Electronic Research Archive

http://www.aimspress.com/journal/era

ERA, 31(3): 1466–1484.
DOI: 10.3934/era.2023074
Received: 18 November 2022
Revised: 20 December 2022
Accepted: 03 January 2023
Published: 11 January 2023

Research article

Structural stability for Forchheimer fluid in a semi-infinite pipe

Zhiqing Li, Wenbin Zhang and Yuanfei Li*

School of Data Science, Guangzhou Huashang College, Guangdong 511300, China

* **Correspondence:** Email: liqfd@163.com.

Abstract: In this paper, it is assumed that the Forchheimer flow goes through a semi-infinite cylinder. The nonlinear boundary condition is satisfied on the finite end of the cylinder, and the homogeneous boundary condition is satisfied on the side of the cylinder. Using the method of energy estimate, the structural stability of the solution in the semi-infinite cylinder is obtained.

Keywords: Forchheimer flow; structural stability; energy estimate

1. Introduction

The Forchheimer model equation describes the flow in a polar medium, which is widely used in fluid mechanics (see [1, 2]). Increasing scholars have studied the spatial properties of the solution to the fluid equation defined on a semi-infinite cylinder, and a large number of results have emerged (see [3–9]).

In 2002, Payne and Song [3] have studied the following Forchheimer model

$$b|\boldsymbol{u}|u_i + (1 + \gamma T)u_i = -p_{,i} + g_iT, \ in\ \Omega \times \{t > 0\}, \tag{1.1}$$

$$u_{i,i} = 0, \ in\ \Omega \times \{t > 0\}, \tag{1.2}$$

$$\partial_t T + u_i T_{,i} = \Delta T, \ in\ \Omega \times \{t > 0\}, \tag{1.3}$$

where $i = 1, 2, 3$. u_i, p, T represent the velocity, pressure, and temperature of the flow, respectively. g_i is a known function. Δ is the Laplace operator, $\gamma > 0$ is a constant, and b is the Forchheimer coefficient. For simplicity, we assume that

$$g_i g_i \leq 1.$$

In (1.1)–(1.3), Ω is defined as

$$\Omega = \Big\{(x_1, x_2, x_3)\Big|(x_1, x_2) \in D,\ x_3 \geq 0\Big\},$$

where D is a bounded simply-connected region on (x_1, x_2)-plane.

In this paper, the comma is used to indicate partial differentiation and the usual summation convection is employed, with repeated Latin subscripts summed from 1 to 3, e.g., $u_{i,j}u_{i,j} = \sum_{i,j=1}^{3}\left(\frac{\partial u_i}{\partial x_j}\right)^2$. We also use the summation convention summed from 1 to 2, e.g., $u_{\alpha,\beta}u_{\alpha,\beta} = \sum_{\alpha,\beta=1}^{2}\left(\frac{\partial u_\alpha}{\partial x_\beta}\right)^2$.

The Eqs (1.1)–(1.3) also satisfy the following initial-boundary conditions

$$u_i(x_1, x_2, x_3, t) = 0,\ T(x_1, x_2, x_3, t) = 0,\ on\ \partial D \times \{x_3 > 0\} \times \{t > 0\}, \tag{1.4}$$

$$u_i(x_1, x_2, 0, t) = f_i(x_1, x_2, t),\ on\ D \times \{t > 0\}, \tag{1.5}$$

$$T(x_1, x_2, 0, t) = H(x_1, x_2, t),\ on\ D \times \{t > 0\}, \tag{1.6}$$

$$T(x_1, x_2, x_3, 0) = 0,\ (x_1, x_2, x_3) \in \Omega, \tag{1.7}$$

$$|\boldsymbol{u}|, |T| = O(1),\ |u_3|, |\nabla T|, |p| = o(x_3^{-1}),\ as\ x_3 \to \infty. \tag{1.8}$$

where f_i and H are differentiable functions.

In this paper, we will study the structural stability of Eqs (1.1)–(1.8) on Ω by using the spatial decay results obtained in [3]. Since the concept of structural stability was proposed by Hirsch and Smale [10], the structural stability of various types of partial differential equations defined in a bounded domain has received sufficient attention(see [11–19]). Some perturbations are inevitable in the process of model establishment and simplification, so it is necessary to study that whether such small perturbations of the equations themselves will cause great changes in the solutions. This gives rise to the phenomenon of structural stability.

If the bounded domain is replaced by a semi-infinite pipe, the structural stability of the partial differential equations is very interesting and has begun to attract attention. Li and Lin [20] considered the continuous dependence on the Forchheimer coefficient of Forchheimer equations in a semi-infinite pipe. Different from the studies of [11–19], we should consider not only the time variable but also the space variable. Therefore, the methods in the literature cannot be directly applied to the semi-infinite region. Compared with [3], we not only reconfirmed the spatial decay result of [3], but also proved the structural stability of the solution to b and γ.

We also introduce the notations:

$$\Omega_z = \Big\{(x_1, x_2, x_3)|(x_1, x_2) \in D, x_3 \geq z \geq 0\Big\},$$

$$D_z = \Big\{(x_1, x_2, x_3)|(x_1, x_2) \in D, x_3 = z \geq 0\Big\},$$

where z is a running variable along the x_3 axis.

2. A prior bounds

First, to obtain the main result, we shall make frequent use of the following three inequalities.

Lemma 2.1.(see [21]) If ϕ is a Dirichlet integrable function on Ω and $\int_\Omega \phi dx = 0$, then there exists a Dirichlet integrable function $\boldsymbol{w} = (w_1, w_2, w_3)$ such that

$$w_{i,i} = \phi,\ in\ \Omega,\ w_i = 0,\ on\ \partial\Omega,$$

and a positive constant k_1 depends only on the geometry of Ω such that

$$\int_\Omega w_{i,j}w_{i,j}dx \le k_1 \int_\Omega (w_{i,i})^2 dx.$$

Lemma 2.2.(see [3,4]) If $\phi\Big|_{\partial D} = 0$, then

$$\lambda \int_D \phi^2 dA \le \int_D \phi_{,\alpha}\phi_{,\alpha} dA,$$

where λ is the smallest positive eigenvalue of

$$\Delta_2\vartheta + \lambda\vartheta = 0, \ in\ D, \ \vartheta = 0, \ on\ \partial D.$$

Here Δ_2 is a two-dimensional Laplace operator.

Now, we give a lemma which has been proved by Horgan and Wheeler [4] and has been used by Payne and Song [6].

Lemma 2.3.(see [3,4]) If ϕ is a Dirichlet integrable function and $\phi\Big|_{\partial D} = 0, \phi \to \infty$ (as $x_3 \to \infty$),

$$\int_{\Omega_z} |\phi|^4 dx \le k_2 \Big(\int_{\Omega_z} \phi_{,j}\phi_{,j} dx \Big)^2,$$

where $k_2 > 0$.

Lemma 2.4. If $\phi \in C_0^1(\Omega)$, then

$$\int_{\Omega_z} |\phi|^6 dx \le \Lambda \Big(\int_{\Omega_z} \phi_{,i}\phi_{,i} dx \Big)^3,$$

where [22,23] have proved that the optimal value of Λ is determined to be $\Lambda = \frac{1}{27}\left(\frac{3}{4}\right)^4$.

Using the maximum principle for the temperature T, we can have the following lemma which has been used in Song [5].

Lemma 2.5. Assume that $H \in L^\infty(\Omega)$, then

$$\sup_{\Omega\times\{t>0\}} |T| \le T_M,$$

where $T_M = \sup_{\Omega\times\{t>0\}} H$.

Second, we list some useful results which have been derived by Payne and Song [3].

Payne and Song have established a function

$$P(z,t) = \int_0^t \int_{\Omega_z} (\xi - z) T_{,i} T_{,i} dx d\eta + a_1 \int_0^t \int_{\Omega_z} |\boldsymbol{u}|^3 dx d\eta + a_2 \int_0^t \int_{\Omega_z} (1 + \gamma T)|\boldsymbol{u}|^2 dx d\eta, \tag{2.1}$$

where a_1 and a_2 are positive constants. From Eqs (3.27) and (3.36) of [3], we know that

$$P(z,t) \le P(0,t)e^{-\frac{z}{k_3}}, \ P(0,t) \le k_4(t), \tag{2.2}$$

where k_3 is a positive constant and $k_4(t)$ is a function related to the boundary values.

Combining Eqs (2.1) and (2.2), we have the following lemma.

Lemma 2.6. Assume that $H \in L^\infty(\Omega)$ and $\int_D f dA = 0$, then

$$a_1 \int_0^t \int_{\Omega_z} |\boldsymbol{u}|^3 dx d\eta + a_2 \int_0^t \int_{\Omega_z} (1+\gamma T)|\boldsymbol{u}|^2 dx d\eta \le k_4(t) e^{-\frac{z}{k_3}}.$$

In order to derive the main result, we need bounds for $\|\boldsymbol{u}\|^2_{L^2(\Omega)}$ and $\|\boldsymbol{u}\|^3_{L^2(\Omega)}$.

Lemma 2.7. Assume that $f_i \in H^1(\Omega), H, \widetilde{H} \in L^\infty(\Omega)$, $\int_D f_3 dA = 0$ and $f_{\alpha,\alpha} - \gamma f_3 = 0$ then

$$b\int_\Omega |\boldsymbol{u}|^3 dx + \int_\Omega |\boldsymbol{u}|^2 dx \le k_5(t),$$

where $k_6(t)$ is a positive function.

