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Research article

The critical exponents for a semilinear fractional pseudo-parabolic equation with nonlinear memory in a bounded domain

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Abstract: This paper considers blow-up and global existence for a semilinear space-time fractional pseudo-parabolic equation with nonlinear memory in a bounded domain. We determine the critical exponents of the Cauchy problem when $\alpha < \gamma$ and $\alpha \geq \gamma$, respectively. The results obtained in this study are noteworthy extension to the results of time-fractional differential equation. The critical exponent is consistent with the corresponding Cauchy problem for the time-fractional differential equation with nonlinear memory, which illustrates that the diffusion effect of the third order term is not strong enough to change the critical exponents.

Keywords: fractional pseudo-parabolic equation; critical exponent; global existence; blow-up

1. Introduction

This paper concerns the blow-up and global existence of solutions to the following space-time fractional pseudo-parabolic equation with nonlinear memory

$$\begin{cases} \mathbb{D}_t^{\alpha}(u - m\Delta u)(x,t) + (-\Delta)^{\beta/2}u(x,t) = {}_0I_t^{1-\gamma}\left(|u|^{p-1}u\right), & x \in \Omega, t > 0 \\ u(x,t) = 0, & x \in \partial\Omega, t > 0 \\ u(x,0) = u_0(x), & x \in \Omega \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $u_0 \in C_0(\Omega), 0 < \alpha < 1, 0 < \beta \leq 2, 0 \leq \gamma < 1, p > 1$ and $m > 0$. The symbol $\mathbb{D}_t^{\alpha}$ denotes the Caputo time fractional derivative, which is defined by $\mathbb{D}_t^{\alpha}u = \frac{\partial}{\partial t}\left[{}_0I_t^{1-\alpha}\left(u(t,x) - u_0(x)\right)\right]$, where ${}_0I_t^{1-\alpha}u = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{u(s)}{(t-s)^{\alpha}}ds$. $(-\Delta)^{\beta/2}$ is the fractional Laplace operator, which may be defined as

$$(-\Delta)^{\beta/2}v(x,t) = \mathcal{F}^{-1}\left(|\xi|^{\beta}\mathcal{F}(v)(\xi)\right)(x,t),$$

where $\mathcal{F}$ denotes the Fourier transform and $\mathcal{F}^{-1}$ represents the inverse Fourier transform in $L^2(\mathbb{R}^{\mathbb{N}})$.

Recently, we find that space-time fractional differential equations have been used in lots of applications, such as memory effect, anomalous diffusion, quantum mechanics, Levy flights in physics etc. (see [1–5]). It describes some physical phenomena more accurate than classical integral differential equations [6–8]. On the other hand, the pseudo-parabolic equation is also called as the nonclassical diffusion equation, which is a significant mathematical model used to depict physical phenomena, such as non-Newtonian, solid mechanics, and heat conduction(see [9, 10]). Some practical problems such as the power-law memory [11, 12] in time and space require us to consider the space-time fractional pseudo-parabolic model, for example, [13] considered the case of pseudo-parabolic equations with fractional derivatives both in time and space.

If $\alpha = 1, m > 0, \beta = 2, \gamma = 1$, Problem (1.1) becomes classical pseudo-parabolic equation, Cao et al. [14] considered the following semilinear pseudo-parabolic equation

$$u_t - m\triangle u_t - \triangle u = u^p$$

They investigated the necessary existence, uniqueness for mild solutions and they also studied the large time behavior of solutions. Ji et al. [15] considered the Cauchy problem of the following space-fractional pseudo-parabolic equation

$$u_t - m\triangle u_t + (-\triangle)^\sigma u = u^p$$

They considered the global existence, time-decay rates and the large time behavior of the solutions. There are also many recent results on the behavior of the solutions for the Cauchy problem of fractional nonclassical diffusion equations [16–19].

In [20, 21], Zhang and Li considered the following nonlinear time-fractional equation in $\mathbb{R}^N$ and a bounded domain respectively,

$$\mathbb{D}_t^\alpha u - \Delta u = {}_0I_t^{1-\gamma}\left(|u|^{p-1}u\right), \quad t > 0, \tag{1.2}$$

where $p > 1, 0 < \alpha < 1$, and $0 \le \gamma < 1$. They obtained the critical exponent of problem (1.2) for $\alpha \ge \gamma$ and $\alpha < \gamma$, respectively.

In [22], Tuan et al. investigated the following two fractional pseudo-parabolic equations

$$\begin{cases} \mathbb{D}_t^\alpha (u - m\Delta u)(x,t) + (-\Delta)^\sigma u(x,t) = \mathcal{N}(u), & x \in \Omega, \\ u(x,t) = 0, & x \in \partial\Omega, t > 0 \\ u(x,0) = u_0(x), & x \in \Omega \end{cases} \tag{1.3}$$

$$\begin{cases} \mathbb{D}_t^\alpha (u - m\Delta u)(x,t) - \Delta u(x,t) = \mathcal{N}(u), & x \in \mathbb{R}^N, t > 0 \\ u(0,x) = u_0(x), & x \in \mathbb{R}^N \end{cases} \tag{1.4}$$

where $0 < \alpha < 1$, $m > 0$ and $\mathcal{N}(u)$ has Lipschitz properties. They established the local well-posedness results including existence, uniqueness and regularity of the local solution for the problem (1.3) and proved the global existence theorem of problem (1.4).

Motivated by the results we have mentioned, in this article, we obtain sharp blow-up and global existence results of problem (1.1) on the condition that $\gamma \le \alpha$ and $\gamma > \alpha$.

We get the following conclusions when $\gamma \le \alpha$.

