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Research article

Existence of a positive radial solution for semilinear elliptic problem involving variable exponent

Changmu Chu*, **Shan Li and Hongmin Suo**

School of preparatory education, Guizhou Minzu University, Guizhou 550025, China

* **Correspondence:** Email:gzmychuchangmu@sina.com.

Abstract: This paper consider that the following semilinear elliptic equation

$$\begin{cases} -\Delta u = u^{q(x)-1}, & \text{in } B_1, \\ u > 0, & \text{in } B_1, \\ u = 0, & \text{in } \partial B_1, \end{cases} \tag{0.1}$$

where B_1 is the unit ball in $\mathbb{R}^N(N \geq 3)$, $q(x) = q(|x|)$ is a continuous radial function satisfying $2 \leq q(x) < 2^* = \frac{2N}{N-2}$ and $q(0) > 2$. Using variational methods and a priori estimate, the existence of a positive radial solution for (0.1) is obtained.

Keywords: semilinear elliptic problem; variable exponent; mountain pass lamma; a priori estimate; positive radial solution

1. Introduction and main result

In recent years, the following nonlinear elliptic equation

$$\begin{cases} -\Delta_{p(x)} u = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

was studied due to the fact that it can be applied to fluid mechanics and the field of image processing (see [1, 2]), where $\Omega \subset \mathbb{R}^N(N \geq 3)$ is a bounded smooth domain, $p \in C(\overline{\Omega}, \mathbb{R})$, $1 < p^- = \min\limits_{x\in\overline{\Omega}} p(x) \leq p(x) \leq \max\limits_{x\in\overline{\Omega}} p(x) = p^+ < N$, $\Delta_{p(x)}u := div(|\nabla u|^{p(x)-2}\nabla u)$ and $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$.

In 2003, Fan and Zhang in [3] gave several sufficient conditions for the solvability of nontrivial solutions for problem (1.1). These conditions include either the sublinear growth condition

$$|f(x,t)| \leq C\left(1 + |t|^{p^-}\right), \quad \text{for } x \in \Omega \text{ and } t \in \mathbb{R}$$

or Ambrosetti-Rabinowitz type growth condition ((AR)-condition, for short): there is $\theta > p^+$ such that

$$f(x,t)t \geq \theta F(x,t) > 0, \quad \text{for all } x \in \Omega \text{ and } |t| \text{ large enough},$$

where $C > 0$, $F(x,t) = \int_0^t f(x,s)\,ds$, and $|f(x,t)t| \leq C(1+|t|^{p^*(x)})$ with $p^*(x) = \frac{Np(x)}{N-p(x)}$. Subsequently, Chabrowski and Fu in [4] discussed problem (1.1) in a more general setting than that in [3].

As is well known, the (AR)-condition ensure the boundedness of Palais-Smale sequence of the corresponding function. However, there are some papers considering the nonlinearity without (AR)-condition. [5] proveed the existence of strong solutions of problem (1.1) without the growth condition of the well-known AmbrosettiCRabinowitz type. Subsequently, [6] extended the results of [5]. Under no AmbrosettiCRabinowitzs superquadraticity conditions, [7] and [8] obtained the existence and multiplicity of the solution of problem (1.1) by different methods. In addition, [9] and [10] pointed out the importance of the Cerami condition. In fact, these papers still require nonlinearity to satisfy superlinear growth condition:

$$f(x,t)t > p(x)F(x,t), \quad \text{for all } x \in \Omega \text{ and } |t| \text{ is large enough}.$$

As far as we know, there are few results in the case $f(x,t)t = p(x)F(x,t)$ for some $x \in \Omega$ and $|t|$ large enough. In addition to the eigenvalue problem was studied in [11] and [12], we only see that [13] and [14] discussed the multiplicity of nontrivial solutions and sign-changing solutions, respectively. As described in [15], there are new difficulties in dealing with this situation.

Let $S_N = \inf\limits_{0\neq u\in H_0^1(B_1)} \frac{\int_{B_1} |\nabla u|^2 dx}{\left(\int_{B_1} |u|^{2^*} dx\right)^{\frac{2}{2^*}}}$ be the best Sobolev constant and B_1 be the unit ball in $\mathbb{R}^N (N \geq 3)$,we consider the following elliptic problem

$$\begin{cases} -\Delta u = u^{q(x)-1}, & \text{in } B_1, \\ u > 0, & \text{in } B_1, \\ u = 0, & \text{in } \partial B_1, \end{cases} \tag{1.2}$$

where $q(x) = q(|x|)$ is a continuous radial function satisfying $2 \leq q(x) < 2^* = \frac{2N}{N-2}$ and $q(0) > 2$.

