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Research article

Algebraic Schouten solitons associated to the Bott connection on three-dimensional Lorentzian Lie groups

Jinguo Jiang*

School of Mathematical science, Nankai University, Tianjin 300071, China

* **Correspondence:** Email: jiangjinguo@mail.nankai.edu.cn.

Abstract: In this paper, I define and classify the algebraic Schouten solitons associated with the Bott connection on three-dimensional Lorentzian Lie groups with three different distributions.

Keywords: Pseudo-Riemannian metric; algebraic Schouten solitons; Bott connection; Lorentzian Lie groups

1. Introduction

Einstein metrics are pivotal in numerous domains of mathematical physics and differential geometry. They are also of interest in pure mathematics, particularly in the fields of geometric analysis and algebraic geometry. An Einstein metric can be regarded as a fixed solution (up to diffeomorphism and scaling) of the Hamilton Ricci flow equation. On a Riemannian manifold (M, g), a Ricci soliton is said to exist when there is a smooth vector field X and a constant λ in the reals that satisfy the condition stated below:

$$R_{ij} + \frac{1}{2}(\mathcal{L}_X g)_{ij} = \lambda g_{ij},$$

where R_{ij} is the Ricci curvature tensor, and $\mathcal{L}_X g$ stands for the Lie derivative of g with respect to the vector field X. This concept was introduced by Hamilton in [1], and later utilized by Perelman in his proof of the long-standing Poincare conjecture [2]. Lauret further generalized the notion of Einstein metrics to algebraic Ricci solitons in the Riemannian context, introducing them as a natural extension in [3]. Subsequently, Onda and Batat applied this framework to pseudo-Riemannian Lie groups, achieving a complete classification of algebraic Ricci solitons in three-dimensional Lorentzian Lie groups in [4]. Additionally, they proved that within the framework of pseudo-Riemannian manifolds, there is algebraic Ricci solitons that are not of the conventional Ricci soliton type.

In [5], Etayo and Santamaria explored the concept of distinguished connections on $(J^2 = \pm 1)$-metric manifolds. This sparked interest among mathematicians in studying Ricci solitons linked to various affine connections, which can be found in [6–8]. The Bott connection was introduced in earlier works (see [9–11]). In [12], the authors developed a theory on geodesic variations under metric changes in a geodesic foliation, with the Bott connection serving as the primary natural connection respecting the foliation's structure. In [13], Wu and Wang studied affine Ricci solitons and quasi-statistical structures on three-dimensional Lorentzian Lie groups associated with the Bott connection. Furthermore, in [14, 15], the authors examined the algebraic schouten solitons and affine Ricci solitons concerning various affine connections.

Inspired by Lauret's work, Wears introduced algebraic T-solitons, linking them to T-solitons in [16]. Later, in [7], Azami introduced Schouten solitons, as a new type of generalized Ricci soliton. In this paper, I focus on algebraic Schouten solitons concerning the Bott connection with three distributions, aiming to classify and describe them on three-dimensional Lorentzian Lie groups.

In Section 2, I introduce the fundamental notions associated with three-dimensional Lie groups and algebraic Schouten soliton. In Sections 3–5, I discuss and present algebraic Schouten solitons concerning the Bott connection on three-dimensional Lorentzian Lie groups, each focusing on a different type of distribution.

2. Preliminaries

In [17], Milnor conducted a survey of both classical and recent findings on left-invariant Riemannian metrics on Lie groups, particularly on three-dimensional unimodular Lie groups. Furthermore, in [18], Rahmani classified three-dimensional unimodular Lie groups in the Lorentzian context. The non-unimodular cases were handled in [19, 20]. Throughout this paper, I use $\{G_i\}_{i=1}^7$ for connected, simply connected three-dimensional Lie groups equipped with a left-invariant Lorentzian metric g. Their corresponding Lie algebras are denoted by $\{\mathfrak{g}_i\}_{i=1}^7$, and each possess a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$ (with e_3 timelike, see [4]). Let ∇^{LC} and R^{LC} denote the Levi-Civita connection and curvature tensor of G_i, respectively, then

$$R^{LC}(X,Y)Z = \nabla^{LC}_X \nabla^{LC}_Y Z - \nabla^{LC}_Y \nabla^{LC}_X Z - \nabla^{LC}_{[X,Y]} Z.$$

The Ricci tensor of (G_i, g) is defined as follows:

$$\rho^{LC}(X,Y) = -g(R^{LC}(X,e_1)Y,e_1) - g(R^{LC}(X,e_2)Y,e_2) + g(R^{LC}(X,e_3)Y,e_3).$$

Moreover, I have the expression for the Ricci operator Ric^{LC}:

$$\rho^{LC}(X,Y) = g(Ric^{LC}(X),Y).$$

Next, recall the Bott connection, denoted by ∇^B. Consider a smooth manifold (M, g) that is equipped with the Levi-Civita connection ∇, and let TM represent its tangent bundle, spanned by $\{e_1, e_2, e_3\}$. I introduce a distribution D spanned by $\{e_1, e_2\}$ and its orthogonal complement $D^\perp$, which is spanned by $\{e_3\}$. The Bott connection ∇^B associated with distribution D is then defined as follows:

$$\nabla^B_X Y = \begin{cases} \pi_D(\nabla^{LC}_X Y), & X, Y \in \Gamma^\infty(D), \\ \pi_D([X,Y]), & X \in \Gamma^\infty(D^\perp),\ Y \in \Gamma^\infty(D), \\ \pi_{D^\perp}([X,Y]), & X \in \Gamma^\infty(D),\ Y \in \Gamma^\infty(D^\perp), \\ \pi_{D^\perp}(\nabla^{LC}_X Y), & X, Y \in \Gamma^\infty(D^\perp), \end{cases} \tag{2.1}$$

where π_D (respectively, $\pi_{D^\perp}$) denotes the projection onto D (respectively, $D^\perp$). For a more detailed discussion, refer to [9–11, 21]. I denote the curvature tensor of the Bott connection ∇^B by R^B, which is defined as follows:

$$R^B(X,Y)Z = \nabla^B_X\nabla^B_Y Z - \nabla^B_Y\nabla^B_X Z - \nabla^B_{[X,Y]}Z. \tag{2.2}$$

The Ricci tensor ρ^B associated to the connection ∇^B is defined as:

$$\rho^B(X,Y) = \frac{B(X,Y)+B(Y,X)}{2},$$

where

$$B(X,Y) = -g(R^B(X,e_1)Y,e_1) - g(R^B(X,e_2)Y,e_2) + g(R^B(X,e_3)Y,e_3).$$

Using the Ricci tensor ρ^B, the Ricci operator Ric^B is given by:

$$\rho^B(X,Y) = g(Ric^B(X),Y). \tag{2.3}$$

Then, I have the definition of the Schouten tensor as follows:

$$S^B(e_i,e_j) = \rho^B(e_i,e_j) - \frac{s^B}{4}g(e_i,e_j),$$

where s^B represents the scalar curvature. Moreover, I generalized the Schouten tensor to:

$$S^B(e_i,e_j) = \rho^B(e_i,e_j) - s^B\lambda_0 g(e_i,e_j),$$

where λ_0 is a real number. By [22], I obtain the expression of s^B as

$$s^B = \rho^B(e_1,e_1) + \rho^B(e_2,e_2) - \rho^B(e_3,e_3).$$

Definition 1. *A manifold (G_i, g) is called an algebraic Schouten soliton with associated to the connection ∇^B if it satisfies:*

$$Ric^B = (s^B\lambda_0 + c)Id + D^B,$$

where c is a constant, and D^B is a derivation of $\mathfrak{g}_i$, i.e.,

$$D^B[X_1,X_2] = [D^BX_1,X_2] + [X_1,D^BX_2], \quad for\ X_1,X_2 \in \mathfrak{g}_i. \tag{2.4}$$

3. Algebraic Schouten soliton concerning connection ∇^B

In this section, I derive the algebraic criterion for the three-dimensional Lorentzian Lie group to exhibit an algebraic Schouten soliton related to the connection ∇^B. Moreover, I indicate that G_4 and G_7 do not have such solitons.

3.1. Algebraic Schouten soliton of G_1

According to [4], I have the expression for $\mathfrak{g}_1$:

$$[e_1,e_2] = \alpha e_1 - \beta e_3, \quad [e_1,e_3] = -\alpha e_1 - \beta e_2, \quad [e_2,e_3] = \beta e_1 + \alpha e_2 + \alpha e_3,$$

where $\alpha \neq 0$. From this, I derive the following theorem.

Theorem 2. *If (G_1, g) constitutes an algebraic Schouten soliton concerning connection ∇^B; then, it fulfills the conditions: $\beta = 0$ and $c = -\frac{1}{2}\alpha^2 + 2(\alpha^2+\beta^2)\lambda_0$.*

Proof. From [23], the expression for Ric^B is as follows:

$$Ric^B \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -(\alpha^2+\beta^2) & \alpha\beta & \frac{1}{2}\alpha\beta \\ \alpha\beta & -(\alpha^2+\beta^2) & -\frac{1}{2}\alpha^2 \\ -\frac{1}{2}\alpha\beta & \frac{1}{2}\alpha^2 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.$$

The scalar curvature is $s^B = -2(\alpha^2+\beta^2)$. Now, I can express D^B as follows:

$$\begin{cases} D^B e_1 = -(\alpha^2+\beta^2+s\lambda_0+c)e_1 + \alpha\beta e_2 + \dfrac{\alpha\beta}{2}e_3, \\ D^B e_2 = \alpha\beta e_1 - (\alpha^2+\beta^2+s\lambda_0+c)e_2 - \dfrac{\alpha^2}{2}e_3, \\ D^B e_3 = -\dfrac{\alpha\beta}{2}e_1 + \dfrac{\alpha^2}{2}e_2 + (2(\alpha^2+\beta^2)\lambda_0 - c)e_3. \end{cases}$$

Therefore, based on Eq (2.4), there is an algebraic Schouten soliton associated with ∇^B on (G_1, g), if and only if the following condition satisfies:

$$\begin{cases} \dfrac{5}{2}\alpha^2\beta + 2\beta^3 - 2\lambda_0\beta(\alpha^2+\beta^2) + c\beta = 0, \\ \dfrac{\alpha^3}{2} + 2\alpha\beta^2 - 2\lambda_0\alpha(\alpha^2+\beta^2) + c\alpha = 0, \\ \alpha^2\beta = 0, \\ 2\alpha^2\beta + \beta^3 - 4\lambda_0\beta(\alpha^2+\beta^2) + 2c\beta = 0, \\ \dfrac{\alpha^3}{2} + \dfrac{3}{2}\alpha\beta^2 - 2\lambda_0\alpha(\alpha^2+\beta^2) + c\alpha = 0. \end{cases}$$

Since $\alpha \neq 0$, I have $\beta = 0$ and $c = -\frac{1}{2}\alpha^2 + 2(\alpha^2+\beta^2)\lambda_0$. □

3.2. Algebraic Schouten soliton of G_2

According to [4], I have the expression for $\mathfrak{g}_2$:

$$[e_1, e_2] = \gamma e_2 - \beta e_3, \quad [e_1, e_3] = -\beta e_2 - \gamma e_3, \quad [e_2, e_3] = \alpha e_1,$$

where $\gamma \neq 0$. From this, I derive the following theorem.

