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Research article

Fixed point of Hardy-Rogers-type contractions on metric spaces with graph

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Abstract: This paper presents a novel concept of G-Hardy-Rogers functional operators on metric spaces endowed with a graph. It investigates sufficient circumstances under which such a mapping becomes a Picard operator. As applications of the principal idea discussed herein, a few important corresponding fixed point results in ordered metric spaces and cyclic operators are pointed out and analyzed. For upcoming research papers in this field, comparative graphical illustrations are created to highlight the pre-eminence of proposed notions with respect to the existing ones.

Keywords: fixed point; picard operator; directed graph; Hardy-Rogers contraction

1. Introduction and preliminaries

The contraction mapping principle due to Banach [1] is among popular results in metric fixed point theory (FPT). Following Banach [1], the investigation of fixed and common fixed points of operators obeying various contractive inequalities attracted several researchers (for example, see [2–6]). In 1973, Hardy and Rogers [7] presented a FP theorem using a more wider contractive expression which improved the FP result due to Reich [8]. The principal result in [7] is the following.

Theorem 1.1. *[7] If (Θ, ρ) is a complete metric space (MS) and the mapping $\tau : \Theta \longrightarrow \Theta$ satisfies:*

$$\rho(\tau\varrho, \tau\sigma) \leq \alpha\rho(\varrho, \sigma) + \beta\rho(\varrho, \tau\varrho) + \gamma\rho(\sigma, \tau\sigma) + \delta\rho(\varrho, \tau\sigma) + \lambda\rho(\sigma, \tau\varrho), \tag{1.1}$$

for all $\varrho, \sigma \in \Theta$, given that $\alpha, \beta, \gamma, \delta, \lambda$ are positive reals obeying $\alpha + \beta + \gamma + \delta + \lambda < 1$, then τ has a unique FP in Θ.

A short time ago, Roldán et al. [9] brought up new FP theorems for a family of contractions depending on two functions and some parameters under the name multiparametric contractions and discussed a significant number of Hardy-Roger's type contractions in the framework of both metric and quasi metric spaces. Later after, more than a handful of researchers have come up with multifarious improvements to Theorem 1.1. For a few of these modifications, one can consult [10–15] and the references therein.

Let (Θ, ρ) be a MS and $\tau : \Theta \longrightarrow \Theta$ be a mapping. Consistent with Petrusel and Rus [16], we call τ a Picard operator (PO, for short), if τ possesses a unique FP u^* and $\lim\limits_{l \longrightarrow \infty} \tau^l \varrho = u^*$ for all $\varrho \in \Theta$. The mapping τ is said to be a weakly Picard operator (WPO, for short), if the sequence $\{\tau^l \varrho\}_{l \in \mathbb{N}}$ converges for all $\varrho \in \Theta$, and the limit is a FP of τ.

Let Ω represents the diagonal of the Cartesian product $\Theta \times \Theta$ (i.e., $\Omega = \Theta \times \Theta$) and let Ψ be a directed graph such that the set $\Delta(\Psi)$ of its nodes coincides with Θ, and the set $H(\Psi)$ of its edges contains all loops; that is, $\Omega \subseteq H(\Psi)$. We presume throughout that Ψ does not contain any parallel edge so as to have $\Psi = (\Delta(\Psi), H(\Psi))$. We also take Ψ as a weighted graph(see [17, P.309]) by allocating to each edge, the distance between its nodes. By Ψ^{-1}, we depict the conversion of Ψ; in other words, the graph derived from Ψ by inverting the orientation of edges. Hence,

$$H(\Psi^{-1}) = \{(\varrho, \sigma) \in \Theta \times \Theta : (\sigma, \varrho) \in H(\Psi)\}.$$

The symbol $\hat{\Psi}$ represents the graph with no orientation gotten from Ψ by overlooking the orientation of edges. Indeed, it is more handy to regard $\hat{\Psi}$ as a graph with orientation in which case the set of its edges is symmetric. Via this observation,

$$H(\hat{\Psi}) = H(\Psi) \cup H(\Psi^{-1}).$$

We say that (Δ', H') is a subgraph of Ψ if $\Delta' \subseteq \Delta(\Psi), H' \subseteq H(\Psi)$ and, for each $(\varrho, \sigma) \in H', \varrho, \sigma \in \Delta'$.

Furthermore, we record some basis of graph connectivity. All of these fundamentals can be found in [17]. Let ϱ, σ be any two nodes of a graph Ψ. Then, a path in Ψ from ϱ to σ of length $L (L \in \mathbb{N})$, is a sequence $\{\varrho_i\}_{i=0}^{L}$ of $L + 1$ nodes such that $\varrho_0 = \varrho, \varrho_L = \sigma$ and $(\varrho_{l-1}, \varrho_l) \in H(\Psi)$ for $i = \overline{1, L}$. A graph Ψ is said to be connected if there is a path between any two nodes. Ψ is called weakly connected if $\hat{\Psi}$ is connected. If Ψ is such that $H(\Psi)$ is symmetric and ϱ is a vertex in Ψ, then the subgraph Ψ_ϱ comprising of all edges and nodes which are contained in some path starting from ϱ is termed the component of Ψ containing ϱ. In this instance, $\Delta(\Psi_\varrho) = [\Psi]_\varrho$, where $[\Psi]_\varrho$ is the equivalence class of the relation $\sim$ given on $\Delta(\Psi)$ by the assignment $\sigma \sim z$ if there is a path in Ψ from σ to z. Obviously, Ψ_ϱ is connected.

If $\tau : \Theta \longrightarrow \Theta$ is an operator, then by $Fix(\tau) = \{\varrho \in \Theta : \varrho = \tau\varrho\}$, we depict the set of all FP of τ. Represent also $\Theta_\tau = \{\varrho \in \Theta : (\varrho, \tau\varrho) \in H(\Psi)\}$.

Definition 1.2. A mapping $\tau : \Theta \longrightarrow \Theta$ is said to be orbitally continuous if for all $\varrho \in \Theta$ and any sequence $\{\zeta_l\}_{l \in \mathbb{N}}$ of positive integers, $\tau^{\zeta_l}\varrho \longrightarrow \sigma$ implies $\tau(\tau^{\zeta_l}\varrho) \longrightarrow \tau\sigma$ as $l \longrightarrow \infty$.

Consistent with Definition 1.2, Bojor [18] gave the following concept.

Definition 1.3. [18] A mapping $\tau : \Theta \longrightarrow \Theta$ is called orbitally Ψ-continuous, if for any $\varrho \in \Theta$ and a sequence $\{\varrho_l\}_{l \in \mathbb{N}}$,

$$\varrho_l \longrightarrow \varrho \text{ and } (\varrho_l, \varrho_{l+1}) \in H(\Psi) \text{ for } l \in \mathbb{N} \text{ implies } \tau\varrho_l \longrightarrow \tau\varrho.$$

Meanwhile, a number of results have been established discussing sufficient criteria for an operator to be a PO, provided (Θ, ρ) is equipped with a graph. The earliest breakthrough in this context was announced by Jachymski [19], who employed graph theoretic jargons in place of partial order to launch the idea of Ψ-contraction in the manner given herewith.

