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Research article

Existence and multiplicity of sign-changing solutions for supercritical quasi-linear Schrödinger equations

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Abstract: This paper focuses on a class of supercritical, quasi-linear Schrödinger equations. Based on the methods of invariant sets, some results about the existence and multiplicity of sign-changing solutions for supercritical equations are obtained.

Keywords: Quasi-linear Schrödinger equations; supercritical growth; variational methods; sign-changing solutions; invariant sets

1. Introduction and main results

1.1. Background

In the past two decades, many attentions have been devoted to the investigation of the quasi-linear Schrödinger equation:

$$i\partial_t z = -\Delta z + W(x)z - l(|z|^2)z + \gamma z\Delta(|z|^2),\ x \in \mathbb{R}^N, \tag{1.1}$$

where $z : \mathbb{R}^N \times \mathbb{R} \to \mathbb{C}$, $W : \mathbb{R}^N \to \mathbb{R}$ is a given potential, γ is a real constant and l is real functions.

Equation (1.1) appears in various fields of physics (see [1, 2]), and is known to be more accurate in many physical phenomena compared with the semi-linear Schrödinger equation.

$$i\partial_t z = -\Delta z + W(x)z - l(|z|^2)z,\ x \in \mathbb{R}^N.$$

The additional term $\gamma z\Delta(|z|^2)$ appears in various physical models and arises due to:
a) the non-locality of the nonlinear interaction for electron [3],
b) the weak nonlocal limit for nonlocal nonlinear Kerr media [4],
c) the surface term for superfluid film [5].

In particular, the standing wave solution of Eq (1.1) is also a solution of the form $z(t, x) := \exp(-iEt)u(x)$ with $E > 0$, we are led to study the following elliptic equation

$$-\Delta u + V(x)u + \gamma\Delta(u^2)u = f(u),\ x \in \mathbb{R}^N, \tag{1.2}$$

where $V(x) = W(x) - E$ and $f(t) := l(|t|^2)t$. The parameter γ represents the strength of each effect and can be assumed to be positive or negative in different situations.

Notice that, for $\tau = 0$, the Eq (1.2) is a Schrödinger-type equation, which is fundamental in modern physics and many other fields, see e.g., [6–11].

1.2. Motivation

In the last decades, scholars have obtained existence and multiplicity of solutions for Eq (1.2) with $\gamma < 0$, based on variational methods. To the best of our knowledge, Poppenberg, Schmitt and Wang proved the existence of positive solutions for the first time in [12] by means of a constrained minimization argument. By using a change of variable and converting the quasi-linear Eq (1.2) into a semi-linear one in an Orlicz space framework, Liu et al. in [13] obtained existence of solutions for a general case. Subsequently, Colin and Jeanjean chose the classical Sobolev space $H^1(\mathbb{R}^N)$ in [14], then they can use a simpler and shorter proof than [13] to get the same conclusion. We refer the readers to [15–22] for more results.

For $\gamma > 0$, in [23], Alves, Wang and Shen used the change of known variables $s = H^{-1}(t)$ for $t \in [0, M]$, where

$$H(s) = \int_0^s \sqrt{1 - \gamma t^2}\mathrm{d}t, \tag{1.3}$$

and $H^{-1}(t) = -H^{-1}(-t)$ for $t \in [-M, 0)$. Since $\gamma > 0$ small enough, Eq (1.3) is well-defined and the inverse function $H^{-1}(t)$ exists. They established the existence of weak solutions for Eq (1.2) based on variational methods, for $\gamma > 0$ small enough and $f(u) = |u|^{p-2}u$ ($p \in (2, \frac{2N}{N-2})$).

For the case $\gamma = 1$, notice that $1 - t^2$ may be negative, for this possibility, the change of variables Eq (1.3) is no longer suitable for dealing with such problems. Recently, in [24] we considered the existence of a positive solution for Eq (1.2) with $\gamma = 1$ and λ large enough:

$$-\Delta u + V(x)u + \Delta(u^2)u = \lambda f(u),\ x \in \mathbb{R}^N, \tag{1.4}$$

where $N \geq 3$, $f(t) \in C(\mathbb{R})$ and superlinear in a neighborhood of $t = 0$. There is a more in-depth study of this idea in [25], where treated the case that $\lambda = 1$. However in [24,25], we were mainly interested in obtaining the existence and multiplicity of solutions for Eq (1.4), leaving nodal properties of solutions unconsidered.