Theorem 1.1. *Assume that* $\gamma \leq \alpha, p > 1$ *and* $u_0 \in C_0(\Omega)$.
(1) If $p\gamma \leq 1$ *and* $u_0 \geq 0, u_0 \not\equiv 0$, *then the weak solutions of* (1.1) *blow up in a finite time in* $C((0,\infty), C_0(\Omega))$.
(2) If $p\gamma > 1$ *and* $\|u_0\|_{L^\infty(\Omega)}$ *is small enough, then the weak solution of* (1.1) *in* $C((0,\infty), C_0(\Omega))$ *exists globally.*

We get the following conclusions when $\gamma > \alpha$.

Theorem 1.2. *Assume that* $\gamma > \alpha, p > 1, \sigma = 1-\gamma$ *and* $u_0 \in C_0(\Omega)$.
(1) If $p < 1 + \frac{\sigma}{\alpha}$ *and* $u_0 \geq 0, u_0 \not\equiv 0$, *then the weak solutions of* (1.1) *blow up in a finite time in* $C((0,\infty), C_0(\Omega))$.
(2) If $p \geq 1 + \frac{\sigma}{\alpha}$ *and* $\|u_0\|_{L^\infty(\Omega)}$ *is small enough, then the weak solutions of* (1.1) *in* $C((0,\infty), C_0(\Omega))$ *exists globally.*

Our proof of blow up results is based on the asymptotic properties of solutions for an ordinary fractional differential inequality. Compared with the results of time-fractional differential equation, the major difference between the space-time fractional Eq (1.1) and Eq (1.2) is that the definition of weak solution and mild solution. The critical exponent is consistent with the corresponding Cauchy problem for the time-fractional differential equation with nonlinear memory [21], which shows that the diffusion effect of the third order term is not strong enough to change the critical exponents.

The structure of this article is as follows. In Section 2, we present some definitions and properties that will be used in the next section. In Section 3, we give the proof of our main results.

2. Preliminaries

This section presents some preliminaries concerning special functions and fractional knowledge that will be used in the next sections.

First, we review some definitions and properties of the fractional knowledge including fractional integrals and fractional derivatives. For $T > 0$ and $u \in L^1((0,T))$, the left and right Riemann-Liouville fractional integrals of the order $\alpha \in (0,1)$ are defined by [3]

$$ {}_0I_t^\alpha u = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(s)}{(t-s)^{1-\alpha}} ds, \quad {}_tI_T^\alpha u = \frac{1}{\Gamma(\alpha)} \int_t^T \frac{u(s)}{(s-t)^{1-\alpha}} ds, $$

where Γ is the Gamma function. If $f \in L^p((0,T)), g \in L^q((0,T))$ and $p, q \geq 1, 1/p + 1/q = 1$, then we have

$$ \int_0^T ({}_0I_t^\alpha f)\, g(t) dt = \int_0^T ({}_tI_T^\alpha g)\, f(t) dt. \tag{2.1} $$

The Caputo fractional derivatives are defined by

$$ \mathbb{D}_t^\alpha f = \frac{d}{dt} {}_0I_t^{1-\alpha}[f(t) - f(0)], {}_t\mathbb{D}_T^\alpha f = -\frac{d}{dt} {}_tI_T^{1-\alpha}[f(t) - f(T)], $$

If $f \in AC([0,T])$, then $\mathbb{D}_t^\alpha f$ and ${}_t\mathbb{D}_T^\alpha f$ exist almost everywhere on $[0,T]$ and $\mathbb{D}_t^\alpha f = {}_0I_t^{1-\alpha} f'(t), {}_t\mathbb{D}_T^\alpha g = -{}_tI_T^{1-\alpha} g'(t)$. Moreover, assuming $f \in C([0,T]), \mathbb{D}_t^\alpha f \in L^1(0,T), g \in AC([0,T])$ and $g(T) = 0$, for all

$T > 0$ and $\alpha \in (0, 1)$, we have

$$\int_0^T g(t)\,(\mathbb{D}_t^\alpha f)\,dt = \int_0^T (f(t) - f(0))_t\mathbb{D}_T^\alpha g dt, \tag{2.2}$$

which is called the formula of integration by parts for fractional derivatives.

Now, we recall the Mittag-Leffler function which is defined by [23]

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \alpha, \beta \in \mathbb{C}, \mathrm{Re}(\alpha) > 0, \quad E_\alpha(z) = E_{\alpha,1}(z), \quad z \in \mathbb{C}, \tag{2.3}$$

and its Riemann-Liouville fractional integral satisfies

$$ {}_0I_t^{1-\alpha}\left(t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)\right) = E_{\alpha,1}(\lambda t^\alpha) \text{ for } \lambda \in \mathbb{C}, 0 < \alpha < 1.$$

Let $\alpha \in (0, 1), \mu \in \mathbb{R}$ and $\frac{\pi\alpha}{2} < \mu < \min\{\pi, \pi\alpha\}$. Then

$$E_{\alpha,\beta}(z) = -\sum_{k=1}^{N} \frac{1}{\Gamma(\beta - \alpha k)} \frac{1}{z^k} + O\left(\frac{1}{z^{N+1}}\right), \quad \mu \le |\arg(z)| \le \pi \tag{2.4}$$

with $|z| \to +\infty$. The Wright type function which was considered by Mainardi [24]

$$\phi_\alpha(z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\Gamma(-\alpha k + 1 - \alpha)} = \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-z)^k \Gamma(\alpha(k+1)) \sin(\pi(k+1)\alpha)}{k!} \tag{2.5}$$

for $0 < \alpha < 1, z \in \mathbb{C}$. It is an entire function and has the following properties (see [1]).