Theorem 3. *If (G_2, g) constitutes an algebraic Schouten soliton concerning the connection ∇^B, then it fulfills the conditions: $\alpha = 0$ and $c = -\beta^2 - \gamma^2 + (\beta^2 + 2\gamma^2)\lambda_0$.*

Proof. From [23], the expression for Ric^B is as follows:

$$Ric^B \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -(\beta^2+\gamma^2) & 0 & 0 \\ 0 & -(\gamma^2+\alpha\beta) & \frac{\alpha\gamma}{2} \\ 0 & -\frac{\alpha\gamma}{2} & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.$$

The scalar curvature is $s^B = -(\beta^2 + 2\gamma^2 + \alpha\beta)$. Now, I can express D^B as follows:

$$\begin{cases} D^B e_1 = -(\beta^2 + \gamma^2 - (\beta^2 + 2\gamma^2 + \alpha\beta)\lambda_0 + c)e_1, \\ D^B e_2 = -(\gamma^2 + \alpha\beta - (\beta^2 + 2\gamma^2 + \alpha\beta)\lambda_0 + c)e_2 + \dfrac{\alpha\gamma}{2} e_3, \\ D^B e_3 = -\dfrac{\alpha\gamma}{2} e_2 + ((\beta^2 + 2\gamma^2 + \alpha\beta)\lambda_0 - c)e_3. \end{cases}$$

Therefore, based on Eq (2.4), there is an algebraic Schouten soliton associated with ∇^B on (G_2, g), if and only if the following condition satisfies:

$$\begin{cases} \alpha\gamma^2 - \beta^3 + \alpha\beta^2 + (\beta^2 + 2\gamma^2 + \alpha\beta)\beta\lambda_0 - c\beta = 0, \\ \gamma(\beta^2 + \gamma^2 + \alpha\beta - (\beta^2 + 2\gamma^2 + \alpha\beta)\lambda_0 + c) = 0, \\ \alpha\gamma^2 - \beta^3 - 2\beta\gamma^2 - \alpha\beta^2 + (\beta^2 + 2\gamma^2 + \alpha\beta)\beta\lambda_0 - c\beta = 0, \\ \alpha(-\beta^2 + \alpha\beta - (\beta^2 + 2\gamma^2 + \alpha\beta)\lambda_0 + c) = 0. \end{cases}$$

Suppose that $\alpha = 0$, then $c = -\beta^2 - \gamma^2 + (\beta^2 + 2\gamma^2)\lambda_0$. If $\alpha \neq 0$, I have

$$\begin{cases} \beta(\beta^2 + \gamma^2 - (\beta^2 + 2\gamma^2 + \alpha\beta)\lambda_0 + c) = 0, \\ \beta^2 + \gamma^2 + \alpha\beta - (\beta^2 + 2\gamma^2 + \alpha\beta)\lambda_0 + c = 0, \\ -\beta^2 + \alpha\beta - (\beta^2 + 2\gamma^2 + \alpha\beta)\lambda_0 + c = 0. \end{cases}$$

Since $\gamma \neq 0$, solving the equations of the above system gives $2\beta^2 + \gamma^2 = 0$, which is a contradiction. □

3.3. Algebraic Schouten soliton of G_3

According to [4], I have the expression for $\mathfrak{g}_3$:

$$[e_1, e_2] = -\gamma e_3, \quad [e_1, e_3] = -\beta e_2, \quad [e_2, e_3] = \alpha e_1.$$

From this, I derive the following theorem.

Theorem 4. *If (G_3, g) is an algebraic Schouten soliton concerning connection ∇^B; then, one of the following cases holds:*

i. $\alpha = \beta = \gamma = 0$, *for all* c;

ii. $\alpha \neq 0, \beta = \gamma = 0, c = 0$;

iii. $\alpha = \gamma = 0, \beta \neq 0, c = 0$;

iv. $\alpha \neq 0, \beta \neq 0, \gamma = 0, c = 0$;

v. $\alpha = \beta = 0, \gamma \neq 0, c = 0$;

vi. $\alpha \neq 0, \beta = 0, \gamma \neq 0, c = -\alpha\gamma + \alpha\gamma\lambda_0$;

vii. $\alpha = 0, \beta \neq 0, \gamma \neq 0, c = -\beta\gamma + \beta\gamma\lambda_0$.

Proof. From [23], the expression for Ric^B is as follows:

$$Ric^B\begin{pmatrix}e_1\\e_2\\e_3\end{pmatrix}=\begin{pmatrix}-\beta\gamma & 0 & 0\\0 & -\alpha\gamma & 0\\0 & 0 & 0\end{pmatrix}\begin{pmatrix}e_1\\e_2\\e_3\end{pmatrix}.$$

The scalar curvature is $s^B = -\gamma(\alpha+\beta)$. Now, I can express D^B as follows:

$$\begin{cases}D^B e_1 = -(\beta\gamma - \gamma(\alpha+\beta)\lambda_0 + c)e_1,\\ D^B e_2 = -(\alpha\gamma - \gamma(\alpha+\beta)\lambda_0 + c)e_2,\\ D^B e_3 = (\gamma(\alpha+\beta)\lambda_0 - c)e_3.\end{cases}$$

Therefore, based on Eq (2.4), there is an algebraic Schouten soliton associated with ∇^B on (G_3, g), if and only if the following condition satisfies:

$$\begin{cases}\beta\gamma^2 + \alpha\gamma^2 - \gamma^2(\alpha+\beta)\lambda_0 + c\gamma = 0,\\ \beta^2\gamma - \beta\gamma(\alpha+\beta)\lambda_0 + c\beta = 0,\\ \alpha^2\gamma - \alpha\beta\gamma - \alpha\gamma(\alpha+\beta)\lambda_0 + c\alpha = 0.\end{cases} \tag{3.1}$$

Assuming that $\gamma = 0$. In this case, (3.1) becomes:

$$\begin{cases}\beta c = 0,\\ \alpha c = 0.\end{cases} \tag{3.2}$$

If $\beta = 0$, for Cases *i* and *ii*, system (3.1) holds. If $\beta \neq 0$, for Cases *iii* and *iv*, system (3.1) holds. Then, I assume that $\gamma \neq 0$. Thus, system (3.1) becomes:

$$\begin{cases}\beta\gamma + \alpha\gamma - \gamma(\alpha+\beta)\lambda_0 + c,\\ \alpha\beta = 0.\end{cases} \tag{3.3}$$

If $\beta = 0$, I have Cases *v* and *vi*. If $\beta \neq 0$, for Case *vii*, system (3.1) holds □

3.4. Algebraic Schouten soliton of G_4

According to [4], $\mathfrak{g}_4$ takes the following form:

$$[e_1, e_2] = -e_2 + (2\eta - \beta)e_3, \quad [e_1, e_3] = e_3 - \beta e_2, \quad [e_2, e_3] = \alpha e_1,$$

where $\eta = 1$ *or* -1. From this, I derive the following theorem.

Theorem 5. (G_4, g) *is not an algebraic Schouten soliton concerning connection* ∇^B.

Proof. According to [23], the expression for Ric^B is derived as follows:

$$Ric^B\begin{pmatrix}e_1\\e_2\\e_3\end{pmatrix}=\begin{pmatrix}-(\beta-\eta)^2 & 0 & 0\\0 & 2\alpha\eta-\alpha\beta-1 & -\frac{1}{2}\alpha\\0 & \frac{1}{2}\alpha & 0\end{pmatrix}\begin{pmatrix}e_1\\e_2\\e_3\end{pmatrix}.$$

The scalar curvature is $s^B = -((\beta-\eta)^2 + \alpha\beta - 2\alpha\eta + 1)$. Now, I can express D^B as follows:

$$\begin{cases} D^B e_1 = -((\beta-\eta)^2 - ((\beta-\eta)^2 + \alpha\beta - 2\alpha\eta + 1)\lambda_0 + c)e_1, \\ D^B e_2 = (2\alpha\eta - \alpha\beta - 1 + ((\beta-\eta)^2 + \alpha\beta - 2\alpha\eta + 1)\lambda_0 - c)e_2 - \dfrac{a}{2}e_3, \\ D^B e_3 = \dfrac{\alpha}{2}e_2 + (((\beta-\eta)^2 + \alpha\beta - 2\alpha\eta + 1)\lambda_0 - c)e_3. \end{cases}$$

Therefore, based on Eq (2.4), there is an algebraic Schouten soliton associated with ∇^B on (G_4, g), if and only if the following condition satisfies:

$$\begin{cases} -(2\eta-\beta)((\beta-\eta)^2 - 2\alpha\eta + \alpha\beta + 1 - ((\beta-\eta)^2 + \alpha\beta - 2\alpha\eta + 1)\lambda_0 + c) = \alpha, \\ \beta((\beta-\eta)^2 + 2\alpha\eta - \alpha\beta - 1 - ((\beta-\eta)^2 + \alpha\beta - 2\alpha\eta + 1)\lambda_0 + c) = \alpha, \\ (\beta-\eta)^2 - ((\beta-\eta)^2 + \alpha\beta - 2\alpha\eta + 1)\lambda_0 + c = \alpha(\eta-\beta), \\ \alpha((\beta-\eta)^2 + 2\alpha\eta - \alpha\beta - 1 + ((\beta-\eta)^2 + \alpha\beta - 2\alpha\eta + 1)\lambda_0 - c) = 0. \end{cases} \tag{3.4}$$

I now analyze the system under different assumptions.

