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Research article

Power series expansion, decreasing property, and concavity related to logarithm of normalized tail of power series expansion of cosine

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Abstract: In this paper, in view of a determinantal formula for higher order derivatives of the ratio of two differentiable functions, we expand the logarithm of the normalized tail of the power series expansion of the cosine function into a Maclaurin power series expansion whose coefficients are expressed in terms of specific Hessenberg determinants, present the decreasing property and concavity of the normalized tail of the Maclaurin power series expansion of the cosine function, deduce a new determinantal expression of the Bernoulli numbers, and verify the decreasing property for the ratio of the logarithms of the first two normalized tails of the Maclaurin power series expansion of the cosine function.

Keywords: Maclaurin power series expansion; normalized tail; cosine; ratio; derivative formula; Hessenberg determinant; decreasing property; concavity; Bernoulli number; logarithm; determinantal expression

1. Motivations

In [1, pp. 42 and 55], we find the Maclaurin power series expansions

$$\cos x = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!} = 1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}+\frac{x^8}{40320}-\cdots, \quad x\in\mathbb{R}$$

and

$$\ln\cos x = -\sum_{k=1}^{\infty}\frac{2^{2k-1}(2^{2k}-1)}{k(2k)!}|B_{2k}|x^{2k} = -\frac{x^2}{2}-\frac{x^4}{12}-\frac{x^6}{45}-\frac{17x^8}{2520}-\cdots \tag{1}$$

for $x^2 < \frac{\pi^2}{4}$, where B_{2k} denotes the Bernoulli numbers which can be generated by

$$\frac{z}{e^z - 1} = \sum_{k=0}^{\infty} B_k \frac{z^k}{k!} = 1 - \frac{z}{2} + \sum_{k=1}^{\infty} B_{2k} \frac{z^{2k}}{(2k)!}, \quad |z| < 2\pi.$$

Let

$$F(x) = \begin{cases} \ln \dfrac{2(1-\cos x)}{x^2}, & 0 < |x| < 2\pi; \\ 0, & x = 0 \end{cases} \tag{2}$$

and

$$R(x) = \begin{cases} \dfrac{\ln \frac{2(1-\cos x)}{x^2}}{\ln \cos x}, & 0 < |x| < \dfrac{\pi}{2}; \\ \dfrac{1}{6}, & x = 0; \\ 0, & x = \pm\dfrac{\pi}{2}. \end{cases} \tag{3}$$

In the recent paper [2], Li and Qi obtained the following two results:

1. The function $F(x)$ defined by (2) can be expanded into the Maclaurin power series expansion

$$F(x) = -\sum_{n=1}^{\infty} \frac{E_{2n}}{(2n)!} x^{2n} = -\frac{x^2}{12} - \frac{x^4}{1440} - \frac{x^6}{90720} - \frac{x^8}{4838400} - \cdots \tag{4}$$

for $|x| < 2\pi$, where

$$E_{2n} = -\begin{vmatrix} A_{2n-1,1} & B_{2n-1,2n-1} \\ \omega_n & C_{1,2n+1} \end{vmatrix}, \quad n \ge 1,$$

$$\omega_k = \frac{(-1)^k}{(k+1)(2k+1)}, \quad k \ge 0,$$

$$A_{2n-1,1} = \begin{pmatrix} 0 \\ \omega_1 \\ 0 \\ \omega_2 \\ \vdots \\ 0 \\ \omega_{n-1} \\ 0 \end{pmatrix} = \left(a_{i,j}\right)_{\substack{1 \le i \le 2n-1 \\ j=1}}, \quad n \ge 1,$$

$$a_{i,1} = \begin{cases} 0, & 1 \le i = 2k-1 \le 2n-1; \\ \omega_k, & 2 \le i = 2k \le 2n-2, \end{cases}$$

$$B_{2n-1,2n-1} = \begin{vmatrix} \binom{0}{0}\omega_0 & 0 & 0 & \cdots & 0 \\ 0 & \binom{1}{1}\omega_0 & 0 & \cdots & 0 \\ \binom{2}{0}\omega_1 & 0 & \binom{2}{2}\omega_0 & \cdots & 0 \\ 0 & \binom{3}{1}\omega_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{2n-4}{0}\omega_{n-2} & 0 & \binom{2n-4}{2}\omega_{n-3} & \cdots & 0 \\ 0 & \binom{2n-3}{1}\omega_{n-2} & 0 & \cdots & 0 \\ \binom{2n-2}{0}\omega_{n-1} & 0 & \binom{2n-2}{2}\omega_{n-1} & \cdots & \binom{2n-2}{2n-2}\omega_0 \end{vmatrix}$$

$$= \left(b_{i,j}\right)_{1\le i,j\le 2n-1}, \quad n \ge 1,$$

$$b_{i,j} = \begin{cases} 0, & 1 \le i < j \le 2n-1; \\ \dbinom{i}{j}\omega_k, & 0 \le i-j = 2k \le 2n-2; \\ 0, & 1 \le i-j = 2k-1 \le 2n-3, \end{cases}$$

$$C_{1,2n-1} = \begin{pmatrix} 0 & \binom{2n-1}{1}\omega_{n-1} & 0 & \binom{2n-1}{3}\omega_{n-2} & \cdots & 0 & \binom{2n-1}{2n-3}\omega_1 & 0 \end{pmatrix}$$

$$= \left(c_{i,j}\right)_{\substack{i=1 \\ 1\le j\le 2n-1}}, \quad n \ge 1,$$

$$c_{1,j} = \begin{cases} 0, & 1 \le j = 2k-1 \le 2n-1; \\ \dbinom{2n-1}{2k-1}\omega_{n-k}, & 2 \le j = 2k \le 2n-2. \end{cases}$$