(1)$\phi_\alpha(\theta) \ge 0$ for $\theta \ge 0$ and $\int_0^\infty \phi_\alpha(\theta)d\theta = 1$.

$$(2) \int_0^\infty \phi_\alpha(\theta)e^{-z\theta}d\theta = E_{\alpha,1}(-z), z \in \mathbb{C}. \tag{2.6}$$

$$(3)\alpha \int_0^\infty \theta\phi_\alpha(\theta)e^{-z\theta}d\theta = E_{\alpha,\alpha}(-z), z \in \mathbb{C}. \tag{2.7}$$

Then we consider the spectral problem (see [25])

$$\begin{cases} (-\Delta)^{\beta/2}\varphi_j(x) = \lambda_j^{\beta/2}\varphi_j(x), & x \in \Omega, \beta \in (0, 2], \\ \varphi_j(x) = 0, & x \in \partial\Omega, \end{cases} \tag{2.8}$$

and the set of the eigenvalues of the spectral problem consists of a sequence

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \ldots \le \lambda_j \le \ldots \nearrow \infty$$

Let $e^{t^\alpha\theta\mathcal{A}}$ denote the semigroup in Ω under the Dirichlet boundary condition where $\mathcal{A} = (-\Delta)^{\frac{\beta}{2}}(m\Delta - I)^{-1}$. We define the operators $P_\alpha(t)$ and $S_\alpha(t)$ as

$$P_\alpha(t)u_0 = \int_0^\infty \phi_\alpha(\theta)e^{t^\alpha\theta\mathcal{A}}u_0 d\theta, \quad S_\alpha(t)u_0 = \alpha\int_0^\infty \theta\phi_\alpha(\theta)e^{t^\alpha\theta\mathcal{A}}u_0 d\theta, t \ge 0,$$

where $\phi_\alpha(\theta)$ is given by (2.5). By [26] and the properties of $P_\alpha(t)$ and $S_\alpha(t)$, we can deduce that

$$\|P_\alpha(t)u_0\|_{L^\infty(\Omega)} \le C\,\|u_0\|_{L^\infty(\Omega)}\,,$$

$$\|S_\alpha(t)u_0\|_{L^\infty(\Omega)} \le C\,\|u_0\|_{L^\infty(\Omega)}\,,$$

$$\|\mathcal{A}P_\alpha(t)u_0\|_{L^\infty(\Omega)} \le \frac{C}{t^\alpha}\,\|u_0\|_{L^\infty(\Omega)}\,, \tag{2.9}$$

$${}_0I_t^{1-\alpha}\left(t^{\alpha-1}S_\alpha(t)\mathcal{A}u_0\right) = P_\alpha(t)\mathcal{A}u_0 = \mathcal{A}P_\alpha(t)u_0. \tag{2.10}$$

Lemma 2.1. *Assume that* $q > 1, f \in L^q\left((0,T), C_0(\Omega)\right)$. *Let* $w(t) = \int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)f(s)ds$, *then*

$${}_0I_t^{1-\alpha}w = \int_0^t P_\alpha(t-s)f(s)ds.$$

Proof. The proof is similar to that of Theorem 2.4 in [26]. By Fubini theorem and (2.10), we have

$$\begin{aligned}
{}_0I_t^{1-\alpha}w &= \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\int_0^s (s-\tau)^{\alpha-1}S_\alpha(s-\tau)\mathcal{G}f(\tau)d\tau ds\\
&= \frac{1}{\Gamma(1-\alpha)}\int_0^t\int_\tau^t (t-s)^{-\alpha}(s-\tau)^{\alpha-1}S_\alpha(s-\tau)\mathcal{G}f(\tau)dsd\tau\\
&= \frac{1}{\Gamma(1-\alpha)}\int_0^t\int_0^{t-\tau} (t-s-\tau)^{-\alpha}s^{\alpha-1}S_\alpha(s)\mathcal{G}f(\tau)dsd\tau\\
&= \int_0^t P_\alpha(t-\tau)\mathcal{G}f(\tau)d\tau.
\end{aligned}$$

Hence, we get the conclusion. □

Remark 2.2. *For* $\alpha = 1, \beta = 2$, *the conclusion of Lemma 2.1 still holds.*

Lemma 2.3. *(see [20])Let* $T > 0, p > 1, 0 \le \gamma < 1, \gamma \le \alpha, \sigma = 1-\gamma, a > 0$, *and* $b > 0$. *If* $w > 0$ *satisfies* $w \in C([0,T]), {}_0I_t^{1-\alpha}(w - w(0)) \in AC([0,T])$ *and, for almost every* $t \in [0,T]$,

$$\mathbb{D}_t^\alpha w + aw \ge b_0 I_t^{1-\gamma}w^p,$$

then the following conclusions hold.
(1) For every $l \ge \frac{p(\alpha+\sigma)}{p-1}$, *we have* $w(0) \le K_1(a,b,\alpha,\gamma,p)T^{\alpha+\sigma-\frac{p\sigma}{p-1}} + K_2(b,\alpha,\gamma,p)T^{-\frac{\alpha+\sigma}{p-1}}$, *where*

$$\begin{aligned}
K_1 &= \tfrac{p-1}{p}\left(\tfrac{2a^p}{bp}\right)^{\frac{1}{p-1}} \frac{\Gamma(l+1)^{\frac{1}{p-1}}\Gamma(l+2-\alpha-\sigma)}{\Gamma(l+1-\sigma)^{\frac{p}{p-1}}}\frac{p-1}{(l+1)(p-1)-p\sigma},\\
K_2 &= \tfrac{p-1}{p}\left(\tfrac{2}{bp}\right)^{\frac{1}{p-1}} \frac{\Gamma(l+1)^{\frac{1}{p-1}}\Gamma(l+2-\alpha-\sigma)}{\Gamma(l+1-\alpha-\sigma)^{\frac{p}{p-1}}}\frac{p-1}{(l+1)(p-1)-p\sigma}.
\end{aligned}$$

(2) If $p\gamma \le 1$, *then we have* $T < +\infty$.

