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Research article

Periodic traveling wave solutions of the Nicholson's blowflies model with delay and advection

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Abstract: The existence, stability and bifurcation direction of periodic traveling waves for the Nicholson's blowflies model with delay and advection are investigated by applying the Hopf bifurcation theorem, center manifold theorem as well as normal form theory. Some numerical simulations are presented to illustrate our main results.

Keywords: Nicholson's blowflies model; delay; advection; periodic traveling waves; numerical simulations

1. Introduction

In population dynamics, the study of traveling waves can help us to understand the mechanisms behind various propagating wave patterns better [1, 2]. The existence and stability of various types of wave solutions have been theoretically proved in many population models [3].

Based on the delayed Nicholson's blowflies equation [4], So and Yang [5] proposed the following diffusive version of delayed Nicholson's blowflies equation

$$\frac{\partial u(x,t)}{\partial t} = d_1 \frac{\partial^2 u(x,t)}{\partial x^2} - \delta u(x,t) + pu(x,t-\tau)e^{-au(x,t-\tau)}, \tag{1.1}$$

which describes the population growth of Nicholson's blowflies with spatial diffusion. So and Yang [5] studied the stability of the steady-state solutions for system (1.1) under the Dirichlet problem. So and Zou [6] proved the existence of traveling wavefronts. Stability of traveling wavefronts were investigated in [7, 8]. Yang [9] studied the existence and stability of periodic traveling waves. In order to describe the individual diffuse phenomenon during the reproductive cycle and more general birth functions, So et al. [10] derived the following reaction-diffusion equation involving nonlocal delay effects

$$\frac{\partial u(x,t)}{\partial t} = d_1 \frac{\partial^2 u(x,t)}{\partial x^2} - \delta u(x,t) + \varepsilon \int_{-\infty}^{+\infty} J_\alpha(y) b(u(x-y, t-\tau)) \mathrm{d}y, \tag{1.2}$$

and obtained Hopf bifurcation of model (1.2). See [11–16] for more progress on various types of traveling waves of system (1.2). In [17], Liang and Wu considered a delayed reaction diffusion equation with advection

$$\frac{\partial u(x,t)}{\partial t} = d_1 \frac{\partial^2 u(x,t)}{\partial x^2} + d_2 \frac{\partial u(x,t)}{\partial x} - \delta u(x,t) + \varepsilon \int_{-\infty}^{+\infty} J_\alpha(y) b(u(x + d_2\tau - y, t-\tau)) \mathrm{d}y, \tag{1.3}$$

which describes the pattern dynamics of a single-species population with two age classes and a fixed maturation period living in a spatial transport field and proved the existence of traveling wavefront. Stability of the traveling wavefronts were investigated in [18–20]. Of particular interest is the influence of advection terms on the propagation of traveling wave solutions. The influence of advection terms on traveling wavefronts were studied by many researchers [19,21]. To the best of our knowledge, however, the work on periodic traveling wave solution in the population model like system (1.3) with advection effect is relatively rare. Moreover, in the previous studies on periodic traveling wave solutions mainly focused on theoretical proof, see [9, 11, 22], which lacked the support of numerical simulation.

The objective of this paper is to derive the influence of advection terms on periodic traveling wave solutions and illustrate our main results by using numerical simulations. In order to make the influence of advection terms more obvious, in model (1.3), we suppose that the diffusion rate $d_1 = 0$ and replace the heat kernel function $J_\alpha(y)$ by the Dirac function to eliminate the influence of diffusion and nonlocal delay. Moreover, we choose the Nicholson's birth function on account of facilitate numerical simulations. Therefore, we consider the Nicholson's blowflies model with delay and advection as follows:

$$\frac{\partial u(x,t)}{\partial t} = d \frac{\partial u(x,t)}{\partial x} - \delta u(x,t) + p u(x, t-\tau) e^{-au(x,t-\tau)}, \tag{1.4}$$

with the following initial condition

$$u(x,t) = u_0(x,t), \quad t \in [-\tau, 0], \quad x \in \mathbb{R}.$$

Here, $u(x,t)$ denotes the total mature population at location $x \in \mathbb{R}$ and time $t \geq 0$; d is the advection rate of the mature represents the velocity of the spatial transport field; δ is the per capita adult death rate; p is the maximum per capita daily egg production rate; τ is a delay represents the maturation time; $\frac{1}{a}$ is the size at which the population reproduces at its maximum rate, where d, δ, p, a and τ are positive constants. Moreover, we assume that $p > \delta$ such that system (1.4) always has two equilibria $u_\star = 0$ and $u_* = \frac{1}{a} \ln \frac{p}{\delta} \in \mathbb{R}_+$.

2. Main results

In this part, the existence, stability and bifurcation direction of periodic traveling wave solutions of system (1.4) is proved theoretically by using the Hopf bifurcation theorem, center manifold theorem as well as normal form theory [23], which are bifurcated from two equilibria, respectively.

For the equilibrium $u_\star = 0$, we have the following result.