Theorem 1.1. *Let $q(x)$ be a continuous radial function satisfying*

$$q(x) = q(|x|), \quad 2 \leq q(x) \leq q^+ < 2^*, \quad q(0) > 2.$$

Suppose that $\Omega_0 = \{x \in B_1 : q(x) = 2\}$ is not empty and the measure satisfies

$$S_N^{-1} |\Omega_0|^{\frac{2^*-2}{2^*}} < \frac{1}{2}.$$

Then problem (1.2) *has at least a positive radial symmetric solution.*

Remark 1.2. *In [15], the authors considered the existence of a nontrivial solution of $-\Delta_p u + u^{p-1} = u^{q(x)-1}$, $u \in W_r^{1,p}(\mathbb{R}^N)$ and $u \geq 0$ in $\mathbb{R}^N$ for $1 < p < N$, where $\Delta_p u := div(|\nabla u|^{p-2}\nabla u)$. They showed that if there exist positive constants R_1, R_2, C_1, C_2 and $0 < l_1,\ l_2 < 1$ such that $ess \inf_{x\in B_{R_1}}\{q(x)\} > p$, $q(x) \geq p + \frac{C_1}{|\log|x||^{l_1}}$ for $x \in \mathbb{R}^N \backslash B_{R_1}$ and $q(x) \leq \frac{Np}{N-p} - \frac{C_2}{|\log|x||^{l_2}}$ for $x \in B_{R_2}$, then there exists a nontrivial solution to this equation. However, our Theorem 1.1 allows $q(x) = 2$ for some $x \in B_1$.*

Remark 1.3. *The hypothesis of Theorem 1.1 can not ensure that problem* (1.2) *satisfies the Ambrosetti-Rabinowitz growth condition. Indeed, for the case* $p(x) \equiv 2$ *and* $f(x,t) = t^{q(x)-1}$, *we have* $p^+ = 2$ *and* $F(x,t) = \frac{1}{q(x)} t^{q(x)}$ *for* $t \geq 0$. *It follows that* $f(x,t)t = p^+ F(x,t)$ *for any* $x \in \Omega_0$.

Remark 1.4. *In our paper, the* L^∞ *estimate is an essential tool that makes the solution go back to the original problem. The condition of radial symmetry plays a major role in the estimation of the solution.*

2. The auxiliary problem

According to $q(x) \geq 2$, It is not easy to determine whether the functional I satisfies the Palais-Smale condition. To apply the mountain pass theorem, the first step is to modify the nonlinearity. By the continuity of $q(x)$, $2 \leq q(x) \leq q^+ < 2^*$ and $q(0) > 2$, we see that there exist $\delta \in (0, \frac{1}{4})$ and $r > 0$ such that

$$q(x) \geq 2 + r, \quad x \in B_{2\delta}; \qquad q^+ + r < 2^*, \quad x \in B_1. \tag{2.1}$$

Let $\psi(t) \in C_0^\infty(\mathbb{R}, [0,1])$ be an even function satisfying $\psi(t) = 1$ for $|t| \leq 1$, $\psi(t) = 0$ for $|t| \geq 2$ and $\psi(t)$ decreases monotonically over $\mathbb{R}^+$. Define

$$b_\mu(t) = \psi(\mu t), \qquad m_\mu(t) = \int_0^t b_\mu(\tau) d\tau,$$

for $\mu \in (0, 1]$. We consider the auxiliary problem

$$\begin{cases} -\Delta u = (1 - Q(x)) \left(\frac{u}{m_\mu(u)}\right)^r u^{q(x)-1} + Q(x) u^{q(x)-1}, & \text{in } B_1, \\ u > 0, & \text{in } B_1, \\ u = 0, & \text{in } \partial B_1, \end{cases} \tag{2.2}$$

where $Q(x) = Q(|x|) \in C(B_1, [0,1])$ satisfies $Q(x) = 1$ for $x \in B_\delta$ and $Q(x) = 0$ for $x \in B_1 \setminus B_{2\delta}$.

Theorem 2.1. *Assume that* $q(x) = q(|x|)$ *is a continuous radial function satisfying* $2 \leq q(x) \leq q^+ < 2^*$ *and* $q(0) > 2$, *the measure of* $\Omega_0 = \{x | q(x) = 2\}$ *satisfies* $S_N^{-1} |\Omega_0|^{\frac{2^*-2}{2^*}} < \frac{1}{2}$. *Then problem* (2.2) *has at least a positive radial symmetric solution for any* $\mu \in (0, 1]$.

