{(p_i-2)(p_i-3), i=1,\cdots,r\}=(p_r-2)(p_r-3).$$

On the other hand, when $(p_1+p_r-5)\leq 2p_r-5\leq 0$, that is to say $1<p_1<\cdots<p_r\leq\frac{5}{2}$, we have

$$\max\{(p_i-2)(p_i-3), i=1,\cdots,r\}=(p_1-2)(p_1-3).$$

In conclusion, we can get

$$\max\{(p_i-2)(p_i-3), i=1,\cdots,r\}=\begin{cases}(p_r-2)(p_r-3) & \text{if } p_r>...>p_1\geq\frac{5}{2},\\ (p_1-2)(p_1-3) & \text{if } 1<p_1<\cdots<p_r\leq\frac{5}{2}.\end{cases} \tag{2.4}$$

Therefore, when

$$1<p_1<\cdots<p_r<\min\left\{\frac{5}{2},\frac{m}{m-1}(p_1-1)^2-(p_1-2)(p_1-3)+1\right\},$$

we have

$$\begin{aligned}
&\frac{m}{m-1}(p_1-1)^2-(p_r-1)-\frac{\sum_{i=1}^{r}(p_i-2)(p_i-3)t^{p_i-2}}{\sum_{i=1}^{r}t^{p_i-2}}\\
\geq&\left(\frac{m}{m-1}\right)(p_1-1)^2-(p_r-1)-(p_1-2)(p_1-3)\\
>&0,
\end{aligned} \tag{2.5}$$

whereas when

$$\max\left\{\frac{5}{2},1+\sqrt{\frac{m-1}{m}((p_r-1)+(p_r-2)(p_r-3))}\right\}<p_1<\cdots<p_r,$$

we have

$$
\begin{aligned}
&\frac{m}{m-1}(p_1-1)^2-(p_r-1)-\frac{\sum_{i=1}^{r}(p_i-2)(p_i-3)\,t^{p_i-2}}{\sum_{i=1}^{r}t^{p_i-2}}\\
\geq&\left(\frac{m}{m-1}\right)(p_1-1)^2-(p_r-1)-(p_r-2)(p_r-3)\\
>&0.
\end{aligned}
\tag{2.6}
$$

This completes the proof of Lemma 2.1. □

Lemma 2.2. *The $m\times m$ matrix*

$$\Theta:=I_m+\frac{\sum_{i=1}^{r}(p_i-2)|\nabla v|^{p_i-2}}{\sum_{i=1}^{r}|\nabla v|^{p_i-2}}|\nabla v|^{-2}dv\otimes dv$$

is positive definite. Here, I_m is the $m\times m$ identity matrix.

Proof. For any $\zeta\in\mathbb{R}^m\backslash\{0\}$, we have

$$\Theta(\zeta,\zeta):=|\zeta|^2+\frac{\sum_{i=1}^{r}(p_i-2)|\nabla v|^{p_i-2}}{\sum_{i=1}^{r}|\nabla v|^{p_i-2}}|\nabla v|^{-2}\langle\zeta,\nabla v\rangle^2.$$

(i) If $\frac{\sum_{i=1}^{r}(p_i-2)|\nabla v|^{p_i-2}}{\sum_{i=1}^{r}|\nabla v|^{p_i-2}}\geq 0$, we obtain $\Theta(\zeta,\zeta)\geq|\zeta|^2>0$.

(ii) If $-1<\frac{\sum_{i=1}^{r}(p_i-2)|\nabla v|^{p_i-2}}{\sum_{i=1}^{r}|\nabla v|^{p_i-2}}<0$, then we have $\langle\zeta,\nabla v\rangle\leq|\zeta||\nabla v|$, and thus

$$\frac{\langle\zeta,\nabla v\rangle^2}{|\nabla v|^2}\leq|\zeta|^2.$$

Consequently, we obtain

$$
\begin{aligned}
\Theta(\zeta,\zeta)&\geq|\zeta|^2+\frac{\sum_{i=1}^{r}(p_i-2)|\nabla v|^{p_i-2}}{\sum_{i=1}^{r}|\nabla v|^{p_i-2}}|\zeta|^2\\
&=|\zeta|^2\left(1+\frac{\sum_{i=1}^{r}(p_i-2)|\nabla v|^{p_i-2}}{\sum_{i=1}^{r}|\nabla v|^{p_i-2}}\right)\\
&>0.
\end{aligned}
$$

Thus, in any case, we have shown that $\Theta>0$. □

Lemma 2.3. *Let $1<p_1<\cdots<p_r$ and $m\geq 2$. Let v_i denote the partial derivative with respect to x_i. Then, we have*

$$
\begin{aligned}
&\left(1+\frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r}v_1^{p_i-2}}\right)v_{11}^2+\sum_{l=2}^{m}v_{ll}^2\\
\geq&\left(\frac{m}{m-1}+\frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r}v_1^{p_i-2}}\right)v_{11}^2-\frac{2}{m-1}v_{11}\Delta v+\frac{1}{m-1}(\Delta v)^2.
\end{aligned}
\tag{2.7}
$$

Proof. We note that

$$\begin{aligned}\left(\sum_{l=2}^{m} v_{ll}\right)^2 &= \sum_{l=2}^{m} v_{ll}^2 + 2\sum_{2\le k<l}^{m} v_{kk}v_{ll}\\ &\le \sum_{l=2}^{m} v_{ll}^2 + \sum_{2\le k<l}^{m}\left(v_{kk}^2 + v_{ll}^2\right)\\ &= \sum_{l=2}^{m} v_{ll}^2 + (m-2)\sum_{l=2}^{m} v_{ll}^2.\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{l=2}^{m} v_{ll}^2 &\ge \frac{1}{m-1}\left(\sum_{l=2}^{m} v_{ll}\right)^2\\ &= \frac{1}{m-1}\left(\Delta v - v_{11}\right)^2\\ &= \frac{1}{m-1}\left((\Delta v)^2 - 2v_{11}\Delta v + v_{11}^2\right).\end{aligned} \tag{2.8}$$

By adding $\left(1 + \frac{\sum_{i=1}^{r}(p_i-2)v_1^{p_i-2}}{\sum_{i=1}^{r} v_1^{p_i-2}}\right)v_{11}^2$ to each side of (2.8), we get

$$\begin{aligned}&\left(1 + \frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r} v_1^{p_i-2}}\right)v_{11}^2 + \sum_{l=2}^{m} v_{ll}^2\\ \ge&\left(\frac{m}{m-1} + \frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r} v_1^{p_i-2}}\right)v_{11}^2 - \frac{2}{m-1}v_{11}\Delta v + \frac{1}{m-1}(\Delta v)^2.\end{aligned}$$

This completes the proof of Lemma 2.3. □

3. Proof of Theorem 1.1

In this section, we employ the Bernstein method to prove Theorem 1.1. We divide the proof into several steps.

Step 1. First, we state a pointwise differential inequality.

Lemma 3.1. *Let v be a positive solution of* (1.1) *in* $\mathbb{R}^m$. *Assume that p_i satisfies* (1.7) *or* (1.8). *Assume that f satisfies* (1.9) *or* (1.10). *Then, the following pointwise differential inequality holds:*

$$\frac{1}{\phi}\left(\Delta\phi + \frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r} v_1^{p_i-2}}\phi_{11}\right) \ge \tilde{A}\frac{|\nabla\phi|^2}{2\phi^2} + \frac{\tilde{B}}{2}\frac{v_1^2}{v^2} - \frac{1}{2}\left(\frac{\tilde{D}^2}{\tilde{C}} + \frac{2\tilde{E}^2}{\tilde{B}}\right)\frac{\phi_1^2}{2\phi^2}, \tag{3.1}$$

where $\phi = \frac{|\nabla v|^2}{v^2}$, and the variables $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$, and $\tilde{E}$ are defined as in (3.27).