Definition 1.4. [19] Let (Θ, ρ) be a MS. A mapping $\tau : \Theta \longrightarrow \Theta$ is called a Banach Ψ-contraction (Ψ-contraction, for short) if τ preserves edges of Ψ in the following sense:

$$\forall \varrho, \sigma \in \Theta \left((\varrho, \sigma) \in H(\Psi) \text{ implies } (\tau\varrho, \tau\sigma) \in H(\Psi) \right),$$

and τ decreases weights of edges of Ψ; that is,

$$\text{there exists } \alpha \in (0,1), \text{ for all } \varrho, \sigma \in \Theta \left((\varrho, \sigma) \in H(\Psi) \text{ implies } \rho(\tau\varrho, \tau\sigma) \leq \alpha\rho(\varrho, \sigma) \right).$$

The following notion is common in the literature.

Definition 1.5. Let (Θ, ρ) be a MS. The operator τ is named a $\grave{C}$iri$\grave{c}$-Reich-Rus operator if there exist positive reals α, β, γ with $\alpha + \beta + \gamma < 1$ such that for all $\varrho, \sigma \in \Theta$,

$$\rho(\tau\varrho, \tau\sigma) \leq \alpha\rho(\varrho, \sigma) + \beta\rho(\varrho, \tau\varrho) + \gamma\rho(\sigma, \tau\sigma).$$

Recently, Bojor [18] examined the existence of FP for $\grave{C}$iri$\grave{c}$-Reich-Rus operator in complete MS equipped with a graph by launching the idea of Ψ-$\grave{C}$iri$\grave{c}$-Reich-Rus operator in the following sense.

Definition 1.6. [18, Definition 7] Let (Θ, ρ) be a MS. The mapping $\tau : \Theta \longrightarrow \Theta$ is called a Ψ-$\grave{C}$iri$\grave{c}$-Reich-Rus operator, if :

(CRR1) $((\varrho, \sigma) \in H(\Psi)$ implies $(\tau\varrho, \tau\sigma) \in H(\Psi))$, for all $\varrho, \sigma \in \Theta$;
(CRR2) there exist nonnegative numbers α, β, γ with $\alpha + \beta + \gamma < 1$ such that for each $(\varrho, \sigma) \in H(\Psi)$,

$$\rho(\tau\varrho, \tau\sigma) \leq \alpha\rho(\varrho, \sigma) + \beta\rho(\varrho, \tau\varrho) + \gamma\rho(\sigma, \tau\sigma).$$

It has been demonstrated in [18, Example 2] that every $\grave{C}$iri$\grave{c}$-Reich-Rus operator is a Ψ-$\grave{C}$iri$\grave{c}$-Reich-Rus operator, but the converse does not always hold.

Definition 1.7. [18, Definition 8] Let (Θ, ρ) be a MS equipped with a graph Ψ and $\tau : \Theta \longrightarrow \Theta$ be an operator. We say that the graph Ψ is τ-connected if for all nodes ϱ, σ of Ψ with $(\varrho, \sigma) \notin H(\Psi)$, there is a path in Ψ, $\{\varrho_i\}_{i=0}^{L}$ from ϱ to σ such that $\varrho_0 = \varrho, \varrho_L = \sigma$ and $(\varrho_i, \tau\varrho_i) \in H(\Psi)$ for all $i = \overline{1, L-1}$. A graph Ψ is weakly τ-connected if $\hat{\Psi}$ is τ-connected.

The principle result in [18] is the following.

Theorem 1.8. *[18, Theorem 6] Let (Θ, ρ) be a complete MS equipped with a graph Ψ and $\tau : \Theta \longrightarrow \Theta$ be a Ψ-$\grave{C}$iri$\grave{c}$-Reich-Rus operator. Suppose further that:*

(B1) Ψ is τ-connected;
(B2) for any sequence $\{\varrho_l\}_{l\in\mathbb{N}}$ in Θ, if $\varrho_l \longrightarrow \varrho$ and $(\varrho_l, \varrho_{l+1}) \in H(\Psi)$ for $l \in \mathbb{N}$, then there exists a subsequence $\{\varrho_{\zeta_l}\}$ with $(\varrho_{\zeta_l}, \varrho) \in H(\Psi)$ for $l \in \mathbb{N}$.

Then τ is a PO.

For some recent advancements to the ideas of Ψ-contractions, one can consult [20–24] and the references therein.

The focus of this paper is to study new conditions for the existence of FP for Hardy-Rogers operator in MS equipped with a graph by initiating the notion of Ψ-Hardy-Rogers operators. As some applications of our principal result, new FP theorems in partially ordered MS and cyclic operators are deduced. Our principal result is further invited to investigate novel conditions for the existence of a solution to an integral equation. Nontrivial comparative examples are provided to show that the ideas proposed herein properly contained a few well-known results in the corresponding literature.

2. Main result

Motivated by Theorem 1.1, we employ in this section the idea of Jachysmki [19] to introduce the concept of Ψ-Hardy-Rogers operator in the following manner.

Definition 2.1. Let (Θ,ρ) be a MS. The mapping $\tau:\Theta\longrightarrow\Theta$ is called a Ψ-Hardy-Rogers operator if it satisfies the following conditions:

(H1) $((\varrho,\sigma)\in H(\Psi)$ implies $(\tau\varrho,\tau\sigma)\in H(\Psi))$, for all $\varrho,\sigma\in\Theta$;
(H2) there exist nonnegative numbers $\alpha,\beta,\gamma,\delta,\zeta$ with $\alpha+\beta+\gamma+\delta+\zeta<1$ such that for all $(\varrho,\sigma)\in H(\Psi)$,

$$\rho(\tau\varrho,\tau\sigma)\leq\alpha\rho(\varrho,\sigma)+\beta\rho(\varrho,\tau\varrho)+\gamma\rho(\sigma,\tau\sigma)+\delta\rho(\varrho,\tau\sigma)+\zeta\rho(\sigma,\tau\varrho). \tag{2.1}$$

Example 2.2. Any Hardy-Rogers operator is a Ψ_0-Hardy-Rogers operator, where the graph Ψ_0 is given by $H(\Psi_0)=\Theta\times\Theta$.

The next example points out that every Ψ-Hardy-Rogers operator needs not be a Hardy-Rogers operator.