3. Finite time blow-up and global existence

This section is dedicated to proving Theorems 1.1 and 1.2 . First, we give the definition of mild solution of (1.1).

Definition 3.1. *Let $u_0 \in C_0(\Omega)$ and $T > 0$, we call that $u \in C([0,T], C_0(\Omega))$ is a mild solution of* (1.1)*, if u satisfies the following integral equation*

$$u = P_\alpha(t)u_0 + \int_0^t (t-\tau)^{\alpha-1} S_\alpha(t-\tau) \mathscr{G}\, {}_0I_t^{1-\gamma}\left(|u|^{p-1}u\right) d\tau$$

where $\mathscr{G} = -(m\Delta - I)^{-1}$.

Theorem 3.2. *Let $p > 1, 0 < \alpha \le 1, 0 < \beta \le 2$, and $0 \le \gamma < 1$, If $u_0 \in C_0(\Omega)$, there exists $T = T(u_0) > 0$ and a unique mild solution $u \in C([0,T], C_0(\Omega))$ to the problem* (1.1)*. The solution u can be extended to a maximal interval $[0, T_{\max})$ and either $T_{\max} = +\infty$ or $T_{\max} < +\infty$ and $\|u\|_{L^\infty((0,T),L^\infty(\Omega))} \to +\infty$ as $T \to T_{\max}^-$. Furthermore, if $u_0 \ge 0, u_0 \not\equiv 0$, then $u(t,x) > 0$ and $u(t,x) \ge P_\alpha(t)u_0$ for $t \in (0, T_{\max})$ and $x \in \Omega$.*

Proof. The proof is similar to that of Theorem 3.2 in [26]. By the contraction mapping principle and the properties of $P_\alpha(t)$ and $S_\alpha(t)$, we can get the conclusion. The main different is that operators $P_\alpha(t)$ and $S_\alpha(t)$ are expressed by semigroup generated by the infinitesimal generator $\mathscr{A} = (-\Delta)^{\frac{\beta}{2}}(m\Delta - I)^{-1}$, but the semigroup in [26] is generated by $-\Delta$. □

Here, we assume $u_0 \in C_0(\Omega)$ for convenience of proof. In fact, if u_0 belongs to Lebesgue space, we can obtain the similar existence results under certain conditions.

Remark 3.3. *Let $0 < \alpha < 1$, $r \in (q, +\infty]$ and $q_c = \frac{N(p-1)}{\beta}$. Let $u_0 \in L^q(\Omega), \alpha q_c < q < +\infty$. Then there exists $T > 0$ such that problem* (1.1) *has a mild solution u in $C([0,T], L^q(\Omega)) \cap C((0,T], L^r(\Omega))$.*

Then we give the definition of weak solution of (1.1) as follows.

Definition 3.4. *Let $u_0 \in L^1(\Omega)$ and $T > 0$, $u \in L^p((0,T), L^p(\Omega))$ is called a weak solution of* (1.1) *if*

$$\int_\Omega \int_0^T \left[{}_0I_t^{1-\gamma}\left(|u|^{p-1}u\right)\varphi + u_0\left({}_t\mathbb{D}_T^\alpha \varphi\right) + mu\left({}_t\mathbb{D}_T^\alpha \Delta\varphi\right)\right] dtdx = \int_\Omega \int_0^T u\,(-\Delta)^{\frac{\beta}{2}}\varphi dtdx$$
$$+ \int_\Omega \int_0^T u\left({}_t\mathbb{D}_T^\alpha \varphi\right) dtdx + \int_\Omega \int_0^T mu_0\left({}_t\mathbb{D}_T^\alpha \Delta\varphi\right) dtdx$$

for every $\varphi \in C^{2,1}\left(\bar{\Omega}, [0,T]\right)$ with $\varphi = 0$ on $\partial\Omega$ and $\varphi(x,T) = 0$ for $x \in \bar{\Omega}$. Moreover, if $T > 0$ can be arbitrarily chosen, u is called a global weak solution of (1.1).

Lemma 3.5. *Let $T > 0$, $u_0 \in C_0(\Omega)$, if $u \in C([0,T], C_0(\Omega))$ is a mild solution of* (1.1)*, then u is a weak solution of* (1.1).

Proof. Suppose that $u \in C([0,T], C_0(\Omega))$ is a mild solution of (1.1), then

$$u - u_0 = P_\alpha(t)u_0 - u_0 + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)\mathscr{G}\, {}_0I_t^{1-\gamma}\left(|u|^{p-1}u\right) ds$$

where $\mathscr{G} = -(m\Delta - I)^{-1}$. Now, noting that by Lemma 2.1,

$$ {}_0I_t^{1-\alpha}\left(\int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)\mathscr{G}\,{}_0I_t^{1-\gamma}\left(|u|^{p-1}u\right)ds\right) = \int_0^t P_\alpha(t-s)\mathscr{G}\,{}_0I_t^{1-\gamma}\left(|u|^{p-1}u\right)ds. $$

so, we have

$$ {}_0I_t^{1-\alpha}(u-u_0) = {}_0I_t^{1-\alpha}(P_\alpha(t)u_0 - u_0) + \int_0^t P_\alpha(t-s)\mathscr{G}\,{}_0I_t^{1-\gamma}\left(|u|^{p-1}u\right)ds $$