Proof. We know that the function $v(x)$ is C^3 in $\{x \in \Omega : |\nabla v| > 0\}$ (see [19]). Let $\phi = \frac{|\nabla v|^2}{v^2}$ and $\{e_1, e_2, \ldots, e_m\}$ be a local orthonormal frame such that $e_1 = \frac{\nabla v}{|\nabla v|}$ at x^*. If we denote $\nabla v = \Sigma_{i=1}^{m} v_i e_i$, it is easy to see that

$$|\nabla v| = v_1, \quad v_l = 0, \quad l = 2, \cdots, m \tag{3.2}$$

at x^*. We observe that for $k, l = 1, 2, \cdots, m$, we have

$$\phi_k = 2v_{kl}v_l v^{-2} - 2|\nabla v|^2 v_k v^{-3} \tag{3.3}$$

and

$$\phi_{kk} = 2v_{kkl}v_l v^{-2} + 2v_{kl}v_{kl}v^{-2} - 8v_{kl}v_k v_l v^{-3} - 2|\nabla v|^2 v_{kk} v^{-3} + 6|\nabla v|^2 v^{-4} v_k^2. \tag{3.4}$$

Utilizing (3.3) and (3.4), we can get

$$\phi_{11} = 2v_{111}v_1 v^{-2} + 2v_{1l}v_{1l}v^{-2} - 10v_{11}v_1^2 v^{-3} + 6v^{-4}v_1^4$$

and

$$\Delta\phi = 2\left(\Delta v\right)_1 v_1 v^{-2} + 2v_{kl}v_{kl}v^{-2} - 8v_{11}v_1^2 v^{-3} - 2v_1^2 v^{-3}\Delta v + 6v^{-4}v_1^4.$$

Through calculations, we obtain the following at x^*:

$$\begin{aligned}
\Delta\phi + \frac{\sum_{i=1}^r \left(p_i - 2\right) v_1^{p_i-2}}{\sum_{i=1}^r v_1^{p_i-2}}\phi_{11} &= 2v_1 v^{-2}\left[\left(\Delta v\right)_1 + \frac{\sum_{i=1}^r \left(p_i - 2\right) v_1^{p_i-2}}{\sum_{i=1}^r v_1^{p_i-2}} v_{111}\right] \\
&\quad + 2v^{-2}\left(v_{kl}v_{kl} + \frac{\sum_{i=1}^r \left(p_i - 2\right) v_1^{p_i-2}}{\sum_{i=1}^r v_1^{p_i-2}} v_{1l}v_{1l}\right) \\
&\quad - 2v_1^2 v^{-3}\left[\Delta v + \left(4 + 5\frac{\sum_{i=1}^r \left(p_i - 2\right) v_1^{p_i-2}}{\sum_{i=1}^r v_1^{p_i-2}}\right) v_{11}\right] \\
&\quad + 6\left(1 + \frac{\sum_{i=1}^r \left(p_i - 2\right) v_1^{p_i-2}}{\sum_{i=1}^r v_1^{p_i-2}}\right) v_1^4 v^{-4}.
\end{aligned} \tag{3.5}$$

Subsequently, we will focus on rearranging equation (3.5). We start by rewriting equation (1.1) in x^*, that is,

$$\sum_{i=1}^r |\nabla v|^{p_i-2}\Delta v + \frac{\sum_{i=1}^r \left(p_i - 2\right)|\nabla v|^{p_i-3}}{|\nabla v|} v_{kl}v_k v_l = -f\left(v\right). \tag{3.6}$$

According to (3.2), at x^*, we obtain the following equation from (3.6):

$$\sum_{i=1}^r v_1^{p_i-2}\Delta v + \sum_{i=1}^r \left(p_i - 2\right) v_1^{p_i-2} v_{11} = -f\left(v\right), \tag{3.7}$$

that is,

$$\Delta v + \frac{\sum_{i=1}^r \left(p_i - 2\right) v_1^{p_i-2}}{\sum_{i=1}^r v_1^{p_i-2}} v_{11} = -\frac{1}{\sum_{i=1}^r v_1^{p_i-2}} f\left(v\right). \tag{3.8}$$

We now differentiate (3.6) with respect to x_1.

$$\begin{aligned}
&\sum_{i=1}^r \left(p_i - 2\right) v_1^{p_i-3} v_{11}\Delta v + \sum_{i=1}^r v_1^{p_i-2}\left(\Delta v\right)_1 + \sum_{i=1}^r \left(p_i - 2\right)\left(p_i - 3\right) v_1^{p_i-3} v_{11}^2 \\
&- \sum_{i=1}^r \left(p_i - 2\right) v_1^{p_i-3} v_{11}^2 + \sum_{i=1}^r \left(p_i - 2\right) v_1^{p_i-2} v_{111} \\
&+ 2\sum_{i=1}^r \left(p_i - 2\right) v_1^{p_i-3}\sum_{l=1}^m v_{1l}^2 + f'\left(v\right) v_1 = 0,
\end{aligned}$$

that is,

$$
\begin{aligned}
&\sum_{i=1}^{r} v_1^{p_i-2}(\Delta v)_1+\sum_{i=1}^{r}(p_i-2)\, v_1^{p_i-2} v_{111}\\
=&-2\sum_{i=1}^{r}(p_i-2)\, v_1^{p_i-3}\sum_{l=2}^{m} v_{1l}^2-\sum_{i=1}^{r}(p_i-2)\, v_1^{p_i-3} v_{11}\Delta v\\
&-\sum_{i=1}^{r}(p_i-2)(p_i-3)\, v_1^{p_i-3} v_{11}^2+\sum_{i=1}^{r}(p_i-2)\, v_1^{p_i-3} v_{11}^2-f'(v)\, v_1.
\end{aligned}
\tag{3.9}
$$

Dividing both sides of (3.9) by $\sum_{i=1}^{r} v_1^{p_i-2}$, replacing Δv in (3.9) with (3.8), and rearranging, we obtain

$$
\begin{aligned}
(\Delta v)_1+\frac{\sum_{i=1}^{r}(p_i-2)\, v_1^{p_i-2}}{\sum_{i=1}^{r} v_1^{p_i-2}} v_{111}=&-2\frac{\sum_{i=1}^{r}(p_i-2)\, v_1^{p_i-3}}{\sum_{i=1}^{r} v_1^{p_i-2}}\sum_{l=2}^{m} v_{1l}^2\\
&-\frac{\sum_{i=1}^{r}(p_i-2)(p_i-3)\, v_1^{p_i-3}}{\sum_{i=1}^{r} v_1^{p_i-2}} v_{11}^2-\frac{f'(v)}{\sum_{i=1}^{r} v_1^{p_i-2}} v_1\\
&+\frac{\sum_{i=1}^{r}(p_i-2)\, v_1^{p_i-3}}{\sum_{i=1}^{r} v_1^{p_i-2}} v_{11}\Bigg(\frac{f(v)}{\sum_{i=1}^{r} v_1^{p_i-2}}\\
&+\left(1+\frac{\sum_{i=1}^{r}(p_i-2)\, v_1^{p_i-2}}{\sum_{i=1}^{r} v_1^{p_i-2}}\right) v_{11}\Bigg).
\end{aligned}
\tag{3.10}
$$