Theorem 2.1. *For each fixed wave velocity c, when the advection rate $d > c$, Eq (1.4) has a branch of periodic traveling waves bifurcating from $u_\star$ for the bifurcation parameter τ near $\tau_n^\star$, with period approximately equals to $\frac{2\pi}{\omega_\star}$. Moreover, if $\kappa_\star > 0$ (resp. < 0), the bifurcation is backward (resp. forward) and periodic traveling waves are unstable (resp. have the same stability as $u_\star$ before the bifurcation), where $\omega_\star$, $\tau_n^\star$ and $\kappa_\star$ are shown in Eqs (2.6), (2.7) and (2.15), respectively.*

Proof. Let the traveling wave variable $s = x + ct$ and $u(x,t) = v(s)$, then Eq (1.4) can be rewriten as

$$(c-d)\dot{v}(s) = -\delta v(s) + pv(s-c\tau)e^{-av(s-c\tau)}. \tag{2.1}$$

Linearizing system (2.1) around $v_\star = u_\star = 0$, we have

$$(c-d)\dot{v}(s) = -\delta v(s) + pv(s-c\tau). \tag{2.2}$$

The characteristic equation of system (2.2) is

$$(c-d)\lambda + \delta - pe^{-c\tau\lambda} = 0. \tag{2.3}$$

When $\tau = 0$, Eq (2.3) reduces to

$$(c-d)\lambda + \delta - p = 0,$$

which has a real root $\lambda = \frac{p-\delta}{c-d}$. If $0 < c < d$, we have $\lambda < 0$; if $c > d$, then $\lambda > 0$.

Assume that there is some $\omega > 0$ such that $\lambda = \mathrm{i}\omega$ is a root of Eq (2.3), i.e.,

$$\mathrm{i}(c-d)\omega + \delta - pe^{-\mathrm{i}c\tau\omega} = 0.$$

From Euler's formula, we obtain

$$\begin{cases} p\sin(c\tau\omega) = (d-c)\omega, \\ p\cos(c\tau\omega) = \delta. \end{cases} \tag{2.4}$$

In view of $\sin^2(c\tau\omega) + \cos^2(c\tau\omega) = 1$, from system (2.4), we have

$$(c-d)^2\omega^2 = p^2 - \delta^2. \tag{2.5}$$

Equation (2.5) have a positive solution

$$\omega_\star = \sqrt{\frac{p^2-\delta^2}{(d-c)^2}}. \tag{2.6}$$

Substituting $\omega_\star$ into Eq (2.4), we obtain

$$\tau_n^\star = \frac{n\pi + \arctan\left(\frac{(d-c)\omega_\star}{\delta}\right)}{c\omega_\star} \begin{pmatrix} n = 0,1,2,\cdots, & \text{if} \quad c < d, \\ n = 1,2,3,\cdots, & \text{if} \quad c > d. \end{pmatrix} \tag{2.7}$$

It implies that Eq (2.3) has a pair of purely imaginary roots $\lambda = \pm\mathrm{i}\omega_\star$ when $\tau = \tau_n^\star$. Moreover, we have

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[(c-d)\lambda + \delta - pe^{-c\tau\lambda}\right]\Big|_{\tau=\tau_n^\star} = c - d + c\delta\tau_n^\star \pm \mathrm{i}(c^2 - cd)\tau_n^\star\omega_\star \neq 0,$$

i.e., $\lambda = \mathrm{i}\omega_\star$ is a simple root of Eq (2.3). Moreover, we need to check the transversality condition in the Hopf bifurcation theorem. To this end, we discuss the sign of

$$\mathrm{Re}\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\bigg|_{\tau=\tau_n^\star}.$$

Taking derivatives of system (2.3) with respect to τ yields that

$$(c-d)\frac{\mathrm{d}\lambda}{\mathrm{d}\tau} + pc\tau e^{-c\tau\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}\tau} + pc\lambda e^{-c\tau\lambda} = 0. \tag{2.8}$$

Noticing that $pe^{-c\tau\lambda} = (c-d)\lambda + \delta$, we rewrite Eq (2.8) as

$$(c-d)\frac{\mathrm{d}\lambda}{\mathrm{d}\tau} + c\tau[\delta + (c-d)\lambda]\frac{\mathrm{d}\lambda}{\mathrm{d}\tau} + c\lambda[\delta + (c-d)\lambda] = 0.$$

Hence, we obtain

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\tau} = -\frac{c\delta\lambda + c(c-d)\lambda^2}{c-d+c\tau\delta+c\tau(c-d)\lambda}.$$

Separating the real part, we yield

$$\mathrm{Re}\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\bigg|_{\tau=\tau_n^\star} = \frac{c(c-d)^2\omega_\star^2}{(c-d+c\delta\tau_n^\star)^2 + [(c^2-cd)\tau_n^\star\omega_\star]^2} > 0.$$

It implies that the purely imaginary roots of Eq (2.3) will fall into the right half plane of the complex plane when the delay τ increases slightly around the critical value $\tau = \tau_n^\star$. Based on the continuity of function $\lambda(\tau)$, we know that only the case of $c < d$ satisfies the transversality condition, i.e., when $c < d$, equation (1.4) has $\frac{2\pi}{\omega_\star}$–periodic traveling waves bifurcating from $u_\star$.