Set $H_{0,r}^1(B_1) = \left\{u \in H_0^1(B_1) \mid u(x) = u(|x|)\right\}$, $\|\cdot\| = \|\nabla(\cdot)\|_{L^2(B_1)}$. $I_\mu : H_{0,r}^1(B_1) \to \mathbb{R}$ by

$$I_\mu(u) = \frac{1}{2} \int_{B_1} |\nabla u|^2 \, dx - \int_{B_1} (1 - Q(x)) K_\mu(u^+) \, dx - \int_{B_1} \frac{Q(x)}{q(x)} (u^+)^{q(x)} \, dx,$$

where $k_\mu(x,t) = k_\mu(t) = \left(\frac{t}{m_\mu(t)}\right)^r t^{q(x)-1}$, $K_\mu(x,t) = K_\mu(t) = \int_0^t k_\mu(s) \, ds$.

Lemma 2.2. $K_\mu(x,t)$ *have the following properties:*

$$K_\mu(x,t) \leq \frac{1}{q(x)} t k_\mu(x,t), \qquad K_\mu(x,t) \leq \frac{1}{q(x) + r} t k_\mu(x,t) + C_\mu,$$

for $t > 0$, *where* $C_\mu > 0$.

Proof. According to the monotonicity of $b_\mu(t)$, one has

$$\frac{d}{dt}\left(\frac{t}{m_\mu(t)}\right) = \frac{m_\mu(t) - tb_\mu(t)}{m_\mu^2(t)} = \frac{t(b_\mu(\xi) - b_\mu(t))}{m_\mu^2(t)} \geq 0,$$

for $t > 0$, where $\xi \in (0, t)$. Therefore, $\frac{t}{m_\mu(t)}$ is monotonically increasing on $\mathbb{R}^+$. Hence, $\frac{k_\mu(x,t)}{t^{q(x)-1}} = \left(\frac{t}{m_\mu(t)}\right)^r$ is also monotonically increasing on $\mathbb{R}^+$. It implies that

$$K_\mu(x, t) = \int_0^t k_\mu(x, \tau)d\tau \leq \int_0^t \frac{k_\mu(x, t)}{t^{q(x)-1}}\tau^{q(x)-1}d\tau = \frac{1}{q(x)}tk_\mu(x, t), \tag{2.3}$$

for $t > 0$. Obviously, $m_\mu(t) = \frac{A}{\mu}$ for $t \geq \frac{2}{\mu}$, where $A = 1 + \int_1^2 \psi(\tau)d\tau$. For $t > \frac{2}{\mu}$, one has

$$\begin{aligned}
K_\mu(x, t) &= \int_0^{\frac{2}{\mu}} k_\mu(x, \tau)\, d\tau + \int_{\frac{2}{\mu}}^t \left(\frac{\mu}{A}\right)^r \tau^{q(x)+r-1}\, d\tau \\
&= \int_0^{\frac{2}{\mu}} \left(k_\mu(x, \tau) - \left(\frac{\mu}{A}\right)^r \tau^{q(x)+r-1}\right) d\tau + \int_0^t \left(\frac{\mu}{A}\right)^r \tau^{q(x)+r-1}\, d\tau \\
&\leq C_\mu + \frac{tk_\mu(x, t)}{q(x) + r}.
\end{aligned} \tag{2.4}$$

Combining (2.3) with (2.4), we obtain $K_\mu(x, t) \leq \frac{1}{q(x)+r}tk_\mu(x, t) + C_\mu$ for $t > 0$.

Lemma 2.3. *Suppose that $q(x) = q(|x|)$ is a continuous radial function satifying $2 \leq q(x) \leq q^+ < 2^*$ and $q(0) > 2$. Then I_μ satisfies the (PS) condition for all $\mu \in (0, 1]$.*

Proof. Let $\{u_n\}$ be a (PS) sequence of I_μ in $H^1_{0,r}(B_1)$. There exists $C > 0$ such that

$$|I_\mu(u_n)| \leq C, \quad I'_\mu(u_n) \to 0 \text{ as } n \to \infty. \tag{2.5}$$

By (2.1) and Lemma 2.2, we have

$$\begin{aligned}
&I_\mu(u_n) - \frac{1}{2 + r}\langle I'_\mu(u_n), u_n\rangle \\
= \;& \frac{r}{2(2 + r)}\|u_n\|^2 + \int_{B_1} (1 - Q(x))\left(\frac{k_\mu(x, u_n^+)u_n^+}{2 + r} - K_\mu(x, u_n^+)\right) dx \\
&+ \int_{B_{2\delta}} \left(\frac{1}{2 + r} - \frac{1}{q(x)}\right) Q(x)(u_n^+)^{q(x)}dx \\
\geq \;& \frac{r}{2(2 + r)}\|u_n\|^2 - C_\mu,
\end{aligned}$$

which implies that $\frac{r}{2(2+r)}\|u_n\|^2 \leq C + C_\mu + o(\|u_n\|)$. We obtain $\{u_n\}$ is bounded in $H^1_{0,r}(B_1)$. Up to a subsequence, we may assume that

$$\begin{cases} u_n \rightharpoonup u, & \text{in } H^1_{0,r}(B_1), \\ u_n \to u, & \text{in } L^s(B_1),\ 1 \leq s < 2^*. \end{cases}$$