Proof. To deal with boundary terms, we set $\boldsymbol{S} = (S_1, S_2, S_3)$, where

$$S_i = f_i e^{-\gamma_1 x_3},\ \gamma_1 > 0. \tag{2.3}$$

Using Eq (1.1), we have

$$\int_\Omega \Big[b|\boldsymbol{u}|u_i + (1+\gamma T)u_i + p_{,i} - g_i T \Big](u_i - S_i) dx = 0.$$

Using the divergence theorem, we have

$$\begin{aligned} b\int_\Omega |\boldsymbol{u}|^3 dx + \int_\Omega (1+\gamma T)|\boldsymbol{u}|^2 dx = b\int_\Omega |\boldsymbol{u}|u_i S_i dx + \int_\Omega (1+\gamma T) u_i S_i dx \\ - \int_\Omega g_i T u_i dx + \int_\Omega g_i T S_i dx. \end{aligned} \tag{2.4}$$

Using the Hölder inequality and Young's inequality, we have

$$\begin{aligned} b\int_\Omega |\boldsymbol{u}|u_i S_i dx &\le b\Big(\int_\Omega |\boldsymbol{u}|^3 dx\Big)^{\frac{2}{3}} \Big(\int_\Omega |\boldsymbol{S}|^3 dx\Big)^{\frac{1}{3}} \\ &\le \frac{2}{3} b\varepsilon_1 \int_\Omega |\boldsymbol{u}|^3 dx + \frac{1}{3} b\varepsilon_1^{-2} \int_\Omega |\boldsymbol{S}|^3 dx, \end{aligned} \tag{2.5}$$

$$\int_\Omega (1+\gamma T)u_i S_i dx \le \frac{1}{4}\int_\Omega (1+\gamma T)|\boldsymbol{u}|^2 dx + (1+\gamma T_M)\int_\Omega |\boldsymbol{S}|^2 dx, \tag{2.6}$$

$$\begin{aligned} -\int_\Omega g_i T u_i dx &\le \sqrt{T_M}\Big(\int_\Omega (1+\gamma T)|\boldsymbol{u}|^2 dx \int_\Omega g_i g_i dx\Big)^{\frac{1}{2}} \\ &\le \frac{1}{4}\int_\Omega (1+\gamma T)|\boldsymbol{u}|^2 dx + \frac{T_M}{\gamma}\int_\Omega g_i g_i dx, \end{aligned} \tag{2.7}$$

$$\int_\Omega g_i T S_i dx \le T_M \int_\Omega |g_i S_i| dx. \tag{2.8}$$

Inserting Eqs (2.5)–(2.8) into Eq (2.4) and choosing that $\varepsilon_1 = \frac{3}{4}$, we obtain

$$b\int_\Omega |\boldsymbol{u}|^3 dx + \int_\Omega (1+\gamma T)|\boldsymbol{u}|^2 dx \le \frac{2}{3} b\varepsilon_1^{-2} \int_\Omega |\boldsymbol{S}|^3 dx + 2(1+\gamma T_M)\int_\Omega |\boldsymbol{S}|^2 dx$$

$$+\frac{2T_M}{\gamma}\int_\Omega g_ig_idx+2T_M\int_\Omega |g_iS_i|dx. \tag{2.9}$$

After choosing

$$\begin{aligned}k_5(t)=&\frac{2}{3}b\varepsilon_1^{-2}\int_\Omega |\boldsymbol{S}|^3dx+2(1+\gamma T_M)\int_\Omega|\boldsymbol{S}|^2dx\\&+\frac{2T_M}{\gamma}\int_\Omega g_ig_idx+2T_M\int_\Omega|g_iS_i|dx,\end{aligned} \tag{2.10}$$

we can complete the proof of Lemma 2.7.

3. Important lemma

In this section, we derive an important lemma which leads to our main result.

Assume that (u_i^*, T^*, p^*) is a solution of Eqs (1.1)–(1.8) when $b=b^*$. If we let

$$\mathcal{D}_i=u_i-u_i^*,\ \Sigma=T-T^*,\ \pi=p-p^*,\ \widetilde{b}=b-b^*,$$

then $(\mathcal{D}_i,\Sigma,\pi)$ satisfies

$$[b_1|\boldsymbol{u}|u_i-b_2|\boldsymbol{u}^*|u_i^*]+(1+\gamma T)\mathcal{D}_i+\gamma\Sigma u_i^*=-\pi_{,i}+g_i\Sigma,\ in\ \Omega\times\{t>0\}, \tag{3.1}$$

$$\mathcal{D}_{i,i}=0,\ in\ \Omega\times\{t>0\}, \tag{3.2}$$

$$\partial_t\Sigma+u_i\Sigma_{,i}+\mathcal{D}_iT_{,i}^*=\Delta\Sigma,\ in\ \Omega\times\{t>0\}, \tag{3.3}$$

$$\mathcal{D}_i=0,\Sigma=0,\ on\ \partial D\times\{x_3>0\}\times\{t>0\}, \tag{3.4}$$

$$\mathcal{D}_i=0,\Sigma=0,\ on\ D\times\{t>0\}, \tag{3.5}$$

$$\Sigma(x_1,x_2,x_3,0)=0,\ in\ \Omega \tag{3.6}$$

$$|\boldsymbol{u}|,|\Sigma|=O(1),|\mathcal{D}_3|,|\nabla\Sigma|,|\pi|=o(x_3^{-1}),\ as\ x_3\to\infty. \tag{3.7}$$

We can have the following lemma.

Lemma 3.1. Assume that $(\mathcal{D}_i,\Sigma,\pi)$ is a solution to Eqs (3.1)–(3.6) with $\int_D f_3dA=0, H\in L^\infty(\Omega)$ and the boundary data (e.g., H) satisfies Eq (3.21), then

$$\Phi(z,t)\le n_6^*\Big[-\frac{\partial}{\partial z}\Phi(z,t)\Big]+n_7(t)\widetilde{b}^2e^{-\frac{z}{k_3}},$$

where n_6^* is the maximum of $n_6(t)$ and $n_6(t), n_7(t)$ will be defined in Eq (3.39).

Proof. We define an auxiliary function

$$\Phi_1(z,t)=\int_0^t\int_{\Omega_z}e^{-\omega\eta}\pi\mathcal{D}_3dxd\eta, \tag{3.8}$$

where $\omega>0$.

Using the divergence theorem and Eq (3.1), we have

$$\Phi_1(z,t)=-\int_0^t\int_{\Omega_z}e^{-\omega\eta}(\xi-z)\pi_{,i}\mathcal{D}_idxd\eta$$

$$= \int_0^t \int_{\Omega_z} e^{-\omega\eta}(\xi - z)\mathcal{D}_i[b_1|\boldsymbol{u}|u_i - b_2|\boldsymbol{u}^*|u_i^*]dxd\eta + \int_0^t \int_{\Omega_z} e^{-\omega\eta}(\xi - z)(1 + \gamma T)\mathcal{D}_i\mathcal{D}_i dxd\eta$$
$$+ \gamma \int_0^t \int_{\Omega_z} e^{-\omega\eta}(\xi - z)\mathcal{D}_i\Sigma u_i^* dxd\eta - \int_0^t \int_{\Omega_z} e^{-\omega\eta}(\xi - z)\mathcal{D}_i g_i \Sigma dxd\eta. \tag{3.9}$$

Since

$$\begin{aligned}\mathcal{D}_i\Big[b_1|\boldsymbol{u}|u_i - b_2|\boldsymbol{u}^*|u_i^*\Big] &= \frac{\widetilde{b}}{2}\mathcal{D}_i\Big[|\boldsymbol{u}|u_i + |\boldsymbol{u}^*|u_i^*\Big] + \frac{b_1 + b_2}{2}\mathcal{D}_i\Big[|\boldsymbol{u}|u_i - |\boldsymbol{u}^*|u_i^*\Big] \\ &= \frac{\widetilde{b}}{2}\Big[|\boldsymbol{u}|u_i + |\boldsymbol{u}^*|u_i^*\Big]\mathcal{D}_i + \frac{b_1 + b_2}{4}\Big[|\boldsymbol{u}| + |\boldsymbol{u}^*|\Big]\mathcal{D}_i\mathcal{D}_i \\ &+ \frac{b_1 + b_2}{4}\Big[|\boldsymbol{u}| - |\boldsymbol{u}^*|\Big]^2\Big[|\boldsymbol{u}| + |\boldsymbol{u}^*|\Big],\end{aligned}$$

from Eq (3.9) we have

$$\begin{aligned}\Phi_1(z,t) = &\frac{b_1 + b_2}{4}\int_0^t \int_{\Omega_z} e^{-\omega\eta}(\xi - z)\Big[|\boldsymbol{u}| + |\boldsymbol{u}^*|\Big]\mathcal{D}_i\mathcal{D}_i dxd\eta + \int_0^t \int_{\Omega_z} e^{-\omega\eta}(\xi - z)(1 + \gamma T)\mathcal{D}_i\mathcal{D}_i dxd\eta \\ &+ \frac{\widetilde{b}}{2}\int_0^t \int_{\Omega_z} e^{-\omega\eta}(\xi - z)\Big[|\boldsymbol{u}|u_i + |\boldsymbol{u}^*|u_i^*\Big]\mathcal{D}_i dxd\eta \\ &+ \frac{b_1 + b_2}{4}\int_0^t \int_{\Omega_z} e^{-\omega\eta}(\xi - z)\Big[|\boldsymbol{u}| - |\boldsymbol{u}^*|\Big]^2\Big[|\boldsymbol{u}| + |\boldsymbol{u}^*|\Big]dxd\eta \\ &+ \gamma \int_0^t \int_{\Omega_z} e^{-\omega\eta}(\xi - z)\mathcal{D}_i\Sigma u_i^* dxd\eta - \int_0^t \int_{\Omega_z} e^{-\omega\eta}(\xi - z)\mathcal{D}_i g_i\Sigma dxd\eta.\end{aligned} \tag{3.10}$$

Using the Hölder inequality, Young's inequality and Lemma 2.6, we obtain

$$\begin{aligned}&\frac{\widetilde{b}}{2}\int_0^t \int_{\Omega_z} e^{-\omega\eta}\Big[|\boldsymbol{u}|u_i + |\boldsymbol{u}^*|u_i^*\Big]\mathcal{D}_i dxd\eta \\ &\geq -\frac{\widetilde{b}}{2}\Big(\int_0^t \int_{\Omega_z} e^{-\omega\eta}|\boldsymbol{u}|\mathcal{D}_i\mathcal{D}_i dxd\eta\Big)^{\frac{1}{2}}\Big(\int_0^t \int_{\Omega_z} e^{-\omega\eta}|\boldsymbol{u}|^3 dxd\eta\Big)^{\frac{1}{2}} \\ &+ -\frac{\widetilde{b}}{2}\Big(\int_0^t \int_{\Omega_z} e^{-\omega\eta}|\boldsymbol{u}^*|\mathcal{D}_i\mathcal{D}_i dxd\eta\Big)^{\frac{1}{2}}\Big(\int_0^t \int_{\Omega_z} e^{-\omega\eta}|\boldsymbol{u}|^3 dxd\eta\Big)^{\frac{1}{2}} \\ &\geq -\frac{b_1 + b_2}{16}\int_0^t \int_{\Omega_z} e^{-\omega\eta}\Big[|\boldsymbol{u}| + |\boldsymbol{u}^*|\Big]\mathcal{D}_i\mathcal{D}_i dxd\eta \\ &- \frac{4\widetilde{b}^2}{b_1 + b_2}\int_0^t \int_{\Omega_z} e^{-\omega\eta}\Big[|\boldsymbol{u}|^3 + |\boldsymbol{u}^*|^3\Big]dxd\eta \\ &\geq -\frac{b_1 + b_2}{16}\int_0^t \int_{\Omega_z} e^{-\omega\eta}\Big[|\boldsymbol{u}| + |\boldsymbol{u}^*|\Big]\mathcal{D}_i\mathcal{D}_i dxd\eta - \frac{8\widetilde{b}^2}{a_1(b_1 + b_2)}k_4(t)e^{-\frac{z}{k_3}}.\end{aligned} \tag{3.11}$$