2. The function $R(x)$ defined by (3) decreasingly maps $[0, \frac{\pi}{2}]$ onto $[0, \frac{1}{6}]$.

We now introduce several new even functions as follows:

1. The first function is

$$F_0(x) = \ln\cos x, \quad x \in \bigcup_{k=0}^{\infty}\left(\pm 2k\pi - \frac{\pi}{2}, \pm 2k\pi + \frac{\pi}{2}\right). \tag{5}$$

2. The second function is

$$F_1(x) = \begin{cases} \ln\dfrac{2(1-\cos x)}{x^2}, & x \in \mathbb{R} \setminus \{\pm 2k\pi, k = 1, 2, \dots\}; \\ 0, & x = 0. \end{cases} \tag{6}$$

It is clear that $F_1(x) = F(x)$ on $(-2\pi, 2\pi)$.

3. Generally, the third function we are introducing is

$$F_n(x) = \begin{cases} \ln\left((-1)^n \dfrac{(2n)!}{x^{2n}}\left[\cos x - \displaystyle\sum_{k=0}^{n-1}(-1)^k \frac{x^{2k}}{(2k)!}\right]\right), & x \ne 0 \\ 0, & x = 0 \end{cases} \tag{7}$$

for $n \ge 2$.

Since the double inequality

$$0 < (-1)^n\left[\cos x - \sum_{k=0}^{n-1}(-1)^k\frac{x^{2k}}{(2k)!}\right] < \frac{x^{2n}}{(2n)!} \tag{8}$$

is valid for $n \geq 2$ and $x \in \mathbb{R} \setminus \{0\}$, see [3, p. 326], the function $F_n(x)$ is significantly defined for $n \geq 2$ and $x \in \mathbb{R}$.

As a stronger version of the double inequality (8), the following positive, nonnegative, decreasing, and concave properties of the normalized tail

$$\operatorname{CosR}_n(x) = \begin{cases} (-1)^n\dfrac{(2n)!}{x^{2n}}\left[\cos x - \displaystyle\sum_{k=0}^{n-1}(-1)^k\frac{x^{2k}}{(2k)!}\right], & x \neq 0 \\ 1, & x = 0 \end{cases} \tag{9}$$

for $x \in (0, \infty)$ and $n \geq 1$ were discovered in the paper [4]:

(a) the normalized tail $\operatorname{CosR}_1(x)$ is nonnegative on $(0, \infty)$ and is decreasing on $[0, 2\pi]$;
(b) the normalized tail $\operatorname{CosR}_n(x)$ for $n \geq 2$ is decreasing and positive on $(0, \infty)$;
(c) the normalized tail $\operatorname{CosR}_1(x)$ is concave on $(0, x_0)$, where $x_0 \in (\frac{\pi}{2}, \pi)$ is the first positive zero of the equation

$$(x^2 - 2)\sin x + 2x\cos x = 0$$

and the normalized remainder $\operatorname{CosR}_n(x)$ for $n \geq 2$ is concave on $(0, \pi)$.

Comparing the definition in (9) with those in (6) and (7) leads to the following conclusions:

(a) the function $F_1(x)$ is decreasing and negative on $(0, 2\pi)$, and is concave on $(0, x_0)$;
(b) the function $F_n(x)$ for $n \geq 2$ is decreasing and negative on $(0, \infty)$, and is concave on $(0, \pi)$.

4. The fourth function $R_{m,n}(x)$ for $n > m \geq 0$ is defined
   (a) when $n > m = 0$, by

$$R_{0,n}(x) = \begin{cases} \dfrac{F_n(x)}{F_0(x)}, & 0 < |x| < \dfrac{\pi}{2}; \\ \dfrac{1}{(n+1)(2n+1)}, & x = 0; \\ 0, & x = \pm\dfrac{\pi}{2}, \end{cases} \tag{10}$$

   (b) when $n > m = 1$, by

$$R_{1,n}(x) = \begin{cases} \dfrac{F_n(x)}{F_1(x)}, & x \in \mathbb{R} \setminus \{\pm 2k\pi, k = 1, 2, \dots\}; \\ \dfrac{6}{(n+1)(2n+1)}, & x = 0; \\ 0, & x \in \{\pm 2k\pi, k = 1, 2, \dots\}, \end{cases}$$

   (c) when $n > m \geq 2$, by

$$R_{m,n}(x) = \begin{cases} \dfrac{F_n(x)}{F_m(x)}, & x \neq 0; \\ \dfrac{(m+1)(2m+1)}{(n+1)(2n+1)}, & x = 0. \end{cases}$$

It is easy to see that $R_{0,1}(x) = R(x)$.

We now propose the following three problems:

1. Is the function $F_n(x)$ for $n \ge 0$ decreasing and concave?
2. What is the Maclaurin power series expansion of $F_n(x)$ for $n \ge 0$ around the origin $x = 0$?
3. Is the function $R_{m,n}(x)$ for $n > m \ge 0$ decreasing?

The first problem for the case $n = 0$ is immediate: the even function $F_0(x) = \ln\cos x$ is decreasing and, by virtue of the series expansion (1), is concave in $x \in (0, \frac{\pi}{2})$. The first problem for the case $n \ge 1$ was solved in the paper [4], as mentioned above. In a word, the first problem has been thoroughly solved.

The second problem for $n = 0$ is just the Maclaurin power series expansion (1). The second problem for $n = 1$ was solved by the Maclaurin series expansion (4), which was established in [2, Section 3].