It implies that

$$\begin{aligned}\|u_i - u_j\|^2 &= \langle I_\mu'(u_i) - I_\mu'(u_j), u_i - u_j\rangle + \int_{B_1} (1 - Q(x))(k_\mu(u_i^+) - k_\mu(u_j^+))(u_i - u_j)dx \\ &\quad + \int_{B_1} Q(x)((u_i^+)^{q(x)-1} - (u_j^+)^{q(x)-1})(u_i - u_j)dx.\end{aligned}$$

It follows from (2.5) that

$$\langle I_\mu'(u_i) - I_\mu'(u_j), u_i - u_j\rangle \to 0, \quad \text{as} \quad i,\ j \to +\infty. \tag{2.6}$$

It is not difficult to see that

$$|k_\mu(t)| \le |t|^{q(x)-1} + \left(\frac{\mu}{A}\right)^r |t|^{q(x)+r-1}.$$

By the Sobolev imbedding theorem and $2 \le q(x) < q(x) + r < q^+ + r < 2^*$, one has

$$\begin{aligned}&\left|\int_{B_1} (1 - Q(x))(k_\mu(u_i^+) - k_\mu(u_j^+))(u_i - u_j)dx\right| \\ \le\ & C\int_{B_1} \left(|u_i| + |u_j| + |u_i|^{q^++r-1} + |u_j|^{q^++r-1}\right)|u_i - u_j| \to 0\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}&\left|\int_{B_1} Q(x)((u_i^+)^{q(x)-1} - (u_j^+)^{q(x)-1})(u_i - u_j)dx\right| \\ \le\ & C\int_{B_1} \left(|u_i| + |u_j| + |u_i|^{q^+-1} + |u_j|^{q^+-1}\right)|u_i - u_j| \to 0\end{aligned} \tag{2.8}$$

as i and j tend to $+\infty$. From (2.6)–(2.8), we have $\|u_i - u_j\| \to 0$ as $i,\ j \to +\infty$, which implies that $\{u_n\}$ contains a strongly convergent subsequence in $H^1_{0,r}(B_1)$. Hence I_μ satisfies the (PS) condition.

Lemma 2.4. *I_μ has the following properties:*
(1) there exist $m,\ \rho > 0$ such that $I_\mu(u) > m$ for any $u \in H^1_{0,r}(B_1)$ with $\|u\| = \rho$;
(2) there exists $w \in H^1_{0,r}(B_1)$ such that $\|w\| > \rho$ and $I_\mu(w) < 0$.

Proof. By definition of the function k_μ, we have

$$|k_\mu(t)| \le |t|^{q(x)-1} + \left(\frac{\mu}{A}\right)^r |t|^{q(x)+r-1}.$$

It follows that

$$|K_\mu(t)| \le \frac{|t|^{q(x)}}{q(x)} + \left(\frac{\mu}{A}\right)^r \frac{|t|^{q(x)+r}}{q(x)+r}.$$

Therefore, there exists $C > 0$ such that

$$\left|\int_{B_1} (1 - Q(x))K_\mu(u^+)\, dx + \int_{B_1} \frac{Q(x)}{q(x)}(u^+)^{q(x)}\, dx\right|$$

$$\leq \int_{B_1} |u|^{q(x)}\, dx + C \int_{B_1} |u|^{q(x)+r}\, dx. \tag{2.9}$$

By the Sobolev imbedding theorem, it implies from $2 \leq q(x) < q(x) + r < 2^*$ that

$$\int_{B_1} |u|^{q(x)+r}\, dx \leq \int_{B_1} (|u|^{2+r} + |u|^{2^*})\, dx \leq C(\|u\|^{2+r} + \|u\|^{2^*}). \tag{2.10}$$