Using the Hölder inequality, Young's inequality and Lemmas 2.3 and 2.7, we obtain

$$\gamma \int_0^t \int_{\Omega_z} e^{-\omega\eta}\mathcal{D}_i\Sigma u_i^* dxd\eta \geq -\gamma \int_0^t e^{-\omega\eta}\Big(\int_{\Omega_z} |\boldsymbol{u}^*|\mathcal{D}_i\mathcal{D}_i dx\Big)^{\frac{1}{2}}\Big(\int_{\Omega_z} |\boldsymbol{u}^*|^2 dx\Big)^{\frac{1}{4}}\Big(\int_{\Omega_z} \Sigma^4 dx\Big)^{\frac{1}{4}} d\eta$$

$$\geq -\gamma\sqrt[4]{k_5(t)k_2}\int_0^t e^{-\omega\eta}\Big(\int_{\Omega_z}|\boldsymbol{u}^*|\mathcal{D}_i\mathcal{D}_i dx\Big)^{\frac{1}{2}}\Big(\int_{\Omega_z}\Sigma_{,i}\Sigma_{,i}dx\Big)^{\frac{1}{2}}d\eta$$
$$\geq -\frac{b_1+b_2}{16}\int_0^t\int_{\Omega_z}e^{-\omega\eta}|\boldsymbol{u}^*|\mathcal{D}_i\mathcal{D}_i dxd\eta$$
$$-\frac{4\gamma^2\sqrt{k_5(t)k_2}}{b_1+b_2}\int_0^t\int_{\Omega_z}e^{-\omega\eta}\Sigma_{,i}\Sigma_{,i}dxd\eta, \tag{3.12}$$

$$-\int_0^t\int_{\Omega_z}e^{-\omega\eta}\mathcal{D}_i g_i\Sigma dxd\eta \geq -\frac{1}{2}\int_0^t\int_{\Omega_z}e^{-\omega\eta}(1+\gamma T)\mathcal{D}_i\mathcal{D}_i dxd\eta - \frac{1}{2\gamma}\int_0^t\int_{\Omega_z}e^{-\omega\eta}\Sigma^2 dxd\eta. \tag{3.13}$$

Calculating the differential of Eq (3.10) and then inserting Eqs (3.11)–(3.13) into Eq (3.10), we have

$$-\frac{\partial}{\partial z}\Phi_1(z,t) \geq \frac{b_1+b_2}{8}\int_0^t\int_{\Omega_z}e^{-\omega\eta}\Big[|\boldsymbol{u}|+|\boldsymbol{u}^*|\Big]\mathcal{D}_i\mathcal{D}_i dxd\eta + \frac{1}{2}\int_0^t\int_{\Omega_z}e^{-\omega\eta}(1+\gamma T)\mathcal{D}_i\mathcal{D}_i dxd\eta$$
$$-\frac{4\gamma^2\sqrt{k_5(t)k_2}}{b_1+b_2}\int_0^t\int_{\Omega_z}e^{-\omega\eta}\Sigma_{,i}\Sigma_{,i}dxd\eta - \frac{1}{2\gamma}\int_0^t\int_{\Omega_z}e^{-\omega\eta}\Sigma^2 dxd\eta$$
$$-\frac{8\widetilde{b}^2}{a_1(b_1+b_2)}k_4(t)e^{-\frac{z}{k_3}}, \tag{3.14}$$

where we have dropped the fourth term of Eq (3.10).

Similarly, we have

$$\Phi_2(z,t) = -\int_0^t\int_{\Omega_z}e^{-\omega\eta}\Sigma\Sigma_{,3}dxd\eta + \frac{1}{2}\int_0^t\int_{\Omega_z}e^{-\omega\eta}u_3\Sigma^2 dxd\eta + \int_0^t\int_{\Omega_z}e^{-\omega\eta}\mathcal{D}_3 T^*\Sigma dxd\eta$$
$$\doteq \Phi_{21}(z,t)+\Phi_{22}(z,t)+\Phi_{23}(z,t). \tag{3.15}$$

Using the divergence theorem and Eq (3.2), we have

$$\Phi_2(z,t) = \frac{1}{2}e^{-\omega t}\int_{\Omega_z}(\xi-z)\Sigma^2 dx + \int_0^t\int_{\Omega_z}e^{-\omega\eta}(\xi-z)\Big[\frac{1}{2}\omega\Sigma^2+\Sigma_{,i}\Sigma_{,i}\Big]dxd\eta$$
$$-\int_0^t\int_{\Omega_z}e^{-\omega\eta}(\xi-z)\mathcal{D}_i\Sigma_{,i}T^* dxd\eta. \tag{3.16}$$

Using the Hölder inequality and Lemma 2.5, we have

$$-\int_0^t\int_{\Omega_z}e^{-\omega\eta}\mathcal{D}_i\Sigma_{,i}T^* dxd\eta \geq -\frac{1}{2}\int_0^t\int_{\Omega_z}e^{-\omega\eta}\Sigma_{,i}\Sigma_{,i}dxd\eta - \frac{1}{2}T_M^2\int_0^t\int_{\Omega_z}e^{-\omega\eta}\mathcal{D}_i\mathcal{D}_i dxd\eta. \tag{3.17}$$

Calculating the differential of Eq (3.16) and then inserting Eq (3.17) into Eq (3.16), we have

$$-\frac{\partial}{\partial z}\Phi_2(z,t) = \frac{1}{2}e^{-\omega t}\int_{\Omega_z}\Sigma^2 dx + \int_0^t\int_{\Omega_z}e^{-\omega\eta}\Big[\frac{1}{2}\omega\Sigma^2+\frac{1}{2}\Sigma_{,i}\Sigma_{,i}\Big]dxd\eta$$
$$-\frac{1}{2\gamma}T_M^2\int_0^t\int_{\Omega_z}e^{-\omega\eta}(1+\gamma T)\mathcal{D}_i\mathcal{D}_i dxd\eta. \tag{3.18}$$

Now, we define

$$-\frac{\partial}{\partial z}\Phi(z,t)=\frac{2}{\gamma}T_M^2\Big[-\frac{\partial}{\partial z}\Phi_1(z,t)\Big]+\Big[-\frac{\partial}{\partial z}\Phi_2(z,t)\Big]. \tag{3.19}$$

Combining Eqs (3.14) and (3.18), we have

$$\begin{aligned}-\frac{\partial}{\partial z}\Phi(z,t)\geq{}&\frac{b_1+b_2}{4\gamma}T_M^2\int_0^t\int_{\Omega_z}e^{-\omega\eta}\Big[|\boldsymbol{u}|+|\boldsymbol{u}^*|\Big]\mathcal{D}_i\mathcal{D}_i dxd\eta+\frac{1}{2\gamma}T_M^2\int_0^t\int_{\Omega_z}e^{-\omega\eta}(1+\gamma T)\mathcal{D}_i\mathcal{D}_i dxd\eta\\&+\frac{1}{2}e^{-\omega t}\int_{\Omega_z}\Sigma^2dx+\int_0^t\int_{\Omega_z}e^{-\omega\eta}\Big[\frac{1}{2}\omega\Sigma^2+\frac{1}{2}\Sigma_{,i}\Sigma_{,i}\Big]dxd\eta\\&-\frac{8\gamma\sqrt{k_5(t)k_2}}{b_1+b_2}\int_0^t\int_{\Omega_z}e^{-\omega\eta}\Sigma_{,i}\Sigma_{,i}dxd\eta-\frac{1}{\gamma^2}T_M^2\int_0^t\int_{\Omega_z}e^{-\omega\eta}\Sigma^2dxd\eta\\&-\frac{16\widetilde{b}^2}{a_1\gamma(b_1+b_2)}T_M^2k_4(t)e^{-\frac{z}{k_3}}.\end{aligned} \tag{3.20}$$

Choosing $\omega>\frac{4}{\gamma^2}T_M^2$ and the boundary data (e.g., H) satisfies

$$\frac{8\gamma\sqrt{k_5(t)k_2}}{b_1+b_2}<\frac{1}{4}, \tag{3.21}$$

from Eq (3.20) we obtain

$$\begin{aligned}-\frac{\partial}{\partial z}\Phi(z,t)\geq{}&\frac{b_1+b_2}{4\gamma}T_M^2\int_0^t\int_{\Omega_z}e^{-\omega\eta}\Big[|\boldsymbol{u}|+|\boldsymbol{u}^*|\Big]\mathcal{D}_i\mathcal{D}_i dxd\eta+\frac{1}{2\gamma}T_M^2\int_0^t\int_{\Omega_z}e^{-\omega\eta}(1+\gamma T)\mathcal{D}_i\mathcal{D}_i dxd\eta\\&+\frac{1}{2}e^{-\omega t}\int_{\Omega_z}\Sigma^2dx+\int_0^t\int_{\Omega_z}e^{-\omega\eta}\Big[\frac{1}{4}\omega\Sigma^2+\frac{1}{4}\Sigma_{,i}\Sigma_{,i}\Big]dxd\eta\\&-\frac{16\widetilde{b}^2}{a_1\gamma(b_1+b_2)}T_M^2k_4(t)e^{-\frac{z}{k_3}}.\end{aligned} \tag{3.22}$$