The third problem for $(m, n) = (0, 1)$ was solved in [2, Section 4], as mentioned above.

In this paper, we will give a full answer to the second problem for all cases $n \ge 2$, solve the first problem on the interval $(0, \frac{\pi}{2})$ once again, and discuss the third problem for the case $n \ge 2$.

2. What is the Maclaurin power series expansion of $F_n(x)$?

In this section, we solve the second problem: what is the Maclaurin power series expansion of $F_n(x)$ for $n \ge 0$ around the origin $x = 0$?

Theorem 1. *For $n \ge 0$, let*

$$e_{i,j}(n) = \begin{cases} (-1)^{i/2}\dfrac{1+(-1)^i}{2}\dfrac{1}{\binom{2n+i}{2n}}, & 1 \le i \le 2m, j = 1 \\ (-1)^{(i-j+1)/2}\dfrac{1+(-1)^{i-j+1}}{2}\dfrac{\binom{i-1}{j-2}}{\binom{2n+i-j+1}{2n}}, & 1 \le i \le 2m, 2 \le j \le 2m \end{cases}$$

and

$$D_{2m}(n) = \left|e_{i,j}(n)\right|_{(2m)\times(2m)}.$$

Then the even function $F_n(x)$ for $n \ge 0$ can be expanded into

$$F_n(x) = -\sum_{m=1}^{\infty}\frac{D_{2m}(n)}{(2m)!}x^{2m}, \quad |x| < \begin{cases} \dfrac{\pi}{2}, & n = 0; \\ 2\pi, & n = 1; \\ \infty, & n \ge 2. \end{cases} \tag{11}$$

Proof. Let $u(x)$ and $v(x) \ne 0$ be two n-time differentiable functions on an interval I for a given integer $n \ge 0$. Then the nth derivative of the ratio $\frac{u(x)}{v(x)}$ is

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[\frac{u(x)}{v(x)}\right] = (-1)^n\frac{\left|W_{(n+1)\times(n+1)}(x)\right|}{v^{n+1}(x)}, \quad n \ge 0, \tag{12}$$

where the matrix

$$W_{(n+1)\times(n+1)}(x) = \begin{pmatrix} U_{(n+1)\times 1}(x) & V_{(n+1)\times n}(x) \end{pmatrix}_{(n+1)\times(n+1)},$$

the matrix $U_{(n+1)\times 1}(x)$ is an $(n+1)\times 1$ matrix whose elements satisfy $u_{k,1}(x) = u^{(k-1)}(x)$ for $1 \le k \le n+1$, the matrix $V_{(n+1)\times n}(x)$ is an $(n+1)\times n$ matrix whose elements are $v_{\ell,j}(x) = \binom{\ell-1}{j-1} v^{(\ell-j)}(x)$ for $1 \le \ell \le n+1$ and $1 \le j \le n$, and the notation $|W_{(n+1)\times(n+1)}(x)|$ denotes the determinant of the $(n+1)\times(n+1)$ matrix $W_{(n+1)\times(n+1)}(x)$. This is a slight reformulation of [5, p. 40, Exercise 5].

Let

$$u_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{\binom{2k+2n+2}{2n}} \frac{x^{2k+1}}{(2k+1)!} \quad \text{and} \quad v_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^{k}}{\binom{2k+2n}{2n}} \frac{x^{2k}}{(2k)!}.$$

Then, straightforward differentiation yields

$$u_n^{(2\ell+1)}(0) = \lim_{x\to 0} \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{\binom{2k+2n+2}{2n}} \langle 2k+1\rangle_{2\ell+1} \frac{x^{2k-2\ell}}{(2k+1)!} = \frac{(-1)^{\ell+1}}{\binom{2\ell+2n+2}{2n}},$$

$$v_n^{(2\ell)}(0) = \lim_{x\to 0} \sum_{k=0}^{\infty} \frac{(-1)^{k}}{\binom{2k+2n}{2n}} \langle 2k\rangle_{2\ell} \frac{x^{2k-2\ell}}{(2k)!} = \frac{(-1)^{\ell}}{\binom{2\ell+2n}{2n}},$$

$$u_n^{(2\ell)}(0) = 0, \quad v_n^{(2\ell+1)}(0) = 0$$

for $\ell \ge 0$. Considering Expression (14) and applying the derivative formula (12) for the ratio of two differentiable functions, we acquire