Set $\Omega_\varepsilon = \{x \in B_1 | 2 \leq q(x) < 2 + \varepsilon\}$. By the Sobolev imbedding theorem and the Hölder inequality, we obtain

$$\begin{aligned}\int_{B_1} |u|^{q(x)}\, dx &= \int_{\Omega_\varepsilon} |u|^{q(x)}\, dx + \int_{B_1\setminus\Omega_\varepsilon} |u|^{q(x)}\, dx \\ &\leq \int_{\Omega_\varepsilon} (|u|^2 + |u|^{2+\varepsilon})\, dx + \int_{B_1\setminus\Omega_\varepsilon} (|u|^{2+\varepsilon} + |u|^{2^*})\, dx \\ &\leq \int_{\Omega_\varepsilon} |u|^2\, dx + \int_{B_1} (|u|^{2+\varepsilon} + |u|^{2^*})\, dx \\ &\leq S_N^{-1} |\Omega_\varepsilon|^{\frac{2^*-2}{2^*}} \|u\|^2 + C(\|u\|^{2+\varepsilon} + \|u\|^{2^*}).\end{aligned} \tag{2.11}$$

Since $S_N^{-1}|\Omega_0|^{\frac{2^*-2}{2^*}} < \frac{1}{2}$, for $\varepsilon > 0$ small enough, one has $S_N^{-1}|\Omega_\varepsilon|^{\frac{2^*-2}{2^*}} < \frac{1}{4} + \frac{1}{2}S_N^{-1}|\Omega_0|^{\frac{2^*-2}{2^*}}$. From (2.9)–(2.11), we obtain

$$I_\mu(u) \geq \left(\frac{1}{4} - \frac{1}{2}S_N^{-1}|\Omega_0|^{\frac{2^*-2}{2^*}}\right)\|u\|^2 - C(\|u\|^{2+\varepsilon} + \|u\|^{2+r} + \|u\|^{2^*}).$$

Therefore, there exist $m,\ \rho > 0$ such that $I_\mu(u) > m$ for any $u \in H^1_{0,r}(B_1)$ with $\|u\| = \rho$.

Fix a nonnegative radial function $v_0 \in H^1_{0,r}(B_\delta)\setminus\{0\}$. We have

$$I_\mu(tv_0) = \frac{t^2}{2}\|v_0\|^2 - \int_{B_\delta} \frac{|tv_0|^{q(x)}}{q(x)} dx \leq \frac{t^2}{2}\|v_0\|^2 - \frac{1}{2^*}\int_{B_\delta} (t^{2+r}|v_0|^{2+r} + t^{2^*}|v_0|^{2^*})\, dx < 0,$$

for $t > 0$ sufficiently large. Choosing $w = tv_0$, we have $\|w\| > \rho$ and $I_\mu(w) < 0$ for $t > 0$ large enough.

Proof of Theorem 2.1. By Lemmas 2.3 and 2.4, we know that I_μ satisfy the (PS) condition and the mountain pass geometry. Define

$$\Gamma = \{\gamma \in C([0,1], H^1_{0,r}(B_1)) |\ \gamma(0) = 0,\ \gamma(1) = w\}, \quad c_\mu = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\mu(\gamma(t)).$$

We obtain that problem (2.2) has a solution u_μ by the mountain pass theorem (see [16]). After a direct calculation, we derive that $\|u_\mu^-\|^2 = \langle I'_\mu(u_\mu), u_\mu^-\rangle = 0$, which implies that $u_\mu^- = 0$. Hence, $u_\mu \geq 0$. Since $I_\mu(u_\mu) > 0 = I(0)$, we have $u_\mu \neq 0$. One has u_μ is a positive solution to problem (2.2) by the Strong Maximum Principle (see [17]).

It follows from (2.1) that

$$c_\mu \leq \max_{t\in[0,1]} I_\mu(tw) \leq \max_{t\in[0,1]} \Big(\frac{t^2}{2}\int_{B_1} |\nabla w|^2 dx - \frac{t^{2+r}}{q^+}\int_{B_\delta} w^{q(x)}\, dx\Big).$$

Therefore, c_μ is uniformly bounded. In other words, we have the following results.

Remark 2.5. *$c_\mu \leq D$, where D is a positive constant independent of μ.*

3. L^∞-estimate and the proof of main results

In this section, we will show that solutions of auxiliary problem (2.2) are indeed solutions of original problem (1.2) for sufficiently small μ.