Assuming that $\alpha = 0$. Then, system (3.4) becomes:

$$\begin{cases} (2\eta-\beta)((\beta-\eta)^2 + 1 - ((\beta-\eta)^2 + 1)\lambda_0 + c) = 0, \\ \beta((\beta-\eta)^2 - 1 - ((\beta-\eta)^2 + 1)\lambda_0 + c) = 0, \\ (\beta-\eta)^2 - ((\beta-\eta)^2 + 1)\lambda_0 + c = 0. \end{cases}$$

Upon direct calculation, it is evident that $(2\eta - \beta) = \beta = 0$, which leads to a contradiction.

If $\alpha \neq 0$, we have

$$\begin{cases} (2\eta-\beta)(\alpha\eta - 1) = \alpha, \\ \beta(3\alpha\eta - 2\alpha\beta - 1) = \alpha, \\ (\beta-\eta)^2 - \alpha(\eta-\beta) - ((\beta-\eta)^2 - 2\alpha\eta + \alpha\beta + 1)\lambda_0 + c = 0, \\ \alpha((\beta-\eta)^2 + 2\alpha\eta - \alpha\beta - 1 + ((\beta-\eta)^2 - 2\alpha\eta + \alpha\beta + 1)\lambda_0 - c) = 0. \end{cases} \tag{3.5}$$

From the last two equations in (3.5), we have $(\beta - \eta)^2 = (1 - \alpha\eta)$. Substituting into the first two equations in (3.5) yields $\alpha = \eta$, which is a contradiction. Therefore, system (3.4) has no solutions. Then, the theorem is true. □

3.5. Algebraic Schouten soliton of G_5

According to [4], we have the expression for $\mathfrak{g}_5$:

$$[e_1, e_2] = 0, \quad [e_1, e_3] = \alpha e_1 + \beta e_2, \quad [e_2, e_3] = \gamma e_1 + \delta e_2,$$

where $\alpha + \delta \neq 0$ and $\alpha\gamma - \beta\delta = 0$. From this, we derive the following theorem.

Theorem 6. *If (G_5, g) constitutes an algebraic Schouten soliton concerning connection ∇^B, then $c = 0$.*

Proof. According to [23], the expression for Ric^B is derived as follows:

$$Ric^B \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.$$

The scalar curvature is $s^B = 0$. Now, I can express D^B as follows:

$$\begin{cases} D^B e_1 = -ce_1, \\ D^B e_2 = -ce_2, \\ D^B e_3 = -ce_3. \end{cases}$$

Hence, by (2.4), I conclude that there is an algebraic Schouten soliton associated with ∇^B on (G_5, g). Furthermore, for this algebraic Schouten soliton, I have $c = 0$. □

3.6. Algebraic Schouten soliton of G_6

According to [4], I have the expression for $\mathfrak{g}_6$:

$$\begin{aligned} [e_1, e_2] &= \alpha e_2 + \beta e_3, \\ [e_1, e_3] &= \gamma e_2 + \delta e_3, \\ [e_2, e_3] &= 0, \end{aligned}$$

where $\alpha + \delta \neq 0$ and $\alpha\gamma - \beta\delta = 0$. From this, I derive the following theorem.

Theorem 7. *If (G_6, g) constitutes an algebraic Schouten soliton concerning connection ∇^B, then one of the following cases holds:*

i. $\alpha = \beta = \gamma = 0$, $\delta \neq 0$, $c = 0$;

ii. $\alpha = \beta = 0$, $\gamma \neq 0$, $\delta \neq 0$, $c = 0$;

iii. $\alpha \neq 0$, $\beta = \gamma = \delta = 0$, $c = -\alpha^2 + 2\alpha^2\lambda_0$;

iv. $\alpha \neq 0$, $\beta = \gamma = 0$, $\delta \neq 0$, $c = -\alpha^2 + 2\alpha^2\lambda_0$.

Proof. From [23], I have the expression for Ric^B as follows:

$$Ric^B \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -(\alpha^2 + \beta\gamma)^2 & 0 & 0 \\ 0 & -\alpha^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.$$

The scalar curvature is $s^B = -(2\alpha^2 + \beta\gamma)$. Now, I can express D^B as follows:

$$\begin{cases} D^B e_1 = -(\alpha^2 + \beta\gamma - (2\alpha^2 + \beta\gamma)\lambda_0 + c)e_1, \\ D^B e_2 = -(\alpha^2 - (2\alpha^2 + \beta\gamma)\lambda_0 + c)e_2, \\ D^B e_3 = ((2\alpha^2 + \beta\gamma)\lambda_0 - c)e_3. \end{cases} \tag{3.6}$$

Therefore, based on Eq (2.4), there is an algebraic Schouten soliton associated with ∇^B on (G_6, g), if and only if the following condition satisfies:

$$\begin{cases}\beta(2\alpha^2+\beta\gamma-(2\alpha^2+\beta\gamma)\lambda_0+c)=0,\\ \alpha(\alpha^2+\beta\gamma-(2\alpha^2+\beta\gamma)\lambda_0+c)=0,\\ \gamma(\beta\gamma-(2\alpha^2+\beta\gamma)\lambda_0+c)=0,\\ \delta(\alpha^2+\beta\gamma-(2\alpha^2+\beta\gamma)\lambda_0+c)=0.\end{cases}$$

From the first equation above, we have either $\beta = 0$ or $\beta \neq 0$. I now analyze the system under different assumptions.

Assuming that $\beta = 0$, I have:

$$\begin{cases}\alpha(\alpha^2-2\alpha^2\lambda_0+c)=0,\\ \gamma(-2\alpha^2\lambda_0+c)=0,\\ \delta(\alpha^2-2\alpha^2\lambda_0+c)=0.\end{cases} \tag{3.7}$$

Given $\alpha\gamma - \beta\delta = 0$ and $\alpha + \delta \neq 0$, assume first that $\alpha = 0$. In this case, system (3.7) can be simplified to:

$$s^B\lambda_0 + c = 0.$$

Then, I have Cases *i* and *ii*. If $\beta \neq 0$, system (3.7) becomes:

$$\alpha^2 + s^B\lambda_0 + c = 0. \tag{3.8}$$

Then, I have Cases *iii* and *iv*.

If $\beta \neq 0$, system (3.7) becomes:

$$\begin{cases}\beta(2\alpha^2+\beta\gamma-(2\alpha^2+\beta\gamma)\lambda_0+c)=0,\\ \alpha(\alpha^2+\beta\gamma-(2\alpha^2+\beta\gamma)\lambda_0+c)=0,\end{cases} \tag{3.9}$$

which is a contradiction. □

3.7. Algebraic Schouten soliton of G_7

According to [4], I have the expression for $\mathfrak{g}_7$:

$$[e_1, e_2] = -\alpha e_1 - \beta e_2 - \beta e_3,$$
$$[e_1, e_3] = \alpha e_1 + \beta e_2 + \beta e_3,$$
$$[e_2, e_3] = \gamma e_1 + \delta e_2 + \delta e_3,$$

where $\alpha + \delta \neq 0$ and $\alpha\delta = 0$. From this, I derive the following theorem.

Theorem 8. *(G_7, g) is not an algebraic Schouten soliton concerning connection ∇^B.*

Proof. From [23], I have the expression for Ric^B as follows:

$$Ric^B\begin{pmatrix}e_1\\e_2\\e_3\end{pmatrix}=\begin{pmatrix}-\alpha^2 & \frac{1}{2}\beta(\delta-\alpha) & -\delta(\alpha+\delta)\\ \frac{1}{2}\beta(\delta-\alpha) & -(\alpha^2+\beta^2+\beta\delta) & -\delta^2-\frac{1}{2}(\beta\gamma+\alpha\delta)\\ \delta(\alpha+\delta) & \delta^2+\frac{1}{2}(\beta\gamma+\alpha\delta) & 0\end{pmatrix}\begin{pmatrix}e_1\\e_2\\e_3\end{pmatrix}.$$

The scalar curvature is $s^B = -(2\alpha^2 + \beta^2 + \beta\delta)$. Now, I can express D^B as follows:

$$\begin{cases} D^B e_1 = -(\alpha^2 - (2\alpha^2 + \beta^2 + \beta\delta)\lambda_0 + c)e_1 + \frac{1}{2}\beta(\delta - \alpha)e_2 - \delta(\alpha + \delta)e_3, \\ D^B e_2 = \frac{1}{2}\beta(\delta - \alpha)e_1 - (\alpha^2 + \beta^2 + \beta\delta - (2\alpha^2 + \beta^2 + \beta\delta)\lambda_0 + c)e_2 - (\delta^2 + \frac{1}{2}(\beta\gamma + \alpha\delta))e_3, \\ D^B e_3 = \delta(\alpha + \delta)e_1 + (\delta^2 + \frac{1}{2}(\beta\gamma + \alpha\delta))e_2 + ((2\alpha^2 + \beta^2 + \beta\delta)\lambda_0 - c)e_3. \end{cases}$$

Therefore, based on Eq (2.4), there is an algebraic Schouten soliton associated with ∇^B on (G_7, g), if and only if the following condition satisfies:

$$\begin{cases} \alpha(\alpha^2 + \beta^2 + \beta\alpha + s^B\lambda_0 + c) + (\gamma + \beta)\delta(\alpha + \delta) + \frac{1}{2}\beta^2(\delta - \alpha) = \alpha(\delta^2 + \frac{1}{2}(\beta\gamma + \alpha\delta)), \\ \beta(\alpha + s^B\lambda_0 + c) + \delta^2(\alpha + \delta) + \frac{1}{2}\alpha\beta(\delta - \alpha) = 0, \\ \beta(2\alpha^2 + \beta^2 + \beta\alpha + s^B\lambda_0 + c) + \delta(\delta - \alpha)(\alpha + \delta) = 2\beta(\delta^2 + \frac{1}{2}(\beta\gamma + \alpha\delta)), \\ \alpha(s^B\lambda_0 + c) + \alpha(\delta^2 + \frac{1}{2}(\beta\gamma + \alpha\delta)) + \frac{1}{2}\beta(\beta - \gamma)(\delta - \alpha) + \beta\delta(\alpha + \delta) = 0, \\ \beta(-\beta^2 - \beta\alpha + s^B\lambda_0 + c) - \frac{1}{2}\beta(\delta - \alpha)^2 + 2\beta(\delta^2 + \frac{1}{2}(\beta\gamma + \alpha\delta)) = 0, \\ \beta(\alpha^2 + s^B\lambda_0 + c) = \alpha\delta(\alpha + \delta) + \frac{1}{2}\beta\delta(\delta - \alpha), \\ \gamma(\beta^2 + \beta\alpha + s^B\lambda_0 + c) = \delta(\alpha - \delta)(\alpha + \delta) - \frac{1}{2}\beta(\delta - \alpha)^2, \\ \delta(s^B\lambda_0 + c) + \delta(\delta^2 + \frac{1}{2}(\beta\gamma + \alpha\delta)) = \beta\delta(\alpha + \delta) + \frac{1}{2}\beta(\delta - \alpha)(\beta - \gamma), \\ \delta(\alpha^2 + \beta^2 + \beta\alpha + s^B\lambda_0 + c) - \delta(\delta^2 + \frac{1}{2}(\beta\gamma + \alpha\delta)) = (\beta + \gamma)\delta(\alpha + \delta) + \frac{1}{2}\beta^2(\delta - \alpha). \end{cases}$$