$$
\begin{aligned}
F_n^{(2m)}(0) &= \lim_{x\to 0} \left[\frac{\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{\binom{2k+2n+2}{2n}} \frac{x^{2k+1}}{(2k+1)!}}{\sum_{k=0}^{\infty} \frac{(-1)^{k}}{\binom{2k+2n}{2n}} \frac{x^{2k}}{(2k)!}} \right]^{(2m-1)} = \lim_{x\to 0} \left[\frac{u_n(x)}{v_n(x)} \right]^{(2m-1)} \\
&= \frac{(-1)^{2m-1}}{v_n^{2m}(0)}
\begin{vmatrix}
u_n(0) & v_n(0) & 0 & \cdots & 0 \\
u_n'(0) & v_n'(0) & \binom{1}{1} v_n(0) & \cdots & 0 \\
u_n''(0) & v_n''(0) & \binom{2}{1} v_n'(0) & \cdots & 0 \\
u_n^{(3)}(0) & v_n^{(3)}(0) & \binom{3}{1} v_n''(0) & \cdots & 0 \\
u_n^{(4)}(0) & v_n^{(4)}(0) & \binom{4}{1} v_n^{(3)}(0) & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
u_n^{(2m-1)}(0) & v_n^{(2m-1)}(0) & \binom{2m-1}{1} v_n^{(2m-2)}(0) & \cdots & \binom{2m-1}{2m-2} v_n'(0)
\end{vmatrix} \\
&= -\left|
\begin{matrix}
0 & \binom{0}{0} & 0 & 0 & 0 \\
\frac{-1}{\binom{2n+2}{2n}} & 0 & \binom{1}{1} & 0 & 0 \\
0 & \frac{-\binom{2}{0}}{\binom{2n+2}{2n}} & 0 & \binom{2}{2} & 0 \\
\frac{1}{\binom{2n+4}{2n}} & 0 & \frac{-\binom{3}{1}}{\binom{2n+2}{2n}} & 0 & \binom{3}{3} \\
0 & \frac{\binom{4}{0}}{\binom{2n+4}{2n}} & 0 & \frac{-\binom{4}{2}}{\binom{2n+2}{2n}} & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\frac{(-1)^{m-1}}{\binom{2n+2m-2}{2n}} & 0 & \frac{(-1)^{m-2}\binom{2m-3}{1}}{\binom{2n+2m-4}{2n}} & 0 & \frac{(-1)^{m-3}\binom{2m-3}{3}}{\binom{2n+2m-6}{2n}} \\
0 & \frac{(-1)^{m-1}\binom{2m-2}{0}}{\binom{2n+2m-2}{2n}} & 0 & \frac{(-1)^{m-2}\binom{2m-2}{2}}{\binom{2n+2m-4}{2n}} & 0 \\
\frac{(-1)^{m}}{\binom{2n+2m}{2n}} & 0 & \frac{(-1)^{m-1}\binom{2m-1}{1}}{\binom{2n+2m-2}{2n}} & 0 & \frac{(-1)^{m-2}\binom{2m-1}{3}}{\binom{2n+2m-4}{2n}}
\end{matrix}
\right.
\end{aligned}
$$

$$
\left.\begin{matrix}
\cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\cdots & \frac{\binom{2m-3}{2m-7}}{\binom{2n+4}{2n}} & 0 & \frac{-\binom{2m-3}{2m-5}}{\binom{2n+2}{2n}} & 0 & \binom{2m-3}{2m-3} & 0 \\
\cdots & 0 & \frac{\binom{2m-2}{2m-6}}{\binom{2n+4}{2n}} & 0 & \frac{-\binom{2m-2}{2m-4}}{\binom{2n+2}{2n}} & 0 & \binom{2m-2}{2m-2} \\
\cdots & \frac{-\binom{2m-1}{2m-7}}{\binom{2n+6}{2n}} & 0 & \frac{\binom{2m-1}{2m-5}}{\binom{2n+4}{2n}} & 0 & \frac{-\binom{2m-1}{2m-3}}{\binom{2n+2}{2n}} & 0
\end{matrix}\right|
$$

for $m \ge 1$. In other words, for $m \ge 1$,

$$
F_n^{(2m)}(0) = -D_{2m}(n) = -\left|e_{i,j}(n)\right|_{(2m)\times(2m)}.
$$

Consequently, the even function $F_n(x)$ can be expanded into

$$
F_n(x) = \sum_{k=0}^{\infty} F_n^{(k)} \frac{x^k}{k!} = \sum_{m=1}^{\infty} F_n^{(2m)} \frac{x^{2m}}{(2m)!} = -\sum_{m=1}^{\infty} \frac{D_{2m}(n)}{(2m)!} x^{2m}.
$$

The proof of Theorem 1 is completed. □

Remark 1. When $n = 0$, a direct computation gives

$$
D_4(0) = \begin{vmatrix} 0 & \binom{0}{0} & 0 & 0 \\ -1 & 0 & \binom{1}{1} & 0 \\ 0 & -\binom{2}{0} & 0 & \binom{2}{2} \\ 1 & 0 & -\binom{3}{1} & 0 \end{vmatrix} = 2 \quad \text{and} \quad D_2(0) = \begin{vmatrix} 0 & \binom{0}{0} \\ -1 & 0 \end{vmatrix} = 1.
$$

Then, the first two terms of the Maclaurin power series expansion of the function $\ln\cos x$ are

$$
-\frac{D_2(0)}{2!}x^2 - \frac{D_4(0)}{4!}x^4 = -\frac{1}{2!}x^2 - \frac{2}{4!}x^4 = -\frac{1}{2}x^2 - \frac{1}{12}x^4,
$$

which coincide with the first two terms in the series expansion (1).

When $n = 1$, straightforward computation shows

$$
D_4(1) = \begin{vmatrix} 0 & \binom{0}{0} & 0 & 0 \\ \frac{-1}{\binom{4}{2}} & 0 & \binom{1}{1} & 0 \\ 0 & \frac{-\binom{2}{0}}{\binom{4}{2}} & 0 & \binom{2}{2} \\ \frac{1}{\binom{6}{2}} & 0 & \frac{-\binom{3}{1}}{\binom{4}{2}} & 0 \end{vmatrix} = \frac{1}{60} \quad \text{and} \quad D_2(1) = \begin{vmatrix} 0 & \binom{0}{0} \\ \frac{-1}{\binom{4}{2}} & 0 \end{vmatrix} = \frac{1}{6}.
$$

Then, the first two terms of the Maclaurin power series expansion of the function $F(x)$ defined by (2) are

$$
-\frac{D_2(1)}{2!}x^2 - \frac{D_4(1)}{4!}x^4 = -\frac{1}{12}x^2 - \frac{1}{1440}x^4,
$$

which coincide with the first two terms in the series expansion (4).