Example 2.3. Let $\Theta=\{1,2,3,4\}$ and $\rho(\varrho,\sigma)=|\varrho-\sigma|$ for all $\varrho,\sigma\in\Theta$. Define the operator $\tau:\Theta\longrightarrow\Theta$ as follows:

$$\tau\varrho=\begin{cases}1, & \text{if } \varrho\in\{1,2\}\\ 2, & \text{if } \varrho\in\{3,4\}.\end{cases}$$

By considering the constants $\alpha=\frac{1}{12},\beta=\frac{1}{12},\gamma=\frac{1}{4},\delta=\frac{1}{3}$ and $\lambda=\frac{1}{12}$, we see that τ is a Ψ-Hardy-Rogers operator, where $\Psi=(\Delta(\Psi),H(\Psi))$ with $\Delta(\Psi)=\Theta$ and

$$H(\Psi)=\{(1,2),(1,3),(1,4),(2,4),(3,4)\}\cup\Omega.$$

However, τ is not a Hardy-Rogers operator, since for $\varrho=2$ and $\sigma=3$, we get

$$\begin{aligned}\rho(\tau2,\tau3) &= 1>\frac{7}{12}\\ &= \frac{1}{12}\rho(2,3)+\frac{1}{12}\rho(2,\tau2)+\frac{1}{4}\rho(3,\tau3)+\frac{1}{3}\rho(2,\tau3)+\frac{1}{12}\rho(3,\tau2).\end{aligned}$$

The graph of Example 2.3 is represented in Figure 1.

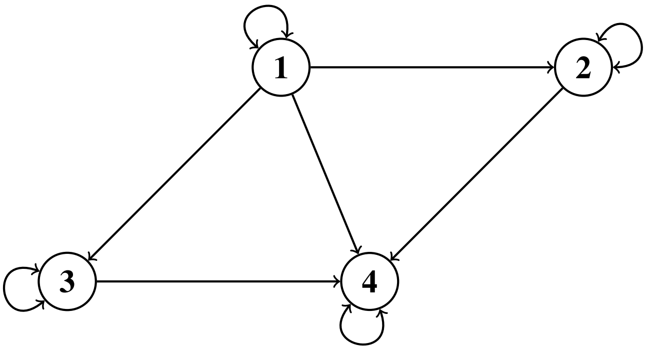

Figure 1. The graph model of Example 2.3.

We approach our main result via the following lemma.

Lemma 2.4. *Let* (Θ, ρ) *be a MS equipped with a graph* Ψ *and* $\tau : \Theta \longrightarrow \Theta$ *be a* Ψ*-Hardy-Rogers operator. If* $\varrho \in \Theta$ *satisfies the condition* $(\varrho, \tau\varrho) \in H(\Psi)$*, then there exists an* $\omega \in (0, 1)$ *such that for all* $l \in \mathbb{N}$*,*

$$\rho(\tau^l\varrho, \tau^{l+1}\varrho) \leq \omega^l \rho(\varrho, \tau\varrho). \tag{2.2}$$

Proof. Let $\varrho \in \Theta$ with $(\varrho, \tau\varrho) \in H(\Psi)$. An easy induction shows that $(\tau^l\varrho, \tau^{l+1}\varrho) \in H(\Psi)$ for all $l \in \mathbb{N}$.Then, for all $l \in \mathbb{N}$,

$$\begin{aligned}
\rho(\tau^l\varrho, \tau^{l+1}\varrho) &\leq \alpha\rho(\tau^{l-1}\varrho, \tau^l\varrho) + \beta\rho(\tau^{l-1}\varrho, \tau^l\varrho) + \gamma\rho(\tau^l\varrho, \tau^{l+1}\varrho) \\
&\quad + \delta\rho(\tau^{l-1}\varrho, \tau^{l+1}\varrho) + \lambda\rho(\tau^l\varrho, \tau^l\varrho) \\
&\leq \alpha\rho(\tau^{l-1}\varrho, \tau^l\varrho) + \beta\rho(\tau^{l-1}\varrho, \tau^l\varrho) + \gamma\rho(\tau^l\varrho, \tau^{l+1}\varrho) \\
&\quad + \delta[\rho(\tau^{l-1}\varrho, \tau^l\varrho) + \rho(\tau^l\varrho, \tau^{l+1}\varrho)],
\end{aligned}$$

from which we have

$$\rho(\tau^l\varrho, \tau^{l+1}\varrho) \leq \left(\frac{\alpha + \beta + \delta}{1 - \gamma - \delta}\right)\rho(\tau^{l-1}\varrho, \tau^l\varrho). \tag{2.3}$$

By symmetry, we can interchange β with γ and δ with λ in (2.3), to have

$$\rho(\tau^l\varrho, \tau^{l+1}\varrho) \leq \left(\frac{\alpha + \gamma + \lambda}{1 - \beta - \lambda}\right)\rho(\tau^{l-1}\varrho, \tau^l\varrho). \tag{2.4}$$

Then

$$\omega = \min\left\{\left(\frac{\alpha + \gamma + \lambda}{1 - \beta - \lambda}\right), \left(\frac{\alpha + \beta + \delta}{1 - \gamma - \delta}\right)\right\} \in (0, 1),$$

and $\rho(\tau^l\varrho, \tau^{l+1}\varrho) \leq \omega\rho(\tau^{l-1}\varrho, \tau^l\varrho)$. Hence, for all $l \in \mathbb{N}$,

$$\rho(\tau^l\varrho, \tau^{l+1}\varrho) \leq \omega^l \rho(\varrho, \tau\varrho).$$

Lemma 2.5. *Let* (Θ, ρ) *be a MS equipped with the graph* Ψ *and* $\tau : \Theta \longrightarrow \Theta$ *be a* Ψ*-Hardy-Rogers operator such that the graph* Ψ *is* τ*-connected. Then, for all* $\varrho \in \Theta$*, the sequence* $\{\tau^l\varrho\}_{l \in \mathbb{N}}$ *is Cauchy.*

Proof. Let $\varrho \in \Theta$ be fixed. We discuss two cases:

Case 1. If $(\varrho, \tau\varrho) \in H(\Psi)$, then by Lemma 2.4, there exists $\omega \in (0,1)$ such that $\rho(\tau^l\varrho, \tau^{l+1}\varrho) \leq \omega^l \rho(\varrho, \tau\varrho)$, for all $l \in \mathbb{N}$. Since $\omega \in (0,1)$, we have

$$\sum_{l=0}^{\infty} \rho(\tau^l\varrho, \tau^{l+1}\varrho) \leq \sum_{l=0}^{\infty} \omega^l \rho(\varrho, \tau\varrho) = \frac{1}{1-\omega}\rho(\varrho, \tau\varrho) < \infty.$$

Hence, a standard argument shows that $\{\tau^l\varrho\}_{l\in\mathbb{N}}$ is a Cauchy sequence.