Next, we study the stability and bifurcation direction of periodic traveling wave solutions. Let $y(s) = v(c\tau s)$ and $\tau = \tau_n^\star + \gamma$, $\gamma \in \mathbb{R}$. Equation (2.1) can be rewritten as

$$\dot{y}(s) = \frac{c(\tau_n^\star+\gamma)}{c-d}\left[-\delta y(s) + py(s-1)e^{-ay(s-1)}\right]. \tag{2.9}$$

From Taylor's formula, we have

$$\dot{y}(s) = L(\gamma)y_s + H(\gamma, y_s), \tag{2.10}$$

where $L(\gamma)$ and $H(\gamma,\cdot)$ are given by

$$L(\gamma)\phi = -\frac{c(\tau_n^\star+\gamma)}{c-d}[\delta\phi(0) - p\phi(-1)],$$

and

$$H(\gamma,\phi) = \frac{pc(\tau_n^\star+\gamma)}{c-d}\left[-a\phi^2(-1) + \frac{a^2}{2}\phi^3(-1)\right] + O(\phi^4(-1)),$$

for $\phi \in C([-1,0],\mathbb{R})$, respectively.

Define a bounded variation function as

$$\eta(\gamma,\varsigma) = -\frac{c(\tau_n^\star+\gamma)}{c-d}[\delta\bar{\delta}(\varsigma) + p\bar{\delta}(\varsigma+1)],$$

such that

$$L(\gamma)\phi = \int_{-1}^{0} \mathrm{d}_{\varsigma}\eta(\gamma,\varsigma)\phi(\varsigma),$$

where $\bar{\delta}(\cdot)$ is the Delta function. Moreover, for $\psi \in C([-1,0],\mathbb{R})$, we define

$$A(\gamma)\psi(\theta) = \begin{cases} \dfrac{\mathrm{d}\psi(\theta)}{\mathrm{d}\theta}, & \text{if} \quad \theta \in [-1,0), \\ \displaystyle\int_{-1}^{0} \mathrm{d}_{\varsigma}\eta(\gamma,\varsigma)\psi(\varsigma), & \text{if} \quad \theta = 0, \end{cases}$$

and

$$N(\gamma)\psi(\theta) = \begin{cases} 0, & \text{if} \quad \theta \in [-1,0), \\ H(\gamma,\psi), & \text{if} \quad \theta = 0. \end{cases}$$

Hence, we can rewrite Eq (2.10) as

$$\dot{y}_s = A(\gamma)y_s + N(\gamma)y_s, \tag{2.11}$$

where $y_s(\theta) = y(s+\theta)$, $\theta \in [-1,0]$.

Define the adjoint operator A^* of $A(0)$ and the bilinear form as

$$A^*\varphi(\xi) = \begin{cases} -\dfrac{\mathrm{d}\varphi(\xi)}{\mathrm{d}\xi}, & \text{if} \quad \xi \in (0,1], \\ \displaystyle\int_{-1}^{0} \mathrm{d}_{\varsigma}\eta(0,\varsigma)\varphi(-\varsigma), & \text{if} \quad \xi = 0, \end{cases}$$

and

$$\langle \varphi, \psi \rangle = \bar{\varphi}(0)\psi(0) - \int_{-1}^{0}\int_{0}^{\varsigma} \bar{\varphi}(\zeta-\varsigma)\mathrm{d}_{\varsigma}\eta(0,\varsigma)\psi(\zeta)\mathrm{d}\zeta,$$

where $\psi \in C([-1,0],\mathbb{R})$ and $\varphi \in C([0,1],\mathbb{R})$. Then, we have $\langle \varphi, A(0)\psi \rangle = \langle A^*\varphi, \psi \rangle$. Hence, we obtain that $q^*(\chi) = De^{\mathrm{i}c\omega_\star\tau_n^\star\chi}$ is the eigenfunction of A^* associated with $-\mathrm{i}c\omega_\star\tau_n^\star$, and $q(\theta) = e^{\mathrm{i}c\omega_\star\tau_n^\star\theta}$ is the eigenfunction of $A(0)$ associated with $\mathrm{i}c\omega_\star\tau_n^\star$, where $D = \frac{1}{1+\frac{c\tau_n^\star\delta}{c-d}-\mathrm{i}c\omega_\star\tau_n^\star}$. So, we yield

$$\langle q^*, \bar{q} \rangle = 0, \quad \langle q^*, q \rangle = 1.$$

As a solution y_s of system (2.11) with $\gamma = 0$, we define $z(s) = \langle q^*, y_s \rangle$ and

$$W(s,\theta) = y_s(\theta) - z(s)q(\theta) - \bar{z}(s)\bar{q}(\theta). \tag{2.12}$$

Then, we yield $\langle q^*, W \rangle = 0$ and

$$W(s,\theta) = W(z,\bar{z},\theta) \triangleq W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots.$$

It follows $\langle q^*, \dot{y}_s \rangle = \langle q^*, A(0)y_s \rangle + \langle q^*, N(0)y_s \rangle$ that

$$\begin{aligned} \dot{z}(s) =& \mathrm{i}c\omega_\star\tau_n^\star z(s) + \bar{D}H(0,y_s) \\ =& \mathrm{i}c\omega_\star\tau_n^\star z(s) + \bar{D}H(0, z(s)q(\theta) + \bar{z}(s)\bar{q}(\theta) + W(z,\bar{z},\theta)). \end{aligned}$$