Remark 2. Comparing the Maclaurin series expansions (1) and (4) with the series expansion (11) reveals

$$|B_{2m}| = \frac{m}{2^{2m-1}(2^{2m}-1)} D_{2m}(0) \quad \text{and} \quad E_{2m} = D_{2m}(1) \tag{13}$$

for $m \geq 1$. The first formula in (13) is a new determinantal expression for the Bernoulli numbers B_{2m} with $m \geq 1$.

Additionally, we point out that, in the papers [2, 6–10], there have been many related results, but different from and more complicated than the first one in (13), and plenty of closely-related references on closed-form formulas and determinantal expressions for the Bernoulli numbers and polynomials B_{2m} and $B_m(x)$ with $m \in \mathbb{N}$.

3. Is the function $F_n(x)$ for $n \geq 0$ decreasing and concave on $(0, \frac{\pi}{2})$?

In this section, we give an alternative and united proof of a modification of the first problem: is the function $F_n(x)$ for $n \geq 0$ decreasing and concave on $(0, \frac{\pi}{2})$?

Theorem 2. *For $n = 0$ and $n \geq 2$, the even function $F_n(x)$ defined by* (5) *and* (7) *is decreasing and concave on $(0, \frac{\pi}{2})$. The even function $F_1(x)$ defined in* (6) *is decreasing on $(0, 2\pi)$ and $(x_k, 2(k+1)\pi)$ for $k \in \mathbb{N}$, while it is increasing on $(2k\pi, x_k)$ for $k \in \mathbb{N}$, where $x_k \in (2k\pi, 2(k+1)\pi)$ for $k \in \mathbb{N}$ is the zero of the equation $\tan \frac{x}{2} = \frac{x}{2}$ on $(0, \infty)$.*

Proof. In the first section of this paper, it has been immediately verified that the function $F_0(x) = \ln \cos x$ is decreasing and concave on $(0, \frac{\pi}{2})$.

The derivative of $F_1(x)$ can be written as

$$F_1'(x) = \frac{1}{\tan \frac{x}{2}} - \frac{1}{\frac{x}{2}}, \quad x \neq \pm 2k\pi, \quad k \in \mathbb{N}.$$

Therefore, the derivative $F_1'(x)$ is negative on $(0, 2\pi)$, is positive on $(2k\pi, x_k)$, and is negative on $(x_k, 2(k+1)\pi)$ for $k \in \mathbb{N}$, where $x_k \in (2k\pi, 2(k+1)\pi)$ for $k \in \mathbb{N}$ is the zero of the equation $\tan \frac{x}{2} = \frac{x}{2}$ on $(0, \infty)$. Accordingly, the function $F_1(x)$ is decreasing on $(0, 2\pi)$ and $(x_k, 2(k+1)\pi)$, while it is increasing on $(2k\pi, x_k)$ for $k \in \mathbb{N}$.

On the interval $(0, \frac{\pi}{2})$ and for $n \geq 2$, the function $F_n(x)$ can be written as

$$F_n(x) = \ln \sum_{k=n}^{\infty} (-1)^{k-n} \frac{(2n)!}{(2k)!} x^{2k-2n} = \ln \sum_{k=0}^{\infty} \frac{(-1)^k}{\binom{2k+2n}{2n}} \frac{x^{2k}}{(2k)!}.$$

Its first derivative is

$$F_n'(x) = \frac{\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{\binom{2k+2n+2}{2n}} \frac{x^{2k+1}}{(2k+1)!}}{\sum_{k=0}^{\infty} \frac{(-1)^k}{\binom{2k+2n}{2n}} \frac{x^{2k}}{(2k)!}}. \tag{14}$$

By virtue of [11, Theorem 7.6], or in view of the results at the site https://math.stackexchange.com/a/477549 (accessed on 18 January 2024), we derive the integral representation

$$\cos x - \sum_{k=0}^{n-1} (-1)^k \frac{x^{2k}}{(2k)!} = (-1)^n x^{2n} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2n+2k)!} x^{2k} = \frac{(-1)^n}{(2n-2)!} \int_0^x (x-t)^{2n-2} \sin t \, \mathrm{d}\, t \tag{15}$$

for $n \ge 1$ and $x \in \mathbb{R}$. From the integral representation (15), it follows that

$$\begin{aligned}
(-1)^n\frac{(2n)!}{x^{2n}}\left[\cos x-\sum_{k=0}^{n-1}\frac{(-1)^k}{(2k)!}x^{2k}\right] &= \sum_{k=0}^{\infty}\frac{(-1)^k}{\binom{2k+2n}{2n}}\frac{x^{2k}}{(2k)!}\\
&= \frac{2n(2n-1)}{x^{2n}}\int_0^x (x-t)^{2n-2}\sin t \,\mathrm{d}\, t\\
&= 2n(2n-1)\int_0^1 (1-u)^{2n-2}\frac{\sin(ux)}{x}\,\mathrm{d}\, u\\
&> 0, \quad 0<x<\pi
\end{aligned} \tag{16}$$

and

$$\begin{aligned}
\left[\sum_{k=0}^{\infty}\frac{(-1)^k}{\binom{2k+2n}{2n}}\frac{x^{2k}}{(2k)!}\right]' &= \sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{\binom{2k+2n+2}{2n}}\frac{x^{2k+1}}{(2k+1)!}\\
&= 2n(2n-1)\int_0^1 (1-u)^{2n-2}\frac{\mathrm{d}}{\mathrm{d}\, x}\left[\frac{\sin(ux)}{x}\right]\mathrm{d}\, u\\
&= 2n(2n-1)\int_0^1 (1-u)^{2n-2}\cos(ux)\frac{(ux)-\tan(ux)}{x^2}\,\mathrm{d}\, u\\
&< 0, \quad 0<x<\frac{\pi}{2}
\end{aligned}$$