Case 2. If $(\varrho, \tau\varrho) \notin H(\Psi)$, then by τ-connectedness of Ψ, we can find a path in Ψ, $\{\varrho_i\}_{i=0}^L$ from ϱ to $\tau\varrho$ such that $\varrho_0 = \varrho, \varrho_L = \tau\varrho$ with $(\varrho_{i-1}, \varrho_i) \in H(\Psi)$ for all $i = \overline{1, L}$ and $(\varrho_i, \tau\varrho_i) \in H(\Psi)$ for all $i = \overline{1, L-1}$. Then, by triangle inequality and (2.2), we get

$$\begin{aligned}
&\rho(\tau^l\varrho, \tau^{l+1}\varrho) \leq \sum_{i=1}^{L} \rho(\tau^l\varrho_{i-1}, \tau^l\varrho_i) \\
\leq\ & \alpha \sum_{i=1}^{L} \rho(\tau^{l-1}\varrho_{i-1}, \tau^{l-1}\varrho_i) + \beta \sum_{i=1}^{L} \rho(\tau^{l-1}\varrho_{i-1}, \tau^l\varrho i-1) \\
&+\gamma \sum_{i=1}^{L} \rho(\tau^{l-1}\varrho_i, \tau^l\varrho_i) + \delta \sum_{i=1}^{L} \rho(\tau^{l-1}\varrho_{i-1}, \tau^l\varrho_i) + \lambda \sum_{i=1}^{L} \rho(\tau^{l-1}\varrho_i, \tau^l\varrho i-1) \\
\leq\ & \alpha \sum_{i=1}^{L} \rho(\tau^{l-1}\varrho_{i-1}, \tau^{l-1}\varrho_i) + \beta\rho(\tau^{l-1}\varrho, \tau^l\varrho) + \beta\omega^{l-1} \sum_{i=2}^{L} \rho(\varrho_{i-1}, \tau\varrho_{i-1}) \\
&+\gamma\rho(\tau^l\varrho, \tau^{l+1}\varrho) + \gamma\omega^{l-1} \sum_{i=2}^{L} \rho(\varrho_i, \tau\varrho_i) \\
&+\delta \sum_{i=1}^{L} \rho(\tau^{l-1}\varrho_{i-1}, \tau^l\varrho_i) + \lambda \sum_{i=1}^{L} \rho(\tau^{l-1}\varrho_i, \tau^l\varrho_i) \\
\leq\ & (\alpha+\beta+\gamma) \sum_{i=1}^{L} \rho(\tau^{l-1}\varrho_{i-1}, \tau^{l-1}\varrho_i) + (\beta+\gamma) \sum_{i=2}^{L} \omega^{l-1}\rho(\varrho_i, \tau\varrho_{i-1}) \\
&+\delta \sum_{i=1}^{L} \rho(\tau^{l-1}\varrho_{i-1}, \tau^l\varrho_i) + \lambda \sum_{i=1}^{L} \rho(\tau^{l-1}\varrho_i, \tau^l\varrho i-1).
\end{aligned}$$

Let $\varrho_l = \sum_{i=1}^{L} \rho(\tau^l\varrho_{i-1}, \tau^l\varrho_i), r(\varrho) = (\beta+\gamma)\sum_{i=2}^{L} \rho(\varrho_i, \tau\varrho_{i-1}), \eta = \frac{\alpha+\beta+\gamma}{1-\delta-\lambda}$. Then,

$$\varrho_l \leq (\alpha+\beta+\gamma)\varrho_{l-1} + (\beta+\gamma)\omega^{l-1}r(\varrho) + (\delta+\lambda)\varrho_l,$$

from which we have

$$\begin{aligned}
\varrho_l \leq\ & \left(\frac{\alpha+\beta+\gamma}{1-\delta-\lambda}\right)\varrho_{l-1} + \left(\frac{\beta+\gamma}{1-\delta-\lambda}\right)\omega^{l-1}r(\varrho) \\
\leq\ & \eta\varrho_{l-1} + \left(\frac{\beta+\gamma}{1-\delta-\lambda}\right)\omega^{l-1}r(\varrho).
\end{aligned} \tag{2.5}$$

Using (2.5) and direct calculation, we obtain

$$\varrho_l \leq l\left(\frac{\beta+\gamma}{1-\delta-\lambda}\right)\omega^{l-1}r(\varrho), \tag{2.6}$$

for all $l \in \mathbb{N}$. Since $\eta \in (0,1)$ and by (2.6),

$$\begin{aligned}\sum_{l=0}^{\infty}\rho(\tau^l\varrho, \tau^{l+1}\varrho) &\leq \sum_{l=0}^{\infty}\varrho_l \\ &\leq \left(\frac{\beta+\gamma}{1-\delta-\lambda}\right)r(\varrho)\sum_{l=0}^{\infty}l\omega^{l-1} \\ &= \frac{\beta+\gamma}{(1-\delta-\lambda)(1-\omega)^2}r(\varrho) < \infty,\end{aligned}$$

and a standard argument reveals that $\{\varrho_l\}_{l\in\mathbb{N}}$ is a Cauchy sequence.

The next result is the principal theorem of this paper.

Theorem 2.6. *Let (Θ, ρ) be a complete MS equipped with a graph Ψ and $\tau : \Theta \longrightarrow \Theta$ be a Ψ-Hardy-Rogers operator. Assume further that*

(G1) Ψ is τ-connected;
(G2) for any sequence $\{\varrho_l\}_{l\in\mathbb{N}}$ in Θ, if $\varrho_l \longrightarrow \varrho$ as $l \longrightarrow \infty$ and $(\varrho_l, \varrho_{l+1}) \in H(\Psi)$ for all $l \in \mathbb{N}$, then there is a subsequence $\{\varrho_{\zeta_l}\}_{\zeta\in\mathbb{N}}$ with $(\varrho_{\zeta_l}, \varrho) \in H(\Psi)$ for all $\zeta \in \mathbb{N}$.

Then τ is a PO.

Proof. By Lemma 2.5, $\{\tau^l\varrho\}_{l\in\mathbb{N}}$ is a Cauchy sequence in Θ for all $\varrho \in \Theta$. And, by hypothesis, it follows that $\{\tau^l\varrho\}_{l\in\mathbb{N}}$ is convergent. Let $\varrho, \sigma \in \Theta$. Then $\tau^l\varrho \longrightarrow u^*$ and $\tau^l\sigma \longrightarrow v^*$ as $l \longrightarrow \infty$. Now, consider the following cases:
Case 1. If $(\varrho, \sigma) \in H(\Psi)$, we have $(\tau^l\varrho, \tau^l\sigma) \in H(\Psi)$ for all $l \in \mathbb{N}$. Then, for all $l \in \mathbb{N}$,

$$\begin{aligned}\rho(\tau^l\varrho, \tau^l\sigma) \leq\ & \alpha\rho(\tau^{l-1}\varrho, \tau^{l-1}\sigma) + \beta\rho(\tau^{l-1}\varrho, \tau^l\varrho) + \gamma\rho(\tau^{l-1}\sigma, \tau^l\sigma) \\ & + \delta\rho(\tau^{l-1}\varrho, \tau^l\sigma) + \lambda\rho(\tau^{l-1}\sigma, \tau^l\varrho).\end{aligned} \tag{2.7}$$