Let

$$f(z,\bar{z})(s) \triangleq \bar{D}H(0, z(s)q(\theta) + \bar{z}(s)\bar{q}(\theta) + W(z,\bar{z},\theta)) = f_{20}\frac{z^2}{2} + f_{11}z\bar{z} + f_{02}\frac{\bar{z}^2}{2} + f_{21}\frac{z^2\bar{z}}{2} + \cdots,$$

where

$$f_{20} = \frac{-2apc\bar{D}\tau_n^\star}{c-d}e^{-2\mathrm{i}c\omega_\star\tau_n^\star},\quad f_{11} = \frac{-2apc\bar{D}\tau_n^\star}{c-d},\quad f_{02} = \frac{-2apc\bar{D}\tau_n^\star}{c-d}e^{2\mathrm{i}c\omega_\star\tau_n^\star},$$

and

$$f_{21} = \frac{apc\bar{D}\tau_n^\star}{c-d}\left[3ae^{-\mathrm{i}c\omega_\star\tau_n^\star} - 4e^{-\mathrm{i}c\omega_\star\tau_n^\star}W_{11}(-1) - 2e^{\mathrm{i}c\omega_\star\tau_n^\star}W_{20}(-1)\right].$$

Next, we compute $W_{11}(\theta)$ and $W_{20}(\theta)$. Taking derivatives of system (2.12) with respect to s yields that

$$\begin{aligned}\dot{W} =&\dot{y}_s - \dot{z}q - \dot{\bar{z}}\bar{q}\\ =&A(0)y_s + N(0)y_s - [\mathrm{i}c\omega_\star\tau_n^\star z + f(z,\bar{z})]q - [-\mathrm{i}c\omega_\star\tau_n^\star\bar{z} + \bar{f}(z,\bar{z})]\bar{q}\\ =&A(0)[zq + \bar{z}\bar{q} + W] + N(0)y_s - [\mathrm{i}c\omega_\star\tau_n^\star z + f(z,\bar{z})]q + [\mathrm{i}c\omega_\star\tau_n^\star\bar{z} - \bar{f}(z,\bar{z})]\bar{q}\\ =&A(0)W + N(0)y_s - f(z,\bar{z})q - \bar{f}(z,\bar{z})\bar{q},\end{aligned}$$

where

$$N(0)y_s = \begin{cases}\dfrac{1}{\bar{D}}f(z,\bar{z}), & \text{if}\quad \theta = 0,\\ 0, & \text{if}\quad \theta\in[-1,0).\end{cases}$$

Moreover, we have

$$\begin{aligned}\dot{W} =&W_z\dot{z} + W_{\bar{z}}\dot{\bar{z}}\\ =&[W_{20}z + W_{11}\bar{z} + \cdots]\dot{z} + [W_{11}z + W_{02}\bar{z} + \cdots]\dot{\bar{z}}\\ =&[W_{20}z + W_{11}\bar{z}][\mathrm{i}c\omega_\star\tau_n^\star z + f(z,\bar{z})] + [W_{11}z + W_{02}\bar{z}][-\mathrm{i}c\omega_\star\tau_n^\star\bar{z} + \bar{f}(z,\bar{z})] + \cdots.\end{aligned}$$

Hence, we yield

$$(2\mathrm{i}c\omega_\star\tau_n^\star I - A(0))W_{20}(\theta) = \begin{cases}-f_{20}q(\theta) - \bar{f}_{02}\bar{q}(\theta), & \text{if}\quad \theta\in[-1,0),\\ \left(\dfrac{1}{\bar{D}} - 1\right)f_{20} - \bar{f}_{02}, & \text{if}\quad \theta = 0,\end{cases}$$

and

$$-A(0)W_{11}(\theta) = \begin{cases}-f_{11}q(\theta) - \bar{f}_{11}\bar{q}(\theta), & \text{if}\quad \theta\in[-1,0),\\ \left(\dfrac{1}{\bar{D}} - 1\right)f_{11} - \bar{f}_{11}, & \text{if}\quad \theta = 0.\end{cases}$$

As $\theta\in[-1,0)$, we have

$$\begin{aligned}\dot{W}_{20}(\theta) =&2\mathrm{i}c\omega_\star\tau_n^\star w_{20}(\theta) + f_{20}q(\theta) + \bar{f}_{02}\bar{q}(\theta),\\ \dot{W}_{11}(\theta) =&f_{11}q(\theta) + \bar{f}_{11}\bar{q}(\theta).\end{aligned}$$

Hence, we have

$$\begin{aligned}W_{20}(\theta) =&\frac{\mathrm{i}f_{20}}{c\omega_\star\tau_n^\star}e^{\mathrm{i}c\omega_\star\tau_n^\star\theta} + \frac{\mathrm{i}\bar{f}_{02}}{3c\omega_\star\tau_n^\star}e^{-\mathrm{i}c\omega_\star\tau_n^\star\theta} + E_1e^{2\mathrm{i}c\omega_\star\tau_n^\star\theta},\\ W_{11}(\theta) =&-\frac{\mathrm{i}f_{11}}{c\omega_\star\tau_n^\star}e^{\mathrm{i}c\omega_\star\tau_n^\star\theta} + \frac{\mathrm{i}\bar{f}_{11}}{c\omega_\star\tau_n^\star}e^{-\mathrm{i}c\omega_\star\tau_n^\star\theta} + E_2.\end{aligned}\tag{2.13}$$