for $n \ge 1$, where we used the inequalities $\cos x > 0$ and $x - \tan x < 0$ in $x \in (0, \frac{\pi}{2})$. This means that

$$F_n'(x) = \frac{\int_0^1 (1-u)^{2n-2}\cos(ux)\frac{(ux)-\tan(ux)}{x^2}\,\mathrm{d}\, u}{\int_0^1 (1-u)^{2n-2}\frac{\sin(ux)}{x}\,\mathrm{d}\, u} < 0, \quad 0<x<\frac{\pi}{2}$$

for $n \ge 1$. In conclusion, the function $F_n(x)$ for $n \ge 1$ is decreasing on $(0, \frac{\pi}{2})$.

It is known that

$$\frac{\sin x}{x} = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k+1)!}, \quad |x|<\infty.$$

Straightforward differentiating and simplifying give

$$\begin{aligned}
\left(\frac{\sin x}{x}\right)'' &= \sum_{k=0}^{\infty}(-1)^{k+1}\frac{(2k+2)(2k+1)}{(2k+3)!}x^{2k}\\
&= -\sum_{k=0}^{\infty}\frac{(4k+4)(4k+3)}{(4k+5)!}\left[\frac{(4k+1)(4k+2)(4k+5)}{(4k+3)}-x^2\right]x^{4k}\\
&< 0, \qquad x \in \left(0, \sqrt{\frac{10}{3}}\right) \supset \left(0, \frac{\pi}{2}\right].
\end{aligned}$$

Therefore, a direct differentiation and simplification yield

$$
\begin{aligned}
F_n''(x) &= \left[\frac{\int_0^1 (1-u)^{2n-2} \frac{\mathrm{d}}{\mathrm{d}\,x}\left(\frac{\sin(ux)}{x}\right) \mathrm{d}\,u}{\int_0^1 (1-u)^{2n-2} \frac{\sin(ux)}{x} \,\mathrm{d}\,u}\right]' \\
&= \frac{-1}{\left[\int_0^1 (1-u)^{2n-2} \frac{\sin(ux)}{x} \,\mathrm{d}\,u\right]^2} \left[\left(\int_0^1 (1-u)^{2n-2} \frac{\mathrm{d}}{\mathrm{d}\,x}\left(\frac{\sin(ux)}{x}\right) \mathrm{d}\,u\right)^2\right. \\
&\quad \left. - \int_0^1 (1-u)^{2n-2} \frac{\mathrm{d}^2}{\mathrm{d}\,x^2}\left(\frac{\sin(ux)}{x}\right) \mathrm{d}\,u \int_0^1 (1-u)^{2n-2} \frac{\sin(ux)}{x} \,\mathrm{d}\,u\right] \\
&= \frac{-1}{\left[\int_0^1 (1-u)^{2n-2} \frac{\sin(ux)}{x} \,\mathrm{d}\,u\right]^2} \left[\left(\int_0^1 u(1-u)^{2n-2} \frac{\mathrm{d}}{\mathrm{d}\,x}\left(\frac{\sin(ux)}{ux}\right) \mathrm{d}\,u\right)^2\right. \\
&\quad \left. - \int_0^1 u(1-u)^{2n-2} \frac{\mathrm{d}^2}{\mathrm{d}\,x^2}\left(\frac{\sin(ux)}{ux}\right) \mathrm{d}\,u \int_0^1 u(1-u)^{2n-2} \frac{\sin(ux)}{ux} \,\mathrm{d}\,u\right] \\
&< 0
\end{aligned}
$$

on $(0, \frac{\pi}{2})$ for $n \ge 1$. Accordingly, the function $F_n(x)$ for $n \ge 1$ is concave on $(0, \frac{\pi}{2})$. The proof of Theorem 2 is thus complete. □

Remark 3. We note that a concave function must be a logarithmically concave function, but the converse is not true. However, a logarithmically convex function must be a convex function, but the converse is not true.

4. Is the function $R_{0,2}(x)$ decreasing?

In [2, Section 4], the function $R_{0,1}(x) = R(x)$ defined by (3) or (10) for $n = 1$ was proved to be decreasing on $[0, \frac{\pi}{2}]$ onto $[0, \frac{1}{6}]$.

Theorem 3. *The even function $R_{0,2}(x)$ defined by* (10) *for the case $n = 2$ is decreasing on* $[0, \frac{\pi}{2}]$.

Proof. For $n \ge 2$, direct differentiation gives

$$
\frac{F_n'(x)}{F_0'(x)} = \frac{[\ln \mathrm{CosR}_n(x)]'}{(\ln \cos x)'} = -\frac{\mathrm{CosR}_n'(x)}{\mathrm{CosR}_n(x)} \frac{\cos x}{\sin x}
$$

and

$$
\begin{aligned}
\left[\frac{F_n'(x)}{F_0'(x)}\right]' &= -\left[\frac{\mathrm{CosR}_n'(x)}{\mathrm{CosR}_n(x)} \frac{\cos x}{\sin x}\right]' \\
&= -\frac{(\mathrm{CosR}_n(x) \,\mathrm{CosR}_n''(x) - [\mathrm{CosR}_n'(x)]^2) \cos x \sin x - \mathrm{CosR}_n(x) \,\mathrm{CosR}_n'(x)}{[\mathrm{CosR}_n(x) \sin x]^2}.
\end{aligned}
$$