Integrating Eq (3.22) from z to ∞, we obtain

$$\begin{aligned}\Phi(z,t)\geq{}&\frac{b_1+b_2}{4\gamma}T_M^2\int_0^t\int_{\Omega_z}e^{-\omega\eta}(\xi-z)\Big[|\boldsymbol{u}|+|\boldsymbol{u}^*|\Big]\mathcal{D}_i\mathcal{D}_i dxd\eta\\&+\frac{1}{2\gamma}T_M^2\int_0^t\int_{\Omega_z}e^{-\omega\eta}(\xi-z)(1+\gamma T)\mathcal{D}_i\mathcal{D}_i dxd\eta\\&+\frac{1}{2}e^{-\omega t}\int_{\Omega_z}(\xi-z)\Sigma^2dx+\int_0^t\int_{\Omega_z}e^{-\omega\eta}(\xi-z)\Big[\frac{1}{4}\omega\Sigma^2+\frac{1}{4}\Sigma_{,i}\Sigma_{,i}\Big]dxd\eta\\&-\frac{16\widetilde{b}^2}{a_1\gamma(b_1+b_2)}k_3T_M^2k_4(t)e^{-\frac{z}{k_3}}.\end{aligned} \tag{3.23}$$

We note that

$$\int_{D_z}\mathcal{D}_3dA=\int_D\mathcal{D}_3dA+\int_0^z\int_{D_\xi}\frac{\partial\mathcal{D}_3}{\partial x_3}dAd\xi=-\int_0^z\int_{D_\xi}\mathcal{D}_{\alpha,\alpha}dAd\xi=0.$$

According to Lemma 2.1, there exists a vector function $\boldsymbol{w} = (w_1, w_2, w_3)$ such that

$$w_{i,i} = \mathcal{D}_3, \ in\ \Omega; \quad w_i = 0, \ on\ \partial\Omega.$$

Therefore, using Eq (3.1) we obtain

$$\begin{aligned}\frac{2}{\gamma}T_M^2\Phi_1(z,t) &= \frac{2}{\gamma}T_M^2\int_0^t\int_{\Omega_z}e^{-\omega\eta}\pi w_{i,i}dxd\eta = -\frac{2}{\gamma}T_M^2\int_0^t\int_{\Omega_z}e^{-\omega\eta}\pi_{,i}w_idxd\eta \\ &= \frac{2}{\gamma}T_M^2\int_0^t\int_{\Omega_z}e^{-\omega\eta}\Big\{[b_1|\boldsymbol{u}|u_i - b_2|\boldsymbol{u}^*|u_i^*] + (1+\gamma T)\mathcal{D}_i + \gamma\Sigma u_i^* - g_i\Sigma\Big\}w_idxd\eta. \qquad (3.24)\end{aligned}$$

Since

$$\begin{aligned}\Big[b_1|\boldsymbol{u}|u_i - b_2|\boldsymbol{u}^*|u_i^*\Big]w_i &= \frac{\widetilde{b}}{2}\Big[|\boldsymbol{u}|u_i + |\boldsymbol{u}^*|u_i^*\Big]w_i + \frac{b_1+b_2}{2}\Big[|\boldsymbol{u}| + |\boldsymbol{u}^*|\Big]\mathcal{D}_iw_i \\ &\quad + \frac{b_1+b_2}{2}\Big[|\boldsymbol{u}| - |\boldsymbol{u}^*|\Big](u_i+u_i^*)w_i \\ &= \frac{\widetilde{b}}{2}\Big[|\boldsymbol{u}|u_i + |\boldsymbol{u}^*|u_i^*\Big]w_i + \frac{b_1+b_2}{2}\Big[|\boldsymbol{u}| + |\boldsymbol{u}^*|\Big]\mathcal{D}_iw_i \\ &\quad + \frac{b_1+b_2}{2}\frac{(u_j-u_j^*)(u_j+u_j^*)}{|\boldsymbol{u}|+|\boldsymbol{u}^*|}(u_i+u_i^*)w_i \\ &\leq \frac{\widetilde{b}}{2}\Big[|\boldsymbol{u}|u_i + |\boldsymbol{u}^*|u_i^*\Big]w_i + \frac{b_1+b_2}{2}\Big[|\boldsymbol{u}| + |\boldsymbol{u}^*|\Big]\mathcal{D}_iw_i \\ &\quad + \frac{b_1+b_2}{2}\Big[|\boldsymbol{u}| + |\boldsymbol{u}^*|\Big]|\boldsymbol{D}||\boldsymbol{w}|,\end{aligned}$$

we have

$$\begin{aligned}\int_0^t\int_{\Omega_z}e^{-\omega\eta}\Big[b_1|\boldsymbol{u}|u_i - b_2|\boldsymbol{u}^*|u_i^*\Big]w_idxd\eta &\leq \frac{\widetilde{b}}{2}\int_0^t\int_{\Omega_z}e^{-\omega\eta}\Big[|\boldsymbol{u}|u_i + |\boldsymbol{u}^*|u_i^*\Big]w_idxd\eta \\ &\quad + \frac{b_1+b_2}{2}\int_0^t\int_{\Omega_z}e^{-\omega\eta}\Big[|\boldsymbol{u}| + |\boldsymbol{u}^*|\Big]\mathcal{D}_iw_idxd\eta \\ &\quad + \frac{b_1+b_2}{2}\int_0^t\int_{\Omega_z}e^{-\omega\eta}\Big[|\boldsymbol{u}| + |\boldsymbol{u}^*|\Big]|\boldsymbol{D}||\boldsymbol{w}|dxd\eta. \qquad (3.25)\end{aligned}$$

Using the Hölder inequality, Lemmas 2.2, 2.3, 2.1, 2.7 and 2.6, and Young's inequality, we obtain

$$\begin{aligned}&\frac{\widetilde{b}}{2}\int_0^t\int_{\Omega_z}e^{-\omega\eta}\Big[|\boldsymbol{u}|u_i + |\boldsymbol{u}^*|u_i^*\Big]w_idxd\eta \\ &\leq \frac{\widetilde{b}}{2}\int_0^t e^{-\omega\eta}\Big[\int_{\Omega_z}\Big[|\boldsymbol{u}|^3 + |\boldsymbol{u}^*|^3\Big]dx\Big]^{\frac{2}{3}}\Big[\int_{\Omega_z}\Big(w_iw_i\Big)^{\frac{3}{2}}dx\Big]^{\frac{1}{3}}d\eta \\ &\leq \frac{\widetilde{b}}{2}\int_0^t e^{-\omega\eta}\Big[\int_{\Omega_z}\Big[|\boldsymbol{u}|^3 + |\boldsymbol{u}^*|^3\Big]dx\Big]^{\frac{2}{3}}\Big[\int_{\Omega_z}w_iw_idx\Big]^{\frac{1}{6}} \\ &\quad\cdot\Big[\int_{\Omega_z}\Big(w_iw_i\Big)^2dx\Big]^{\frac{1}{6}}d\eta\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\widetilde{b}\sqrt[6]{k_2}}{2\sqrt[6]{\lambda}}\int_0^t e^{-\omega\eta}\Big[\int_{\Omega_z}\Big[|\boldsymbol{u}|^3+|\boldsymbol{u}^*|^3\Big]dx\Big]^{\frac{2}{3}}\Big[\int_{\Omega_z} w_{i,\alpha}w_{i,\alpha}dx\Big]^{\frac{1}{6}}\\
&\cdot\Big[\int_{\Omega_z} w_{i,j}w_{i,j}dx\Big]^{\frac{1}{3}}d\eta\\
&\leq \frac{\widetilde{b}\sqrt[6]{k_2}}{2\sqrt[6]{\lambda}}\int_0^t e^{-\omega\eta}\Big[\int_{\Omega_z}\Big[|\boldsymbol{u}|^3+|\boldsymbol{u}^*|^3\Big]dx\Big]^{\frac{2}{3}}\Big[\int_{\Omega_z} w_{i,j}w_{i,j}dx\Big]^{\frac{1}{2}}d\eta\\
&\leq \frac{\widetilde{b}\sqrt[6]{k_2}\sqrt{k_1}}{2\sqrt[6]{\lambda}}\int_0^t e^{-\omega\eta}\Big[\int_{\Omega}\Big[|\boldsymbol{u}|^3+|\boldsymbol{u}^*|^3\Big]dx\Big]^{\frac{1}{6}}\\
&\cdot\Big[\int_{\Omega_z}\Big[|\boldsymbol{u}|^3+|\boldsymbol{u}^*|^3\Big]dx\Big]^{\frac{1}{2}}\Big[\int_{\Omega_z}\mathcal{D}_3^2dx\Big]^{\frac{1}{2}}d\eta\\
&\leq \frac{\widetilde{b}^2\sqrt[3]{2k_5(t)k_2}k_1}{4\sqrt[3]{b\lambda}}\int_0^t\int_{\Omega_z} e^{-\omega\eta}\Big[|\boldsymbol{u}|^3+|\boldsymbol{u}^*|^3\Big]dxd\eta+\frac{1}{2}\int_0^t\int_{\Omega_z} e^{-\omega\eta}\mathcal{D}_3^2dxd\eta\\
&\leq \frac{\widetilde{b}^2\sqrt[3]{2k_5(t)k_2}k_1k_4(t)}{2a\sqrt[3]{b\lambda}}e^{-\frac{z}{k_3}}+\frac{1}{2}\int_0^t\int_{\Omega_z} e^{-\omega\eta}(1+\gamma T)\mathcal{D}_3^2dxd\eta. \qquad (3.26)
\end{aligned}
$$