When $\theta = 0$, we obtain

$$\begin{aligned}\int_{-1}^{0} \mathrm{d}_{\varsigma}\eta(0,\varsigma)W_{20}(\varsigma) &= 2\mathrm{i}c\omega_{\star}\tau_n^{\star}W_{20}(0) + f_{20} + \bar{f}_{02} - \frac{1}{\bar{D}}f_{20},\\ \int_{-1}^{0} \mathrm{d}_{\varsigma}\eta(0,\varsigma)W_{11}(\varsigma) &= f_{11} + \bar{f}_{11} - \frac{1}{\bar{D}}f_{11}.\end{aligned} \tag{2.14}$$

Substituting Eq (2.13) into (2.14), we yield

$$E_1 = \frac{f_{20}}{\bar{D}\left(2\mathrm{i}c\omega_{\star}\tau_n^{\star} - \int_{-1}^{0}\mathrm{d}_{\varsigma}\eta(0,\varsigma)e^{2\mathrm{i}c\omega_{\star}\tau_n^{\star}\varsigma}\right)}, \quad E_2 = -\frac{f_{11}}{\bar{D}\int_{-1}^{0}\mathrm{d}_{\varsigma}\eta(0,\varsigma)}.$$

It follows [23] that stability and bifurcation direction of periodic orbit is determined by the quantity

$$\kappa_{\star} = \mathrm{Re}\left\{\frac{\mathrm{i}}{2c\omega_{\star}\tau_n^{\star}}\left[f_{11}f_{20} - 2|f_{11}|^2 - \frac{|f_{02}|^2}{3}\right] + \frac{f_{21}}{2}\right\}. \tag{2.15}$$

When $\kappa^{\star} > 0$ (resp. < 0), there exists $\zeta > 0$ such that for $\tau \in (\tau_n - \zeta, \tau_n)$ (resp. $\tau \in (\tau_n, \tau_n + \zeta)$), periodic traveling waves of Eq (1.4) bifurcating from $u_{\star}$ are unstable (resp. have the same stability as $u_{\star}$ before the bifurcation). □

Furthermore, for the positive equilibrium $u_* = \frac{1}{a}\ln\frac{p}{\delta}$, we obtain the similar result.

Theorem 2.2. *For each fixed wave velocity c, when the advection rate $d < c$ and $p > \delta e^2$, there exists a branch of periodic traveling wave solutions of system (1.4) with period approximately equals to $\frac{2\pi}{\omega_*}$, which is bifurcated from u_* for the bifurcation parameter τ near τ_n^*. Moreover, the bifurcation is backward (resp. forward) and periodic traveling waves are unstable (resp. have the same stability as u_* before the bifurcation) if $\kappa_* > 0$ (resp. < 0), where ω_*, τ_n^* and κ_* are shown in Eqs (2.17)–(2.19), respectively.*

In view of the proof of Theorem 2.2 is similar to Theorem 2.1, we only give the simple proof of Theorem 2.2 included some key relations.

Proof of Theorem 2.2. Linearizing Eq (2.1) at $v_* = u_*$, we obtain the following characteristic equation

$$(c-d)\lambda + \delta - \delta\left(1 - \ln\frac{p}{\delta}\right)e^{-c\tau\lambda} = 0. \tag{2.16}$$

When $\tau = 0$, the characteristic equation reduces to $(c-d)\lambda + \delta\ln\frac{p}{\delta} = 0$, which has a real root $\lambda = \frac{\delta}{d-c}\ln\frac{p}{\delta}$. We know that $\lambda > 0$ as $0 < c < d$, and $\lambda < 0$ with $c > d$. Similarly to the above process, we know that when $p > e^2\delta$, Eq (2.16) have a pair of simple purely imaginary roots $\lambda = \pm\mathrm{i}\omega_*$, where

$$\omega_* = \frac{\delta}{|c-d|}\sqrt{\ln\frac{p}{\delta}\left(\ln\frac{p}{\delta} - 2\right)}, \tag{2.17}$$

with

$$\tau = \tau_n^* = \frac{(n+1)\pi + \arctan\left(\frac{(d-c)\omega_*}{\delta}\right)}{c\omega_*}, \quad n = 0, 1, 2, \cdots. \tag{2.18}$$

Moreover, we have

$$\mathrm{Re}\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\bigg|_{\tau=\tau_n^*} = \frac{c(c-d)^2\omega_*^2}{(c-d+c\delta\tau_n^*)^2+[(c^2-cd)\tau_n^*\omega_*]^2} > 0.$$

It implies that when $p > \delta e^2$ and $c > d$, there exists a branch of periodic traveling wave solutions of system (1.4) with period approximately equals to $\frac{2\pi}{\omega_*}$, which is bifurcated from u_* for the bifurcation parameter τ near τ_n^*.