Next, we can get

$$
\begin{aligned}
v_{kl}v_{kl}+\frac{\sum_{i=1}^{r}(p_i-2)\, v_1^{p_i-2}}{\sum_{i=1}^{r} v_1^{p_i-2}} v_{1l}v_{1l}=&\,2\sum_{l=2}^{m} v_{1l}^2+v_{11}^2+\sum_{l=2}^{m} v_{ll}^2+\sum_{k,l\geq 2,k\neq l} v_{kl}^2\\
&+\frac{\sum_{i=1}^{r}(p_i-2)\, v_1^{p_i-2}}{\sum_{i=1}^{r} v_1^{p_i-2}} v_{11}^2\\
&+\frac{\sum_{i=1}^{r}(p_i-2)\, v_1^{p_i-2}}{\sum_{i=1}^{r} v_1^{p_i-2}}\sum_{l=2}^{m} v_{1l}^2\\
\geq&\left(2+\frac{\sum_{i=1}^{r}(p_i-2)\, v_1^{p_i-2}}{\sum_{i=1}^{r} v_1^{p_i-2}}\right)\sum_{l=2}^{m} v_{1l}^2\\
&+\left(1+\frac{\sum_{i=1}^{r}(p_i-2)\, v_1^{p_i-2}}{\sum_{i=1}^{r} v_1^{p_i-2}}\right) v_{11}^2+\sum_{l=2}^{m} v_{ll}^2.
\end{aligned}
\tag{3.11}
$$

Applying (2.7) to (3.11) and rearranging terms, we obtain

$$
\begin{aligned}
v_{kl}v_{kl}+\frac{\sum_{i=1}^{r}(p_i-2)\, v_1^{p_i-2}}{\sum_{i=1}^{r} v_1^{p_i-2}} v_{1l}v_{1l}\geq&\left(2+\frac{\sum_{i=1}^{r}(p_i-2)\, v_1^{p_i-2}}{\sum_{i=1}^{r} v_1^{p_i-2}}\right)\sum_{l=2}^{m} v_{1l}^2\\
&+\left(\frac{m}{m-1}+\frac{\sum_{i=1}^{r}(p_i-2)\, v_1^{p_i-2}}{\sum_{i=1}^{r} v_1^{p_i-2}}\right) v_{11}^2\\
&-\frac{2}{m-1} v_{11}\Delta v+\frac{1}{m-1}(\Delta v)^2.
\end{aligned}
\tag{3.12}
$$

Then, we substitute (3.10) and (3.12) into (3.5) to obtain

$$
\begin{aligned}
&\Delta\phi+\frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r}v_1^{p_i-2}}\phi_{11}\\
\geq&2v_1v^{-2}\Bigg[-2\frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-3}}{\sum_{i=1}^{r}v_1^{p_i-2}}\sum_{l=2}^{m}v_{1l}^2-\frac{\sum_{i=1}^{r}(p_i-2)\,(p_i-3)\,v_1^{p_i-3}}{\sum_{i=1}^{r}v_1^{p_i-2}}v_{11}^2-\frac{f'(v)}{\sum_{i=1}^{r}v_1^{p_i-2}}v_1\\
&+\frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-3}}{\sum_{i=1}^{r}v_1^{p_i-2}}v_{11}\left(\frac{f(v)}{\sum_{i=1}^{r}v_1^{p_i-2}}+\left(1+\frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r}v_1^{p_i-2}}\right)v_{11}\right)\Bigg]\\
&+2v^{-2}\Bigg[\left(2+\frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r}v_1^{p_i-2}}\right)\sum_{l=2}^{m}v_{1l}^2+\left(\frac{m}{m-1}+\frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r}v_1^{p_i-2}}\right)v_{11}^2\\
&-\frac{2}{m-1}v_{11}\Delta v+\frac{1}{m-1}(\Delta v)^2\Bigg]-2v_1^2v^{-3}\Bigg[\Delta v+\left(4+5\frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r}v_1^{p_i-2}}\right)v_{11}\Bigg]\\
&+6\left(1+\frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r}v_1^{p_i-2}}\right)v_1^4v^{-4}.
\end{aligned}
\tag{3.13}
$$

Replacing Δv in (3.13) with (3.8), we get

$$
\begin{aligned}
&\Delta\phi+\frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r}v_1^{p_i-2}}\phi_{11}\\
\geq&2v_1v^{-2}\Bigg[-2\frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-3}}{\sum_{i=1}^{r}v_1^{p_i-2}}\sum_{l=2}^{m}v_{1l}^2-\frac{\sum_{i=1}^{r}(p_i-2)\,(p_i-3)\,v_1^{p_i-3}}{\sum_{i=1}^{r}v_1^{p_i-2}}v_{11}^2-\frac{f'(v)}{\sum_{i=1}^{r}v_1^{p_i-2}}v_1\\
&+\frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-3}}{\sum_{i=1}^{r}v_1^{p_i-2}}v_{11}\left(\frac{f(v)}{\sum_{i=1}^{r}v_1^{p_i-2}}+\left(1+\frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r}v_1^{p_i-2}}\right)v_{11}\right)\Bigg]\\
&+2v^{-2}\Bigg[\left(2+\frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r}v_1^{p_i-2}}\right)\sum_{l=2}^{m}v_{1l}^2+\left(\frac{m}{m-1}+\frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r}v_1^{p_i-2}}\right)v_{11}^2\\
&+\frac{2}{m-1}v_{11}\left(\frac{f(v)}{\sum_{i=1}^{r}v_1^{p_i-2}}+\frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r}v_1^{p_i-2}}v_{11}\right)\\
&+\frac{1}{m-1}\left(\frac{f(v)}{\sum_{i=1}^{r}v_1^{p_i-2}}+\frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r}v_1^{p_i-2}}v_{11}\right)^2\Bigg]\\
&-2v_1^2v^{-3}\Bigg[-\left(\frac{f(v)}{\sum_{i=1}^{r}v_1^{p_i-2}}+\frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r}v_1^{p_i-2}}v_{11}\right)\\
&+\left(4+5\frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r}v_1^{p_i-2}}\right)v_{11}\Bigg]+6\left(1+\frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r}v_1^{p_i-2}}\right)v_1^4v^{-4}.
\end{aligned}
\tag{3.14}
$$

After rearranging the terms in (3.14), we get

$$\begin{aligned}
&\Delta\phi + \frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r} v_1^{p_i-2}}\phi_{11} \\
\geq& 2v^{-2}\Bigg\{\left(2-\frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r} v_1^{p_i-2}}\right)\sum_{l=2}^{m} v_{1l}^2 - \frac{f'(v)\,v_1^2}{\sum_{i=1}^{r} v_1^{p_i-2}} \\
&+\Bigg[\frac{m}{m-1}\left(\frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r} v_1^{p_i-2}}\right)^2 \\
&-\frac{\sum_{i=1}^{r}(p_i-2)(p_i-3)\,v_1^{p_i-2}}{\sum_{i=1}^{r} v_1^{p_i-2}}+\frac{m}{m-1}+\frac{2m}{m-1}\frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r} v_1^{p_i-2}}\Bigg]v_{11}^2 \qquad (3.15)\\
&+\frac{1}{m-1}\left(\frac{1}{\sum_{i=1}^{r} v_1^{p_i-2}}f(v)\right)^2+\Bigg[\frac{2}{m-1}\left(1+\frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r} v_1^{p_i-2}}\right) \\
&+\frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r} v_1^{p_i-2}}\Bigg]\frac{f(v)}{\sum_{i=1}^{r} v_1^{p_i-2}}v_{11}-\Bigg[-\frac{f(v)}{\sum_{i=1}^{r} v_1^{p_i-2}} \\
&+4\left(1+\frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r} v_1^{p_i-2}}\right)v_{11}\Bigg]v_1^2v^{-1}+3\left(1+\frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r} v_1^{p_i-2}}\right)v_1^4v^{-2}\Bigg\}.
\end{aligned}$$

From (3.2) and (3.3), we know that for $l = 2, \cdots, m$, $\phi_l = \frac{2v_{1l}v_1}{v^2}$ and

$$\phi = \frac{|\nabla v|^2}{v^2} = \frac{v_1^2}{v^2}. \tag{3.16}$$

Then, we have

$$v_{1l} = \frac{\phi_l v^2}{2v_1}. \tag{3.17}$$

We note that

$$|\nabla\phi|^2 = \sum_{l=1}^{m}\phi_l^2. \tag{3.18}$$

By applying (3.17), (3.16), and (3.18), we obtain

$$\frac{2}{v^2}\sum_{l=2}^{m} v_{1l}^2 = \frac{\sum_{l=2}^{m}\phi_l^2}{2\left(\frac{v_1}{v}\right)^2} = \frac{\sum_{l=2}^{m}\phi_l^2}{2\phi} \leq \frac{|\nabla\phi|^2}{2\phi}. \tag{3.19}$$