Then, for every $\varphi \in C^{2,1}\left(\bar{\Omega}, [0,T]\right)$ with $\varphi = 0$ on $\partial\Omega$ and $\varphi(\cdot, T) = 0$.

$$ \int_\Omega {}_0I_t^{1-\alpha}(u-u_0)\,\mathscr{G}^{-1}\varphi dx = I_1(t) + I_2(t) \tag{3.1} $$

where

$$ I_1(t) = \int_\Omega {}_0I_t^{1-\alpha}(P_\alpha(t)u_0 - u_0)\,\mathscr{G}^{-1}\varphi dx, \quad I_2(t) = \int_\Omega \int_0^t P_\alpha(t-s)\,{}_0I_t^{1-\gamma}\left(|u|^{p-1}u\right)ds\varphi dx. $$

By (2.9) and the dominated convergence theorem, we get

$$ \frac{dI_1}{dt} = -\int_\Omega P_\alpha(t)u_0\,(-\Delta)^{\frac{\beta}{2}}\varphi dx + \int_\Omega {}_0I_t^{1-\alpha}(P_\alpha(t)u_0 - u_0)\,\mathscr{G}^{-1}\varphi_t dx. \tag{3.2} $$

For arbitrary $h > 0, t \in [0,T)$ and $t + h \le T$, we obtain

$$ \begin{aligned} \frac{1}{h}(I_2(t+h) - I_2(t)) =& \frac{1}{h}\int_0^{t+h}\int_\Omega P_\alpha(t+h-s)\,{}_0I_t^{1-\gamma}\left(|u|^{p-1}u\right)ds\varphi(t+h,x)dx \\ &- \frac{1}{h}\int_0^t\int_\Omega P_\alpha(t-s)\,{}_0I_t^{1-\gamma}\left(|u|^{p-1}u\right)uds\varphi(t,x)dx \\ =& I_3(h) + I_4(h) + I_5(h), \end{aligned} $$

where

$$ I_3(h) = \frac{1}{h}\int_\Omega\int_t^{t+h}\int_0^\infty \phi_\alpha(\theta)T\left((t+h-s)^\alpha\theta\right){}_0I_t^{1-\gamma}\left(|u|^{p-1}u\right)d\theta ds\varphi(t+h,x)dx, $$

$$ I_4(h) = \frac{1}{h}\int_\Omega\int_0^t\int_0^\infty \phi_\alpha(\theta)\left(T\left((t+h-s)^\alpha\theta\right) - T\left((t-s)^\alpha\theta\right)\right){}_0I_t^{1-\gamma}\left(|u|^{p-1}u\right)d\theta ds\varphi(t,x)dx, $$

$$ I_5(h) = \frac{1}{h}\int_\Omega\int_0^t\int_0^\infty \phi_\alpha(\theta)T\left((t+h-s)^\alpha\theta\right){}_0I_t^{1-\gamma}\left(|u|^{p-1}u\right)d\theta ds(\varphi(t+h,x) - \varphi(t,x))dx. $$

Using the dominated convergence theorem, we conclude that

$$ I_3(h) \to \int_\Omega {}_0I_t^{1-\gamma}\left(|u|^{p-1}u\right)\varphi dx \text{ as } h \to 0, $$

$$ I_5(h) \to \int_\Omega\int_0^t\int_0^\infty \phi_\alpha(\theta)T\left((t-s)^\alpha\theta\right){}_0I_t^{1-\gamma}\left(|u|^{p-1}u\right)d\theta ds\varphi_t dx $$

$$= \int_\Omega \int_0^t P_\alpha(t-s)_0 I_t^{1-\gamma}\left(|u|^{p-1}u\right) ds\varphi_t dx \text{ as } h \to 0.$$

Since

$$\begin{aligned}
&I_4(h)\\
&= -\int_\Omega \int_0^t \int_0^\infty \int_0^1 \alpha\theta\phi_\alpha(\theta)(t+\tau h-s)^{\alpha-1}(-\Delta)^{\frac{\beta}{2}}\mathscr{G}\left(T\left((t+\tau h-s)^\alpha\theta\right)\right)\mathcal{N}(u)d\tau d\theta ds\varphi dx\\
&= -\int_\Omega (-\Delta)^{\frac{\beta}{2}} \int_0^t \int_0^\infty \int_0^1 \alpha\theta\phi_\alpha(\theta)(t+\tau h-s)^{\alpha-1}\mathscr{G}T\left((t+\tau h-s)^\alpha\theta\right)\mathcal{N}(u)d\tau d\theta ds\varphi dx\\
&= -\int_\Omega \int_0^t \int_0^\infty \int_0^1 \alpha\theta\phi_\alpha(\theta)(t+\tau h-s)^{\alpha-1}\mathscr{G}T\left((t+\tau h-s)^\alpha\theta\right)\mathcal{N}(u)d\tau d\theta ds(-\Delta)^{\frac{\beta}{2}}\varphi dx
\end{aligned}$$

where $\mathcal{N}(u) = {}_0I_t^{1-\gamma}\left(|u|^{p-1}u\right)$.
Using dominated convergence theorem, we have

$$I_4(h) \to -\int_\Omega \int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)\mathscr{G}{}_0I_t^{1-\gamma}\left(|u|^{p-1}u\right)ds(-\Delta)^{\frac{\beta}{2}}\varphi dx \text{ as } h \to 0.$$