Taking limit as $l \longrightarrow \infty$ in (2.7), yields

$$\begin{aligned}\rho(u^*, v^*) &\leq \alpha\rho(u^*, v^*) + \delta\rho(u^*, v^*) + \lambda\rho(v^*, u^*) \\ &= (\alpha+\delta+\lambda)\rho(u^*, v^*),\end{aligned}$$

from which we get $(1-\alpha-\delta-\lambda)\rho(u^*, v^*) \leq 0$. Since $(1-\alpha-\delta-\lambda) > 0$, it comes up that $u^* = v^*$.
Case 2. If $(\varrho, \sigma) \notin H(\Psi)$, there is a path in Ψ, $\{\varrho_i\}_{i=0}^{L}$ from ϱ to σ such that $\varrho_0 = \varrho, \varrho_L = \sigma$ with $(\varrho_{i-1}, \varrho_i) \in H(\Psi)$ for all $i = \overline{1, L}$ and $(\varrho_i, \tau\varrho_i) \in H(\Psi)$ for all $i = \overline{1, L-1}$. Then $(\tau^l\varrho_{i-1}, \tau^l\varrho_i) \in H(\Psi)$ for all $l \in \mathbb{N}$ and $i = \overline{1, L-1}$. And, by triangle inequality, we get

$$\begin{aligned}\rho(\tau^l\varrho, \tau^n y) \leq\ & \sum_{i=1}^{L}\rho(\tau^l\varrho_{i-1}, \tau^l\varrho_i) \\ \leq\ & \alpha\sum_{i=1}^{L}\rho(\tau^{l-1}\varrho_{i-1}, \tau^{l-1}\varrho_i) + \beta\sum_{i=1}^{L}\rho(\tau^{l-1}\varrho_{i-1}, \tau^l\varrho_{i-1}) + \gamma\rho(\tau^{l-1}\varrho_i, \tau^l\varrho_i) \\ & + \delta\rho(\tau^{l-1}\varrho_{i-1}, \tau^l\varrho_i) + \lambda\rho(\tau^{l-1}\varrho_i, \tau^l\varrho_{i-1}).\end{aligned} \tag{2.8}$$

By Lemma 2.5 and hypothesis, $\{\tau^l \varrho_i\}_{l\in\mathbb{N}}$ is convergent. Hence, using the continuity of the distance ρ, we obtain that $\{\rho(\tau^l \varrho_{i-1}, \tau^l \varrho_i)\}_{l\in\mathbb{N}}$ is convergent. Let $\lim\limits_{l\longrightarrow\infty} \rho(\tau^l \varrho_{i-1}, \tau^l \varrho_i) = \zeta_i$ for all $i = \overline{1, L}$. Then letting $l \longrightarrow \infty$ in (2.8), gives $\rho(u^*, v^*) \leq 0$, which imply that $u^* = v^*$. Whence, for all $\varrho \in \Theta$, there exists $u^* \in \Theta$ such that $\lim\limits_{l\longrightarrow\infty} \tau^l \varrho = u^*$.
Now, we will show that $u^* = \tau u^*$. Since the graph Ψ is τ-connected, there exists at least $\varrho_0 \in \Theta$ such that $(\varrho_0, \tau\varrho_0) \in H(\Psi)$ and so $(\tau^l \varrho_0, \tau^{l+1}\varrho_0) \in H(\Psi)$ for all $l \in \mathbb{N}$. But $\lim\limits_{l\longrightarrow\infty} \tau^l \varrho = u^*$, then by the assumption (iii), there exists a subsequence $\{\tau^{\zeta_l}\varrho_0\}_{l\in\mathbb{N}}$ with $(\tau^{\zeta_l+1}\varrho_0, \tau u^*) \in H(\Psi)$ for all $l \in \mathbb{N}$. Then, for all $l \in \mathbb{N}$, we have

$$\begin{aligned} \rho(u^*, \tau u^*) \leq\ & \rho(u^*, \tau^{\zeta_l+1}\varrho_0) + \rho(\tau^{\zeta_l+1}\varrho_0, \tau u^*) \\ \leq\ & \rho(u^*, \tau^{\zeta_l+1}\varrho_0) + \alpha\rho(\tau^{\zeta_l}\varrho_0, u^*) + \beta\rho(\tau^{\zeta_l}\varrho_0, \tau^{\zeta_l}\varrho_0) + \gamma\rho(u^*, \tau u^*) \\ & + \delta\rho(\tau^{\zeta_l}\varrho_0, \tau u^*) + \lambda\rho(u^*, \tau^{\zeta_l+1}\varrho_0). \end{aligned} \tag{2.9}$$

As $l \longrightarrow \infty$ in (2.9), we have $\rho(u^*, \tau u^*) \leq \gamma\rho(u^*, \tau u^*)$; from which we have $u^* = \tau u^* \in Fix(\tau)$.
If we have $\tau\sigma = \sigma$ for some $\sigma \in \Theta$, then from above, we must get $\tau^n y \longrightarrow u^*$, so $\sigma = u^*$. It follows that τ is a PO.

Example 2.7. Let $\Theta = \{l + 1 : l = \overline{1, 3}\} \cup \{4l : l = \overline{2, 4}\}$ and $\rho(\varrho, \sigma) = |\varrho - \sigma|$, for all $\varrho, \sigma \in \Theta$. Then (Θ, ρ) is a complete MS. Define the operator $\tau : \Theta \longrightarrow \Theta$ as follows:

$$\tau\varrho = \begin{cases} \frac{\varrho}{4}, & \text{if } \varrho \in \{4l : l = \overline{2, 4}\} \\ 2, & \text{if } \varrho \in \{l + 1 : l = \overline{1, 3}\}. \end{cases}$$

Let $\alpha = \frac{11}{100}, \beta = \frac{3}{25}, \gamma = \frac{7}{100}, \delta = \frac{1}{50}$ and $\lambda = \frac{1}{20}$, and consider the graph $\Psi = (\Delta(\Psi), H(\Psi))$, where $\Delta(\Psi) = \Theta$ and

$$H(\Psi) = \{(\varrho, \sigma) \in \Theta \times \Theta \backslash \{(2, 8), (3, 12)\}\} \cup \Omega.$$

Then, it is easy to see that the operator τ is edge-preserving and Ψ is τ-connected.