Taking $n = 2$ and simplifying lead to

$$
\begin{aligned}
\left[\frac{F_2'(x)}{F_0'(x)}\right]' &= -\frac{72}{x^{10}[\mathrm{CosR}_2(x)\sin x]^2}[4x(x^4-6x^2+12)+4(7x^2-16)x\cos x+16x\cos(2x)\\
&\quad -4x^3\cos(3x)+(3x^4+4x^2-16)\sin x+2(x^4-10x^2+12)\sin(2x)\\
&\quad -(x^2-2x-4)(x^2+2x-4)\sin(3x)+4\sin(4x)]\\
&= -\frac{72}{x^{10}[\mathrm{CosR}_2(x)\sin x]^2}\sum_{k=6}^{\infty}(-1)^k Q(k)\frac{x^{2k+1}}{(2k+1)!}\\
&= -\frac{72}{x^{10}[\mathrm{CosR}_2(x)\sin x]^2}\sum_{k=3}^{\infty}\left[\frac{Q(2k)}{Q(2k+1)}\frac{(4k+3)!}{(4k+1)!}-x^2\right]\frac{Q(2k+1)}{(4k+3)!}x^{4k+1},
\end{aligned}
$$

where

$$
\begin{aligned}
Q(k) &= 4^{2k+2}-4(4k^4-28k^3+107k^2+61k+324)3^{2k-3}+(4k^4-4k^3+39k^2+53k+64)2^{2k}\\
&\quad +4(12k^4-68k^3-7k^2-17k-20), \quad k\ge 6.
\end{aligned}
$$

From the facts that

$$
\begin{aligned}
12k^4-68k^3-7k^2-17k-20 &= 12(k-6)^4+220(k-6)^3+1361(k-6)^2+2923(k-6)+490\\
&\ge 490, \quad k\ge 6,\\
4k^4-4k^3+39k^2+53k+64 &= 4(k-6)^4+92(k-6)^3+822(k-6)^2+3437(k-6)+5782\\
&\ge 5782, \quad k\ge 6,
\end{aligned}
$$

and, by induction,

$$
\begin{aligned}
&4^{2k+2}-4(4k^4-28k^3+107k^2+61k+324)3^{2k-3}\\
&\qquad = 16\times 3^{2k}\left[\left(\frac{4}{3}\right)^{2k}-\frac{4k^4-28k^3+107k^2+61k+324}{108}\right]>0, \quad k\ge 7,
\end{aligned}
$$

we conclude, together with $Q(6)=3871296$, that $Q(k)\ge 3871296$ for $k\ge 6$.

Let

$$
\mathcal{Q}(k)=\frac{Q(2k)}{Q(2k+1)}\frac{(4k+3)!}{(4k+1)!}, \quad k\ge 3.
$$

The inequality

$$
\mathcal{Q}(k+1)>\mathcal{Q}(k), \quad k\ge 3
$$

is equivalent to

$$
(2k+3)(4k+7)Q(2k+1)Q(2k+2)>(2k+1)(4k+3)Q(2k)Q(2k+3)
$$

for $k\ge 3$, that is,

$$
\begin{aligned}
\mathfrak{Q}(k) &= 216[73728(k-2)^9+1508352(k-2)^8+13174784(k-2)^7\\
&\quad +63787136(k-2)^6+185680928(k-2)^5+329304964(k-2)^4\\
&\quad +345900612(k-2)^3+210339955(k-2)^2+94117995(k-2)
\end{aligned}
$$

$$
\begin{aligned}
&+ 42704050] + 27 \times 2^{4k}\{(142293 \times 2^{4k+9} + 93239336)k^5 \\
&+ 4(228963 \times 2^{4k+7} - 315163)k^4 + (1730269 \times 2^{4k+5} - 77141534)k^3 \\
&+ (590587 \times 2^{4k+5} - 61162271)k^2 + 31(39633 \times 2^{4k+3} - 705961)k \\
&+ 72(32581 \times 16^k - 50315) + 4096(57 \times 2^{4k+3} + 847)k^8 \\
&+ 2048(2^{4k+7} - 540k - 1635)k^9 + 32k^6[2^{4k+6}(3328k - 3377) \\
&+ 1151464k + 2801469]\} + 27 \times 2^{12k+7}[256k(2^{4k+5} - 45k^5) \\
&+ 9 \times 2^{4k+10} + 12352k^5 + 24016k^4 + 33804k^3 + 78984k^2 + 84261k \\
&+ 33915] + 16 \times 3^{4k}[159784960(k-4)^9 + 2908177408(k-4)^8 \\
&+ 31203008512(k-4)^7 + 218332899456(k-4)^6 \\
&+ 1039043941024(k-4)^5 + 3394763402820(k-4)^4 \\
&+ 7477586709444(k-4)^3 + 10528669786283(k-4)^2 \\
&+ 8416641010791(k-4) + 2806689073644] + 8 \times 3^{8k+3}(8192k^9 \\
&+ 9216k^8 + 30720k^7 + 148608k^6 + 179232k^5 + 77220k^4 + 1027780k^3 \\
&+ 2247363k^2 + 1992375k + 751032) + 6^{4k}[107481088(k-4)^9 \\
&+ 1948553216(k-4)^8 + 20892092672(k-4)^7 + 146632542496(k-4)^6 \\
&+ 702801488840(k-4)^5 + 2321298590084(k-4)^4 \\
&+ 5179870312810(k-4)^3 + 7367488384493(k-4)^2 \\
&+ 5849176490661(k-4) + 1817172122508] + 2^{8k+3}3^{4k}[313600(k-13)^6 \\
&+ 20524992(k-13)^5 + 540906544(k-13)^4 + 7219464756(k-13)^3 \\
&+ 49643220604(k-13)^2 + 151463526603(k-13) + 98734469508] \\
&> 0, \quad k \geq 3.
\end{aligned}
$$