Similarly to the above calculation, we get the quantity determined stability and bifurcation direction of periodic orbit bifurcating from u_* as follows:

$$\kappa_* = \mathrm{Re}\left\{\frac{\mathrm{i}}{2c\omega_*\tau_n^*}\left[g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right] + \frac{g_{21}}{2}\right\}, \tag{2.19}$$

where

$$\begin{aligned}
g_{20} =& \frac{ac\delta\tau_n^*\left(\ln\frac{p}{\delta}-2\right)}{(c-d)+c\tau_n^*\delta+\mathrm{i}c(c-d)\omega_*\tau_n^*}e^{-2\mathrm{i}c\omega_*\tau_n^*},\\
g_{11} =& \frac{ac\delta\tau_n^*\left(\ln\frac{p}{\delta}-2\right)}{(c-d)+c\tau_n^*\delta+\mathrm{i}c(c-d)\omega_*\tau_n^*},\\
g_{02} =& \frac{ac\delta\tau_n^*\left(\ln\frac{p}{\delta}-2\right)}{(c-d)+c\tau_n^*\delta+\mathrm{i}c(c-d)\omega_*\tau_n^*}e^{2\mathrm{i}c\omega_*\tau_n^*},
\end{aligned}$$

and

$$\begin{aligned}
g_{21} &= \frac{ac\delta\tau_n^* e^{-\mathrm{i}c\omega_*\tau_n^*}}{(c-d)+c\tau_n^*\delta+\mathrm{i}c(c-d)\omega_*\tau_n^*}\left[\left(\ln\frac{p}{\delta}-2\right)\left(2M_{11}(-1)+e^{2\mathrm{i}c\omega_*\tau_n^*}M_{20}(-1)\right)+a\left(\ln\frac{p}{\delta}-3\right)\right],\\
M_{11}(-1) &= -\frac{\mathrm{i}g_{11}}{c\omega_*\tau_n^*}e^{-\mathrm{i}c\omega_*\tau_n^*} + \frac{\mathrm{i}\bar{g}_{11}}{c\omega_*\tau_n^*}e^{\mathrm{i}c\omega_*\tau_n^*} + \frac{[c-d+c\tau_n^*\delta+\mathrm{i}c(c-d)\omega_*\tau_n^*]\,g_{11}}{c\tau_n^*\delta\ln\frac{p}{\delta}},\\
M_{20}(-1) &= \frac{\mathrm{i}g_{20}}{c\omega_*\tau_n^*}e^{-\mathrm{i}c\omega_*\tau_n^*} + \frac{\mathrm{i}\bar{g}_{02}}{3c\omega_*\tau_n^*}e^{\mathrm{i}c\omega_*\tau_n^*} + \frac{[c-d+c\tau_n^*\delta+\mathrm{i}c(c-d)\omega_*\tau_n^*]\,g_{20}e^{-2\mathrm{i}c\omega_*\tau_n^*}}{2\mathrm{i}c(c-d)\omega_*\tau_n^*+c\delta\tau_n^*\left[1-\left(1-\ln\frac{p}{\delta}\right)e^{-2\mathrm{i}c\omega_*\tau_n^*}\right]}.
\end{aligned}$$

When $\kappa_* > 0$ (resp. < 0), the bifurcation is backward (resp. forward) and periodic traveling waves are unstable (resp. have the same stability as u_* before the bifurcation). □

3. Numerical simulations

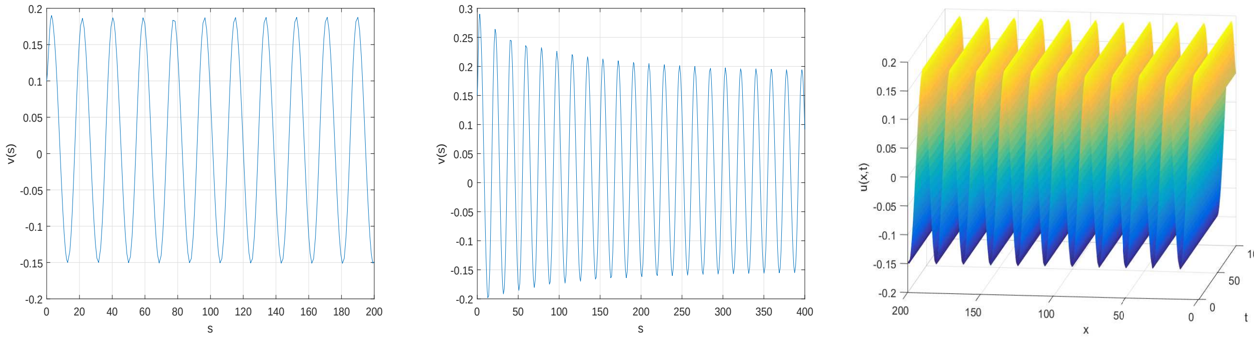
In this section, we illustrate our results by showing some numerical simulations. For Theorem 2.1, we choose the parameters $\delta = 1$, $p = 8$, $a = 1$ and $d = 27$. Set the wave speed $c = 4 < d$. Hence, we have the following system

$$\frac{\partial u(x,t)}{\partial t} = 27\frac{\partial u(x,t)}{\partial x} - u(x,t) + 8u(x,t-\tau)e^{-u(x,t-\tau)}. \tag{3.1}$$

Substituting the traveling wave variable $s = x + 4t$ and $u(x,t) = v(s)$ into Eq (3.1), we have

$$-23\dot{v}(s) = -v(s) + 8v(s-4\tau)e^{-v(s-4\tau)}. \tag{3.2}$$

Here, we only consider the case of the first critical bifurcation parameter $\tau_0^\star$. Based on Section 2, we know that system (3.2) has periodic solution bifurcating from $u^\star = 0$ with τ near $\tau_0^\star \approx 1.0471$, whose period approximately equal to $\frac{2\pi}{\omega_\star} \approx 18.2068$. Moreover, by using Mathematica 12.0, we can calculate that $\kappa^\star \approx -1.8848$, i.e., the periodic solution is stable, whose bifurcation direction is forward. Let the initial function $v_1(s) = \frac{\cos(s)}{10}$. In Figure 1(a), we see that the numerical periodic solution of system (3.2) is bifurcated from $u^\star$ when $\tau = 1.0662 > \tau_0^\star$. It follows Figure 1(b) that solution of (3.2) converges to the periodic solution. Figure 1(c) shows the numerical periodic traveling waves of Eq (3.1) after translating back to the original variable.