Hence, the right derivative of I_2 on [0, T) is

$$\begin{aligned}
\int_\Omega {}_0I_t^{1-\gamma}\left(|u|^{p-1}u\right)\varphi dx - \int_\Omega \int_0^t (t-s)^{\alpha-1}\mathscr{G}S_\alpha(t-s)_0I_t^{1-\gamma}\left(|u|^{p-1}u\right)ds(-\Delta)^{\frac{\beta}{2}}\varphi dx\\
+\int_\Omega \int_0^t P_\alpha(t-s)_0I_t^{1-\gamma}\left(|u|^{p-1}u\right)ds\varphi_t dx
\end{aligned}$$

and it is continuous in $[0, T)$. Therefore,

$$\begin{aligned}
\frac{dI_2}{dt} &= \int_\Omega {}_0I_t^{1-\gamma}\left(|u|^{p-1}u\right)\varphi dx - \int_\Omega \int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)\mathscr{G}{}_0I_t^{1-\gamma}\left(|u|^{p-1}u\right)ds(-\Delta)^{\frac{\beta}{2}}\varphi dx\\
&\quad + \int_\Omega \int_0^t P_\alpha(t-s)_0I_t^{1-\gamma}\left(|u|^{p-1}u\right)ds\varphi_t dx\\
&= \int_\Omega {}_0I_t^{1-\gamma}\left(|u|^{p-1}u\right)\varphi dx - \int_\Omega \int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)\mathscr{G}{}_0I_t^{1-\gamma}\left(|u|^{p-1}u\right)ds(-\Delta)^{\frac{\beta}{2}}\varphi dx\\
&\quad + \int_\Omega {}_0I_t^{1-\alpha}\left(\int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)\mathscr{G}{}_0I_t^{1-\gamma}\left(|u|^{p-1}u\right)ds\right)\mathscr{G}^{-1}\varphi_t dx, t \in [0, T).
\end{aligned} \tag{3.3}$$

Thus, combining (3.1)–(3.3), we conclude that

$$\begin{aligned}
0 &= \int_0^T \frac{d}{dt}\int_\Omega I_t^{1-\alpha}(u-u_0)\varphi dx dt = \int_0^T \frac{dI_1}{dt} + \frac{dI_2}{dt}dt\\
&= -\int_0^T \int_\Omega P_\alpha(t)u_0(-\Delta)^{\frac{\beta}{2}}\varphi dx dt + \int_0^T \int_\Omega {}_0I_t^{1-\alpha}\left(P_\alpha(t)u_0 - u_0\right)\mathscr{G}^{-1}\varphi_t dx dt\\
&\quad + \int_0^T \int_\Omega {}_0I_t^{1-\gamma}\left(|u|^{p-1}u\right)\varphi dx dt
\end{aligned}$$

$$
\begin{aligned}
&-\int_0^T\int_\Omega\int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)\mathscr{G}\,{}_0I_t^{1-\gamma}\left(|u|^{p-1}u\right)ds\,(-\Delta)^{\frac{\beta}{2}}\varphi dxdt\\
&+\int_0^T\int_\Omega {}_0I_t^{1-\alpha}\left(\int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)\mathscr{G}\,{}_0I_t^{1-\gamma}\left(|u|^{p-1}u\right)ds\right)\mathscr{G}^{-1}\varphi_t dxdt,\\
=&-\int_0^T\int_\Omega u\,(-\Delta)^{\frac{\beta}{2}}\varphi dxdt-\int_0^T\int_\Omega (u-u_0)\,{}_t\mathbb{D}_T^\alpha\mathscr{G}^{-1}\varphi dxdt\\
&+\int_0^T\int_\Omega {}_0I_t^{1-\gamma}\left(|u|^{p-1}u\right)\varphi dxdt.
\end{aligned}
$$

so, we can get the following equation

$$
\begin{aligned}
\int_\Omega\int_0^T\left[{}_0I_t^{1-\gamma}\left(|u|^{p-1}u\right)\varphi+u_0\left({}_t\mathbb{D}_T^\alpha\varphi\right)+mu\left({}_t\mathbb{D}_T^\alpha\Delta\varphi\right)\right]dtdx=&\int_\Omega\int_0^T u\,(-\Delta)^{\frac{\beta}{2}}\varphi dtdx\\
&+\int_\Omega\int_0^T u\left({}_t\mathbb{D}_T^\alpha\varphi\right)dtdx+\int_\Omega\int_0^T mu_0\left({}_t\mathbb{D}_T^\alpha\Delta\varphi\right)dtdx.
\end{aligned}
$$

Hence, this completes the proof. □

Now, we prove Theorem 1.1.

Proof. (1) We proof is given by contraction. Let $\lambda_1 > 0$ be the first eigenvalue of $-\Delta$ and φ_1 denote the corresponding positive eigenfunction with $\int_\Omega \varphi_1(x)dx = 1$. From the regularity theory of elliptic equations, one has $\varphi_1 \in C^2(\bar{\Omega})$ and $\varphi_1(x) = 0$ for $x \in \partial\Omega$. Suppose that u is a global mild solution to (1.1). Then we get that u is also a global weak solution of(1.1) by Theorem 3.2 and Lemma 3.5. Let $\psi_T \in C^1([0,T])$ with $\psi_T \geq 0, \psi_T(T) = 0$. Then, from definition 3.4 and taking $\varphi(x,t) = \varphi_1(x)\psi_T(t)$ as a test function, we have

$$
\begin{aligned}
\int_\Omega\int_0^T\left[{}_0I_t^{1-\gamma}\left(u^p\right)\varphi_1\psi_T+u_0\varphi_1\left({}_t\mathbb{D}_T^\alpha\psi_T\right)-m\lambda_1 u\varphi_1\left({}_t\mathbb{D}_T^\alpha\psi_T\right)\right]dtdx=&\int_\Omega\int_0^T \lambda_1^{\frac{\beta}{2}}u\varphi_1\psi_T dtdx\\
&+\int_\Omega\int_0^T u\varphi_1\left({}_t\mathbb{D}_T^\alpha\psi_T\right)dtdx-\int_\Omega\int_0^T m\lambda_1 u_0\varphi_1\left({}_t\mathbb{D}_T^\alpha\psi_T\right)dtdx. \qquad (3.4)
\end{aligned}
$$