Motivated by [23–25] mentioned above, in this paper, we focus on the existence and multiplicity of sign-changing solutions for Eq (1.4).

Compared with [24], the aim of this paper is two-fold. The first purpose is to investigate the existence of a sign-changing solution for Eq (1.4). The second aim is to obtain infinite sign-changing solutions for Eq (1.4) with symmetric condition.

1.3. Our problem and main results

In this paper, we try to consider sign-changing solutions for the following one-parameter supercritical quasi-linear Schrödinger equations:

$$-\Delta u + V(x)u + \Delta(u^2)u = \lambda f(u),\ x \in \mathbb{R}^N,$$

where $N \geq 3$, $\lambda > 0$ and $V \in C(\mathbb{R}^N, \mathbb{R})$ satisfying:
(V_0): $V(x) \geq V_0 > 0$ for all $x \in \mathbb{R}^N$;
(V_1): $V(x) \leq V_\infty := \lim\limits_{|x|\to\infty} V(x) = +\infty$.

We assume that the nonlinearity satisfies the following conditions: $f(t) \in C(\mathbb{R})$;
(f_1): there exists $\alpha \in (2, 2^*)$ such that

$$\limsup_{t\to 0} \frac{f(t)}{|t|^{\alpha-2}t} < +\infty;$$

(f_2): there exists $\beta \in (2, 2^*)$ with $\beta > \alpha$ such that

$$\liminf_{t\to 0} \frac{F(t)}{|t|^\beta} > 0,$$

where $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent and $F(t) = \int_0^t f(s)ds$;
(f_3): there exists $\theta \in (2, 2^*)$ such that

$$0 < \theta F(t) \leq t f(t),\ \text{ for } |t| \text{ small};$$

(f_4): $f(-t) = -f(t)$, for $|t|$ small.

Remark 1.1. *An example of the nonlinearity satisfying* $(f_1) - (f_3)$ *can be taken as*

$$f(t) = C_1|t|^{\alpha-2}t + C_2|t|^{q-2}t,$$

with $2 < \alpha < \beta < 2^* < q$ *and* C_1, C_2 *are positive constants. Notice that* $q > 2^*$, *hence our method in this paper can be used to deal with the supercritical problems.*

Inspired by Costa, Wang [26] and Huang, Jia [24], we establish a sign-changing solution for the following quasi-linear Schrödinger equation

$$-\text{div}(h^2(u)\nabla u) + h(u)h'(u)|\nabla u|^2 + V(x)u = \lambda \widetilde{f}(u),\ x \in \mathbb{R}^N, \tag{1.5}$$

where $h(t) = \sqrt{1-2t^2}$, for $|t| \leq \sqrt{1/6}$ and $\widetilde{f}(t)$ is a modified nonlinearity such that Eq (1.5) possess variational framework. Next, we show Eq (1.5) has a sign-changing solution by using the methods of invariant sets. Then, a regularity argument shows an L^∞-estimate for this sign-changing solution which depends on parameter λ. Finally, take λ large enough such that the solution of Eq (1.5) is the solution of the original Eq (1.4).

Our main results are as follows.

Theorem 1.1. *Assume that* (V_0), (V_1), $(f_1) - (f_3)$ *hold. Then Eq (1.4) possesses at least one sign-changing solution* $u \in E$ *for all sufficiently large* λ.

Theorem 1.2. *Assume that* (V_0), (V_1), $(f_1) - (f_4)$ *hold. For any given* $n \geq 1$, *then Eq (1.4) possesses at least* $n-1$ *pairs sign-changing solutions* $u \in E$ *for all sufficiently large* λ.

From our results, we obtain the existence and multiplicity of sign-changing solutions for supercritical problems.

1.4. Outline of this paper

The outline of this paper is as follows. In Section 2, we describe the modified equation associated with the Eq (1.4). We are devoted to the proofs of Theorems 1.1 and 1.2 in Section 3.

2. The modified problem

When viewed from the perspective of variational, one of the difficulties in treating Eq (1.4) lies in without the behavior of nonlinearity at infinity. Hence, we first give the precise definition of the modified problem.