Using the Hölder inequality, Lemmas 2.3, 2.1 and 2.7, and Young's inequality, we obtain

$$
\begin{aligned}
&\frac{b_1+b_2}{2}\int_0^t\int_{\Omega_z} e^{-\omega\eta}\Big[|\boldsymbol{u}|+|\boldsymbol{u}^*|\Big]\mathcal{D}_iw_idxd\eta\\
&\leq \frac{b_1+b_2}{2}\int_0^t e^{-\omega\eta}\Big[\int_{\Omega_z}|\boldsymbol{u}|\mathcal{D}_i\mathcal{D}_idx\Big]^{\frac{1}{2}}\Big[\int_{\Omega_z}|\boldsymbol{u}|^2dx\Big]^{\frac{1}{4}}\Big[\int_{\Omega_z}(w_iw_i)^2dx\Big]^{\frac{1}{4}}d\eta\\
&+\frac{b_1+b_2}{2}\int_0^t e^{-\omega\eta}\Big[\int_{\Omega_z}|\boldsymbol{u}^*|\mathcal{D}_i\mathcal{D}_idx\Big]^{\frac{1}{2}}\Big[\int_{\Omega_z}|\boldsymbol{u}^*|^2dx\Big]^{\frac{1}{4}}\Big[\int_{\Omega_z}(w_iw_i)^2dx\Big]^{\frac{1}{4}}d\eta\\
&\leq \frac{b_1+b_2}{2}\sqrt[4]{k_2k_5(t)}\int_0^t e^{-\omega\eta}\Big[\int_{\Omega_z}|\boldsymbol{u}|\mathcal{D}_i\mathcal{D}_idx\Big]^{\frac{1}{2}}\Big[\int_{\Omega_z}w_{i,j}w_{i,j}dx\Big]^{\frac{1}{2}}d\eta\\
&+\frac{b_1+b_2}{2}\sqrt[4]{k_2k_5(t)}\int_0^t e^{-\omega\eta}\Big[\int_{\Omega_z}|\boldsymbol{u}^*|\mathcal{D}_i\mathcal{D}_idx\Big]^{\frac{1}{2}}\Big[\int_{\Omega_z}w_{i,j}w_{i,j}dx\Big]^{\frac{1}{2}}d\eta\\
&\leq \frac{b_1+b_2}{4}\sqrt[4]{k_1k_2k_5(t)}\int_0^t\int_{\Omega_z} e^{-\omega\eta}\Big[|\boldsymbol{u}|+|\boldsymbol{u}^*|\Big]\mathcal{D}_i\mathcal{D}_idxd\eta\\
&+\frac{b_1+b_2}{2}\sqrt[4]{k_1k_2k_5(t)}\int_0^t\int_{\Omega_z} e^{-\omega\eta}(1+\gamma T)\mathcal{D}_3^2dxd\eta. \qquad (3.27)
\end{aligned}
$$

Using the Hölder inequality, Lemmas 2.4, 2.1 and 2.7, and Young's inequality, we obtain

$$
\begin{aligned}
&\frac{b_1+b_2}{2}\int_0^t\int_{\Omega_z} e^{-\omega\eta}\Big[|\boldsymbol{u}|+|\boldsymbol{u}^*|\Big]|\boldsymbol{D}||\boldsymbol{w}|dxd\eta\\
&\leq \frac{b_1+b_2}{2}\int_0^t e^{-\omega\eta}\Big[\int_{\Omega_z}|\mathcal{D}_i\mathcal{D}_idx\Big]^{\frac{1}{2}}\Big[\int_{\Omega_z}|\boldsymbol{u}|^3dx\Big]^{\frac{1}{3}}\Big[\int_{\Omega_z}(w_iw_i)^3dx\Big]^{\frac{1}{6}}d\eta\\
&+\frac{b_1+b_2}{2}\int_0^t e^{-\omega\eta}\Big[\int_{\Omega_z}|\mathcal{D}_i\mathcal{D}_idx\Big]^{\frac{1}{2}}\Big[\int_{\Omega_z}|\boldsymbol{u}^*|^3dx\Big]^{\frac{1}{3}}\Big[\int_{\Omega_z}(w_iw_i)^3dx\Big]^{\frac{1}{6}}d\eta
\end{aligned}
$$

$$\leq (b_1+b_2)\sqrt[3]{\frac{k_5(t)}{b}}\sqrt[6]{\Lambda}\int_0^t e^{-\omega\eta}\Big[\int_{\Omega_z}\mathcal{D}_i\mathcal{D}_i dx\Big]^{\frac{1}{2}}\Big[\int_{\Omega_z} w_{i,j}w_{i,j}dx\Big]^{\frac{1}{2}}d\eta$$
$$\leq \frac{b_1+b_2}{2}\sqrt[3]{\frac{k_1k_5(t)}{b}}\sqrt[6]{\Lambda}\int_0^t\int_{\Omega_z} e^{-\omega\eta}(1+\gamma T)\mathcal{D}_i\mathcal{D}_i dxd\eta. \tag{3.28}$$

Inserting Eqs (3.26)–(3.28) into Eq (3.25), we have

$$\int_0^t\int_{\Omega_z} e^{-\omega\eta}\Big[b_1|\boldsymbol{u}|u_i - b_2|\boldsymbol{u}^*|u_i^*\Big]w_i dxd\eta \leq \frac{\widetilde{b}^2\sqrt[3]{2k_5(t)k_2}k_1k_4(t)}{2a\sqrt[3]{b\lambda}}e^{-\frac{z}{k_3}}$$
$$+\frac{b_1+b_2}{4}\sqrt[4]{k_2k_5(t)}\varepsilon_3\int_0^t\int_{\Omega_z} e^{-\omega\eta}\Big[|\boldsymbol{u}|+|\boldsymbol{u}^*|\Big]\mathcal{D}_i\mathcal{D}_i dxd\eta$$
$$+\Big[\frac{b_1+b_2}{2}\sqrt[3]{\frac{k_1k_5(t)}{b}}\sqrt[6]{\Lambda}+\frac{b_1+b_2}{2}\sqrt[4]{k_2k_5(t)}+\frac{1}{2}\Big]$$
$$\cdot\int_0^t\int_{\Omega_z} e^{-\omega\eta}(1+\gamma T)\mathcal{D}_i\mathcal{D}_i dxd\eta. \tag{3.29}$$

Using the Hölder inequality, Young's inequality and Lemmas 2.5, 2.2, 2.1, 2.7 and 2.3, we have

$$\int_0^t\int_{\Omega_z} e^{-\omega\eta}(1+\gamma T)\mathcal{D}_i w_i dxd\eta \leq (1+\gamma T_M)\Big[\int_0^t\int_{\Omega_z} e^{-\omega\eta}(1+\gamma T)\mathcal{D}_i\mathcal{D}_i dxd\eta\int_0^t\int_{\Omega_z} e^{-\omega\eta}w_iw_i dxd\eta\Big]^{\frac{1}{2}}$$
$$\leq \frac{(1+\gamma T_M)}{\sqrt{\lambda}}\Big[\int_0^t\int_{\Omega_z} e^{-\omega\eta}(1+\gamma T)\mathcal{D}_i\mathcal{D}_i dxd\eta\int_0^t\int_{\Omega_z} e^{-\omega\eta}w_{i,\alpha}w_{i,\alpha}dxd\eta\Big]^{\frac{1}{2}}$$
$$\leq \frac{(1+\gamma T_M)\sqrt{k_1}}{\sqrt{\lambda}}\Big[\int_0^t\int_{\Omega_z} e^{-\omega\eta}(1+\gamma T)\mathcal{D}_i\mathcal{D}_i dxd\eta\int_0^t\int_{\Omega_z} e^{-\omega\eta}\mathcal{D}_3^2 dxd\eta\Big]^{\frac{1}{2}}$$
$$\leq \frac{(1+\gamma T_M)\sqrt{k_1}}{\sqrt{\lambda}}\int_0^t\int_{\Omega_z} e^{-\omega\eta}(1+\gamma T)\mathcal{D}_i\mathcal{D}_i dxd\eta, \tag{3.30}$$

$$\gamma\int_0^t\int_{\Omega_z} e^{-\omega\eta}w_i\Sigma u_i^* dxd\eta \leq \gamma\int_0^t e^{-\omega\eta}\Big(\int_{\Omega_z}(u_3^*)^2dx\Big)^{\frac{1}{2}}\Big(\int_{\Omega_z}\Sigma^4 dx\Big)^{\frac{1}{4}}\Big(\int_{\Omega_z}(w_iw_i)^2dx\Big)^{\frac{1}{4}}d\eta$$
$$\leq \gamma\sqrt{k_5(t)k_2}\int_0^t e^{-\omega\eta}\Big(\int_{\Omega_z}\Sigma_{,i}\Sigma_{,i}dx\Big)^{\frac{1}{2}}\Big(\int_{\Omega_z} w_{i,j}w_{i,j}dx\Big)^{\frac{1}{2}}d\eta$$
$$\leq \gamma\sqrt{k_5(t)k_2k_1}\int_0^t e^{-\omega\eta}\Big(\int_{\Omega_z}\Sigma_{,i}\Sigma_{,i}dx\Big)^{\frac{1}{2}}\Big(\int_{\Omega_z}\mathcal{D}_3^2dx\Big)^{\frac{1}{2}}d\eta$$
$$\leq \frac{\sqrt{\gamma k_5(t)k_2k_1}}{T_M}\Big[\frac{1}{4}\int_0^t\int_{\Omega_z} e^{-\omega\eta}e^{-\omega\eta}\Sigma_{,i}\Sigma_{,i}dxd\eta$$
$$+\frac{1}{\gamma}T_M^2\int_0^t\int_{\Omega_z} e^{-\omega\eta}(1+\gamma T)\mathcal{D}_3^2 dxd\eta\Big], \tag{3.31}$$

$$\int_0^t\int_{\Omega_z} e^{-\omega\eta}w_i\Sigma g_i dxd\eta \leq \sqrt{\frac{k_1\gamma}{T_M\omega}}\Big[\frac{1}{\gamma}T_M^2\int_0^t\int_{\Omega_z} e^{-\omega\eta}(1+\gamma T)\mathcal{D}_3^2 dxd\eta$$
$$+\frac{1}{4}\omega\int_0^t\int_{\Omega_z} e^{-\omega\eta}\Sigma^2 dxd\eta\Big]. \tag{3.32}$$