Using (3.19) and (3.15), we obtain

$$
\begin{aligned}
&\Delta\phi + \frac{\sum_{i=1}^{r}(p_i-2)v_1^{p_i-2}}{\sum_{i=1}^{r}v_1^{p_i-2}}\phi_{11}\\
\geq & -\left|2-\frac{\sum_{i=1}^{r}(p_i-2)v_1^{p_i-2}}{\sum_{i=1}^{r}v_1^{p_i-2}}\right|\frac{|\nabla\phi|^2}{2\phi} + 2v^{-2}\Bigg\{-\frac{f'(v)v_1^2}{\sum_{i=1}^{r}v_1^{p_i-2}}\\
&+\left[\frac{m}{m-1}\left(\frac{\sum_{i=1}^{r}(p_i-2)v_1^{p_i-2}}{\sum_{i=1}^{r}v_1^{p_i-2}}\right)^2-\frac{\sum_{i=1}^{r}(p_i-2)(p_i-3)v_1^{p_i-2}}{\sum_{i=1}^{r}v_1^{p_i-2}}+\frac{m}{m-1}\right.\\
&\left.+\frac{2m}{m-1}\frac{\sum_{i=1}^{r}(p_i-2)v_1^{p_i-2}}{\sum_{i=1}^{r}v_1^{p_i-2}}\right]v_{11}^2+\frac{1}{m-1}\left(\frac{1}{\sum_{i=1}^{r}v_1^{p_i-2}}f(v)\right)^2\\
&+\left[\frac{2}{m-1}\left(1+\frac{\sum_{i=1}^{r}(p_i-2)v_1^{p_i-2}}{\sum_{i=1}^{r}v_1^{p_i-2}}\right)+\frac{\sum_{i=1}^{r}(p_i-2)v_1^{p_i-2}}{\sum_{i=1}^{r}v_1^{p_i-2}}\right]\frac{f(v)}{\sum_{i=1}^{r}v_1^{p_i-2}}v_{11}\\
&-\left[-\frac{f(v)}{\sum_{i=1}^{r}v_1^{p_i-2}}+4\left(1+\frac{\sum_{i=1}^{r}(p_i-2)v_1^{p_i-2}}{\sum_{i=1}^{r}v_1^{p_i-2}}\right)v_{11}\right]v_1^2v^{-1}\\
&+3\left(1+\frac{\sum_{i=1}^{r}(p_i-2)v_1^{p_i-2}}{\sum_{i=1}^{r}v_1^{p_i-2}}\right)v_1^4v^{-2}\Bigg\}.
\end{aligned}
\tag{3.20}
$$

Based on (3.3), we obtain $\phi_1 = 2v_{11}v_1v^{-2} - 2v_1^3v^{-3}$, and further get

$$
v_{11} = \frac{\phi_1 v^2}{2v_1} + \frac{v_1^2}{v}. \tag{3.21}
$$

By inserting (3.21) into (3.20), we obtain

$$
\begin{aligned}
&\Delta\phi + \frac{\sum_{i=1}^{r}(p_i-2)v_1^{p_i-2}}{\sum_{i=1}^{r}v_1^{p_i-2}}\phi_{11}\\
\geq & -\left|2-\frac{\sum_{i=1}^{r}(p_i-2)v_1^{p_i-2}}{\sum_{i=1}^{r}v_1^{p_i-2}}\right|\frac{|\nabla\phi|^2}{2\phi} + 2v^{-2}\Bigg\{-\frac{f'(v)v_1^2}{\sum_{i=1}^{r}v_1^{p_i-2}}+\left[\frac{m}{m-1}\left(\frac{\sum_{i=1}^{r}(p_i-2)v_1^{p_i-2}}{\sum_{i=1}^{r}v_1^{p_i-2}}\right)^2\right.\\
&\left.-\frac{\sum_{i=1}^{r}(p_i-2)(p_i-3)v_1^{p_i-2}}{\sum_{i=1}^{r}v_1^{p_i-2}}+\frac{m}{m-1}+\frac{2m}{m-1}\frac{\sum_{i=1}^{r}(p_i-2)v_1^{p_i-2}}{\sum_{i=1}^{r}v_1^{p_i-2}}\right]\left(\frac{\phi_1v^2}{2v_1}+\frac{v_1^2}{v}\right)^2\\
&+\frac{1}{m-1}\left(\frac{f(v)}{\sum_{i=1}^{r}v_1^{p_i-2}}\right)^2+\left[\frac{2}{m-1}\left(1+\frac{\sum_{i=1}^{r}(p_i-2)v_1^{p_i-2}}{\sum_{i=1}^{r}v_1^{p_i-2}}\right)+\frac{\sum_{i=1}^{r}(p_i-2)v_1^{p_i-2}}{\sum_{i=1}^{r}v_1^{p_i-2}}\right]\\
&\frac{f(v)}{\sum_{i=1}^{r}v_1^{p_i-2}}\left(\frac{\phi_1v^2}{2v_1}+\frac{v_1^2}{v}\right)-\left[-\frac{f(v)}{\sum_{i=1}^{r}v_1^{p_i-2}}+4\left(1+\frac{\sum_{i=1}^{r}(p_i-2)v_1^{p_i-2}}{\sum_{i=1}^{r}v_1^{p_i-2}}\right)\left(\frac{\phi_1v^2}{2v_1}+\frac{v_1^2}{v}\right)\right]v_1^2v^{-1}\\
&+3\left(1+\frac{\sum_{i=1}^{r}(p_i-2)v_1^{p_i-2}}{\sum_{i=1}^{r}v_1^{p_i-2}}\right)v_1^4v^{-2}\Bigg\}.
\end{aligned}
\tag{3.22}
$$

Through rearrangement, we find that

$$
\begin{aligned}
&\Delta\phi + \frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r}v_1^{p_i-2}}\phi_{11} \\
\geq &-\left|2-\frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r}v_1^{p_i-2}}\right|\frac{|\nabla\phi|^2}{2\phi}+\Bigg[\frac{m}{m-1}\left(\frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r}v_1^{p_i-2}}\right)^2+\frac{m}{m-1} \\
&-\frac{\sum_{i=1}^{r}(p_i-2)\,(p_i-3)\,v_1^{p_i-2}}{\sum_{i=1}^{r}v_1^{p_i-2}}+\frac{2m}{m-1}\frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r}v_1^{p_i-2}}\Bigg]\left(\frac{2\phi_1v_1}{v}+\frac{2v_1^4}{v^4}\right) \\
&+\frac{2}{m-1}\left(\frac{f(v)}{v\sum_{i=1}^{r}v_1^{p_i-2}}\right)^2+\Bigg[\frac{2}{m-1}\left(1+\frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r}v_1^{p_i-2}}\right) \\
&+\frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r}v_1^{p_i-2}}\Bigg]\frac{\phi_1f(v)}{v_1\sum_{i=1}^{r}v_1^{p_i-2}}-4\left(1+\frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r}v_1^{p_i-2}}\right)\frac{\phi_1v_1}{v} \\
&-2\left(1+\frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r}v_1^{p_i-2}}\right)v_1^4v^{-4}+2\Bigg[\frac{m}{m-1}\left(\frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r}v_1^{p_i-2}}\right)^2 \\
&-\frac{\sum_{i=1}^{r}(p_i-2)\,(p_i-3)\,v_1^{p_i-2}}{\sum_{i=1}^{r}v_1^{p_i-2}}+\frac{m}{m-1}+\frac{2m}{m-1}\frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r}v_1^{p_i-2}}\Bigg]\left(\frac{\phi_1v^2}{2vv_1}\right)^2 \\
&+2v^{-2}\Bigg[-f'(v)+\frac{m+1}{m-1}\left(1+\frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r}v_1^{p_i-2}}\right)\frac{f(v)}{v}\Bigg]\frac{v_1^2}{\sum_{i=1}^{r}v_1^{p_i-2}}.
\end{aligned}
\tag{3.23}
$$