Recall that $\alpha + \delta \neq 0$ and $\alpha\gamma = 0$. I now analyze the system under different assumptions:

Assume first that $\alpha \neq 0$, $\gamma = 0$. Then, the above system becomes:

$$\begin{cases} \alpha(\alpha^2 + \beta^2 + \beta\alpha + s\lambda_0 + c) + \beta\delta(\alpha + \delta) + \frac{1}{2}\beta^2(\delta - \alpha) = \alpha(\delta^2 + \frac{1}{2}\alpha\delta), \\ \beta(\alpha + s\lambda_0 + c) + \delta^2(\alpha + \delta) + \frac{1}{2}\alpha\beta(\delta - \alpha) = 0, \\ \beta(2\alpha^2 + \beta^2 + \beta\alpha + s\lambda_0 + c) + \delta(\delta - \alpha)(\alpha + \delta) = 2\beta(\delta^2 + \frac{1}{2}\alpha\delta), \\ \alpha(s\lambda_0 + c) + \alpha(\delta^2 + \frac{1}{2}\alpha\delta) + \frac{1}{2}\beta^2(\delta - \alpha) + \beta\delta(\alpha + \delta) = 0, \\ \beta(-\beta^2 - \beta\alpha + s\lambda_0 + c) - \frac{1}{2}\beta(\delta - \alpha)^2 + 2\beta(\delta^2 + \frac{1}{2}\alpha\delta) = 0, \\ \beta(\alpha^2 + s\lambda_0 + c) = \alpha\delta(\alpha + \delta) + \frac{1}{2}\beta\delta(\delta - \alpha), \\ \delta(\alpha - \delta)(\alpha + \delta) - \frac{1}{2}\beta(\alpha - \delta)^2 = 0, \\ \beta\delta(\alpha + \delta) + \frac{1}{2}\beta^2(\delta - \alpha) = \delta(s\lambda_0 + c) + \delta(\delta^2 + \frac{1}{2}\alpha\delta), \\ \beta\delta(\alpha + \delta) + \frac{1}{2}\beta^2(\delta - \alpha) = \delta(\alpha^2 + \beta^2 + \beta\alpha + s\lambda_0 + c) - \delta(\delta^2 + \frac{1}{2}\alpha\delta). \end{cases} \tag{3.10}$$

Next, suppose that $\beta = 0$, I have:

$$\begin{cases} \alpha(\alpha^2 + s\lambda_0 + c) - \alpha(\delta^2 + \frac{1}{2}\alpha\delta) = 0, \\ \delta^2(\alpha + \delta) = 0. \end{cases} \tag{3.11}$$

Which is a contradiction.

If $\beta \neq 0$, we further assume that $\delta = 0$. Under this assumption, the last equation in (3.10) yields $\alpha^2\beta = 0$, which leads to a contradiction. If we presume $\alpha = \delta$, then the equations in (3.10) imply that $\alpha\delta(\alpha + \delta) = -\delta^2(\alpha + \delta)$, which is a contradiction. Additionally, if I assume that $\delta \neq 0$ and $\delta \neq -\alpha$, then from equation system (3.10), I have the following equation:

$$\begin{cases} \alpha^2 + s\lambda_0 + c = \dfrac{(\delta - \alpha)^2}{2}, \\ -\beta^2 - \alpha\beta = \dfrac{(\delta - \alpha)^2}{2} - (\delta^2 + \dfrac{\alpha\delta}{2}). \end{cases} \tag{3.12}$$

Substituting (3.12) into the third equation in system (3.10) yields $\alpha^2 = -\frac{1}{2}(\delta-\alpha)^2$, which is a contradiction.

Second, let $\alpha = 0$, $\gamma \neq 0$. Then, if $\beta = 0$, the second equation in (3.10) reduces to $\delta^3 = 0$, which contradicts with $\alpha+\delta \neq 0$. On the other hand, if $\beta \neq 0$, I can derive from the second and sixth equations in system (3.10) that $\delta + \frac{1}{2}\beta = 0$. Substituting into the fourth equation in (3.10) yields $\gamma = 0$, which is a contradiction. □

4. Algebraic Schouten solitons concerning connection ∇^{B_1}

In this section, I formulate the algebraic criterion necessary for a three-dimensional Lorentzian Lie group to have an algebraic Schouten soliton related to the Bott connection ∇^{B_1}. Recall the Bott connection, denoted by ∇^{B_1}, with the second distribution. Consider a smooth manifold (M, g) that is equipped with the Levi-Civita connection ∇, and let TM represent its tangent bundle, spanned by $\{e_1, e_2, e_3\}$. I introduce a distribution D_1 spanned by $\{e_1, e_3\}$ and its orthogonal complement $D_1^{\perp}$, which is spanned by $\{e_2\}$. The Bott connection ∇_1^B associated with the distribution D_1 is then defined as follows:

$$\nabla_X^{B_1} Y = \begin{cases} \pi_{D_1}(\nabla_X^{LC} Y), & X, Y \in \Gamma^\infty(D_1), \\ \pi_{D_1}([X, Y]), & X \in \Gamma^\infty(D_1^{\perp}),\ Y \in \Gamma^\infty(D_1), \\ \pi_{D_1^{\perp}}([X, Y]), & X \in \Gamma^\infty(D_1),\ Y \in \Gamma^\infty(D_1^{\perp}), \\ \pi_{D_1^{\perp}}(\nabla_X^{LC} Y), & X, Y \in \Gamma^\infty(D_1^{\perp}), \end{cases}$$

where π_{D_1} (respectively, $\pi_{D_1^{\perp}}$) denotes the projection onto D_1 (respectively, $D_1^{\perp}$).

4.1. Algebraic Schouten soliton of G_1

Lemma 9. *[13] The Ricci tensor ρ^{B_1} concerning connection ∇^{B_1} of (G_1, g) is given by:*

$$\rho^{B_1}(e_i, e_j) = \begin{pmatrix} \alpha^2-\beta^2 & \frac{1}{2}\alpha\beta & -\alpha\beta \\ \frac{1}{2}\alpha\beta & 0 & \frac{1}{2}\alpha^2 \\ -\alpha\beta & \frac{1}{2}\alpha^2 & \beta^2-\alpha^2 \end{pmatrix}. \tag{4.1}$$

From this, I derive the following theorem.

Theorem 10. *If (G_1, g) constitutes an algebraic Schouten soliton concerning connection ∇^{B_1}, then it fulfills the conditions: $\beta = 0$ and $c = \frac{1}{2}\alpha^2 - 2\alpha^2\lambda_0$.*

Proof. According to (4.1), the expression for Ric^{B_1} is derived as follows:

$$Ric^{B_1}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} \alpha^2-\beta^2 & \frac{1}{2}\alpha\beta & \alpha\beta \\ \frac{1}{2}\alpha\beta & 0 & -\frac{1}{2}\alpha^2 \\ -\alpha\beta & \frac{1}{2}\alpha^2 & \alpha^2-\beta^2 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.$$

The scalar curvature is $s^{B_1} = 2(\alpha^2-\beta^2)$. Now, I can express D^{B_1} as follows:

$$\begin{cases} D^{B_1}e_1 = (\alpha^2-\beta^2-2(\alpha^2-\beta^2)\lambda_0-c)e_1 + \dfrac{1}{2}\alpha\beta e_2 + \alpha\beta e_3, \\ D^{B_1}e_2 = \dfrac{1}{2}\alpha\beta e_1 - (2(\alpha^2-\beta^2)\lambda_0+c)e_2 - \dfrac{\alpha^2}{2}e_3, \\ D^{B_1}e_3 = -\alpha\beta e_1 + \dfrac{1}{2}\alpha^2 e_2 + (\alpha^2-\beta^2-2(\alpha^2-\beta^2)\lambda_0-c)e_3. \end{cases}$$

Therefore, based on Eq (2.4), there is an algebraic Schouten soliton associated with ∇^{B_1} on (G_1, g), if and only if the following condition satisfies:

$$
\begin{cases}
\alpha(2(\alpha^2-\beta^2)\lambda_0+c)+2\alpha\beta^2-\frac{1}{2}\alpha^3=0,\\
\alpha^2\beta=0,\\
\beta(2(\alpha^2-\beta^2)\lambda_0+c)-2\alpha^2\beta=0,\\
\alpha(\alpha^2-\beta^2-2(\alpha^2-\beta^2)\lambda_0-c)-\alpha\beta^2-\frac{1}{2}\alpha^3=0,\\
\beta(\alpha^2-2\beta^2-2(\alpha^2-\beta^2)\lambda_0-c)=0,\\
\alpha(2(\alpha^2-\beta^2)\lambda_0+c)-\frac{1}{2}\alpha^3+2\alpha\beta^2=0.
\end{cases}
\tag{4.2}
$$

Since $\alpha \neq 0$, the second equation in (4.2) yields $\beta = 0$. Then, I have $c = \frac{1}{2}\alpha^2 - 2\alpha^2\lambda_0$. □

4.2. Algebraic Schouten soliton of G_2

Lemma 11. *[13] The Ricci tensor ρ^{B_1} concerning the connection ∇^{B_1} of (G_2, g) is given by:*

$$
\rho^{B_1}(e_i,e_j)=\begin{pmatrix}-(\beta^2+\gamma^2) & 0 & 0\\ 0 & 0 & -\frac{1}{2}\alpha\gamma\\ 0 & -\frac{1}{2}\alpha\gamma & \alpha\beta+\gamma^2\end{pmatrix}.
\tag{4.3}
$$

From this, I derive the following theorem.