Inserting Eqs (3.29)–(3.32) into Eq (3.24), we obtain

$$\begin{aligned}\frac{2}{\gamma}T_M^2\Phi_1(z,t) \le n_1(t)\widetilde{b}^2e^{-\frac{z}{k_3}} + n_2(t)\cdot\frac{(b_1+b_2)T_M^2}{4\gamma}\int_0^t\int_{\Omega_z}e^{-\omega\eta}\big[|\boldsymbol{u}|+|\boldsymbol{u}^*|\big]\mathcal{D}_i\mathcal{D}_i dxd\eta\\ + n_3(t)\cdot\frac{T_M^2}{2\gamma}\int_0^t\int_{\Omega_z}e^{-\omega\eta}(1+\gamma T)\mathcal{D}_i\mathcal{D}_i dxd\eta\\ + n_4(t)\cdot\frac{1}{4}\int_0^t\int_{\Omega_z}e^{-\omega\eta}e^{-\omega\eta}\Sigma_{,i}\Sigma_{,i}dxd\eta + n_5(t)\cdot\frac{1}{4}\omega\int_0^t\int_{\Omega_z}e^{-\omega\eta}\Sigma^2 dxd\eta.\end{aligned} \tag{3.33}$$

where

$$\begin{aligned}n_1(t) &= \frac{2}{\gamma}T_M^2\frac{\sqrt[3]{2k_5(t)k_2}k_1k_4(t)}{2a\sqrt[3]{b\lambda}}, n_2(t) = 2\sqrt[4]{k_2k_5(t)},\\ n_3(t) &= 2\Big[\frac{b_1+b_2}{2}\sqrt[3]{\frac{k_1k_5(t)}{b}}\sqrt[6]{\Lambda} + \frac{b_1+b_2}{2}\sqrt[4]{k_2k_5(t)} + \frac{1}{2}\Big]\\ &\quad + 2\frac{(1+\gamma T_M)\sqrt{k_1}}{\sqrt{\lambda}} + \frac{2\sqrt{\gamma k_5(t)k_2k_1}T_M}{\gamma} + \frac{1}{\gamma}T_M^2\sqrt{\frac{k_1\gamma}{T_M\omega}},\\ n_4(t) &= \frac{2\sqrt{\gamma k_5(t)k_2k_1}T_M}{\gamma}, n_5(t) = \frac{1}{\gamma}T_M^2\sqrt{\frac{k_1\gamma}{T_M\omega}}.\end{aligned}$$

Now, we begin to derive a bound of $\Phi_2(z,t)$ which has been defined in Eq (3.15). Using the Hölder inequality, Young's inequality, Lemmas 2.7 and 2.3, we have

$$\Phi_{21}(z,t) \le \frac{1}{\sqrt{\omega}}\Big[\frac{1}{4}\int_0^t\int_{\Omega_z}e^{-\omega\eta}\Sigma_{,3}^2dxd\eta + \frac{1}{4}\omega\int_0^t\int_{\Omega_z}e^{-\omega\eta}\Sigma^2dxd\eta\Big], \tag{3.34}$$

$$\begin{aligned}\Phi_{22}(z,t) &\le \frac{1}{2}\int_0^t e^{-\omega\eta}\Big(\int_{\Omega_z}u_3^2dx\Big)^{\frac{1}{2}}\Big(\int_{\Omega_z}\Sigma^4dx\Big)^{\frac{1}{2}}d\eta\\ &\le \sqrt{2k_5(t)k_2}\cdot\frac{1}{4}\int_0^t\int_{\Omega_z}e^{-\omega\eta}\Sigma_{,i}\Sigma_{,i}dxd\eta,\end{aligned} \tag{3.35}$$

$$\Phi_{23}(z,t) \le \sqrt{\frac{2\gamma}{\omega}}\Big[\frac{1}{2\gamma}T_M^2\int_0^t\int_{\Omega_z}e^{-\omega\eta}(1+\gamma T)\mathcal{D}_3^2dxd\eta + \frac{1}{4}\omega\int_0^t\int_{\Omega_z}e^{-\omega\eta}\Sigma^2dxd\eta\Big]. \tag{3.36}$$

Inserting Eqs (3.34)–(3.36) into Eq (3.15), we obtain

$$\begin{aligned}\Phi_2(z,t) \le \Big[\frac{1}{\sqrt{\omega}} + \sqrt{2k_5(t)k_2}\Big]\cdot\frac{1}{4}\int_0^t\int_{\Omega_z}e^{-\omega\eta}\Sigma_{,i}\Sigma_{,i}dxd\eta\\ + \Big[\frac{1}{\sqrt{\omega}} + \sqrt{\frac{2\gamma}{\omega}}\Big]\cdot\frac{1}{4}\omega\int_0^t\int_{\Omega_z}e^{-\omega\eta}\Sigma^2dxd\eta\\ + \sqrt{\frac{2\gamma}{\omega}}\cdot\frac{1}{2\gamma}T_M^2\int_0^t\int_{\Omega_z}e^{-\omega\eta}(1+\gamma T)\mathcal{D}_3^2dxd\eta.\end{aligned} \tag{3.37}$$

Combining Eqs (3.19), (3.22), (3.33) and (3.37), we obtain

$$\Phi(z,t) \le n_6(t)\Big[-\frac{\partial}{\partial z}\Phi(z,t)\Big] + n_7(t)\widetilde{b}^2e^{-\frac{z}{k_3}}, \tag{3.38}$$

where

$$n_6(t) = \max\Big\{n_2(t), n_3(t) + \sqrt{\frac{2\gamma}{\omega}}, n_5(t) + \frac{1}{\sqrt{\omega}} + \sqrt{\frac{2\gamma}{\omega}}, n_4(t) + \frac{1}{\sqrt{\omega}} + \sqrt{2k_5(t)k_2}\Big\},$$
$$n_7(t) = \frac{16}{a_1\gamma(b_1+b_2)}T_M^2 k_4(t) n_6(t) + n_1(t). \tag{3.39}$$

4. Continuous dependence on the coefficient b

In this section, we will analysis Lemma 3.1 to derive the following theorem.

Theorem 4.1. Let (u_i, T, p) and (u_i^*, T^*, p^*) be solutions of the Eqs (1.1)–(1.8) in Ω, corresponding to b_1 and b_2, respectively. If $\int_D f_3 dA = 0$, Equation (3.21) holds and $f_{\alpha,\alpha} - \gamma_1 f_3 = 0, H \in L^\infty(\Omega \times \{t > 0\})$, then

$$(u_i, T) \to (u_i^*, T^*), \ as\ b_1 \to b_2.$$

Specifically, either the inequality

$$\begin{aligned}
&\frac{b_1+b_2}{4\gamma}T_M^2\int_0^t\int_{\Omega_z} e^{-\omega\eta}(\xi - z)\big[|\boldsymbol{u}| + |\boldsymbol{u}^*|\big]\mathcal{D}_i\mathcal{D}_i dx d\eta \\
&\quad + \frac{1}{2\gamma}T_M^2\int_0^t\int_{\Omega_z} e^{-\omega\eta}(\xi - z)(1+\gamma T)\mathcal{D}_i\mathcal{D}_i dx d\eta \\
&\quad + \frac{1}{2}e^{-\omega t}\int_{\Omega_z}(\xi - z)\Sigma^2 dx + \int_0^t\int_{\Omega_z} e^{-\omega\eta}(\xi - z)\Big[\frac{1}{4}\omega\Sigma^2 + \frac{1}{4}\Sigma_{,i}\Sigma_{,i}\Big]dx d\eta \\
&\quad \le \frac{16\widetilde{b}^2}{a_1\gamma(b_1+b_2)}k_3 T_M^2 k_4(t) e^{-\frac{z}{k_3}} + \widetilde{b}^2 n_7(t) e^{-\frac{1}{n_6^*}z} + \widetilde{b}^2\frac{n_7(t)}{n_6^*} z e^{-\frac{1}{n_6^*}z}
\end{aligned}$$

holds, or the inequality

$$\begin{aligned}
&\frac{b_1+b_2}{4\gamma}T_M^2\int_0^t\int_{\Omega_z} e^{-\omega\eta}(\xi - z)\big[|\boldsymbol{u}| + |\boldsymbol{u}^*|\big]\mathcal{D}_i\mathcal{D}_i dx d\eta \\
&\quad + \frac{1}{2\gamma}T_M^2\int_0^t\int_{\Omega_z} e^{-\omega\eta}(\xi - z)(1+\gamma T)\mathcal{D}_i\mathcal{D}_i dx d\eta \\
&\quad + \frac{1}{2}e^{-\omega t}\int_{\Omega_z}(\xi - z)\Sigma^2 dx + \int_0^t\int_{\Omega_z} e^{-\omega\eta}(\xi - z)\Big[\frac{1}{4}\omega\Sigma^2 + \frac{1}{4}\Sigma_{,i}\Sigma_{,i}\Big]dx d\eta \\
&\quad \le \frac{16\widetilde{b}^2}{a_1\gamma(b_1+b_2)}k_3 T_M^2 k_4(t) e^{-\frac{z}{k_3}} + \widetilde{b}^2 n_7(t) e^{-\frac{1}{n_6^*}z} + \widetilde{b}^2\frac{n_7(t)}{n_6^*(\frac{1}{n_6^*} - \frac{1}{k_3})} b_3(t)[e^{-\frac{1}{k_3}z} - e^{-\frac{1}{n_6^*}z}]
\end{aligned}$$

holds.

Proof. Using Lemma 3.1, we have

$$\frac{\partial}{\partial z}\Big\{\Phi(z,t)e^{\frac{1}{n_6^*}z}\Big\} \le \widetilde{b}^2\frac{n_7(t)}{n_6^*}e^{(\frac{1}{n_6^*} - \frac{1}{k_3})z}, \ z \ge 0. \tag{4.1}$$

Now, we consider (4.1) for two cases.