To see that the inequality (2.1) is satisfied, we examine the following cases:

Case 1: $\varrho, \sigma \in \{4l : l = \overline{2, 4}\}$, $\varrho = \sigma$;
Case 2: $\varrho, \sigma \in \{4l : l = \overline{2, 4}\}$, $\varrho \neq \sigma$;
Case 3: $\varrho, \sigma \in \{l + 1 : l = \overline{1, 3}\}$, $\varrho = \sigma$;
Case 4: $\varrho, \sigma \in \{l + 1 : l = \overline{1, 3}\}$, $\varrho \neq \sigma$;
Case 5: $\varrho \in \{4l : l = \overline{2, 4}\}$ and $\sigma \in \{l + 1 : l = \overline{1, 3}\}$;
Case 6: $\varrho \in \{l + 1 : l = \overline{1, 3}\}$ and $\sigma \in \{4l : l = \overline{2, 4}\}$.

We show via the following Table 1 that the inequality (2.1) is valid under the above Cases 1–6.

Table 1. Table of values for cases 1–6.

Cases	ϱ	σ	$LHS\ of$ 2.1	$RHS\ of$ 2.1
Case 1	8	8	0	4.68
	12	12	0	7.02
	16	16	0	9.36
Case 2	8	12	1	6.4
	8	16	2	8.12
	12	8	1	6.5
	12	16	1	8.74
	16	8	2	8.32
	16	12	1	8.84
Case 3	2	2	0	0
	3	3	0	0.78
	4	4	0	1.56
Case 4	2	3	0	0.55
	2	4	0	1.1
	3	2	0	0.53
	3	4	0	1.33
	4	2	0	1.06
	4	3	0	1.31
Case 5	8	3	0	3.43
	8	4	0	3.68
	12	2	1	5.2
	12	14	1	8.6
	16	2	2	7.22
	16	3	2	7.27
	16	4	2	7.32
Case 6	2	12	1	5.38
	2	16	2	7.46
	3	8	0	3.53
	3	16	2	7.33
	4	8	0	3.76
	4	12	1	5.48
	4	16	2	7.2

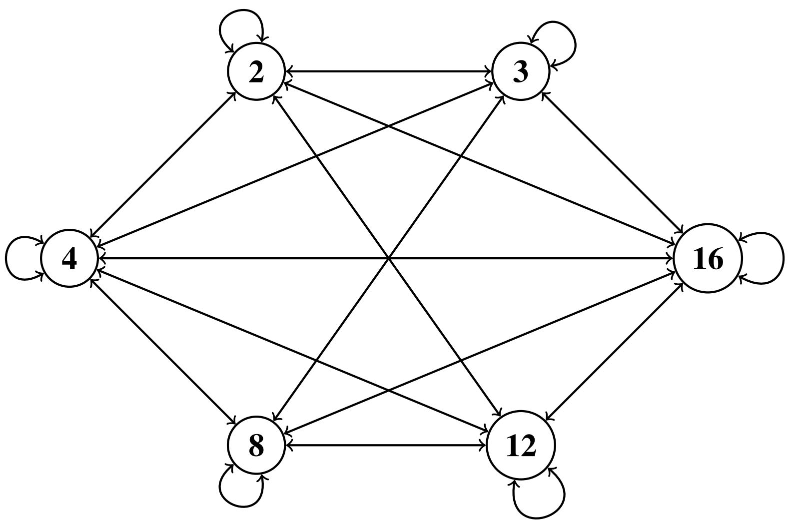

Figure 2. The symmetric graph $\hat{\Psi}$ given in Example 2.7.

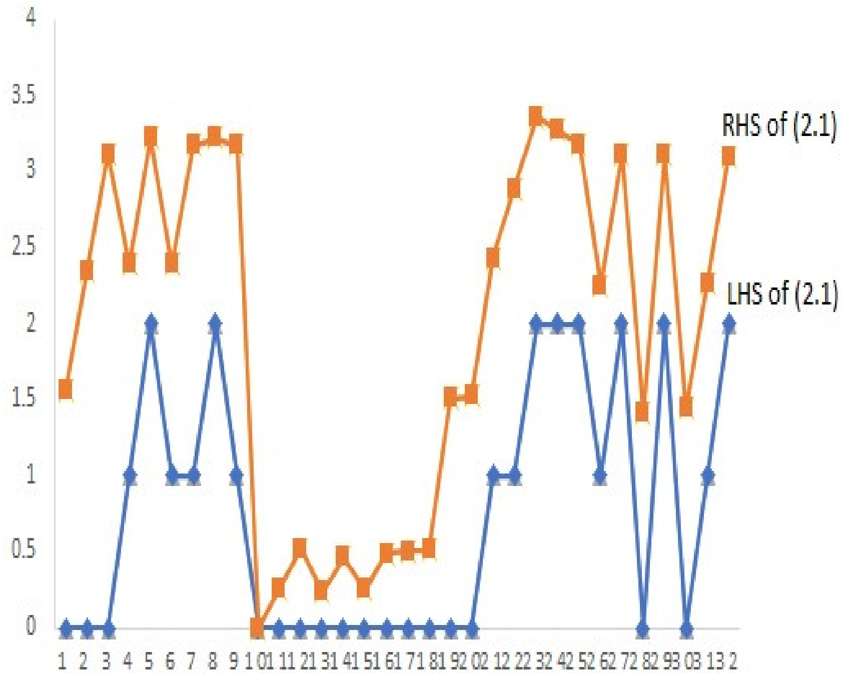

Figure 3. Illustration of the contractive inequality 2.1.

Hence, all the hypotheses of Theorem 2.6 are valid, and thus τ has a unique FP $u^* = 2$ and $\lim_{l \longrightarrow \infty} \tau^l \varrho = 2$ for all $\varrho \in \Theta$. Therefore, τ is a *PO*.

Corollary 1. *Let (Θ, ρ) be a complete MS equipped with a graph Ψ and $\tau : \Theta \longrightarrow \Theta$ be an operator. Assume further that:*

(C1) Ψ is weakly connected;
(C2) there exist nonnegative numbers p, q, r obeying $p + 2q + 2r < 1$ such that for each $(\varrho, \sigma) \in H(\Psi)$,

$$\begin{aligned} \rho(\tau\varrho, \tau\sigma) \quad \leq \quad & p\rho(\varrho, \sigma) + q[\rho(\varrho, \tau\varrho) + \rho(\sigma, \tau\sigma)] \\ & r[\rho(\varrho, \tau\sigma) + \rho(\sigma, \tau\varrho)]; \end{aligned}$$

(C3) for any sequence $\{\varrho_l\}_{l \in \mathbb{N}}$ in Θ, if $\varrho_l \longrightarrow \varrho$ and $(\varrho_l, \varrho_{l+1}) \in H(\Psi)$ for each $l \in \mathbb{N}$, then there is a subsequence $\{\varrho_{\zeta_l}\}_{\zeta \in \mathbb{N}}$ with $(\varrho_{\zeta_l}, \varrho) \in H(\Psi)$ for $l \in \mathbb{N}$.

Then τ is a PO.

Proof. Obviously, the mapping τ is a $\hat{\Psi}$-Hardy-Rogers operator. Hence, taking $\beta = \gamma$ and $\delta = \lambda$ in Theorem 2.6, completes the proof.