By virtue of the above expression for $\mathfrak{Q}(k)$, we see that the sequence $\mathfrak{Q}(k)$ is positive for $k \geq 13$. On the other hand, it is straightforward that

$$
\begin{aligned}
\mathfrak{Q}(3) &= 352780960860822574080, \\
\mathfrak{Q}(4) &= 24939853492306689220057804 8, \\
\mathfrak{Q}(5) &= 613054235585401521504739001856 00, \\
\mathfrak{Q}(6) &= 866542654913503459187937658637767680 0, \\
\mathfrak{Q}(7) &= 896760345938903204692495757195177830886400, \\
\mathfrak{Q}(8) &= 776790402144023813922876138484790955626721331 20, \\
\mathfrak{Q}(9) &= 609282672256574529637611513770725141086968048678092 8, \\
\mathfrak{Q}(10) &= 453017430243005416828735039714559150312936188925226332160, \\
\mathfrak{Q}(11) &= 327491341507070110534568598341631706451926357265243153598720 00, \\
\mathfrak{Q}(12) &= 233207374467955539681287586414418119055421927303274987429365552640 0.
\end{aligned}
$$

Consequently, the sequence $\mathfrak{Q}(k)$ is positive for all $k \geq 3$. As a result, the sequence $Q(k)$ is increasing in

$k \geq 3$. It is immediate that $Q(3) = \frac{423}{115} = 3.678\ldots$. Hence, we acquire

$$Q(k) \geq \frac{423}{115} = 3.678\ldots, \quad k \geq 3.$$

Accordingly, when

$$0 < x \leq \frac{\pi}{2} = 1.570\cdots < \sqrt{\frac{423}{115}} = 3.678\cdots,$$

the derivative $\left[\frac{F_2'(x)}{F_0'(x)}\right]'$ is negative, and then the derivative ratio $\frac{F_2'(x)}{F_0'(x)}$ is decreasing in $x \in (0, \frac{\pi}{2}]$.

In [12, pp. 10–11, Theorem 1.25], a monotonicity rule for the ratio of two functions was established as follows.

> For $a, b \in \mathbb{R}$ with $a < b$, let $p(x)$ and $q(x)$ be continuous on $[a, b]$, differentiable on (a, b), and $q'(x) \neq 0$ on (a, b). If the ratio $\frac{p'(x)}{q'(x)}$ is increasing on (a, b), then both $\frac{p(x)-p(a)}{q(x)-q(a)}$ and $\frac{p(x)-p(b)}{q(x)-q(b)}$ are increasing in $x \in (a, b)$.

With the help of this monotonicity rule and in view of the decreasing property of the derivative ratio $\frac{F_2'(x)}{F_0'(x)}$ in $x \in (0, \frac{\pi}{2}]$, we derive that the ratio $\frac{F_2(x)}{F_0(x)} = R_{0,2}(x)$ is decreasing in $x \in (0, \frac{\pi}{2}]$. The required proof of Theorem 3 is completed. □

Remark 4. How to verify the decreasing property of the function $R_{0,n}(x)$ for $n \geq 3$ on $(0, \frac{\pi}{2}]$, of the function $R_{1,n}(x)$ on $(0, 2\pi)$, and of the function $R_{m,n}(x)$ for $n > m \geq 2$ on $(0, \infty)$? The ideas, approaches, techniques, and methods used in the proof of Theorem 3 should not be valid again, so we need to discover new ideas, approaches, techniques, and methods for verifying the decreasing property mentioned above.

Remark 5. Let

$$f_\alpha(x) = \int_0^1 (1-u)^\alpha \cos(ux)\,\mathrm{d}u, \quad \alpha \in \mathbb{R}.$$

Prove that the function $f_\alpha(x)$ is positive in $x \in (0, \infty)$ if and only if $\alpha > 1$, while it is decreasing in $x \in (0, \infty)$ if and only if $\alpha \geq 2$.

This paper and the articles [2, 13, 14] are siblings, because some results in [2] have been generalized in this paper, and the results in [13, 14] are about the Maclaurin power series expansions of logarithmic expressions involving normalized tails of the tangent and sine functions.

5. Conclusions

In this paper, we presented the following results.

1. The function $F_n(x)$ for $n \geq 0$ was expanded into the Maclaurin power series expansion (11) in Theorem 1.
2. The function $F_n(x)$ defined by (7) for $n \geq 0$ was proved in Theorem 2 to be decreasing and concave on $(0, \frac{\pi}{2})$.
3. A new determinantal expression (13) of the Bernoulli numbers B_{2m} for $m \geq 1$ was derived.
4. The ratio $R_{0,2}(x)$ was proved in Theorem 3 to be decreasing in $x \in [0, \frac{\pi}{2}]$.

In order to verify the decreasing property of the function $R_{0,n}(x)$ for $n \geq 3$ on $(0, \frac{\pi}{2})$, of the function $R_{1,n}(x)$ on $(0, 2\pi)$, and of the function $R_{m,n}(x)$ for $n > m \geq 2$ on $(0, \infty)$, we need new ideas, novel approaches, creative techniques, and innovative methods.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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