I. If $n_6^* = k_3$, we integrate Eq (4.1) from 0 to z to obtain

$$\Phi(z,t) \leq \Phi(0,t)e^{-\frac{1}{n_6^*}z} + \widetilde{b}^2 \frac{n_7(t)}{n_6^*} z e^{-\frac{1}{n_6^*}z}. \tag{4.2}$$

II. If $n_6^* \neq k_3$, we integrate Eq (4.1) from 0 to z to obtain

$$\Phi(z,t) \leq \Phi(0,t)e^{-\frac{1}{n_6^*}z} + \widetilde{b}^2 \frac{n_7(t)}{n_6^*(\frac{1}{n_6^*} - \frac{1}{k_3})} b_3(t)[e^{-\frac{1}{k_3}z} - e^{-\frac{1}{n_6^*}z}]. \tag{4.3}$$

From Eqs (4.2) and (4.3), to obtain the main result, we can conclude that we have to derive a bound for $\Phi(0,t)$. We choose $z = 0$ in Lemma 3.1 to obtain

$$\Phi(0,t) \leq n_6^*\Big[-\frac{\partial \Phi}{\partial z}(0,t)\Big] + \widetilde{b}^2 n_7(t). \tag{4.4}$$

Clearly, if we want to derive a bound for $\Phi(0,t)$, we only need derive a bound for $-\frac{\partial \Phi}{\partial z}(0,t)$. To do this, choosing $z = 0$ in Eq (3.19) and combining Eqs (3.8) and (3.15), we have

$$\begin{aligned} -\frac{\partial \Phi}{\partial z}(0,t) = {} & \frac{2}{\gamma} T_M^2 \int_0^t \int_D e^{-\omega\eta} \pi \mathcal{D}_3 dA d\eta - \int_0^t \int_D e^{-\omega\eta} \Sigma \Sigma_{,3} dA d\eta \\ & + \frac{1}{2} \int_0^t \int_D e^{-\omega\eta} u_3 \Sigma^2 dA d\eta + \int_0^t \int_D e^{-\omega\eta} \mathcal{D}_3 T^* \Sigma dA d\eta. \end{aligned} \tag{4.5}$$

In light of the boundary conditions (3.4)–(3.6), from Eq (4.5) we can know that

$$-\frac{\partial \Phi}{\partial z}(0,t) = 0. \tag{4.6}$$

Inserting Eq (4.6) into Eq (4.4), we obtain

$$\Phi(0,t) \leq \widetilde{b}^2 n_7(t). \tag{4.7}$$

Therefore, from Eqs (4.2), (4.3) and (4.7) we have

$$\Phi(z,t) \leq \widetilde{b}^2 n_7(t) e^{-\frac{1}{n_6^*}z} + \widetilde{b}^2 \frac{n_7(t)}{n_6^*} z e^{-\frac{1}{n_6^*}z}, \text{ if } n_6^* = k_3, \tag{4.8}$$

$$\Phi(z,t) \leq \widetilde{b}^2 n_7(t) e^{-\frac{1}{n_6^*}z} + \widetilde{b}^2 \frac{n_7(t)}{n_6^*(\frac{1}{n_6^*} - \frac{1}{k_3})} b_3(t)[e^{-\frac{1}{k_3}z} - e^{-\frac{1}{n_6^*}z}], \text{ if } n_6^* \neq k_3. \tag{4.9}$$

Combining Eqs (3.24), (4.8) and (4.9) we can complete the proof of Theorem 4.1.

Remark 4.1 Theorem 1 shows that the small perturbation of Forchheimer coefficient will not cause great changes to the solution of Eqs (1.1)–(1.8). Meanwhile, Theorem 1 also shows that the solutions of Eqs (2.12)–(2.21) decay exponentially as the space variable $z \to \infty$.

5. Continuous dependence on the coefficient γ

This section shows how to use the prior estimates in Section 2 and the method in Section 3 to derive the continuous dependence of the solution on γ. Assume that (u_i^*, T^*, p^*) is a solution of Eqs (1.1)–(1.8) with $\gamma = \gamma^*$.

If we also let

$$\mathcal{D}_i = u_i - u_i^*,\ \Sigma = T - T^*,\ \pi = p - p^*,\ \widetilde{\gamma} = \gamma - \gamma^*,$$

then $(\mathcal{D}_i, \Sigma, \pi)$ satisfies

$$b[|\boldsymbol{u}|u_i - |\boldsymbol{u}^*|u_i^*] + \widetilde{\gamma} T u_i + \gamma_2 \Sigma u_i + (1 + \gamma T^*)\mathcal{D}_i + \gamma \Sigma u_i^* = -\pi_{,i} + g_i \Sigma,\ in\ \Omega \times \{t > 0\}, \tag{5.1}$$

$$\mathcal{D}_{i,i} = 0,\ in\ \Omega \times \{t > 0\}, \tag{5.2}$$

$$\partial_t \Sigma + u_i \Sigma_{,i} + \mathcal{D}_i T_{,i}^* = \Delta \Sigma,\ in\ \Omega \times \{t > 0\}, \tag{5.3}$$

$$\mathcal{D}_i = 0, \Sigma = 0,\ on\ \partial D \times \{x_3 > 0\} \times \{t > 0\}, \tag{5.4}$$

$$\mathcal{D}_i = 0, \Sigma = 0,\ on\ D \times \{t > 0\}, \tag{5.5}$$

$$\Sigma(x_1, x_2, x_3, 0) = 0,\ in\ \Omega \tag{5.6}$$

$$|\boldsymbol{u}|, |\Sigma| = O(1), |\mathcal{D}_i|, |\nabla \Sigma|, |\pi| = o(x_3^{-1}),\ as\ x_3 \to \infty. \tag{5.7}$$

We also define $\Phi_1(z, t)$ as that in Eq (3.8). Similar to Eq (3.10), we have

$$\begin{aligned}
\Phi_1(z, t) = {} & \frac{b}{2}\int_0^t \int_{\Omega_z} e^{-\omega\eta}(\xi - z)\Big[|\boldsymbol{u}| + |\boldsymbol{u}^*|\Big]\mathcal{D}_i\mathcal{D}_i dx d\eta + \int_0^t \int_{\Omega_z} e^{-\omega\eta}(\xi - z)(1 + \gamma_2 T^*)\mathcal{D}_i\mathcal{D}_i dx d\eta \\
& + \frac{b}{2}\int_0^t \int_{\Omega_z} e^{-\omega\eta}(\xi - z)\Big[|\boldsymbol{u}| - |\boldsymbol{u}^*|\Big]^2\Big[|\boldsymbol{u}| + |\boldsymbol{u}^*|\Big] dx d\eta \\
& + \gamma_2 \int_0^t \int_{\Omega_z} e^{-\omega\eta}(\xi - z)\mathcal{D}_i \Sigma u_i^* dx d\eta - \int_0^t \int_{\Omega_z} e^{-\omega\eta}(\xi - z)\mathcal{D}_i g_i \Sigma dx d\eta \\
& + \widetilde{\gamma}\int_0^t \int_{\Omega_z} e^{-\omega\eta}(\xi - z) T u_i \mathcal{D}_i dx d\eta.
\end{aligned} \tag{5.8}$$

Using the Hölder inequality, Young's inequality and Lemmas 2.5 and 2.6, we obtain

$$\begin{aligned}
\widetilde{\gamma}\int_0^t \int_{\Omega_z} e^{-\omega\eta} T u_i \mathcal{D}_i dx d\eta \geq {} & -\frac{1}{2} T_M^2 \widetilde{\gamma}^2 \int_0^t \int_{\Omega_z} e^{-\omega\eta} u_i u_i dx d\eta - \frac{1}{2}\int_0^t \int_{\Omega_z} e^{-\omega\eta}(1 + \gamma_2 T^*)\mathcal{D}_i\mathcal{D}_i dx d\eta \\
\geq {} & -\frac{k_4(t)}{2a_2} T_M^2 \widetilde{\gamma}^2 e^{-\frac{z}{k_3}} - \frac{1}{2}\int_0^t \int_{\Omega_z} e^{-\omega\eta}(1 + \gamma_2 T^*)\mathcal{D}_i\mathcal{D}_i dx d\eta,
\end{aligned} \tag{5.9}$$

Combining Eqs (3.12), (3.13), (5.8) and (5.9), we obtain

$$\begin{aligned}
-\frac{\partial}{\partial z}\Phi_1(z, t) \geq {} & \frac{b}{2}\int_0^t \int_{\Omega_z} e^{-\omega\eta}\Big[|\boldsymbol{u}| + |\boldsymbol{u}^*|\Big]\mathcal{D}_i\mathcal{D}_i dx d\eta + \frac{1}{2}\int_0^t \int_{\Omega_z} e^{-\omega\eta}(1 + \gamma_2 T^*)\mathcal{D}_i\mathcal{D}_i dx d\eta \\
& - \frac{4\gamma_2^2\sqrt{k_5(t)k_2}}{b}\int_0^t \int_{\Omega_z} e^{-\omega\eta}\Sigma_{,i}\Sigma_{,i} dx d\eta - \frac{1}{2\gamma}\int_0^t \int_{\Omega_z} e^{-\omega\eta}\Sigma^2 dx d\eta \\
& - \frac{k_4(t)}{2a_2} T_M^2 \widetilde{\gamma}^2 e^{-\frac{z}{k_3}}.
\end{aligned} \tag{5.10}$$

Inserting Eqs (3.18) and (5.10) into Eq (3.19), choosing $\omega > \frac{4T_M^2}{\gamma^2}$ and the boundary data satisfies

$$\frac{8(\gamma^*)^2\sqrt{k_5(t)k_2}}{b} \le \frac{1}{4}, \tag{5.11}$$

we have

$$\begin{aligned} -\frac{\partial}{\partial z}\Phi(z,t) \ge{}& \frac{b}{\gamma^*}T_M^2\int_0^t\int_{\Omega_z} e^{-\omega\eta}\Big[|\boldsymbol{u}|+|\boldsymbol{u}^*|\Big]\mathcal{D}_i\mathcal{D}_i dxd\eta + \frac{1}{(\gamma^*)^2}T_M^2\int_0^t\int_{\Omega_z} e^{-\omega\eta}(1+\gamma_2T^*)\mathcal{D}_i\mathcal{D}_i dxd\eta \\ &+\frac{1}{2}e^{-\omega t}\int_{\Omega_z}\Sigma^2dx + \int_0^t\int_{\Omega_z} e^{-\omega\eta}\Big[\frac{1}{4}\omega\Sigma^2+\frac{1}{4}\Sigma_{,i}\Sigma_{,i}\Big]dxd\eta \\ &-\frac{k_4(t)}{2a_2\gamma^*}T_M^4\widetilde{\gamma}^2e^{-\frac{z}{k_3}}. \end{aligned} \tag{5.12}$$

Integrating Eq (5.12) from z to ∞, we obtain

$$\begin{aligned} \Phi(z,t) \ge{}& \frac{b}{\gamma^*}T_M^2\int_0^t\int_{\Omega_z} e^{-\omega\eta}(\xi-z)\Big[|\boldsymbol{u}|+|\boldsymbol{u}^*|\Big]\mathcal{D}_i\mathcal{D}_i dxd\eta + \frac{1}{(\gamma^*)^2}T_M^2\int_0^t\int_{\Omega_z} e^{-\omega\eta}(1+\gamma_2T^*)\mathcal{D}_i\mathcal{D}_i dxd\eta \\ &+\frac{1}{2}e^{-\omega t}\int_{\Omega_z}(\xi-z)\Sigma^2dx + \int_0^t\int_{\Omega_z} e^{-\omega\eta}(\xi-z)\Big[\frac{1}{4}\omega\Sigma^2+\frac{1}{4}\Sigma_{,i}\Sigma_{,i}\Big]dxd\eta \\ &-\frac{k_4(t)}{2a_2\gamma^*}k_3T_M^4\widetilde{\gamma}^2e^{-\frac{z}{k_3}}. \end{aligned} \tag{5.13}$$

Similar to the calculation in Eqs (3.33) and (3.37), we can get

$$\Phi(z,t) \le n_6'(t)\Big[-\frac{\partial}{\partial z}\Phi(z,t)\Big] + n_7'(t)\widetilde{b}^2e^{-\frac{z}{k_3}}, \tag{5.14}$$

for $n_6'(t), n_7'(t) > 0$.