Theorem 12. *If (G_2, g) constitutes an algebraic Schouten soliton concerning connection ∇^{B_1}; then, it fulfills the conditions: $\alpha = \beta = 0$ and $c = -\gamma^2 + \gamma^2\lambda_0$.*

Proof. According to (4.3), the expression for Ric^{B_1} is derived as follows:

$$
Ric^{B_1}\begin{pmatrix}e_1\\e_2\\e_3\end{pmatrix}=\begin{pmatrix}-(\beta^2+\gamma^2) & 0 & 0\\ 0 & 0 & \frac{1}{2}\alpha\gamma\\ 0 & -\frac{1}{2}\alpha\gamma & -\alpha\beta-\gamma^2\end{pmatrix}\begin{pmatrix}e_1\\e_2\\e_3\end{pmatrix}.
$$

The scalar curvature is $s^{B_1} = -(\beta^2 + \gamma^2 + \alpha\beta)$. Now, I can express D^{B_1} as follows:

$$
\begin{cases}
D^{B_1}e_1=-(\beta^2+\gamma^2-(\beta^2+\gamma^2+\alpha\beta)\lambda_0+c)e_1,\\
D^{B_1}e_2=((\beta^2+\gamma^2+\alpha\beta)\lambda_0-c)e_2+\frac{1}{2}\alpha\gamma e_3,\\
D^{B_1}e_3=-\frac{1}{2}\alpha\gamma e_2-(\alpha\beta+\gamma^2-(\beta^2+\gamma^2+\alpha\beta)\lambda_0+c)e_3.
\end{cases}
$$

Therefore, based on Eq (2.4), there is an algebraic Schouten soliton associated with ∇^{B_1} on (G_2, g), if and only if the following condition satisfies:

$$\begin{cases} \alpha(\beta^2+\gamma^2-(\beta^2+\gamma^2+\alpha\beta)\lambda_0+c)+\frac{1}{2}\alpha\beta\gamma=0, \\ \beta((\beta^2+\gamma^2+\alpha\beta)\lambda_0-c)+\alpha\gamma^2=0, \\ \beta(\beta^2+2\gamma^2+\alpha\beta-(\beta^2+\gamma^2+\alpha\beta)\lambda_0+c)-\alpha\gamma^2=0, \\ \gamma(\beta^2+\gamma^2-(\beta^2+\gamma^2+\alpha\beta)\lambda_0+c)+\frac{1}{2}\alpha\beta\gamma=0, \\ \alpha(\alpha\beta-\beta^2-(\beta^2+\gamma^2+\alpha\beta)\lambda_0+c)=0. \end{cases} \tag{4.4}$$

By solving (4.4), I get $\alpha=\beta=0$, $c=-\gamma^2+\gamma^2\lambda_0$. □

4.3. Algebraic Schouten soliton of G_3

Lemma 13. *[13] The Ricci tensor ρ^{B_1} concerning connection ∇^{B_1} of (G_3,g) is given by:*

$$\rho^{B_1}(e_i,e_j)=\begin{pmatrix} -\beta\gamma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha\beta \end{pmatrix}. \tag{4.5}$$

From this, I derive the following theorem.

Theorem 14. *If (G_3,g) constitutes an algebraic Schouten soliton concerning connection ∇^{B_1}; then, one of the following cases holds:*

i. $\alpha=\beta=\gamma=0$, *for all c;*

ii. $\alpha\neq0,\beta=\gamma=0,\ c=0$;

iii. $\alpha=0,\beta\neq0,\ \gamma=0,\ c=0$;

iv. $\alpha\neq0,\beta\neq0,\ \gamma=0,\ c=\alpha\beta\lambda_0$;

v. $\alpha=\beta=0,\ \gamma\neq0,\ c=0,$;

vi. $\alpha\neq0,\beta=0,\ \gamma\neq0,\ c=0$;

vii. $\alpha=0,\beta\neq0,\ \gamma\neq0,\ c=-\beta\gamma+\beta\gamma\lambda_0$.

Proof. According to (4.5), the expression for Ric^{B_1} is derived as follows:

$$Ric^{B_1}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}=\begin{pmatrix} -\beta\gamma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\alpha\beta \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.$$

The scalar curvature is $s^{B_1}=-(\beta\gamma+\alpha\beta)$. Now, I can express D^{B_1} as follows:

$$\begin{cases} D^{B_1}e_1=-(\beta\gamma-(\beta\gamma+\alpha\beta)\lambda_0+c)e_1, \\ D^{B_1}e_2=((\beta\gamma+\alpha\beta)\lambda_0-c)e_2, \\ D^{B_1}e_3=-(\alpha\beta-(\beta\gamma+\alpha\beta)\lambda_0+c)e_3. \end{cases}$$

Therefore, based on Eq (2.4), there is an algebraic Schouten soliton associated with ∇^{B_1} on (G_3, g), if and only if the following condition satisfies:

$$\begin{cases} \gamma(\beta\gamma - \alpha\beta - (\beta\gamma + \alpha\beta)\lambda_0 + c) = 0, \\ \beta(\alpha\beta + \beta\gamma - (\beta\gamma + \alpha\beta)\lambda_0 + c) = 0, \\ \alpha(\alpha\beta - \beta\gamma - (\beta\gamma + \alpha\beta)\lambda_0 + c) = 0. \end{cases} \tag{4.6}$$

Assume first that $\gamma = 0$; then, (4.6) reduces to:

$$\begin{cases} \beta(\alpha\beta - \alpha\beta\lambda_0 + c) = 0, \\ \alpha(\alpha\beta - \alpha\beta\lambda_0 + c) = 0. \end{cases}$$

Then, I have Cases *i–iv*.

Now, let $\gamma \neq 0$. From the last two equations in (4.6), I obtain $\alpha\beta\gamma = 0$. If $\beta = 0$, then it follows that $c = 0$. Therefore, Cases *v* and *vi* are valid. If $\beta \neq 0$, we deduce $c = -\beta\gamma + \beta\gamma\lambda_0$; then, for Case *vii*, system (4.6) holds. □

4.4. Algebraic Schouten soliton of G_4

Lemma 15. *[13] The Ricci tensor ρ^{B_1} concerning connection ∇^{B_1} of (G_4, g) is given by:*

$$\rho^{B_1}(e_i, e_j) = \begin{pmatrix} -(\beta-\eta)^2 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\alpha \\ 0 & \frac{1}{2}\alpha & \alpha\beta + 1 \end{pmatrix}. \tag{4.7}$$

From this, I derive the following theorem.

Theorem 16. *If (G_4, g) constitutes an algebraic Schouten soliton concerning connection ∇^{B_1}; then, it fulfills the conditions: $\beta = \eta$, $\alpha = -\beta$ and $c = 0$.*

Proof. According to (4.7), the expression for Ric^{B_1} is derived as follows:

$$Ric^{B_1}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -(\beta-\eta)^2 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}\alpha \\ 0 & \frac{1}{2}\alpha & -\alpha\beta - 1 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.$$

The scalar curvature is $s^{B_1} = -((\beta-\eta)^2 + \alpha\beta + 1)$. Now, I can express D^{B_1} as follows:

$$\begin{cases} D^{B_1}e_1 = -((\beta-\eta)^2 - ((\beta-\eta)^2 + \alpha\beta + 1)\lambda_0 + c)e_1, \\ D^{B_1}e_2 = (((\beta-\eta)^2 + \alpha\beta + 1)\lambda_0 - c)e_2 - \dfrac{1}{2}\alpha e_3, \\ D^{B_1}e_3 = \dfrac{1}{2}\alpha - (\alpha\beta + 1 - ((\beta-\eta)^2 + \alpha\beta + 1)\lambda_0 + c)e_3. \end{cases}$$

Therefore, based on Eq (2.4), there is an algebraic Schouten soliton associated with ∇^{B_1} on (G_4, g), if the following condition satisfies:

$$\begin{cases} (\beta-\eta)^2 - ((\beta-\eta)^2 + \alpha\beta + 1)\lambda_0 + c - \alpha(\eta - \beta) = 0, \\ (2\eta - \beta)((\beta-\eta)^2 - \alpha\beta - 1 - ((\beta-\eta)^2 + \alpha\beta + 1)\lambda_0 + c) + \alpha = 0, \\ \beta((\beta-\eta)^2 + \alpha\beta + 1 - ((\beta-\eta)^2 + \alpha\beta + 1)\lambda_0 + c) - \alpha = 0, \\ \alpha(\alpha\beta + 1 - (\beta-\eta)^2 - ((\beta-\eta)^2 + \alpha\beta + 1)\lambda_0 + c) = 0. \end{cases} \tag{4.8}$$

By solving the above system, I obtain the solutions $\beta = \eta$, $\alpha = -\beta$ and $c = 0$. In this case, the theorem is true. □

4.5. Algebraic Schouten soliton of G_5

Lemma 17. *[13] The Ricci tensor ρ^{B_1} concerning connection ∇^{B_1} of (G_5, g) is given by:*

$$\rho^{B_1}(e_i, e_j) = \begin{pmatrix} \alpha^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -(\beta\gamma + \alpha^2) \end{pmatrix}. \tag{4.9}$$

From this, I derive the following result.

Theorem 18. *If (G_5, g) constitutes an algebraic Schouten soliton concerning connection ∇^{B_1}; then, one of the following cases holds:*

i. $\alpha = \beta = \gamma = 0$, $c = 0$;

ii. $\alpha = \beta = 0$, $\gamma \neq 0$, $c = 0$;

iii. $\alpha = 0, \beta \neq 0$, $\gamma \neq 0$, $c = -\beta\gamma + \beta\gamma\lambda_0$;

iv. $\alpha \neq 0, \beta = \delta = \gamma = 0$, $c = -\alpha^2 - 2\alpha^2\lambda_0$;

v. $\alpha \neq 0, \beta = \gamma = 0$, $\delta \neq 0$, $c = \alpha^2 - 2\alpha^2\lambda_0$.