Corollary 2. *[18, Theorem 6] Let (Θ, ρ) be a complete MS equipped with a graph Ψ and $\tau : \Theta \longrightarrow \Theta$ be an operator. Assume further that:*

(D1) Ψ is τ-connected;

(D2) *there exist nonnegative numbers* p, q, r *obeying* $p + q + r < 1$ *such that for each* $(\varrho, \sigma) \in H(\Psi)$,

$$\rho(\tau\varrho, \tau\sigma) \leq p\rho(\varrho, \sigma) + q\rho(\varrho, \tau\varrho) + r\rho(\sigma, \tau\sigma);$$

(D3) *for any sequence* $\{\varrho_l\}_{l\in\mathbb{N}}$ *in* Θ*, if* $\varrho_l \longrightarrow \varrho$ *and* $(\varrho_l, \varrho_{l+1}) \in H(\Psi)$ *for each* $l \in \mathbb{N}$*, there exists a subsequence* $\{\varrho_{\zeta_l}\}_{\zeta\in\mathbb{N}}$ *with* $(\varrho_{\zeta_l}, \varrho) \in H(\Psi)$ *for each* $l \in \mathbb{N}$.

Then τ *is a PO.*

Proof. By condition (D2), τ is a Ψ-Ćirič-Reich-Rus operator. Hence, τ is a Ψ-Hardy-Rogers operator with the constant $\alpha = p, \beta = q, \gamma = r$ and $\lambda = 0$. Thus, by Theorem 2.6, τ is a PO.

3. Applications in partially ordered MS and cyclic operators

In this section, we apply Theorem 2.6 to derive some of its analogues in the bodywork of partially ordered MS and cyclic operators.

Theorem 3.1. *Let* $(\Theta, \preceq)$ *be a partially ordered set and* ρ *be a metric on* Θ *such that the MS* (Θ, ρ) *is complete. Let* $\tau : \Theta \longrightarrow \Theta$ *be an increasing operator such that the following conditions are obeyed:*

(P1) *there exist nonnegative numbers* $\alpha, \beta, \gamma, \delta, \lambda$ *with* $\alpha + \beta + \gamma + \delta + \lambda < 1$ *such that for all* $\varrho, \sigma \in \Theta$*, with* $\varrho \preceq \sigma$*, we have*

$$\begin{aligned}\rho(\tau\varrho, \tau\sigma) \;\; \leq \;\; & \alpha\rho(\varrho, \sigma) + \beta\rho(\varrho, \tau\varrho) + \gamma\rho(\sigma, \tau\sigma) \\ & + \delta\rho(\varrho, \tau\sigma) + \lambda\rho(\sigma, \tau\varrho);\end{aligned}$$

(P2) *for each* $\varrho, \sigma \in \Theta$*, incomparable elements of* $(\varrho, \preceq)$*, there exists* $z \in \Theta$ *such that* $\varrho \preceq z, \sigma \preceq z$ *and* $z \preceq \tau z$*;*

(P3) *if an increasing sequence* $\{\varrho_l\}_{l\in\mathbb{N}}$ *converges to* $\varrho \in \Theta$*, then* $\varrho_l \preceq \varrho$ *for each* $l \in \mathbb{N}$.

Then τ *is a Picard operator.*

Proof. Consider the graph Ψ with $\Delta(\Psi) = \Theta$, and

$$H(\Psi) = \{(\varrho, \sigma) \in \Theta \times \Theta : \varrho \preceq \sigma\}.$$

Since the mapping τ is increasing, and $(P1)$ is satisfied, it follows that τ is a Ψ-Hardy-Rogers operator. By (P2), the graph Ψ is τ-connected, and the assumption (P3) implies that Condition (G2) of Theorem 2.6. Whence, the conclusion can be deduced from Theorem 2.6.

Hereunder, we discuss FP theorem for cyclic operators. Let $\zeta \geq 2$ and $\{D_i\}_{i=1}^{\zeta}$ be nonempty closed subsets of a complete MS Θ. A mapping $\tau : \bigcup_{i=1}^{\zeta} D_i \longrightarrow \bigcup_{i=1}^{\zeta} D_i$ is called a cyclic operator, if

$$\tau(D_i) \subseteq D_{i+1}, \tag{3.1}$$

for all $i \in \{1, 2, \cdots, \zeta\}$, where $D_{\zeta+1} = D_1$.

Theorem 3.2. *Let $D_1, D_2, \cdots, D_\zeta, D_{\zeta+1} = D_1$ be nonempty closed subsets of a complete MS (Θ, ρ) and suppose that $\tau : \bigcup_{i=1}^{\zeta} D_i \longrightarrow \bigcup_{i=1}^{\zeta} D_i$ be a cyclic operator. If there exist nonnegative numbers $\alpha, \beta, \gamma, \delta, \lambda$ obeying $\alpha + \beta + \gamma + \delta + \lambda < 1$ such that for each pair $(\varrho, \sigma) \in D_i \times D_{i+1}, i = \overline{1, \zeta}$, we have*

$$\begin{aligned} \rho(\tau\varrho, \tau\sigma) &\leq \alpha\rho(\varrho, \sigma) + \beta\rho(\varrho, \tau\varrho) + \gamma\rho(\sigma, \tau\sigma) \\ &\quad + \delta\rho(\varrho, \tau\sigma) + \lambda\rho(\sigma, \tau\varrho), \end{aligned}$$

then τ is a PO.

Proof. Take $\Upsilon = \bigcup_{i=1}^{\zeta} D_i$. Then (Υ, ρ) is a complete MS. Consider the graph Ψ with $\Delta(\Psi) = \Upsilon$, and

$$H(\Psi) = \{(\varrho, \sigma) \in \Upsilon \times \Upsilon : \text{ there exists } i \in \{1, \cdots, \zeta\} \text{ such that } \varrho \in D_i \text{ and } \sigma \in D_{i+1}\}.$$

Since τ is a cyclic operator, we get $(\tau\varrho, \tau\sigma) \in H(\Psi)$, for all $(\varrho, \sigma) \in H(\Psi)$. Now, by hypothesis, τ is a Ψ-Hardy-Rogers operator and the graph Ψ is τ-connected. Let $\{\varrho_l\}_{l\in\mathbb{N}}$ be a sequence in Θ such that $\varrho_l \longrightarrow \varrho$ and $(\varrho_l, \varrho_{l+1}) \in H(\Psi)$ for each $l \in \mathbb{N}$. Then there exists $j \in \{1, 2, \cdots, l\}$ such that $\varrho \in \underline{D_j}$. But in respect of (3.1), the sequence $\{\varrho_l\}_{l\in\mathbb{N}}$ has an infinite number of terms in each D_i, for all $i \in \overline{1, \zeta}$. The subsequence of the sequence $\{\varrho_l\}_{l\in\mathbb{N}}$ formed by the terms which is in D_{j-1} obeys condition (G2) of Theorem 2.6. Consequently, τ is a PO.