It is obvious that for $t > 0$, since $1 < p_1 < \cdots < p_r$, the inequality

$$
(p_1-1)\leq 1+\frac{\sum_{i=1}^{r}(p_i-2)t^{p_i-2}}{\sum_{i=1}^{r}t^{p_i-2}}\leq(p_r-1) \tag{3.24}
$$

holds. Using (1.9) and (1.10), we obtain

(i) When $f \geq 0$,

$$
\begin{aligned}
&-f'(v)+\frac{m+1}{m-1}\left(1+\frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r}v_1^{p_i-2}}\right)\frac{f(v)}{v} \\
\geq&-f'(v)+(p_1-1)\frac{m+1}{m-1}\frac{f(v)}{v} \\
\geq&0.
\end{aligned}
\tag{3.25}
$$

(ii) When $f \leq 0$,

$$
\begin{aligned}
&-f'(v)+\frac{m+1}{m-1}\left(1+\frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r}v_1^{p_i-2}}\right)\frac{f(v)}{v} \\
\geq&-f'(v)+(p_r-1)\frac{m+1}{m-1}\frac{f(v)}{v} \\
\geq&0.
\end{aligned}
\tag{3.26}
$$

For the convenience of calculation, we introduce the following symbols:

$$
\begin{aligned}
\tilde{A}(t) &:= -\left|2 - \frac{\sum_{i=1}^{r}(p_i-2)\,t^{p_i-2}}{\sum_{i=1}^{r} t^{p_i-2}}\right|,\\
\tilde{B}(t) &:= 2\Bigg[\frac{m}{m-1}\left(\frac{\sum_{i=1}^{r}(p_i-2)\,t^{p_i-2}}{\sum_{i=1}^{r} t^{p_i-2}}\right)^2 - \frac{\sum_{i=1}^{r}(p_i-2)(p_i-3)\,t^{p_i-2}}{\sum_{i=1}^{r} t^{p_i-2}} + \frac{1}{m-1}\\
&\qquad + \frac{m+1}{m-1}\frac{\sum_{i=1}^{r}(p_i-2)\,t^{p_i-2}}{\sum_{i=1}^{r} t^{p_i-2}}\Bigg],\\
\tilde{C} &:= \frac{2}{m-1},\\
\tilde{D}(t) &:= \frac{2}{m-1}\left(1 + \frac{\sum_{i=1}^{r}(p_i-2)\,t^{p_i-2}}{\sum_{i=1}^{r} t^{p_i-2}}\right) + \frac{\sum_{i=1}^{r}(p_i-2)\,t^{p_i-2}}{\sum_{i=1}^{r} t^{p_i-2}},\\
\tilde{E}(t) &:= 2\Bigg[\frac{m}{m-1}\left(\frac{\sum_{i=1}^{r}(p_i-2)\,t^{p_i-2}}{\sum_{i=1}^{r} t^{p_i-2}}\right)^2 - \frac{\sum_{i=1}^{r}(p_i-2)(p_i-3)\,t^{p_i-2}}{\sum_{i=1}^{r} t^{p_i-2}} + \frac{2-m}{m-1}\\
&\qquad + \frac{2}{m-1}\frac{\sum_{i=1}^{r}(p_i-2)\,t^{p_i-2}}{\sum_{i=1}^{r} t^{p_i-2}}\Bigg],\\
\tilde{F}(t) &:= 2\Bigg[\frac{m}{m-1}\left(\frac{\sum_{i=1}^{r}(p_i-2)\,t^{p_i-2}}{\sum_{i=1}^{r} t^{p_i-2}}\right)^2 - \frac{\sum_{i=1}^{r}(p_i-2)(p_i-3)\,t^{p_i-2}}{\sum_{i=1}^{r} t^{p_i-2}} + \frac{m}{m-1}\\
&\qquad + \frac{2m}{m-1}\frac{\sum_{i=1}^{r}(p_i-2)\,t^{p_i-2}}{\sum_{i=1}^{r} t^{p_i-2}}\Bigg].
\end{aligned}
\tag{3.27}
$$

Using (3.27) in (3.23), dividing both sides of (3.23) by ϕ, and rearranging, we obtain

$$
\begin{aligned}
&\frac{1}{\phi}\left(\Delta\phi + \frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r} v_1^{p_i-2}}\phi_{11}\right)\\
\geq& -\left|2 - \frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r} v_1^{p_i-2}}\right|\frac{|\nabla\phi|^2}{2\phi^2} + 2\Bigg[\frac{m}{m-1}\left(\frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r} v_1^{p_i-2}}\right)^2 + \frac{1}{m-1}\\
&- \frac{\sum_{i=1}^{r}(p_i-2)(p_i-3)\,v_1^{p_i-2}}{\sum_{i=1}^{r} v_1^{p_i-2}} + \frac{m+1}{m-1}\frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r} v_1^{p_i-2}}\Bigg]\frac{v_1^2}{v^2} + \frac{2}{m-1}\left(\frac{f(v)}{v_1\sum_{i=1}^{r} v_1^{p_i-2}}\right)^2\\
&+ \left[\frac{2}{m-1}\left(1 + \frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r} v_1^{p_i-2}}\right) + \frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r} v_1^{p_i-2}}\right]\frac{\phi_1 f(v)}{\phi v_1\sum_{i=1}^{r} v_1^{p_i-2}}\\
&+ 2\Bigg[\frac{m}{m-1}\left(\frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r} v_1^{p_i-2}}\right)^2 - \frac{\sum_{i=1}^{r}(p_i-2)(p_i-3)\,v_1^{p_i-2}}{\sum_{i=1}^{r} v_1^{p_i-2}} + \frac{2-m}{m-1}\\
&+ \frac{2}{m-1}\frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r} v_1^{p_i-2}}\Bigg]\frac{\phi_1 v_1}{\phi v} + 2\Bigg[\frac{m}{m-1}\left(\frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r} v_1^{p_i-2}}\right)^2\\
&- \frac{\sum_{i=1}^{r}(p_i-2)(p_i-3)\,v_1^{p_i-2}}{\sum_{i=1}^{r} v_1^{p_i-2}} + \frac{m}{m-1} + \frac{2m}{m-1}\frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r} v_1^{p_i-2}}\Bigg]\frac{\phi_1^2}{4\phi^2}.
\end{aligned}
\tag{3.28}
$$

Then, we have

$$
\begin{aligned}
&\frac{1}{\phi}\left(\Delta\phi+\frac{\sum_{i=1}^{r}\left(p_i-2\right)v_1^{p_i-2}}{\sum_{i=1}^{r}v_1^{p_i-2}}\phi_{11}\right)\\
\geq&\tilde{A}\left(v_1\right)\frac{|\nabla\phi|^2}{2\phi^2}+\tilde{B}\left(v_1\right)\frac{v_1^2}{v^2}+\tilde{C}\left(\frac{f\left(v\right)}{v_1\sum_{i=1}^{r}v_1^{p_i-2}}\right)^2+\tilde{D}\left(v_1\right)\left(\frac{\phi_1 f\left(v\right)}{\phi v_1\sum_{i=1}^{r}v_1^{p_i-2}}\right)\\
&+\tilde{E}\left(v_1\right)\frac{\phi_1 v_1}{\phi v}+\tilde{F}\left(v_1\right)\frac{\phi_1^2}{4\phi^2}.
\end{aligned}
\tag{3.29}
$$