Proof. From (4.9), I have the expression for Ric^{B_1} as follows:

$$Ric^{B_1}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} \alpha^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (\beta\gamma + \alpha^2) \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.$$

The scalar curvature is $s^{B_1} = (2\alpha^2 + \beta\gamma)$. Now, I can express D^{B_1} as follows:

$$\begin{cases} D^{B_1}e_1 = (\alpha^2 - (2\alpha^2 + \beta\gamma)\lambda_0 - c), \\ D^{B_1}e_2 = -((2\alpha^2 + \beta\gamma)\lambda_0 + c)e_2 \\ D^{B_1}e_3 = (\beta\gamma + \alpha^2 - (2\alpha^2 + \beta\gamma)\lambda_0 - c)e_3. \end{cases}$$

Therefore, based on Eq (2.4), there exists an algebraic Schouten soliton associated to ∇^{B_1} on (G_5, g), if and only if the following condition satisfies:

$$\begin{cases} \alpha(\beta\gamma + \alpha^2 - (\beta\gamma + 2\alpha^2)\lambda_0 - c) = 0, \\ \beta(\beta\gamma + 2\alpha^2 - (\beta\gamma + 2\alpha^2)\lambda_0 - c) = 0, \\ \gamma(\beta\gamma - (\beta\gamma + 2\alpha^2)\lambda_0 - c) = 0, \\ \delta(\beta\gamma + \alpha^2 - (\beta\gamma + 2\alpha^2)\lambda_0 - c) = 0. \end{cases} \tag{4.10}$$

Assume first that $\alpha = 0$, so I have:

$$\begin{cases} \beta(\beta\gamma - \beta\gamma\lambda_0 - c) = 0, \\ \gamma(\beta\gamma - \beta\gamma\lambda_0 - c) = 0, \\ \delta(\beta\gamma - \beta\gamma\lambda_0 - c) = 0. \end{cases} \tag{4.11}$$

Then, for Cases *i–iii*, system (4.10) holds.

Now, I let $\alpha \neq 0$. The second equation in (4.10) leads to $\beta = 0$, then system (4.10) reduces to:

$$\begin{cases} \alpha(\alpha^2 - 2\alpha^2\lambda_0 - c) = 0, \\ \gamma(2\alpha^2\lambda_0 + c) = 0, \\ \delta(\alpha^2 - 2\alpha^2\lambda_0 - c) = 0. \end{cases} \tag{4.12}$$

This proves that Cases *iv* and *v* hold. □

4.6. Algebraic Schouten soliton of G_6

Lemma 19. *[13] The Ricci tensor ρ^{B_1} concerning connection ∇^{B_1} of (G_6, g) is given by:*

$$\rho^{B_1}(e_i, e_j) = \begin{pmatrix} -(\delta^2 + \beta\gamma) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \delta^2 \end{pmatrix}. \tag{4.13}$$

From this, I derive the following result.

Theorem 20. *If (G_6, g) constitutes an algebraic Schouten soliton concerning connection ∇^{B_1}; then, one of the following cases holds:*

1) $\alpha = \beta = \gamma = 0$, $\delta \neq 0$, $c = -\delta^2 + 2\delta^2\lambda_0$;

2) $\alpha \neq 0, \beta = \delta = \gamma = 0$, $c = 0$;

3) $\alpha \neq 0, \beta \neq 0$, $\delta = \gamma = 0$, $c = 0$;

4) $\alpha \neq 0, \beta = \gamma = 0$, $\delta \neq 0$, $c = -\delta^2 + 2\delta^2\lambda_0$.

Proof. From (4.13), I have the expression for Ric^{B_1} as follows:

$$Ric^{B_1}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -(\delta^2 + \beta\gamma) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\delta^2 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.$$

The scalar curvature is $s^{B_1} = -(2\delta^2 + \beta\gamma)$. Now, I can express D^{B_1} as follows:

$$\begin{cases} D^{B_1}e_1 = -(\delta^2 + \beta\gamma - (2\delta^2 + \beta\gamma)\lambda_0 + c)e_1, \\ D^{B_1}e_2 = ((2\delta^2 + \beta\gamma)\lambda_0 - c)e_2, \\ D^{B_1}e_3 = -(\delta^2 - (2\delta^2 + \beta\gamma)\lambda_0 + c). \end{cases}$$

Therefore, based on Eq (2.4), there is an algebraic Schouten soliton associated with ∇^{B_1} on (G_6, g), if and only if the following condition satisfies:

$$\begin{cases} \alpha(\delta^2 + \beta\gamma - (2\delta^2 + \beta\gamma)\lambda_0 + c) = 0, \\ \beta(\beta\gamma - (2\delta^2 + \beta\gamma)\lambda_0 + c) = 0, \\ \gamma(2\delta^2 + \beta\gamma - (2\delta^2 + \beta\gamma)\lambda_0 + c) = 0, \\ \delta(\delta^2 + \beta\gamma - (2\delta^2 + \beta\gamma)\lambda_0 + c) = 0. \end{cases} \tag{4.14}$$

Assume first $\alpha = 0$. Then, $\alpha + \delta \neq 0$ and $\alpha\gamma - \beta\delta = 0$ leads to $\beta = 0$ and $\delta \neq 0$. Therefore, system (4.14) reduces to:

$$\begin{cases} \gamma(2\delta^2 - 2\delta^2\lambda_0 + c) = 0, \\ \delta(\delta^2 - 2\delta^2\lambda_0 + c) = 0. \end{cases} \tag{4.15}$$

Then, for Case 1), system (4.14) holds.

Now, let $\alpha \neq 0$. Suppose $\delta = 0$, from the equations in (4.14) I can derive that $\gamma = 0$. Then, I have $c = 0$. Consequently, I have Cases 2) and 3). If $\delta \neq 0$, the equations in (4.14) imply that $\gamma = 0$. Substituting into the second equation in (4.14) leads to $\beta = 0$. Then, we have 4). □

4.7. Algebraic Schouten soliton of G_7

Lemma 21. *[13] The Ricci tensor ρ^{B_1} concerning connection ∇^{B_1} of (G_7, g) is given by:*

$$\rho^{B_1}(e_i, e_j) = \begin{pmatrix} \alpha^2 & \beta(\alpha+\delta) & \frac{1}{2}\beta(\delta-\alpha) \\ \beta(\alpha+\delta) & 0 & \delta^2 + \frac{1}{2}(\beta\gamma + \alpha\delta) \\ \frac{1}{2}\beta(\delta-\alpha) & \delta^2 + \frac{1}{2}(\beta\gamma+\alpha\delta) & \beta^2 - \alpha^2 - \beta\gamma \end{pmatrix}. \tag{4.16}$$

From this, we derive the following theorem.

Theorem 22. *If (G_7, g) constitutes an algebraic Schouten soliton concerning connection ∇^{B_1}; then, it fulfills the conditions: $\alpha = 2\delta$, $\beta = \gamma = 0$, $c = \frac{\alpha^2}{2} - 2\alpha^2\lambda_0$.*

Proof. From (4.16), I have the expression for Ric^{B_1} as follows:

$$Ric^{B_1}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} \alpha^2 & \beta(\alpha+\delta) & -\frac{1}{2}\beta(\delta-\alpha) \\ \beta(\alpha+\delta) & 0 & -\delta^2 - \frac{1}{2}(\beta\gamma+\alpha\delta) \\ \frac{1}{2}\beta(\delta-\alpha) & \delta^2 + \frac{1}{2}(\beta\gamma+\alpha\delta) & -\beta^2 + \alpha^2 + \beta\gamma \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.$$

The scalar curvature is $s^{B_1} = 2\alpha^2 - \beta^2 + \beta\gamma$. Now, I can express D^{B_1} as follows:

$$\begin{cases} D^{B_1}e_1 = (\alpha^2 - (2\alpha^2 - \beta^2 + \beta\gamma)\lambda_0 - c)e_1 + \beta(\alpha+\delta)e_2 - \frac{1}{2}\beta(\delta-\alpha)e_3, \\ D^{B_1}e_2 = \beta(\alpha+\delta)e_1 - ((2\alpha^2 - \beta^2 + \beta\gamma)\lambda_0 + c)e_2 - (\delta^2 + \frac{1}{2}(\beta\gamma+\alpha\delta))e_3, \\ D^{B_1}e_3 = \frac{1}{2}\beta(\delta-\alpha)e_1 + (\delta^2 + \frac{1}{2}(\beta\gamma+\alpha\delta))e_2 + (\alpha^2 - \beta^2 + \beta\gamma - (2\alpha^2 - \beta^2 + \beta\gamma)\lambda_0 - c)e_3. \end{cases}$$

Therefore, based on Eq (2.4), there is an algebraic Schouten soliton associated with ∇^{B_1} on (G_7, g), if and only if the following condition satisfies:

$$\begin{cases} \alpha(s^{B_1}\lambda_0 + c) - \frac{1}{2}\beta(\beta+\gamma)(\delta-\alpha) - \beta^2(\alpha+\delta) - \alpha(\delta^2 + \frac{1}{2}(\beta\gamma+\alpha\delta)) = 0, \\ \beta(-\alpha^2 + s^{B_1}\lambda_0 + c) + \frac{1}{2}\beta\delta(\delta-\alpha) + \alpha\beta(\alpha+\delta) = 0, \\ \beta(-\beta^2 + \beta\gamma + s^{B_1}\lambda_0 + c) + \frac{1}{2}\beta(\delta-\alpha)^2 - 2\beta(\delta^2 + \frac{1}{2}(\beta\gamma+\alpha\delta)) = 0, \\ \alpha(\alpha^2 + \beta\gamma - \beta^2 - s^{B_1}\lambda_0 - c) + \beta(\gamma-\beta)(\alpha+\delta) - \frac{\beta^2(\delta-\alpha)}{2} - \alpha(\delta^2 + \frac{\beta\gamma+\alpha\delta}{2}) = 0, \\ \beta(2\alpha^2 + \beta\gamma - \beta^2 - s^{B_1}\lambda_0 - c) + \beta(\delta-\alpha)(\alpha+\delta) - 2\beta(\delta^2 + \frac{\beta\gamma+\alpha\delta}{2}) = 0, \\ \beta(\alpha^2 - s^{B_1}\lambda_0 - c) + \beta\delta(\alpha+\delta) + \frac{1}{2}\alpha\beta(\delta-\alpha) = 0, \\ \gamma(-\beta^2 + \beta\gamma - s^{B_1}\lambda_0 - c) + \beta(\alpha-\delta)(\alpha+\delta) - \frac{1}{2}\beta(\alpha-\delta)^2 = 0, \\ \delta(\alpha^2 - \beta^2 + \beta\gamma - s^{B_1}\lambda_0 - c) + \beta(\beta-\gamma)(\alpha+\delta) + \frac{\beta^2(\delta-\alpha)}{2} - \delta(\delta^2 + \frac{\beta\gamma+\alpha\delta}{2}) = 0, \\ -\delta(s^{B_1}\lambda_0 + c) + \beta^2(\alpha+\delta) + \frac{1}{2}\beta(\beta+\gamma)(\delta-\alpha) + \delta(\delta^2 + \frac{1}{2}(\beta\gamma+\alpha\delta)) = 0. \end{cases} \tag{4.17}$$

Since $\alpha\gamma = 0$ and $\alpha + \delta \neq 0$, I now analyze the system under different assumptions.