Lemma 3.1. *If v is a positive critical point of I_μ with $I_\mu(v) = c_\mu$, then $\int_{B_{\frac{\delta}{2}}}(|\nabla v|^2 + v^2)\,dx \leq L$, where L is a positive constant independent of μ.*

Proof. From (2.1) and Lemma 2.2, one has

$$\begin{aligned} c_\mu &= I_\mu(v) - \frac{1}{2}\langle I_\mu^{'}(v), v\rangle \\ &= \int_{B_1}(1-Q(x))\left(\frac{k_\mu(x,v)v}{2} - K_\mu(x,v)\right)dx + \int_{B_{2\delta}}\left(\frac{1}{2} - \frac{1}{q(x)}\right)Q(x)v^{q(x)}dx \\ &\geq \frac{r}{2(2+r)}\int_{B_{2\delta}} Q(x)v^{q(x)}dx \\ &\geq \frac{r}{2(2+r)}\int_{B_\delta} v^{q(x)}dx. \end{aligned} \tag{3.1}$$

Let $\varphi \in C_0^\infty(B_\delta, \mathbb{R})$ satisfies $|\varphi(x)| \leq 1$, $\varphi(x) = 1$ for $|x| \leq \frac{\delta}{2}$ and $|\nabla\varphi| \leq \frac{4}{\delta}$. Multiply problem (2.2) by $v\varphi^2$ and integrate to obtain

$$\begin{aligned} \int_{B_\delta}\nabla v\cdot\nabla(v\varphi^2)dx &= \int_{B_\delta}\left((1-Q(x))\left(\frac{v}{m_\mu(v)}\right)^r v^{q(x)} + Q(x)v^{q(x)}\right)\varphi^2 dx \\ &= \int_{B_\delta} v^{q(x)}\varphi^2 dx. \end{aligned} \tag{3.2}$$

According to (3.1) and (3.2), we have

$$\begin{aligned} \int_{B_{\frac{\delta}{2}}}(|\nabla v|^2 + v^2)\,dx &\leq \int_{B_\delta}|\nabla v|^2\varphi^2\,dx + \int_{B_{\frac{\delta}{2}}} v^2\,dx \\ &\leq 2\int_{B_\delta}\nabla v\cdot\nabla(v\varphi^2)dx + 4\int_{B_\delta}|\nabla\varphi|^2 v^2 dx + \int_{B_{\frac{\delta}{2}}} v^2\,dx \\ &\leq 2\int_{B_\delta}\nabla v\cdot\nabla(v\varphi^2)dx + \frac{8+\delta^2}{\delta^2}\int_{B_\delta} v^2 dx \\ &\leq 2\int_{B_\delta} v^{q(x)}\varphi^2 dx + \frac{8+\delta^2}{\delta^2}\int_{B_\delta}\left(1 + v^{q(x)}\right)dx \\ &\leq \frac{8+\delta^2}{\delta^2}|B_\delta| + \left(2 + \frac{8+\delta^2}{\delta^2}\right)\int_{B_\delta} v^{q(x)}dx \\ &\leq \frac{8+\delta^2}{\delta^2}|B_\delta| + \left(2 + \frac{8+\delta^2}{\delta^2}\right)\frac{2(2+r)c_\mu}{r}. \end{aligned}$$

It implies from Remark 2.5 that $\int_{B_{\frac{\delta}{2}}}\left(|\nabla v|^2 + v^2\right)\,dx \leq L$, where L is a positive constant independent of μ.

Lemma 3.2. *If v is a positive radial symmetric critical point of I_μ with $I_\mu(v) = c_\mu$, then $\|v\|_{L^\infty(B_1)} \leq M$, where M is a positive constant independent of μ.*

Proof. Let $\alpha > 2$ and $\zeta \in C_0^\infty(B_{\frac{\delta}{2}}, \mathbb{R})$. On the one hand, by the Young inequality, we have

$$\begin{aligned}
-\int_{B_{\frac{\delta}{2}}} \zeta^2 v^{\alpha-1}\Delta v\, dx &= (\alpha-1)\int_{B_{\frac{\delta}{2}}} \zeta^2 v^{\alpha-2}|\nabla v|^2 dx + 2\int_{B_{\frac{\delta}{2}}} \zeta v^{\alpha-1}\nabla v\cdot\nabla\zeta dx \\
&= \frac{4(\alpha-1)}{\alpha^2}\int_{B_{\frac{\delta}{2}}} \zeta^2|\nabla v^{\frac{\alpha}{2}}|^2 dx + 2\int_{B_{\frac{\delta}{2}}} \zeta v^{\frac{\alpha}{2}}\nabla v^{\frac{\alpha}{2}}\cdot\nabla\zeta dx \\
&\geq \frac{2(\alpha-1)}{\alpha^2}\int_{B_{\frac{\delta}{2}}} \zeta^2|\nabla v^{\frac{\alpha}{2}}|^2 dx - \frac{\alpha^2}{2(\alpha-1)}\int_{B_{\frac{\delta}{2}}} v^\alpha|\nabla\zeta|^2 dx \\
&\geq \frac{1}{\alpha}\int_{B_{\frac{\delta}{2}}} \zeta^2|\nabla v^{\frac{\alpha}{2}}|^2 dx - \alpha\int_{B_{\frac{\delta}{2}}} v^\alpha|\nabla\zeta|^2 dx. \qquad (3.3)
\end{aligned}$$