If, in Theorem 3.2, $\beta = \gamma$ and $\delta = 0$, we derive the main result of Petric [16]. In similar steps, more consequences of Theorems 2.6, 3.1, 3.2 can be pointed out and discussed.

4. Applications to existence of solutions of integral equations

Integral equations are found to be of great usefulness in studying dynamical systems and stochastic processes. Some examples are in the areas of oscillation problems, sweeping process, granular systems, control problems, an so on. Integral equations arise in several phenomena in mathematical physics, bio-mathematics, control theory, critical point theory for non-smooth energy functionals, differential variational inequalities, fuzzy set arithmetic, traffic problems, to mention but a few. Commonly, the first most concerned problem in the study of differential or integral equation is the conditions for existence of its solutions. Along this direction, many authors have employed different FP approaches to obtain existence results of differential or integral equations in abstract spaces (e.g., see [25, 26]).

In this section, Theorem 2.6 is applied to study new conditions for the existence of a solution to the integral equation:

$$\varrho(t) = g(t) + \int_0^T A(t, s)K(s, \varrho(s))ds. \tag{4.1}$$

Note that if, in (4.1), $g(t) = 0$, then Problem (4.1) represents an integral reformulation of physical phenomenon such as the motion of a spring that is under the influence of a frictional force or a damping force.

Let $\Theta = C([0, T], \mathbb{R})$ be the set of real-valued continuous functions defined on $[0, T]$, and let $\rho : \Theta \times \Theta \longrightarrow \mathbb{R}$ be defined by

$$\rho(\varrho, \sigma) = \max\{|\varrho(t) - \sigma(\sigma)| : t \in [0, T]\}, \text{ for all } \varrho, \sigma \in \Theta.$$

Then (Θ, ρ) is a complete MS. Suppose also that (Θ, ρ) is equipped with a graph G. Moreover, let $L : \Theta \longrightarrow \Theta$ be defined by

$$L(\varrho)(t) = g(t) + \int_0^T A(t, s)K(s, \varrho(s))ds. \tag{4.2}$$

Assume that:

(P1) $K : [0, T] \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous;
(P2) $g : [0, T] \longrightarrow \mathbb{R}$ is continuous;
(P3) $A : [0, T] \times \mathbb{R} \longrightarrow [0, \infty)$ is continuous;
(P4) $((\varrho, \sigma) \in E(G)$ implies $(L\varrho, L\sigma) \in E(G))$, for all $\varrho, \sigma \in \Theta$;
(P5) there exist nonnegative reals $\alpha, \beta, \gamma, \delta, \zeta$ with $\alpha + \beta + \gamma + \delta + \zeta < 1$ such that for all $(\varrho, \sigma) \in E(G)$, and, for all $s \in [0, T]$,

$$\begin{aligned} |K(s, \varrho(s)) - K(s, \sigma(s))| \quad \leq \quad & \alpha|\varrho(s) - \sigma(s)| + \beta|\varrho(s) - L(\varrho(s))| + \gamma|\sigma(s) - L(\sigma(s))| \\ & + \delta|\varrho(s) - L(\sigma(s))| + \zeta|\sigma(s) - L(\varrho(s))|; \end{aligned}$$

(P6) for any sequence $\{\varrho_n\}_{n\in\mathbb{N}}$, if $\varrho_n \longrightarrow \varrho$ as $n \longrightarrow \infty$ and $(\varrho_n, \varrho_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, then there is a subsequence $\{\varrho_{\zeta_n}\}_{\zeta\in\mathbb{N}}$ with $(\varrho_{\zeta_n}, \varrho) \in E(G)$ for all $n \in \mathbb{N}$;
(P7) $\max\limits_{t\in[0,T]} \int_0^T A(t, s)ds \leq 1$.

Theorem 4.1. *Under the hypotheses (P1)–(P7), the integral Eq* (4.1) *has a solution in* Θ.

Proof. Consider the operator $L : \Theta \longrightarrow \Theta$ given by (4.3). Let $(\varrho, \sigma) \in E(G)$. Then, from (P5), we obtain

$$\begin{aligned} & |L(\varrho)(t) - L(\sigma)(t)| \\ = \quad & \left| \int_0^T A(t, s)[K(s, \varrho(s)) - K(s, \sigma(s))]ds \right| \\ \leq \quad & \int_0^T A(t, s)|K(s, \varrho(s)) - K(s, \sigma(s))|ds \\ \leq \quad & \int_0^T A(t, s)ds \Big\{ \alpha|\varrho(s) - \sigma(s)| + \beta|\varrho(s) - L(\varrho(s))| + \gamma|\sigma(s) - L(\sigma(s))| \\ & + \delta|\varrho(s) - L(\sigma(s))| + \zeta|\sigma(s) - L(\varrho(s))| \Big\}. \end{aligned} \tag{4.3}$$

From (4.3),

$$\begin{aligned} & \max_{t\in[0,T]} |L(\varrho)(t) - L(\sigma)(t)| \\ \leq \quad & \max_{t\in[0,T]} \Big\{ \alpha|\varrho(s) - \sigma(s)| + \beta|\varrho(s) - L(\varrho(s))| + \gamma|\sigma(s) - L(\sigma(s))| \\ & + \delta|\varrho(s) - L(\sigma(s))| + \zeta|\sigma(s) - L(\varrho(s))| \Big\}. \end{aligned}$$

The above expression is equivalent to

$$\begin{aligned}\rho(L\varrho, L\sigma) \quad \leq \quad & \alpha\rho(\varrho,\sigma) + \beta\rho(\varrho, L\varrho) + \gamma\rho(\sigma, L\sigma) \\ & +\delta\rho(\varrho, L\sigma) + \zeta\rho(\sigma, L\varrho).\end{aligned}$$

Hence, all the assumptions of Theorem 2.6 are satisfied, and consequently the operator L has a fixed point in Θ, which corresponds to the solution of the integral Eq (4.1).

5. Conclusions

In this note, the notion of Ψ-Hardy-Rogers operator has been introduced and new conditions for such mapping to be a PO have been examined. A few important special cases in the framework of partially ordered MS and cyclic operators have been pointed out and discussed. As an additional application, Theorem 2.6 is employed to discuss novel conditions for the existence of a solution to an integral equation. The main idea presented herein is an extension of the results in [7, 19]. The concepts of set-valued Ψ-Hardy-Rogers operator $\tau : \Theta \longrightarrow 2^{\Theta}$ and fuzzy set-valued Ψ-Hardy-Rogers operator $\tau : \Theta \longrightarrow I^{\Theta}$, where I^{Θ} is the family of fuzzy sets in Θ, would be some appreciable future investigations in the domain of multivalued mappings.

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Conflict of interest

The authors declare that they have no competing interests.

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