Using Lemma 2.1 and (3.24), if

$$
\sup_{t>0}\frac{\sum_{i=1}^{r}\left(p_i-2\right)\left(p_i-3\right)t^{p_i-2}}{\sum_{i=1}^{r}t^{p_i-2}}<\left(\frac{m}{m-1}\right)\left(p_1-1\right)^2-\left(p_r-1\right)
$$

holds, then we have

$$
\begin{aligned}
&\sup_{t>0}\frac{\sum_{i=1}^{r}\left(p_i-2\right)\left(p_i-3\right)t^{p_i-2}}{\sum_{i=1}^{r}t^{p_i-2}}\\
<&\frac{m}{m-1}\left(1+\frac{\sum_{i=1}^{r}\left(p_i-2\right)t^{p_i-2}}{\sum_{i=1}^{r}t^{p_i-2}}\right)^2-\left(1+\frac{\sum_{i=1}^{r}\left(p_i-2\right)t^{p_i-2}}{\sum_{i=1}^{r}t^{p_i-2}}\right).
\end{aligned}
\tag{3.30}
$$

In addition, we have

$$
\begin{aligned}
\tilde{B}=&2\Big[\frac{m}{m-1}\left(\frac{\sum_{i=1}^{r}\left(p_i-2\right)t^{p_i-2}}{\sum_{i=1}^{r}t^{p_i-2}}\right)^2-\frac{\sum_{i=1}^{r}\left(p_i-2\right)\left(p_i-3\right)t^{p_i-2}}{\sum_{i=1}^{r}t^{p_i-2}}\\
&+\frac{1}{m-1}+\frac{m+1}{m-1}\frac{\sum_{i=1}^{r}\left(p_i-2\right)t^{p_i-2}}{\sum_{i=1}^{r}t^{p_i-2}}\Big]\\
=&2\Big[\frac{m}{m-1}\left(1+\frac{\sum_{i=1}^{r}\left(p_i-2\right)t^{p_i-2}}{\sum_{i=1}^{r}t^{p_i-2}}\right)^2-\left(1+\frac{\sum_{i=1}^{r}\left(p_i-2\right)t^{p_i-2}}{\sum_{i=1}^{r}t^{p_i-2}}\right)\\
&-\frac{\sum_{i=1}^{r}\left(p_i-2\right)\left(p_i-3\right)t^{p_i-2}}{\sum_{i=1}^{r}t^{p_i-2}}\Big]
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{F}=&2\Big[\frac{m}{m-1}\left(\frac{\sum_{i=1}^{r}\left(p_i-2\right)t^{p_i-2}}{\sum_{i=1}^{r}t^{p_i-2}}\right)^2-\frac{\sum_{i=1}^{r}\left(p_i-2\right)\left(p_i-3\right)t^{p_i-2}}{\sum_{i=1}^{r}t^{p_i-2}}\\
&+\frac{m}{m-1}+\frac{2m}{m-1}\frac{\sum_{i=1}^{r}\left(p_i-2\right)t^{p_i-2}}{\sum_{i=1}^{r}t^{p_i-2}}\Big]\\
=&2\Big[\frac{m}{m-1}\left(1+\frac{\sum_{i=1}^{r}\left(p_i-2\right)t^{p_i-2}}{\sum_{i=1}^{r}t^{p_i-2}}\right)^2-\frac{\sum_{i=1}^{r}\left(p_i-2\right)\left(p_i-3\right)t^{p_i-2}}{\sum_{i=1}^{r}t^{p_i-2}}\Big].
\end{aligned}
$$

Based on (3.30), we can derive that

$$
\frac{m}{m-1}\left(1+\frac{\sum_{i=1}^{r}\left(p_i-2\right)t^{p_i-2}}{\sum_{i=1}^{r}t^{p_i-2}}\right)^2-\left(1+\frac{\sum_{i=1}^{r}\left(p_i-2\right)t^{p_i-2}}{\sum_{i=1}^{r}t^{p_i-2}}\right)-\frac{\sum_{i=1}^{r}\left(p_i-2\right)\left(p_i-3\right)t^{p_i-2}}{\sum_{i=1}^{r}t^{p_i-2}}>0.
$$

That is to say, $\tilde{B} > 0$. Furthermore, we obtain

$$\begin{aligned}\tilde{F} - \tilde{B} &= 2\left(1 + \frac{\sum_{i=1}^{r} (p_i - 2)\, t^{p_i-2}}{\sum_{i=1}^{r} t^{p_i-2}}\right) \\ &= 2\frac{\sum_{i=1}^{r} (p_i - 1)\, t^{p_i-2}}{\sum_{i=1}^{r} t^{p_i-2}}.\end{aligned}$$

Since $p_i > 1$, we can derive that $\tilde{F} > \tilde{B} > 0$. Then, we get

$$\begin{aligned}&\frac{1}{\phi}\left(\Delta\phi + \frac{\sum_{i=1}^{r} (p_i - 2)\, v_1^{p_i-2}}{\sum_{i=1}^{r} v_1^{p_i-2}}\phi_{11}\right) \\ \geq& \tilde{A}\,(v_1)\, \frac{|\nabla\phi|^2}{2\phi^2} + \tilde{B}\,(v_1)\, \frac{v_1^2}{v^2} + \tilde{C}\left(\frac{f\,(v)}{v_1 \sum_{i=1}^{r} v_1^{p_i-2}}\right)^2 \\ &+ \tilde{D}\,(v_1)\left(\frac{\phi_1 f\,(v)}{\phi v_1 \sum_{i=1}^{r} v_1^{p_i-2}}\right) + \tilde{E}\,(v_1)\, \frac{\phi_1 v_1}{\phi v}.\end{aligned} \tag{3.31}$$

Using the Cauchy-Schwarz inequality, we estimate (3.31) as follows:

$$\begin{aligned}&\frac{1}{\phi}\left(\Delta\phi + \frac{\sum_{i=1}^{r} (p_i - 2)\, v_1^{p_i-2}}{\sum_{i=1}^{r} v_1^{p_i-2}}\phi_{11}\right) \\ \geq& \tilde{A}\frac{|\nabla\phi|^2}{2\phi^2} + \tilde{B}\frac{v_1^2}{v^2} + \tilde{C}\left(\frac{f\,(v)}{v_1 \sum_{i=1}^{r} v_1^{p_i-2}}\right)^2 - \frac{\tilde{D}^2}{2\tilde{C}}\left(\frac{\phi_1^2}{2\phi^2}\right) - \tilde{C}\left(\frac{f\,(v)}{v_1 \sum_{i=1}^{r} v_1^{p_i-2}}\right)^2 \\ &- \frac{\tilde{E}^2}{\tilde{B}}\left(\frac{\phi_1^2}{2\phi^2}\right) - \frac{\tilde{B}}{2}\frac{v_1^2}{v^2} \\ =& \tilde{A}\frac{|\nabla\phi|^2}{2\phi^2} + \frac{\tilde{B}}{2}\frac{v_1^2}{v^2} - \frac{1}{2}\left(\frac{\tilde{D}^2}{C} + \frac{2\tilde{E}^2}{\tilde{B}}\right)\frac{\phi_1^2}{2\phi^2}.\end{aligned}$$

Thus, we have completed the proof of Lemma 3.1. □

Step 2. Below, we employ the maximal principle to (3.1). Let $x_0 \in \mathbb{R}^m$ be an arbitrary but fixed point. Given $R > 0$, we set

$$H\,(x) := \left(R^2 - d^2(x)\right)^2 \phi, \quad \text{where} \quad \phi = \frac{|\nabla v|^2}{v^2} \quad \text{and} \quad d(x) = |x - x_0|. \tag{3.32}$$