The conditions (f_1) and (f_2) imply that there exist positive constants $\delta \in (0, \frac{1}{2})$, A and B such that for $-2\delta \le t \le 2\delta$,

$$F(t) \le A|t|^{\alpha} \text{ and } F(t) \ge B|t|^{\beta}. \tag{2.1}$$

For fixed $\delta > 0$, let $d(t) \in C^1(\mathbb{R}, \mathbb{R})$ be a cut-off function satisfying:

$$d(t) = \begin{cases} 1, & \text{if } |t| \le \delta, \\ 0, & \text{if } |t| \ge 2\delta, \end{cases}$$

$|td'(t)| \le \frac{2}{\delta}$ and $0 \le d(t) \le 1$ for $t \in \mathbb{R}$. Using the truncation argument introduced by Costa and Wang [26], we define

$$\widetilde{F}(t) = d(t)F(t) + (1 - d(t))F_{\infty}(t),$$

where

$$F_{\infty}(t) = A|t|^{\alpha}.$$

And $\widetilde{f}(t) = \widetilde{F}'(t)$. In what follows, we recall the properties of $\widetilde{f}(t)$:

Lemma 2.1. *[26] If (f_1)-(f_3) are satisfied, then we get*
(1) $\widetilde{f} \in C(\mathbb{R}, \mathbb{R})$ and $\widetilde{f}(t) = o(1)$ as $t \to 0$;
(2) $\lim\limits_{t \to +\infty} \frac{\widetilde{f}(t)}{t} = +\infty$;
(3) there exists $C > 0$ such that $|\widetilde{f}(t)| \le C|t|^{\alpha-1}$, for all $t \in \mathbb{R}$;
(4) for all $\delta \in (0, 1)$, there exists a constant $C_\delta > 0$ such that $|\widetilde{f}(t)| \le \delta|t| + C_\delta|t|^{2^*-1}$, where $C_\delta = C\delta^{\frac{\alpha-2^*}{\alpha-2}}$;
(5) for all $t \neq 0$, it implies $0 < \kappa\widetilde{F}(t) \le t\widetilde{f}(t)$, where $\kappa = \min\{\alpha, \theta\}$.

The technique to prove our main results deeply relies on the work of [23, 24, 26]. It should be pointed out that we need to modify the equation as follows in order to adapt to the variational method:

$$-\mathrm{div}(h^2(u)\nabla u) + h(u)h'(u)|\nabla u|^2 + V(x)u = \lambda\widetilde{f}(u), \ x \in \mathbb{R}^N, \tag{2.2}$$

where $h(t) : \mathbb{R} \to \mathbb{R}$ is given by

$$h(t) = \begin{cases} -\frac{1}{6t} + \frac{1}{\sqrt{6}} & \text{if } t \le -\frac{1}{\sqrt{6}}, \\ \sqrt{1 - 2t^2} & \text{if } |t| < \frac{1}{\sqrt{6}}, \\ \frac{1}{6t} + \frac{1}{\sqrt{6}} & \text{if } t \ge \frac{1}{\sqrt{6}}. \end{cases}$$

Next, we define

$$H(t) = \int_0^t h(s)\mathrm{d}s.$$

Then, we will state the properties of the variable $H^{-1}(t)$ after it changes, which plays an important role in proving our main conclusions.

Lemma 2.2. *[23] (1)* $\lim_{t\to 0}\frac{H^{-1}(t)}{t}=1$;
(2) $\lim_{t\to+\infty}\frac{H^{-1}(t)}{t}=\sqrt{6}$;
(3) $t\le H^{-1}(t)\le\sqrt{6}t$, *for all* $t\ge 0$, $\sqrt{6}t\le H^{-1}(t)\le t$, *for all* $t\le 0$;
(4) $-\frac{1}{2}\le\frac{t}{h(t)}h'(t)\le 0$, *for all* $t\in\mathbb{R}$.

Direct calculations show that if $|u|_\infty<\min\{\delta,\sqrt{1/6}\}$, then $h(u)=\sqrt{1-2u^2}$ and $\widetilde{f}(u)=f(u)$. Therefore, our mission is to prove the existence of sign-changing solution u for Eq (2.2) satisfying $|u|_\infty<\min\{\delta,\sqrt{1/6}\}$.