First, if $\alpha = 0$ and $\gamma \neq 0$, from the equations above, and after simple calculation, we have $\beta = 0$. Furthermore, the seventh and eighth equations imply that $\delta^3 = 0$, which leads to a contradiction.

Second, if $\alpha \neq 0$ and $\gamma = 0$. the seventh equation gives rise to three possible subcases: $\beta = 0$, $\alpha = \delta$, or $\alpha + 3\delta = 0$. Initially, let's assume $\beta = 0$. In this case, the equations in system (4.17), imply that $(\alpha - 2\delta)(\alpha + \delta) = 0$, leading to $\alpha = 2\delta$, and the theorem is true. Next, I consider the subcase where $\alpha = \delta$ and $\beta \neq 0$. Then, the fifth and sixth equations result in $4\alpha^2 + \beta^2 = 0$, which leads to a contradiction. Additionally, I consider that $\alpha + 3\delta = 0$ and $\beta \neq 0$. The first and last equations provide $(2\alpha^2 - \beta^2 + \beta\gamma)\lambda_0 + c = 0$. Substituting this into the third equation derives $\beta = 0$, which leads to a contradiction.

Finally, if $\alpha = \gamma = 0$. The first equation in system (4.17) leads to $\beta = 0$. Then, using the equations in (4.17), we have $\delta^3 = 0$, which leads to a contradiction. □

5. Algebraic Schouten solitons concerning connection ∇^{B_2}

In this section, I formulate the algebraic criterion necessary for a three-dimensional Lorentzian Lie group to have an algebraic Schouten soliton related to the given Bott connection ∇^{B_2}. Let us recall the Bott connection with the third distribution, denoted by ∇^{B_2}. Consider a smooth manifold (M, g) that is equipped with the Levi-Civita connection ∇, and let TM represent its tangent bundle, spanned by $\{e_1, e_2, e_3\}$. I introduce a distribution D_2 spanned by $\{e_2, e_3\}$ and its orthogonal complement $D_2^\perp$, which is spanned by $\{e_1\}$. The Bott connection ∇_2^B associated with the distribution D_2 is then defined as follows:

$$\nabla_X^{B_2} Y = \begin{cases} \pi_{D_2}(\nabla_X^{LC} Y), & X, Y \in \Gamma^\infty(D_2), \\ \pi_{D_2}([X, Y]), & X \in \Gamma^\infty(D_2^\perp),\ Y \in \Gamma^\infty(D_2), \\ \pi_{D_2^\perp}([X, Y]), & X \in \Gamma^\infty(D_2),\ Y \in \Gamma^\infty(D_2^\perp), \\ \pi_{D_2^\perp}(\nabla_X^{LC} Y), & X, Y \in \Gamma^\infty(D_2^\perp), \end{cases} \tag{5.1}$$

where π_{D_2} (respectively, $\pi_{D_2^\perp}$) denotes the projection onto D_2 (respectively, $D_2^\perp$).

5.1. Algebraic Schouten soliton of G_1

Lemma 23. *[13] The Ricci tensor ρ^{B_2} concerning connection ∇^{B_2} of (G_1, g) is given by:*

$$\rho^{B_2}(e_i, e_j) = \begin{pmatrix} 0 & \frac{1}{2}\alpha\beta & -\frac{1}{2}\alpha\beta \\ \frac{1}{2}\alpha\beta & -\beta^2 & 0 \\ -\frac{1}{2}\alpha\beta & 0 & \beta^2 \end{pmatrix}. \tag{5.2}$$

From this, I derive the following theorem.

Theorem 24. *If (G_1, g) constitutes an algebraic Schouten soliton concerning connection ∇^{B_2}; then, it fulfills the conditions: $\alpha \neq 0$, $\beta = 0$ and $c = 0$.*

Proof. According to (5.2), the expression for Ric^{B_2} is derived as follows:

$$Ric^{B_2} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2}\alpha\beta & \frac{1}{2}\alpha\beta \\ \frac{1}{2}\alpha\beta & -\beta^2 & 0 \\ -\frac{1}{2}\alpha\beta & 0 & -\beta^2 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.$$

The scalar curvature is $s^{B_2} = -2\beta^2$. Now, I can express D^{B_2} as follows:

$$\begin{cases} D^{B_2}e_1 = (2\beta^2\lambda_0 - c)e_1 + \dfrac{1}{2}\alpha\beta e_2 + \dfrac{1}{2}\alpha\beta e_3, \\ D^{B_2}e_2 = \dfrac{1}{2}\alpha\beta e_1 - (\beta^2 - 2\beta^2\lambda_0 + c)e_2, \\ D^{B_2}e_3 = -\dfrac{1}{2}\alpha\beta e_1 - (\beta^2 - 2\beta^2\lambda_0 + c)e_3. \end{cases}$$

Therefore, based on Eq (2.4), there is an algebraic Schouten soliton associated with ∇^{B_2} on (G_1, g), if and only if the following condition satisfies:

$$\begin{cases} \alpha(\beta^2 - 2\beta^2\lambda_0 + c) + \alpha\beta^2 = 0, \\ \alpha^2\beta = 0, \\ \alpha(-2\beta^2\lambda_0 + c) - \alpha\beta^2 = 0, \\ \beta(-2\beta^2\lambda_0 + c) + \alpha^2\beta = 0. \end{cases} \tag{5.3}$$

By solving (5.3), I have $\alpha \neq 0, \beta = 0$ and $c = 0$. □

5.2. Algebraic Schouten soliton of G_2

Lemma 25. *[13] The Ricci tensor ρ^{B_2} concerning connection ∇^{B_2} of (G_2, g) is given by:*

$$\rho^{B_2}(e_i, e_j) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\alpha\beta & -\alpha\gamma \\ 0 & -\alpha\gamma & \alpha\beta \end{pmatrix}. \tag{5.4}$$

From this, I derive the following theorem.

Theorem 26. *If (G_2, g) constitutes an algebraic Schouten soliton concerning connection ∇^{B_2}; then, one of the following cases holds:*

1) $\alpha = 0, \beta = 0, c = 0$;

2) $\alpha = 0, \beta \neq 0, c = 0$.

Proof. From (5.4), I have the expression for Ric^{B_2} as follows:

$$Ric^{B_2}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\alpha\beta & \alpha\gamma \\ 0 & -\alpha\gamma & -\alpha\beta \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.$$

The scalar curvature is $s^{B_2} = -2\alpha\beta$. Now, I can express D^{B_2} as follows:

$$\begin{cases} D^{B_2}e_1 = (2\alpha\beta\lambda_0 - c)e_1, \\ D^{B_2}e_2 = -(\alpha\beta - 2\alpha\beta\lambda_0 + c)e_2 + \alpha\gamma e_3, \\ D^{B_2}e_3 = -\alpha\gamma e_2 - (\alpha\beta - 2\alpha\beta\lambda_0 + c)e_3. \end{cases}$$

Therefore, based on Eq (2.4), there is an algebraic Schouten soliton associated with ∇^{B_2} on (G_2, g), if and only if the following condition satisfies:

$$\begin{cases} \gamma(-2\alpha\beta\lambda_0 + c) - 2\alpha\beta\gamma = 0, \\ \beta(-2\alpha\beta\lambda_0 + c) - 2\alpha\gamma^2 = 0, \\ \alpha(2\alpha\beta - 2\alpha\beta\lambda_0 + c) = 0. \end{cases} \tag{5.5}$$

Since $\gamma \neq 0$, I assume first that $\alpha = 0$. Under this assumption, the first two equations in (5.5) yield $c = 0$. Therefore, Cases 1) and 2) hold. Now, let $\alpha \neq 0$, then I have $\beta = 0$, and the second equation in (5.5) becomes $2\alpha\gamma^2 = 0$, which is a contradiction. □

5.3. Algebraic Schouten soliton of G_3

Lemma 27. *[13] The Ricci tensor ρ^{B_2} concerning connection ∇^{B_2} of (G_3, g) is given by:*

$$\rho^{B_2}(e_i, e_j) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha\beta \end{pmatrix}. \tag{5.6}$$

From this, I derive the following theorem.

Theorem 28. *If (G_3, g) constitutes an algebraic Schouten soliton concerning connection ∇^{B_2}; then, one of the following cases holds:*

1) $\alpha = \beta = \gamma = 0$, *for all c;*

2) $\alpha = \gamma = 0$, $\beta \neq 0$, $c = 0$*;*

3) $\alpha \neq 0$, $\beta = \gamma = 0$, $c = 0$*;*

4) $\alpha \neq 0$, $\beta \neq 0$, $\gamma = 0$, $c = -\alpha\beta + \alpha\beta\lambda_0$*;*

5) $\alpha = \beta = 0$, $\gamma \neq 0$, $c = 0$.

Proof. According to (5.6), the expression for Ric^{B_2} is derived as follows:

$$Ric^{B_2}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\alpha\beta \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.$$

The scalar curvature is $s^{B_2} = -\alpha\beta$. Now, I can express D^{B_2} as follows:

$$\begin{cases} D^{B_2}e_1 = (\alpha\beta\lambda_0 - c)e_1, \\ D^{B_2}e_2 = (\alpha\beta\lambda_0 - c)e_2, \\ D^{B_2}e_3 = -(\alpha\beta - \alpha\beta\lambda_0 + c)e_3. \end{cases} \tag{5.7}$$

Therefore, based on Eq (2.4), there is an algebraic Schouten soliton associated with ∇^{B_2} on (G_3, g), if and only if the following condition satisfies:

$$\begin{cases} \gamma(-\alpha\beta - \alpha\beta\lambda_0 + c) = 0, \\ \beta(\alpha\beta - \alpha\beta\lambda_0 + c) = 0, \\ \alpha(\alpha\beta - \alpha\beta\lambda_0 + c) = 0. \end{cases} \tag{5.8}$$

Assuming that $\gamma = 0$, I have

$$\begin{cases} \beta(\alpha\beta - \alpha\beta\lambda_0 + c) = 0, \\ \alpha(\alpha\beta - \alpha\beta\lambda_0 + c) = 0. \end{cases} \tag{5.9}$$

Then, for Cases 1)–4), system (5.8) holds. Now, let $\gamma \neq 0$, then we have $\alpha = \beta = 0$. Then, the Case 5) is true. □

5.4. Algebraic Schouten soliton of G_4

Lemma 29. *[13] The Ricci tensor ρ^{B_2} concerning connection ∇^{B_2} of (G_4, g) is given by:*

$$\rho^{B_2}(e_i, e_j) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha(2\eta - \beta) & \alpha \\ 0 & \alpha & \alpha\beta \end{pmatrix}. \tag{5.10}$$

From this, I derive the following theorem.