We observe that within $B := B(x_0, R)$, $H \geq 0$ and $H\,|_{|x-x_0|=R} = 0$. Consequently, H attains its maximum value at some interior point $x^* \in B$. Assuming $\nabla v(x^*) = 0$, it follows that ϕ (and hence H) will be zero at x^*. But then H (and therefore ϕ) would be zero throughout $\overline{B}$. This means $|\nabla v| = 0$ in $\overline{B}$, and, specifically, $\nabla v(x_0) = 0$. Therefore, we assume that $|\nabla v| > 0$ at x^*. Now, at x^*, we have

$$0 = H_k = -2\left(R^2 - d^2(x)\right)\left(d^2(x)\right)_k \phi + \left(R^2 - d^2(x)\right)^2 \phi_k, \quad k = 1, \cdots, m \tag{3.33}$$

and $H_{kl} \leq 0$. According to Lemma 2.2, we have $\Theta H_{kl} \leq 0$ at x^*. Thus, we can conclude that

$$\Delta H + \frac{\sum_{i=1}^{r} (p_i - 2)\, |\nabla v|^{p_i-2}}{\sum_{i=1}^{r} |\nabla v|^{p_i-2}} H_{kl} v_k v_l |\nabla v|^{-2} \leq 0 \tag{3.34}$$

holds at x^*. By using (3.32), we obtain

$$\left(d^2(x)\right)_{11} = 2.$$

Therefore, from (3.34) we obtain

$$\begin{aligned}
&\Delta H + \frac{\sum_{i=1}^r (p_i - 2)\, v_1^{p_i-2}}{\sum_{i=1}^r v_1^{p_i-2}} H_{11}\\
=&2\left|\nabla\left(d^2(x)\right)\right|^2 \phi - 2\left(R^2 - d^2(x)\right)\Delta\left(d^2(x)\right)\phi - 4\left(R^2 - d^2(x)\right)\nabla\left(d^2(x)\right)\cdot\nabla\phi\\
&+\left(R^2 - d^2(x)\right)^2 \Delta\phi + \frac{\sum_{i=1}^r (p_i - 2)\, v_1^{p_i-2}}{\sum_{i=1}^r v_1^{p_i-2}}\Big[2\left(\left(d^2(x)\right)_1\right)^2\phi - 4\left(R^2 - d^2(x)\right)\phi\\
&-4\left(R^2 - d^2(x)\right)\left(d^2(x)\right)_1 \phi_1 + \left(R^2 - d^2(x)\right)^2 \phi_{11}\Big]\\
\le& 0.
\end{aligned}$$

We note that

$$\left|\nabla\left(d^2(x)\right)\right|^2 = 4d^2(x) \quad \text{and} \quad \Delta\left(d^2(x)\right) = 2m.$$

Therefore, using the two equations above for further calculation, we obtain

$$\begin{aligned}
&\Delta\phi + \frac{\sum_{i=1}^r (p_i - 2)\, v_1^{p_i-2}}{\sum_{i=1}^r v_1^{p_i-2}}\phi_{11}\\
\le& -\frac{2\left|\nabla\left(d^2(x)\right)\right|^2\phi}{(R^2 - d^2(x))^2} + \frac{2\Delta\left(d^2(x)\right)\phi}{R^2 - d^2(x)} + \frac{4\nabla\left(d^2(x)\right)\cdot\nabla\phi}{R^2 - d^2(x)}\\
&-\frac{\sum_{i=1}^r (p_i - 2)\, v_1^{p_i-2}}{\sum_{i=1}^r v_1^{p_i-2}}\Big[\frac{2\left(\left(d^2(x)\right)_1\right)^2\phi}{(R^2 - d^2(x))^2} - \frac{4\phi}{R^2 - d^2(x)} - \frac{4\left(d^2(x)\right)_1\phi_1}{R^2 - d^2(x)}\Big]\\
\le& -\frac{8d^2(x)}{(R^2 - d^2(x))^2}\phi + \frac{4m}{R^2 - d^2(x)}\phi + \frac{4\nabla\left(d^2(x)\right)\cdot\nabla\phi}{R^2 - d^2(x)}\\
&+\frac{\sum_{i=1}^r (p_i - 2)\, v_1^{p_i-2}}{\sum_{i=1}^r v_1^{p_i-2}}\Big[-\frac{2\left(\left(d^2(x)\right)_1\right)^2}{(R^2 - d^2(x))^2}\phi + \frac{4}{R^2 - d^2(x)}\phi + \frac{4\left(d^2(x)\right)_1}{R^2 - d^2(x)}\phi_1\Big].
\end{aligned} \tag{3.35}$$

In addition, according to (3.33), we obtain

$$\phi_1 = \frac{2\left(d^2(x)\right)_1}{R^2 - d^2(x)}\phi \quad \text{and} \quad \nabla\phi = \frac{2\phi}{R^2 - d^2(x)}\nabla\left(d^2(x)\right).$$

Substituting the above two equations into (3.35) yields

$$\begin{aligned}
&\Delta\phi + \frac{\sum_{i=1}^r (p_i - 2)\, v_1^{p_i-2}}{\sum_{i=1}^r v_1^{p_i-2}}\phi_{11}\\
\le&\phi\Big\{\frac{24d^2(x)}{(R^2 - d^2(x))^2} + \frac{4m}{R^2 - d^2(x)} + \frac{\sum_{i=1}^r (p_i - 2)\, v_1^{p_i-2}}{\sum_{i=1}^r v_1^{p_i-2}}\Big[\frac{6\left(\left(d^2(x)\right)_1\right)^2}{(R^2 - d^2(x))^2} + \frac{4}{R^2 - d^2(x)}\Big]\Big\}.
\end{aligned}$$

Since $\left(\left(d^2(x)\right)_1\right)^2 \leq \left|\nabla\left(d^2(x)\right)\right|^2 = 4d^2(x)$, we obtain

$$\begin{aligned}&\frac{1}{\phi}\left(\Delta\phi + \frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r} v_1^{p_i-2}}\phi_{11}\right)\\ \leq&\frac{24d^2(x)}{(R^2-d^2(x))^2} + \frac{4m}{R^2-d^2(x)} + \left|\frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r} v_1^{p_i-2}}\right|\left(\frac{24d^2(x)}{(R^2-d^2(x))^2} + \frac{4}{R^2-d^2(x)}\right).\end{aligned}$$

Moreover, since $d^2(x) < R^2$ holds in $B(x_0, R)$, it follows that

$$\begin{aligned}&\frac{1}{\phi}\left(\Delta\phi + \frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r} v_1^{p_i-2}}\phi_{11}\right)\\ \leq&\frac{24R^2}{(R^2-d^2(x))^2} + \frac{4mR^2}{(R^2-d^2(x))^2} + \left|\frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r} v_1^{p_i-2}}\right|\left(\frac{24R^2}{(R^2-d^2(x))^2} + \frac{4R^2}{(R^2-d^2(x))^2}\right).\end{aligned}\tag{3.36}$$

Step 3. By combining (3.1) and (3.36), we get

$$\begin{aligned}&\tilde{A}\frac{|\nabla\phi|^2}{2\phi^2} + \frac{\tilde{B}}{2}\frac{v_1^2}{v^2} - \frac{1}{2}\left(\frac{\tilde{D}^2}{C} + \frac{2\tilde{E}^2}{\tilde{B}}\right)\frac{\phi_1^2}{2\phi^2}\\ \leq&\frac{24R^2}{(R^2-d^2(x))^2} + \frac{4mR^2}{(R^2-d^2(x))^2} + \left|\frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r} v_1^{p_i-2}}\right|\left(\frac{24R^2}{(R^2-d^2(x))^2} + \frac{4R^2}{(R^2-d^2(x))^2}\right).\end{aligned}\tag{3.37}$$