On the other hand, one has

$$\begin{aligned}
&\int_{B_{\frac{\delta}{2}}} \left((1-Q(x))\left(\frac{v}{m_\mu(v)}\right)^r v^{q(x)-1} + Q(x)v^{q(x)-1}\right) v^{\alpha-1}\zeta^2 dx \\
&= \int_{B_{\frac{\delta}{2}}} v^{q(x)+\alpha-2}\zeta^2 dx \\
&\leq \int_{B_{\frac{\delta}{2}}} v^\alpha\zeta^2 dx + \int_{B_{\frac{\delta}{2}}} v^{q^++\alpha-2}\zeta^2 dx. \qquad (3.4)
\end{aligned}$$

Combining (3.3) with (3.4), and noticing that v is a solution to problem (2.2), we obtain

$$\int_{B_{\frac{\delta}{2}}} \zeta^2|\nabla v^{\frac{\alpha}{2}}|^2 dx \leq \alpha\left(\alpha\int_{B_{\frac{\delta}{2}}} v^\alpha|\nabla\zeta|^2 dx + \int_{B_{\frac{\delta}{2}}} v^\alpha\zeta^2 dx + \int_{B_{\frac{\delta}{2}}} v^{q^++\alpha-2}\zeta^2 dx\right). \qquad (3.5)$$

Set $\delta_k = \frac{\delta}{4}\left(1+\frac{1}{2^k}\right)$. Let $\zeta_k \in C_0^\infty(B_{\delta_k}, \mathbb{R})$ satisfies the following properties: $0 \leq \zeta_k \leq 1$, $\zeta_k = 1$ for $x \in B_{\delta_{k+1}}$ and $|\nabla\zeta_k| \leq \frac{1}{4(\delta_k-\delta_{k+1})} = \frac{2^{k+1}}{\delta}$. $B_{\frac{\delta}{2}}$ and ζ are taken to be B_{δ_k} and ζ_k in inequality (3.5), respectively. Using the Sobolev embedding theorem, the Hölder inequality and Lemma 3.1, we obtain

$$\begin{aligned}
&\left(\int_{B_{\delta_{k+1}}} v^{\frac{2^*\alpha}{2}} dx\right)^{\frac{2}{2^*}} \\
&\leq \left(\int_{B_{\delta_k}} \left(\zeta_k v^{\frac{\alpha}{2}}\right)^{2^*} dx\right)^{\frac{2}{2^*}} \\
&\leq C\int_{B_{\delta_k}} |\nabla(\zeta_k v^{\frac{\alpha}{2}})|^2 dx \\
&\leq C\left(\int_{B_{\delta_k}} \zeta_k^2|\nabla v^{\frac{\alpha}{2}}|^2 dx + \int_{B_{\delta_k}} v^\alpha|\nabla\zeta_k|^2 dx\right)
\end{aligned}$$

$$\leq C\alpha\left(\left(\alpha+\frac{1}{\alpha}\right)\int_{B_{\delta_k}} v^{\alpha}|\nabla\zeta_k|^2dx+\int_{B_{\delta_k}} v^{\alpha}\zeta_k^2dx+\int_{B_{\delta_k}} v^{q^++\alpha-2}\zeta_k^2dx\right)$$
$$\leq C\alpha\left(\left(\left(\alpha+\frac{1}{\alpha}\right)\frac{4^{k+1}}{\delta^2}+1\right)\int_{B_{\delta_k}} v^{\alpha}dx+\int_{B_{\delta_k}} v^{q^++\alpha-2}dx\right)$$
$$\leq C\alpha\left(\frac{\alpha 4^{k+2}}{\delta^2}|B_{\delta_k}|^{\frac{q^+-2}{2^*}}+\left(\int_{B_{\delta_k}} v^{2^*}dx\right)^{\frac{q^+-2}{2^*}}\right)\left(\int_{B_{\delta_k}} v^{\frac{2^*\alpha}{2^*-q^++2}}dx\right)^{\frac{2^*-q^++2}{2^*}}$$
$$\leq C\alpha\left(\frac{\alpha 4^{k+2}}{\delta^2}|B_{\delta_k}|^{\frac{q^+-2}{2^*}}+C\left(\int_{B_{\delta_k}}\left(|\nabla v|^2+v^2\right)dx\right)^{\frac{q^+-2}{2}}\right)\left(\int_{B_{\delta_k}} v^{\frac{2^*\alpha}{2^*-q^++2}}dx\right)^{\frac{2^*-q^++2}{2^*}}$$
$$\leq C\alpha\left(\frac{\alpha 4^{k+2}}{\delta^2}|B_{\delta_k}|^{\frac{q^+-2}{2^*}}+C(2L)^{\frac{q^+-2}{2}}\right)\left(\int_{B_{\delta_k}} v^{\frac{2^*\alpha}{2^*-q^++2}}dx\right)^{\frac{2^*-q^++2}{2^*}}$$
$$\leq C\alpha^2 4^{k+1}\left(\int_{B_{\delta_k}} v^{\frac{2^*\alpha}{2^*-q^++2}}dx\right)^{\frac{2^*-q^++2}{2^*}}.$$