After similar analysis as in the previous section, we can get the following theorem from Eq (5.14).

Theorem 5.1. Let (u_i, T, p) and (u_i^*, T^*, p^*) be solutions of the Eqs (1.1)–(1.8) in Ω, corresponding to b_1 and b_2, respectively. If $\int_D f_3 dA = 0$, Equation (5.11) holds and $f_{\alpha,\alpha} - \gamma_1 f_3 = 0, H \in L^\infty(\Omega\times\{t>0\})$, then

$$(u_i, T) \to (u_i^*, T^*),\ as\ b_1 \to b_2.$$

Specifically, either the inequality

$$\begin{aligned} &\frac{b}{\gamma^*}T_M^2\int_0^t\int_{\Omega_z} e^{-\omega\eta}(\xi-z)\Big[|\boldsymbol{u}|+|\boldsymbol{u}^*|\Big]\mathcal{D}_i\mathcal{D}_i dxd\eta \\ &\quad+\frac{1}{(\gamma^*)^2}T_M^2\int_0^t\int_{\Omega_z} e^{-\omega\eta}(\xi-z)(1+\gamma^*T^*)\mathcal{D}_i\mathcal{D}_i dxd\eta \\ &\quad+\frac{1}{2}e^{-\omega t}\int_{\Omega_z}(\xi-z)\Sigma^2dx + \int_0^t\int_{\Omega_z} e^{-\omega\eta}(\xi-z)\Big[\frac{1}{4}\omega\Sigma^2+\frac{1}{4}\Sigma_{,i}\Sigma_{,i}\Big]dxd\eta \\ &\quad\le \frac{k_4(t)}{2a_2\gamma^*}k_3T_M^4\widetilde{\gamma}^2e^{-\frac{z}{k_3}} + \widetilde{\gamma}^2n_7'(t)e^{-\frac{1}{n_6^*}z} + \widetilde{\gamma}^2\frac{n_7'(t)}{n_6^*}ze^{-\frac{1}{n_6^*}z} \end{aligned}$$

holds, or the inequality

$$\begin{aligned}
&\frac{b}{\gamma^*}T_M^2\int_0^t\int_{\Omega_z} e^{-\omega\eta}(\xi-z)\Big[|\boldsymbol{u}|+|\boldsymbol{u}^*|\Big]\mathcal{D}_i\mathcal{D}_i dxd\eta\\
&\quad+\frac{1}{(\gamma^*)^2}T_M^2\int_0^t\int_{\Omega_z} e^{-\omega\eta}(\xi-z)(1+\gamma^*T^*)\mathcal{D}_i\mathcal{D}_i dxd\eta\\
&\quad+\frac{1}{2}e^{-\omega t}\int_{\Omega_z}(\xi-z)\Sigma^2dx+\int_0^t\int_{\Omega_z} e^{-\omega\eta}(\xi-z)\Big[\frac{1}{4}\omega\Sigma^2+\frac{1}{4}\Sigma_{,i}\Sigma_{,i}\Big]dxd\eta\\
&\quad\le\frac{k_4(t)}{2a_2\gamma^*}k_3T_M^4\widetilde{\gamma}^2e^{-\frac{z}{k_3}}+\widetilde{\gamma}^2n_7'(t)e^{-\frac{1}{n_6^*}z}+\widetilde{\gamma}^2\frac{n_7'(t)}{n_6^*(\frac{1}{n_6^*}-\frac{1}{k_3})}b_3(t)[e^{-\frac{1}{k_3}z}-e^{-\frac{1}{n_6^*}z}]
\end{aligned}$$

holds.

6. Conclusions

In this paper, using a priori estimates of the solutions, we show how to control the nonlinear term, and obtain the structural stability of the solution of the Forchheimer equation in a semi-infinite cylinder. Meanwhile, the spatial decay results of the solution are also obtained. The methods in this paper can bring some inspiration for the structural stability of other nonlinear partial differential equations.

Acknowledgments

The authors express their heartfelt thanks to the editors and referees who have provided some important suggestions. This work is supported by the Tutor System Rroject of Guangzhou Huashang College (2021HSDS13) and the Key projects of universities in Guangdong Province (NATURAL SCIENCE) (2019KZDXM042).

Conflict of interest

The authors declare there is no conflict of interest. Conceptualization, and validation, Z. Li.; formal analysis, Z W. Zhang; investigation, Y. Li. All authors have read and agreed to the published version of the manuscript.

References

1. D. A. Nield, A. Bejan, *Convection in Porous Media*, Springer Press, New York, 1992.
2. B. Straughan, *Mathematical Aspects of Penetrative Convection*, Pitman Research Notes in Mathematics Series, CRC Press: Boca Raton, FL, USA, (1993), 288.
3. L. E. Payne, J. C. Song, Spatial decay bounds for the Forchheimer equations, *Int. J. Eng. Sci.*, **40** (2002), 943–956. https://doi.org/10.1016/S0020-7225(01) 00102-1
4. C. O. Horgan, L. T. Wheeler, Spatial decay estimates for the Navier-Stokes equations with application to the problem of entry flow, *SIAM J. Appl. Math.,* **35** (1978), 97–116. https://doi.org/10.1137/0135008

5. J. C. Song, Spatial decay estimates in time-dependent double-diffusive darcy plane flow, *J. Math. Anal. Appl.*, **267** (2002), 76–88. https://doi.org/10.1006/ jmaa.2001.7750

6. L. E. Payne, J. C. Song, Spatial decay bounds for double diffusive convection in Brinkman flow, *J. Differ. Equations*, **244** (2008), 413–430. https://doi.org/10.1016 /j.jde.2007.10.003

7. Y. F. Li, X. J. Chen, Phragmén-Lindelöf type alternative results for the solutions to generalized heat conduction equations, *Phys. Fluids*, **34** (2022), 091901. https://doi.org/10.1063 /5.0118243

8. X. J. Chen, Y. F. Li, Spatial properties and the influence of the Soret coefficient on the solutions of time-dependent double-diffusive Darcy plane flow, *Electron. Res. Arch.*, **31** (2022), 421–441. https://doi.org/10.3934/era.2023021

9. Y. F. Li, X. J. Chen, Phragmén-Lindelöf alternative results in time-dependent double-diffusive Darcy plane flow, *Math. Methods Appl. Sci.*, **45** (2022), 6982–6997. https://doi.org/10.1002/mma.8220

10. M. W. Hirsch, S. Smale, *Differential Equations, Dynamical Systems, and Linear Algebra*, Academic Press, New York, 1974.

11. N. L. Scott, Continuous dependence on boundary reaction terms in a porous medium of Darcy type, *J. Math. Anal. Appl.*, **399** (2013), 667–675. https://doi.org/10.1016/j.jmaa.2012.10.054

12. Y. Liu, S. Z. Xiao, Structural stability for the Brinkman fluid interfacing with a Darcy fluid in an unbounded domain, *Nonlinear Anal. Real World Appl.*, **42** (2018), 308–333. https://doi.org/10.1016/j.nonrwa.2018.01.007

13. B. Straughan, Heated and salted below porous convection with generalized temperature and solute boundary conditions, *Transp. Porous Media*, **131** (2020), 617–631. https://doi.org/ 10.1007/s11242-019-01359-y

14. Y. F. Li, S. Z. Xiao, P. Zeng, The applications of some basic mathematical inequalities on the convergence of the primitive equations of moist atmosphere, *J. Math. Inequal.*, **15** (2021), 293–304. https://doi.org/10.1007/s11242-019-01359-y

15. Y. Liu, X. L. Qin, J. C. Shi, W. J. Zhi, Structural stability of the Boussinesq fluid interfacing with a Darcy fluid in a bounded region in $\mathbb{R}^2$, *Appl. Math. Comput.*, **411** (2021), 126488. https://doi.org/10.1016/j.amc.2021.126488

16. N. L. Scott, B. Straughan, Continuous dependence on the reaction terms in porous convection with surface reactions, *Q. Appl. Math.*, **71** (2013), 501–508. https://doi.org/10.1090/S0033-569X-2013-01289-X

17. M. Gentile, B. Straughan, Structural stability in resonant penetrative convection in a Forchheimer porous material, *Nonlinear Anal. Real World Appl.*, **14** (2013), 397–401. https://doi.org/10.1016/j.nonrwa.2012.07.003

18. R. Quintanilla, Convergence and structural stability in thermoelasticity, *Appl. Math. Comput.*, **135** (2003), 287–300. https://doi.org/10.1016/S0096-3003(01)00331-9

19. B. Straughan, Continuous dependence and convergence for a Kelvin-Voigt fluid of order one, *Ann. Univ. Ferrara*, **68** (2022), 49–61. https://doi.org/10.1007/s11565-021-00381-7

20. Y. F. Li, C. H. Lin, Continuous dependence for the nonhomogeneous Brinkman-Forchheimer equations in a semi-infinite pipe, *Appl. Math. Comput.*, **244** (2014), 201–208. https://doi.org/10.1016/j.amc.2014.06.082

21. J. C. Song, Spatial decay bounds for a temperature dependent Stokes flow, *J. Korean Math. Soc.*, **49** (2012), 1163–1174. http://dx.doi.org/10.4134/JKMS.2012.49.6.1163

22. H. A. Levine, An estimate for the best constant in a Sobolev inequality involving three integral norms, *Ann. Mat. Pura Appl.*, **4** (1980), 181–197. https://doi.org/10.1007/BF01795392

23. G. Talenti, Best constant in Sobolev inequality, *Ann. Mat. Pura Appl.*, **110** (1976), 353–372. https://doi.org/10.1007/BF02418013

AIMS Press