Theorem 30. *If (G_4, g) constitutes an algebraic Schouten soliton concerning connection ∇^{B_2}, then one of the following cases holds:*

1) $\alpha = 0$, $c = 0$;

2) $\alpha \neq 0$, $\beta = \eta$, $c = 0$.

Proof. From (5.10), I have the expression for Ric^{B_2} as follows:

$$Ric^{B_2} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha(2\eta - \beta) & -\alpha \\ 0 & \alpha & -\alpha\beta \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.$$

The scalar curvature is $s^{B_2} = \alpha(2\eta - \beta) - \alpha\beta$. Now, I can express D^{B_2} as follows:

$$\begin{cases} D^{B_2}e_1 = -((\alpha(2\eta - \beta) - \alpha\beta)\lambda_0 + c)e_1, \\ D^{B_2}e_2 = (\alpha(2\eta - \beta) - (\alpha(2\eta - \beta) - \alpha\beta)\lambda_0 - c)e_2 - \alpha e_3, \\ D^{B_2}e_3 = \alpha e_2 - (\alpha\beta + (\alpha(2\eta - \beta) - \alpha\beta)\lambda_0 + c)e_3. \end{cases}$$

Therefore, based on Eq (2.4), there is an algebraic Schouten soliton associated with ∇^{B_2} on (G_4, g), if and only if the following condition satisfies:

$$\begin{cases} 2\alpha(\eta - \beta) - ((\alpha(2\eta - \beta) - \alpha\beta)\lambda_0 + c) = 0, \\ (2\eta - \beta)(2\alpha\eta - (\alpha(2\eta - \beta) - \alpha\beta)\lambda_0 - c) - 2\alpha = 0, \\ \beta(2\alpha\eta + (\alpha(2\eta - \beta) - \alpha\beta)\lambda_0 + c) - 2\alpha = 0, \\ \alpha(\alpha(2\eta - \beta) - \alpha\beta - (\alpha(2\eta - \beta) - \alpha\beta)\lambda_0 - c) = 0. \end{cases} \tag{5.11}$$

Assume first that $\alpha = 0$, then system (5.11) holds trivially. Therefore, Case 1) holds. Now, let $\alpha \neq 0$ I have $\beta = \eta$; then, for Case 2), system (5.11) holds. □

5.5. Algebraic Schouten soliton of G_5

Lemma 31. *[13] The Ricci tensor ρ^{B_2} concerning connection ∇^{B_2} of (G_5, g) is given by:*

$$\rho^{B_2}(e_i, e_j) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \delta^2 & 0 \\ 0 & 0 & -(\beta\gamma + \delta^2) \end{pmatrix}. \tag{5.12}$$

From this, I derive the following theorem.

Theorem 32. *If (G_5, g) constitutes an algebraic Schouten soliton concerning connection ∇^{B_2}, then one of the following cases holds:*

i. $\alpha = \beta = \gamma = 0$, $\delta \neq 0$, $c = \delta^2 - 2\delta^2\lambda_0$;

ii. $\alpha \neq 0$, $\beta = \delta = \gamma = 0$, $c = 0$;

iii. $\alpha \neq 0$, $\beta \neq 0$, $\delta = \gamma = 0$, $c = 0$;

iv. $\alpha \neq 0$, $\beta = \gamma = 0$, $\delta \neq 0$, $c = \delta^2 - 2\delta^2\lambda_0$.

Proof. From (5.12), I have the expression for Ric^{B_2} as follows:

$$Ric^{B_2} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \delta^2 & 0 \\ 0 & 0 & \beta\gamma + \delta^2 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.$$

The scalar curvature is $s^{B_2} = \beta\gamma + 2\delta^2$. Now, I can express D^{B_2} as follows:

$$\begin{cases} D^{B_2}e_1 = -((\beta\gamma + 2\delta^2)\lambda_0 + c)e_1, \\ D^{B_2}e_2 = (\delta^2 - (\beta\gamma + 2\delta^2)\lambda_0 - c)e_2, \\ D^{B_2}e_3 = (\beta\gamma + \delta^2 - (\beta\gamma + 2\delta^2)\lambda_0 - c)e_3. \end{cases}$$

Therefore, based on Eq (2.4), there is an algebraic Schouten soliton associated with ∇^{B_2} on (G_5, g), if and only if the following condition satisfies:

$$\begin{cases} \alpha(\beta\gamma + \delta^2 - (\beta\gamma + 2\delta^2)\lambda_0 - c) = 0, \\ \beta(\beta\gamma - (\beta\gamma + 2\delta^2)\lambda_0 - c) = 0, \\ \gamma(\beta\gamma + 2\delta^2 - (\beta\gamma + 2\delta^2)\lambda_0 - c) = 0, \\ \delta(\beta\gamma + \delta^2 - (\beta\gamma + 2\delta^2)\lambda_0 - c) = 0. \end{cases} \tag{5.13}$$

Let $\alpha = 0$, then I have $\beta = 0$ and $\delta \neq 0$. In this case, (5.13) reduces to:

$$\begin{cases} \gamma(2\delta^2 - 2\delta^2\lambda_0 - c) = 0, \\ \delta(\delta^2 - 2\delta^2\lambda_0 - c) = 0. \end{cases} \tag{5.14}$$

Therefore, I conclude that $\gamma = 0$, and we have Case *i*.

Next, I consider the case where $\alpha \neq 0$. By combining the first and third equations from (5.13), we obtain $\gamma\delta^2 = 0$. If $\gamma = \delta = 0$, then $c = 0$. Therefore, Cases *ii* and *iii* hold. If $\gamma = 0$ and $\delta \neq 0$, then $\beta = 0$. Therefore, Case *iv* holds. □

5.6. Algebraic Schouten soliton of G_6

Lemma 33. *[13] The Ricci tensor ρ^{B_2} concerning connection ∇^{B_2} of (G_6, g) is given by:*

$$\rho^{B_2}(e_i, e_j) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.15}$$

From this, I derive the following theorem.

Theorem 34. *If (G_6, g) constitutes an algebraic Schouten soliton concerning connection ∇^{B_2}, then we have $c = 0$.*

Proof. According to (5.15), the expression for Ric^{B_2} is derived as follows:

$$Ric^{B_2} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.$$

The scalar curvature is $s^{B_2} = 0$. Now, I can express D^{B_2} as follows:

$$\begin{cases} D^{B_2} e_1 = -ce_1, \\ D^{B_2} e_2 = -ce_2, \\ D^{B_2} e_3 = -ce_3. \end{cases}$$

Hence, by (2.4), I have $c = 0$. □

5.7. Algebraic Schouten soliton of G_7

Lemma 35. *[13] The Ricci tensor ρ^{B_2} concerning connection ∇^{B_2} of (G_7, g) is given by:*

$$\rho^{B_2}(e_i, e_j) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\beta\gamma & \beta\gamma \\ 0 & \beta\gamma & -\beta\gamma \end{pmatrix}. \tag{5.16}$$

From this, I derive the following theorem.

Theorem 36. *If (G_7, g) constitutes an algebraic Schouten soliton concerning connection ∇^{B_2}, then one of the following cases holds:*

1) $\alpha = \beta = 0$*,* $\gamma \neq 0$*,* $c = 0$*;*

2) $\alpha \neq 0$*,* $\gamma = 0$*,* $c = 0$*;*

3) $\alpha = \gamma = 0$*,* $c = 0$.

Proof. From (5.16), I have the expression for Ric^{B_2} as follows:

$$Ric^{B_2} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\beta\gamma & -\beta\gamma \\ 0 & \beta\gamma & \beta\gamma \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.$$

The scalar curvature is $s^{B_2} = 0$. Now, I can express D^{B_2} as follows:

$$\begin{cases} D^{B_2}e_1 = -(s\lambda_0 + c)e_1, \\ D^{B_2}e_2 = -(\beta\gamma + s\lambda_0 + c)e_2 - \beta\gamma e_3, \\ D^{B_2}e_3 = \beta\gamma e_2 + (\beta\gamma - s\lambda_0 - c)e_3. \end{cases}$$

Therefore, based on Eq (2.4), there is an algebraic Schouten soliton associated with ∇^{B_2} on (G_7, g), if and only if the following condition satisfies:

$$\begin{cases} \alpha(\beta\gamma + c) - \alpha\beta\gamma = 0, \\ \beta c = 0, \\ \beta(2\beta\gamma + c) - 2\beta^2\gamma = 0, \\ \alpha(\beta\gamma - c) = \alpha\beta\gamma, \\ \beta c + 2\beta^2\gamma = 0, \\ \gamma c = 0, \\ \delta(-\beta\gamma + c) + \beta\gamma\delta = 0, \\ \delta(\beta\gamma + c) - \beta\gamma\delta = 0. \end{cases} \tag{5.17}$$

Since $\alpha\gamma = 0$ and $\alpha + \delta \neq 0$, I now analyze the system under different assumptions.

First, if $\alpha = 0$ and $\gamma \neq 0$, under this assumption, the fifth and sixth equations of (5.17) jointly imply that $\beta = c = 0$. Therefore, Case 1) holds.

Second, if $\alpha \neq 0$ and $\gamma = 0$, then the first equation of (5.17) gives $c = 0$, and for Case 2), system (5.17) holds.

Finally, if $\alpha = \gamma = 0$, then $\delta \neq 0$, and the last equation of (5.17) gives $c = 0$. Therefore, Case 3) holds. □

6. Conclusions

I present algebraic conditions for three-dimensional Lorentzian Lie groups to be an algebraic Schouten soliton associated with the Bott connection, considering three distributions. The main result indicates that G_4 and G_7 do not have such solitons with the first distribution, while the result for G_5 with the first distribution is trivial, and the other cases all possess algebraic Schouten solitons. In the future, we will explore algebraic Schouten solitons in higher dimensions, as in [24, 25].

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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