(a) (b) (c)

Figure 1. (a). Periodic solution of (3.2) for $s \in [0, 200]$; (b). Solution of (3.2) converges to the periodic solution; (c). Periodic traveling wave solution of (3.1) for $t \in [0, 100]$ and $x \in [0, 200]$.

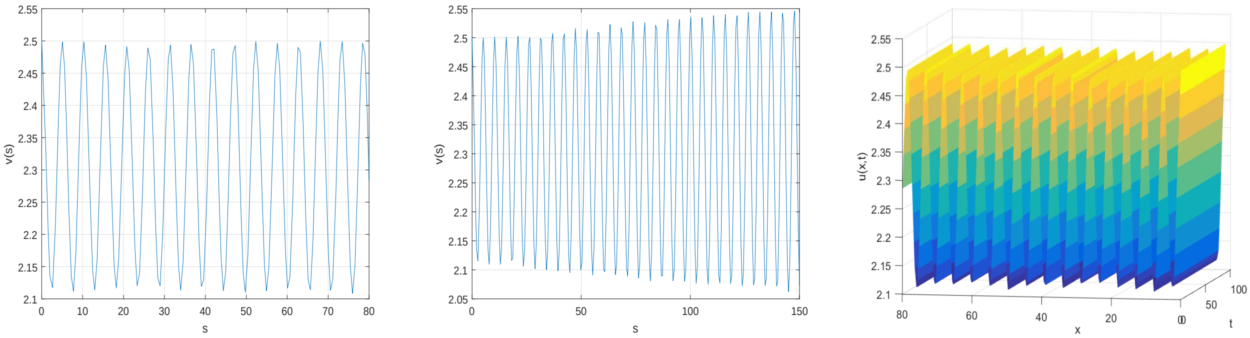
For Theorem 2.2, choosing the parameters $d = 0.5$, $a = 1$, $\delta = 1$ and $p = 10$ such that $p > \delta e^2$. Let the wave speed $c = 1.2 > d = 0.5$. Hence, we obtain the following system

$$\frac{\partial u(x,t)}{\partial t} = 0.5\frac{\partial u(x,t)}{\partial x} - u(x,t) + 10u(x,t-\tau)e^{-u(x,t-\tau)}. \tag{3.3}$$

The differential equation with wave profile of system (3.3) is

$$0.7\dot{v}(s) = -v(s) + 10v(s - 1.2\tau)e^{-v(s-1.2\tau)}. \tag{3.4}$$

It follows Theorem 2.2 that system (3.4) has periodic solution bifurcating from $u^* = \ln 10 \approx 2.303$ with τ near $\tau_0^* \approx 1.709$, where the period approximately equal to $\frac{2\pi}{\omega_*} \approx 5.269$. Moreover, we yield $\kappa^* \approx 0.1851$, i.e., the periodic solution is unstable, whose bifurcation direction is backward. Set the initial function $v_2(s) = \ln 10 + \frac{\cos(s)}{5}$. The numerical periodic solution of system (3.4) with $\tau = 1.685 < \tau_0^*$ bifurcating from u^* is shown in Figure 2(a). In Figure 2(b), we see that the periodic solution is unstable. By translating back to the original variable, the numerical periodic traveling wave solution of Eq (3.3) bifurcating from u^* is shown in Figure 2(c).



(a) (b) (c)

Figure 2. (a). Periodic solution of (3.4) for $s \in [0, 80]$; (b). Solution of (3.4) is away from the periodic solution; (c). Periodic traveling wave solution of (3.3) for $t \in [0, 100]$ and $x \in [0, 80]$.

4. Conclusions

In this paper, we investigate the existence, stability and bifurcation direction of periodic traveling waves for the delayed Nicholson's blowflies model with advection. By using traveling wave transformation, the construction of periodic traveling wave solutions of the original model (1.4) is equivalent to looking for periodic solutions of the delayed Nicholson's blowflies model (2.1). When introducting the advection term into the system, we obtain the influence of advection term on the bifurcation position, stability and bifurcation direction of periodic solutions (see Theorems 2.1 and 2.2). Moreover, we illustrate our theoretical results by using numerical simulations.

There exist some natural extensions of this article. Indeed, we take the diffusion coefficient of system (1.3) to be zero in this paper, which greatly reduces the difficulty of our research. In practice, we also derive the theoretical results of system (1.3) similar to Theorems 2.1 and 2.2 by using the perturbation technique [22]. However, the numerical simulations of periodic traveling waves for reaction-diffusion models is very difficult, because the equilibrium points of the corresponding delayed differential system become hyperbolic in this case. How to use mathematical software to simulate periodic solutions bifurcated from hyperbolic equilibria of delayed differential equations? These raise us the future direction.