Let $f(t) = \int_\Omega u\varphi_1 dx$, $\sigma = 1-\gamma$. We have $f \in C([0,T])$. According to Jensen's inequality and (3.4), (2.1), we deduce that

$$
\int_0^T f^p\left({}_tI_T^\sigma\psi_T\right)dt+(1+m\lambda_1)f(0)\int_0^T\left({}_t\mathbb{D}_T^\alpha\psi_T\right)dt\leq\lambda_1^{\frac{\beta}{2}}\int_0^T f\psi_T dt+(1+m\lambda_1)\int_0^T f\left({}_t\mathbb{D}_T^\alpha\psi_T\right)dt. \tag{3.5}
$$

Moreover, $\mathbb{D}_t^\alpha f$ exists for $t \in [0,T]$ and $\mathbb{D}_t^\alpha f \in C([0,T])$. Thus, using (3.5), (2.1) and (2.2), we conclude that

$$
\begin{aligned}
\int_0^T\left({}_0I_t^\sigma f^p\right)\psi_T dt&\leq\lambda_1^{\frac{\beta}{2}}\int_0^T f\psi_T dt+(1+m\lambda_1)\int_0^T[f(t)-f(0)]{}_t\mathbb{D}_T^\alpha\psi_T dt\\
&=\lambda_1^{\frac{\beta}{2}}\int_0^T f\psi_T dt+(1+m\lambda_1)\int_0^T\mathbb{D}_t^\alpha f\psi_T dt.
\end{aligned}
$$

By the arbitrariness of ψ_T, we obtain

$$(1+m\lambda_1)\mathbb{D}_t^\alpha f + \lambda_1^{\frac{\beta}{2}} f \geq {}_0I_t^\sigma f^p, \quad t \in [0,T]. \tag{3.6}$$

It is easy to see that $f(0) > 0$, then (3.6) is in contradiction with Lemma 2.3 (2). The proof of Theorem 1.1 is finished.

(2)We proof the global existence by the contraction mapping principle. For arbitrary $T > 0$, we defined the space $Y = \{u \in L^\infty((0,\infty), L^\infty(\Omega)) \mid \|u\|_Y < \infty\}$, where $\|u\|_Y = \sup_{t>0}(1+t)^{\frac{\sigma}{p-1}}\|u(t)\|_{L^\infty(\Omega)}$. Given $u \in Y$, $t \geq 0$., let's set

$$\Psi(u)(t) = P_\alpha(t)u_0 + \frac{1}{\Gamma(\sigma)}\int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)\mathscr{G}\int_0^s (s-\tau)^{-\gamma}|u|^{p-1}u(\tau)d\tau ds,$$

Denote $E = \{u \in Y \mid \|u\|_Y \leq M\}$, where $M > 0$ is small enough. According to $\gamma \leq \alpha$ and $p > \frac{1}{1-\sigma}$, we have that $p > \frac{1}{1-\sigma} \geq 1 + \frac{\sigma}{\alpha}$, $\frac{\sigma}{p-1} < \alpha$ and $\frac{p\sigma}{p-1} < 1$. Hence, (2.4) implies that there is a constant $C > 0$ such that for any $u \in E$ and $t \geq 0$,

$$\begin{aligned}(1+t)^{\frac{\sigma}{p-1}}\|P_\alpha(t)u_0\|_{L^\infty(\Omega)} &\leq C(1+t)^{\frac{\sigma}{p-1}}\int_0^{+\infty}\phi_\alpha(\theta)e^{-\lambda_1^{\frac{\beta}{2}}(1+m\lambda_1)^{-1}t^\alpha\theta}d\theta\,\|u_0\|_{L^\infty(\Omega)} \\ &= C(1+t)^{\frac{\sigma}{p-1}}E_\alpha\left(-\lambda_1^{\frac{\beta}{2}}(1+m\lambda_1)^{-1}t^\alpha\right)\|u_0\|_{L^\infty(\Omega)} \\ &\leq C(1+t)^{\frac{\sigma}{p-1}-\alpha}\|u_0\|_{L^\infty(\Omega)},\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}&(1+t)^{\frac{\sigma}{p-1}}\|\Psi(u) - P_\alpha(t)u_0\|_{L^\infty(\Omega)} \\ &\leq C(1+t)^{\frac{\sigma}{p-1}}\int_0^t\int_0^s (t-s)^{\alpha-1}(s-\tau)^{-\gamma}\int_0^{+\infty}\theta\phi_\alpha(\theta)e^{-\lambda_1^{\frac{\beta}{2}}(1+m\lambda_1)^{-1}(t-s)^\alpha\theta}d\theta\|u(\tau)\|^p_{L^\infty(\Omega)}d\tau ds \\ &\leq C(1+t)^{\frac{\sigma}{p-1}}\int_0^t\int_0^s (t-s)^{\alpha-1}(s-\tau)^{-\gamma}E_{\alpha,\alpha}\left(-\lambda_1^{\frac{\beta}{2}}(1+m\lambda_1)^{-1}(t-s)^\alpha\right)\|u(\tau)\|^p_{L^\infty(\Omega)}d\tau ds \\ &\leq CM^p(1+t)^{\frac{\sigma}{p-1}}\int_0^t\int_0^s (t-s)^{\alpha-1}(s-\tau)^{-\gamma}E_{\alpha,\alpha}\left(-\lambda_1^{\frac{\beta}{2}}(1+m\lambda_1)^{-1}(t-s)^\alpha\right)(1+\tau)^{-\frac{p\sigma}{p-1}}d\tau ds \\ &\leq CM^p(1+t)^{\frac{\sigma}{p-1}}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_1^{\frac{\beta}{2}}(1+m\lambda_1)^{-1}(t-s)^\alpha\right)\int_0^s (s-\tau)^{-\gamma}\tau^{-\frac{p\sigma}{p-1}}d\tau ds \\ &= CM^p(1+t)^{\frac{\sigma}{p-1}}B\left(\sigma, 1-\frac{p\sigma}{p-1}\right)\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_1^{\frac{\beta}{2}}(1+m\lambda_1)^{-1}(t-s)^\alpha\right)s^{1-\gamma-\frac{p\sigma}{p-1}}ds\end{aligned} \tag{3.8}$$