In other words, we have

$$0 \leq \frac{v_1^2}{v^2} \leq \frac{2}{\tilde{B}}\left[\frac{4R^2(6+m)}{(R^2-d^2(x))^2} + \left|\frac{\sum_{i=1}^{r}(p_i-2)\,v_1^{p_i-2}}{\sum_{i=1}^{r} v_1^{p_i-2}}\right|\frac{28R^2}{(R^2-d^2(x))^2} + \left|\frac{1}{2}\left(\frac{\tilde{D}^2}{\tilde{C}} + \frac{2\tilde{E}^2}{\tilde{B}}\right) - \tilde{A}\right|\frac{|\nabla\phi|^2}{2\phi^2}\right].\tag{3.38}$$

From (3.33), at x^* we have

$$\frac{\nabla\phi}{\phi} = \frac{2\nabla d^2(x)}{R^2-d^2(x)}, \quad \text{so that} \quad \frac{|\nabla\phi|^2}{2\phi^2} = \frac{8d^2(x)}{(R^2-d^2(x))^2}.\tag{3.39}$$

Therefore, at x^* we have

$$0 \leq \frac{|\nabla v|^2}{v^2} \leq \frac{\tilde{C}_0(v_1)R^2}{(R^2-d^2(x))^2},\tag{3.40}$$

where

$$\tilde{C}_0(v_1) := \frac{8}{\tilde{B}}\left[6 + m + 7p_r + 2\left|\frac{1}{2}\left(\frac{\tilde{D}^2(v_1)}{\tilde{C}} + \frac{2\tilde{E}^2(v_1)}{\tilde{B}(v_1)}\right) - \tilde{A}(v_1)\right|\right].\tag{3.41}$$

From Lemma 2.1, $\tilde{C}_0(v_1)$ is bounded by a positive constant M. Moreover, we observe that

$$H(x^*) = \frac{|\nabla v|^2}{v^2}\left(R^2 - d^2(x)\right)^2 \leq MR^2.$$

Since $H(x_0) \leq H(x^*)$, we conclude that

$$\frac{|\nabla v|^2}{v^2}R^4 = H(x_0) \leq H(x^*) \leq MR^2$$

holds at x_0. Thus, at x_0, we have the estimate

$$\frac{|\nabla v|^2}{v^2} \leq \frac{M}{R^2}.$$

Letting $R \to \infty$, we observe that $|\nabla v| = 0$ at the point x_0. The arbitrariness of x_0 leads us to conclude that $\nabla v \equiv 0$ in $\mathbb{R}^m$. Therefore, the proof of Theorem 1.1 is complete.

4. Proofs of related Corollaries

Proof of Corollary 1.2. We see that $f(v) = -\sum_{i=1}^{k} a_i v^{\alpha_i} \leq 0$. According to Theorem 1.1, we need to verify condition (1.10), namely

$$f'(t) \leq (p_r - 1)\frac{m+1}{m-1}\frac{f(t)}{t}.$$

After substituting $f(t) = -\sum_{i=1}^{k} a_i t^{\alpha_i}$ and $f'(t) = -\sum_{i=1}^{k} \alpha_i a_i t^{\alpha_i - 1}$ into the above inequality, we obtain

$$\sum_{i=1}^{k} \alpha_i a_i v^{\alpha_i - 1} \geq (p_r - 1)\frac{m+1}{m-1}\sum_{i=1}^{k} a_i v^{\alpha_i - 1}.$$

It is sufficient to have $\alpha_i \geq (p_r - 1)\frac{m+1}{m-1}$ for all i, i.e., $\min\limits_{1\leq i\leq k}\{\alpha_i\} \geq (p_r - 1)\frac{m+1}{m-1}$. This completes the proof of Corollary 1.2.

Proof of Corollary 1.3. Note that $f(v) = \sum_{i=1}^{k} a_i v^{\alpha_i} \geq 0$. By Theorem 1.1, it is necessary to verify condition (1.9), that is,

$$f'(t) \leq (p_1 - 1)\frac{m+1}{m-1}\frac{f(t)}{t}.$$

After plugging $f(t) = \sum_{i=1}^{k} a_i t^{\alpha_i}$ and $f'(t) = \sum_{i=1}^{k} \alpha_i a_i t^{\alpha_i - 1}$ into the above inequality, we obtain

$$\sum_{i=1}^{k} \alpha_i a_i v^{\alpha_i - 1} \leq (p_1 - 1)\frac{m+1}{m-1}\sum_{i=1}^{k} a_i v^{\alpha_i - 1}.$$

It is sufficient to have $\alpha_i \leq (p_1 - 1)\frac{m+1}{m-1}$ for all i, i.e., $\max\limits_{1\leq i\leq k} \alpha_i \leq (p_1 - 1)\frac{m+1}{m-1}$. This completes the proof of Corollary 1.3.

Proof of Corollary 1.4. (i) When $f(v) = \sum_{i=1}^{k} a_i v^{\alpha_i} - \sum_{j=1}^{l} b_j v^{\beta_j} \geq 0$, according to Theorem 1.1, one needs to verify condition (1.9). After plugging $f(t) = \sum_{i=1}^{k} a_i t^{\alpha_i} - \sum_{j=1}^{l} b_j t^{\beta_j}$ and $f'(t) = \sum_{i=1}^{k} \alpha_i a_i t^{\alpha_i - 1} - \sum_{j=1}^{l} \beta_j b_j t^{\beta_j - 1}$ into (1.9), we obtain

$$\sum_{i=1}^{k} \alpha_i a_i v^{\alpha_i - 1} - \sum_{j=1}^{l} \beta_j b_j v^{\beta_j - 1} \leq (p_1 - 1)\frac{m+1}{m-1}\left(\sum_{i=1}^{k} a_i v^{\alpha_i - 1} - \sum_{j=1}^{l} b_j v^{\beta_j - 1}\right).$$

A sufficient condition for this to hold is $\alpha_i \leq (p_1 - 1)\frac{m+1}{m-1} \leq \beta_j$ for all i, j, i.e., $\max\limits_{1\leq i\leq k}\{\alpha_i\} \leq (p_1 - 1)\frac{m+1}{m-1} \leq \min\limits_{1\leq j\leq l}\{\beta_j\}$.

(ii) When $f(v) = \sum_{i=1}^{k} a_i v^{\alpha_i} - \sum_{j=1}^{l} b_j v^{\beta_j} \leq 0$, according to Theorem 1.1, one needs to verify condition (1.10). After substituting $f(t)$ and $f'(t)$ into (1.10), we obtain

$$\sum_{i=1}^{k} \alpha_i a_i v^{\alpha_i - 1} - \sum_{j=1}^{l} \beta_j b_j v^{\beta_j - 1} \leq (p_r - 1)\frac{m+1}{m-1}\left(\sum_{i=1}^{k} a_i v^{\alpha_i - 1} - \sum_{j=1}^{l} b_j v^{\beta_j - 1}\right),$$

A sufficient condition for this to hold is $\alpha_i \leq (p_r - 1)\frac{m+1}{m-1} \leq \beta_j$ for all i, j, i.e., $\max\limits_{1\leq i\leq k}\{\alpha_i\} \leq (p_r - 1)\frac{m+1}{m-1} \leq \min\limits_{1\leq j\leq l}\{\beta_j\}$. This completes the proof of Corollary 1.4.

5. Conclusions

Using the Bernstein method, we proved Liouville-type theorems for positive solutions to $\Delta_{p_1,\cdots,p_r} v + f(v) = 0$ in $\mathbb{R}^m$, where $\Delta_{p_1,\cdots,p_r} v := \mathrm{div}(\sum_{i=1}^{r} |\nabla v|^{p_i - 2}\nabla v)$ was $(p_1, \cdots, p_r)$-Laplacian. The particular cases of $\Delta_{p_1,\cdots,p_r}$ were the p-Laplacian and the (p, q)-Laplacian. The nonlinear item $f(v)$ was not required to be positive, which made our results applicable to many classical equations. Our results improved and included many previously known results as special cases.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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