It implies that

$$\|v\|_{L^{\frac{2^*\alpha}{2}}(B_{\delta_{k+1}})}\leq\left(C\alpha^2 4^{k+1}\right)^{\frac{1}{\alpha}}\|v\|_{L^{\frac{2^*\alpha}{2^*-q^++2}}(B_{\delta_k})}. \tag{3.6}$$

Set $\beta_k=2(\frac{2^*-q^++2}{2})^k$ for $k=0,1,\cdots$. Then $\frac{2}{2^*-q^++2}\beta_{k+1}=\beta_k$. By (3.6), we have

$$\|v\|_{L^{2^*\beta_{k+1}}(B_{\delta_{k+1}})}\leq\left(C\beta_{k+1}^2 4^{k+2}\right)^{\frac{1}{2\beta_{k+1}}}\|v\|_{L^{2^*\beta_k}(B_{\delta_k})}.$$

Doing iteration yields

$$\|v\|_{L^{2^*\beta_k}(B_{\delta_k})}\leq C^{\sum_{j=1}^k\frac{1}{2\beta_j}}\cdot\Pi_{j=1}^k\beta_j^{\frac{1}{\beta_j}}\cdot 4^{\sum_{j=1}^k\frac{j+1}{\beta_j}}\|v\|_{L^{2^*}\left(B_{\frac{\delta}{2}}\right)}$$
$$\leq(4C)^{\frac14\sum_{j=1}^k\left(\frac{2}{\beta_1}\right)^j}\cdot\left(\frac{\beta_1}{2}\right)^{\sum_{j=1}^k\frac{j}{2}\left(\frac{2}{\beta_1}\right)^j}\cdot 2^{\sum_{j=1}^k\frac{j+1}{2}\left(\frac{2}{\beta_1}\right)^j}\|v\|_{L^{2^*}\left(B_{\frac{\delta}{2}}\right)}.$$

Since $\beta_1>2$, the series $\sum_{j=1}^{\infty}\left(\frac{2}{\beta_1}\right)^j$ and $\sum_{j=1}^{\infty}j\left(\frac{2}{\beta_1}\right)^j$ are convergent. Letting $k\to\infty$, we conclude that

$$\|v\|_{L^\infty\left(B_{\frac{\delta}{4}}\right)}\leq C\|v\|_{L^{2^*}\left(B_{\frac{\delta}{2}}\right)}\leq C\left(\int_{B_{\frac{\delta}{2}}}(|\nabla v|^2+v^2)\,dx\right)^{\frac12}\leq M.$$

Set $\rho=|x|$. Since v is positive radially symmetric, one has

$$-\frac{1}{\rho^{N-1}}\frac{d}{d\rho}\left(\rho^{N-1}\frac{dv}{d\rho}\right)=(1-Q(\rho))\left(\frac{v}{m_\mu(v)}\right)^{\rho}v^{q(\rho)-1}+Q(\rho)v^{q(\rho)-1}\geq 0,$$

which implies that $\frac{d}{d\rho}\left(\rho^{N-1}\frac{dv}{d\rho}\right)\leq 0$. Notice that $\rho^{N-1}\frac{dv}{d\rho}|_{\rho=0}=0$, we have $\rho^{N-1}\frac{dv}{d\rho}\leq 0$. That is $\frac{dv}{d\rho}\leq 0$. Hence,

$$\|v\|_{L^\infty(B_1)}\leq\|v\|_{L^\infty\left(B_{\frac{\delta}{4}}\right)}\leq M.$$

Proof of Theorem 1.1. By definition of the function m_μ, we have $m_\mu(t) = t$ for $t \leq \frac{1}{\mu}$. It is easy to see problem (2.2) reduce to problem (1.2) for $|u| \leq \frac{1}{\mu}$. Let $\mu > \frac{1}{M}$. We see that a positive solution u_μ problem (2.2) is indeed a positive solution of problem (1.2).

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Conflict of interest

The authors declared that there was no competition of interests.

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