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Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

1. R. Fisher, The wave of advance of advantageous genes, *Ann. Hum. Genet.*, **7** (1937), 355–369. https://dx.doi.org/10.1111/j.1469-1809.1937.tb02153.x
2. I. Lundström, Mechanical wave propagation on nerve axons, *J. Theor. Biol.*, **45** (1974), 487–499. https://doi.org/10.1016/0022-5193(74)90127-1
3. J. Murray, *Mathematical Biology. I. An Introduction*, 3nd edition, New York: Springer-Verlag, 2002. http://dx.doi.org/10.1023/A:1022616217603
4. W. Gurney, S. Blythe, R. Nisbet, Nicholson's blowflies revisited, *Nature*, **287** (1980), 17–21. https://doi.org/10.1038/287017a0
5. J. W. H. So, Y. Yang, Dirichlet problem for the diffusive Nicholson's blowflies equation, *J. Differ. Equations*, **150** (1998), 317–348. https://doi.org/10.1006/jdeq.1998.3489
6. J. W. H. So, X. Zou, Traveling waves for the diffusive Nicholson's blowflies equation, *Appl. Math. Comput.*, **122** (2001), 385–392. https://doi.org/10.1016/S0096-3003(00)00055-2
7. C. Lin, M. Mei, On travelling wavefronts of Nicholson's blowflies equation with diffusion, *P. Roy. Soc. Edinb. A.*, **140** (2010), 135–152. https://doi.org/10.1017/S0308210508000784
8. M. Mei, J. W. H. So, M. Li, S. Shen, Asymptotic stability of travelling waves for Nicholson's blowflies equation with diffusion, *P. Roy. Soc. Edinb. A.*, **134** (2004), 579–594. https://doi.org/10.1017/S0308210500003358
9. G. Yang, Hopf bifurcation of traveling wave of delayed Nicholson's blowflies equation, *J. Biomath.*, **26** (2011), 81–86. https://doi.org/10.1016/j.amc.2013.06.051
10. J. W. H. So, J. Wu, Y. Yang, Numerical steady state and Hopf bifurcation analysis on the diffusive Nicholson's blowflies equation, *Appl. Math. Comput.*, **111** (2000), 53–69. https://doi.org/10.1016/S0096-3003(99)00047-8
11. D. Duehring, W. Huang, Periodic traveling waves for diffusion equations with time delayed and non-local responding reaction, *J. Dyn. Differ. Equations*, **19** (2007), 457–477. https://doi.org/10.1007/s10884-006-9048-8
12. M. Mei, C. Lin, C. Lin, J. W. H. So, Traveling wavefronts for time-delayed reaction-diffusion equation: (I) local nonlinearity, *J. Differ. Equations*, **247** (2009), 495–510. https://doi.org/10.1016/j.jde.2008.12.026
13. M. Mei, C. Ou, X. Zhao, Global stability of monostable traveling waves for nonlocal time-delayed reaction-diffusion equations, *SIAM J. Math. Anal.*, **42** (2010), 2762–2790. https://doi.org/10.1137/090776342
14. Z. Yang, G. Zhang, Global stability of traveling wavefronts for nonlocal reaction-diffusion equation with time delay, *Acta Math. Sci.*, **38** (2018), 289–302. https://doi.org/10.3969/j.issn.0252-9602.2018.01.018

15. J. Zhang, Y. Peng, Travelling waves of the diffusive Nicholson's blowflies equation with strong generic delay kernel and non-local effect, *Nonlinear Anal.-Theor.*, **68** (2008), 1263–1270. https://doi.org/10.1016/j.na.2006.12.019

16. C. Huang, B. Liu, Traveling wave fronts for a diffusive Nicholsons Blowflies equation accompanying mature delay and feedback delay, *Appl. Math. Lett.*, **134** (2022), 108321. https://doi.org/10.1016/j.aml.2022.108321

17. D. Liang, J. Wu, Travelling waves and numerical approximations in a reaction-diffusion equation with nonlocal delayed effect, *J. Nonlinear Sci.*, **13** (2008), 289–310. https://doi.org/10.1007/s00332-003-0524-6

18. L. Liu, Y. Yang, S. Zhang, Stability of traveling fronts in a population model with nonlocal delay and advection, *Malaya J. Mat.*, **3** (2015), 498–510.

19. Z. Wang, W. Li, S. Ruan, Existence and stability of traveling wave fronts in reaction advection diffusion equations with nonlocal delay, *J. Differ. Equations*, **238** (2007), 153–200. https://doi.org/10.1016/j.jde.2007.03.025

20. S. Wu, W. Li, S. Liu, Exponential stability of traveling fronts in monostable reaction-advection-diffusion equations with non-local delay, *Discrete Contin. Dyn. Syst. Ser. B.*, **17** (2012), 347–366. https://doi.org/10.3934/dcdsb.2012.17.347

21. H. Berestycki, *The Influence of Advection on The Propagation of Fronts for Reaction-Diffusion Equations*, Springer Netherlands, 2002. https://doi.org/10.1007/978-94-010-0307-02

22. D. Li, S. Guo, Periodic traveling waves in a reaction-diffusion model with chemotaxis and nonlocal delay effect, *J. Math. Anal. Appl.*, **467** (2018), 1080–1099. https://doi.org/10.1016/j.jmaa.2018.07.050

23. J. Hale, *Theory of Functional Differential Equations*, Springer-Verlag, New York, 1977. https://doi.org/10.1007/978-1-4612-9892-2