where we have applied (2.6) and (2.7). Moreover, from similar calculations of the above proof, we know that there is a constant $C > 0$ such that for any $u, v \in E$ and $t \geq 0$,

$$\begin{aligned}&(1+t)^{\frac{\sigma}{p-1}}\|\Psi(u) - \Psi(v)\|_{L^\infty(\Omega)} \\ &\leq CM^{p-1}(1+t)^{\frac{\sigma}{p-1}}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_1^{\frac{\beta}{2}}(1+m\lambda_1)^{-1}(t-s)^\alpha\right)\int_0^s (s-\tau)^{-\gamma}\tau^{-\frac{p\sigma}{p-1}}d\tau ds\|u-v\|_Y \\ &\leq CM^{p-1}(1+t)^{\frac{\sigma}{p-1}}B\left(\beta, 1-\frac{p\sigma}{p-1}\right)\end{aligned}$$

$$\times \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_1^{\frac{\beta}{2}} (1+m\lambda_1)^{-1} (t-s)^{\alpha}\right) s^{1-\gamma-\frac{p\sigma}{p-1}} ds \|u-v\|_Y. \tag{3.9}$$

It follows from (2.3) and the fact that $\frac{\sigma}{p-1} < 1$ and $E_{\alpha,\alpha}(z)$ is an entire function that

$$\begin{aligned}
&\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_1^{\frac{\beta}{2}} (1+m\lambda_1)^{-1} (t-s)^{\alpha}\right) s^{1-\gamma-\frac{p\beta}{p-1}} ds \\
=&\sum_{k=0}^{\infty} \int_0^t \frac{\left(-\lambda_1^{\frac{\beta}{2}} (1+m\lambda_1)^{-1}\right)^k (t-s)^{\alpha k+\alpha-1} s^{1-\gamma-\frac{p\sigma}{p-1}}}{\Gamma(\alpha k+\alpha)} ds \\
=&\Gamma\left(1-\frac{\sigma}{p-1}\right) t^{\alpha-\frac{\sigma}{p-1}} E_{\alpha,\alpha+1-\frac{\sigma}{p-1}}\left(-\lambda_1^{\frac{\beta}{2}} (1+m\lambda_1)^{-1} t^{\alpha}\right).
\end{aligned}$$

Note that $\frac{\sigma}{p-1} \leq \alpha$ and $\frac{p\sigma}{p-1} < 1$. Therefore, from (2.4), (3.7), (3.8) and (3.9), we know Ψ is a strict contraction on E if $\|u_0\|_{L^\infty(\Omega)}$ and we choose M sufficiently small. Then by the contraction mapping principle, there exists a unique fixed point $u \in E$. Obviously, $u \in C([0,\infty), C_0(\Omega))$. It means that (1.1) admits a global mild solution. Hence, the proof is completed. □

Finally, we proveTheorem 1.2.

Proof. (1) Suppose that u is a mild solution of (1.1). In the same way as the proof of Theorem 1.1(1), we obtain that inequality (3.6) still holds in this case. Hence, we obtain the conclusion.
(2) It follows the assumption that $p \geq 1 + \frac{\sigma}{\alpha}$ and $\gamma > \alpha$, we deduce that $p \geq 1 + \frac{\sigma}{\alpha} > \frac{1}{1-\sigma}$, $\frac{p\sigma}{p-1} < 1$ and $\frac{\sigma}{p-1} \leq \alpha$. Then, in the same way as the proof of Theorem 1.1(ii), we can get the conclusion. □

4. Conclusions

In this work, inspired by the method in [20], we obtained blow-up and global existence results for a semilinear fractional pseudo-parabolic equation with nonlinear memory in a bounded domain. First, we define a solution operator which is expressed by a semigroup and discussed its properties. Based on these properties, we obtained local existence of mild solutions and proved that a mild solution is also a weak solution. Then, we used the integral representation and the contraction-mapping principle to prove the global existence results for solutions of the Cauchy problem (1.1). Finally, we used test function method to prove the blow-up of solutions. Of course, these global existence and blow-up conclusions and their proof also fit the case $m = 0$. However, due to the appearance of the third order term for the Cauchy problem (1.1), the integral representation and the proof are more complicated than that for the case $m = 0$. It is noted that the critical exponent is consistent with the corresponding Cauchy problem for the time-fractional differential equation with nonlinear memory, and we also illustrated that the diffusion effect of the third order term is not strong enough to change the critical exponents.

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Conflict of interest

The authors declare there is no conflicts of interest.

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