Note that Eq (2.2) is the Euler-Lagrange equation associated to the natural energy functional

$$\widetilde{I}_\lambda(u)=\frac{1}{2}\int_{\mathbb{R}^N}h^2(u)|\nabla u|^2\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}^N}V(x)|u|^2\mathrm{d}x-\lambda\int_{\mathbb{R}^N}\widetilde{F}(u)\mathrm{d}x. \tag{2.3}$$

Taking the change variable

$$v=H(u)=\int_0^u h(s)\mathrm{d}s,$$

we observe that the functional $\widetilde{I}_\lambda(u)$ can be written by the following way

$$J_\lambda(v)=\frac{1}{2}\int_{\mathbb{R}^N}|\nabla v|^2\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}^N}V(x)|H^{-1}(v)|^2\mathrm{d}x-\lambda\int_{\mathbb{R}^N}\widetilde{F}(H^{-1}(v))\mathrm{d}x.$$

From Lemmas 2.1 and 2.2, we can get that $J_\lambda(v)$ is well-defined in E, $J_\lambda\in C^1(E,\mathbb{R})$ and

$$\langle J_\lambda'(v),\varphi\rangle=\int_{\mathbb{R}^N}\nabla v\nabla\varphi\mathrm{d}x+\int_{\mathbb{R}^N}V(x)\frac{H^{-1}(v)}{h(H^{-1}(v))}\varphi\mathrm{d}x-\lambda\int_{\mathbb{R}^N}\frac{\widetilde{f}(H^{-1}(v))}{h(H^{-1}(v))}\varphi\mathrm{d}x,\ \text{for all }\varphi\in E,$$

where

$$E=\{u\in D^{1,2}(\mathbb{R}^N):\int_{\mathbb{R}^N}V(x)u^2\mathrm{d}x<\infty\}$$

with the norm $\|u\|_E=\left(\int_{\mathbb{R}^N}(|\nabla u|^2+V(x)u^2)\mathrm{d}x\right)^{\frac{1}{2}}$.

Remark 2.1. *From condition* (V_1), *it implies that embedding* $E\hookrightarrow L^q(\mathbb{R}^N)(2\le q<2^*)$ *is compact. This compact result was firstly introduced by Bartsch, Pankov and Wang [27].*

Lemma 2.3. *If* $v\in E$ *is a critical point of* J_λ, *then* $u=H^{-1}(v)\in E$ *and this* u *is a weak solution for Eq (2.2).*

Proof. Using the fact that $H^{-1}(v)\in C^2$ and Lemma 2.2, we can show that $u=H^{-1}(v)\in E$ through a direct computation. If v is a critical point for J_λ, we have that

$$\int_{\mathbb{R}^N}\nabla v\nabla\varphi\mathrm{d}x+\int_{\mathbb{R}^N}V(x)\frac{H^{-1}(v)}{h(H^{-1}(v))}\varphi\mathrm{d}x-\lambda\int_{\mathbb{R}^N}\frac{\widetilde{f}(H^{-1}(v))}{h(H^{-1}(v))}\varphi\mathrm{d}x=0,\ \text{for all }\varphi\in E.$$

Taking $\varphi = h(u)\psi$, where $\psi \in C_0^\infty(\mathbb{R}^N)$, in the above equation to get

$$\int_{\mathbb{R}^N} \nabla v \nabla u h'(u)\psi \mathrm{d}x + \int_{\mathbb{R}^N} \nabla v \nabla \psi h(u)\mathrm{d}x + \int_{\mathbb{R}^N} V(x)u\psi \mathrm{d}x - \lambda \int_{\mathbb{R}^N} \widetilde{f}(u)\psi \mathrm{d}x = 0,$$

or

$$\int_{\mathbb{R}^N} \Big(-\mathrm{div}(h^2(u)\nabla u) + h(u)h'(u)|\nabla u|^2 + V(x)u - \lambda \widetilde{f}(u)\Big)\psi \mathrm{d}x = 0.$$

This ends the proof.

To find the sign-changing solutions of Eq (2.2), it is sufficient to discuss the existence of the sign-changing solutions of the following equation

$$-\Delta v + V(x)\frac{H^{-1}(v)}{h(H^{-1}(v))} = \lambda \frac{\widetilde{f}(H^{-1}(v))}{h(H^{-1}(v))},\ x \in \mathbb{R}^N. \tag{2.4}$$

3. Proofs of Theorem 1.1 and Theorem 1.2

In this section, we shall use two abstract critical point theorems based on classical Mountain Pass theorem and Symmetric Mountain Pass theorem to prove the existence and multiplicity of sign-changing solutions for Eq (1.4). The two abstract critical point theorems are developed by Liu, Liu and Wang in [28]. In order to prove Theorem 1.1, we make use of the following notations. Let E be a Banach space, $I \in C^1(E, \mathbb{R})$, $P, Q \subset E$ be open sets, $M = P \cap Q$, $\Sigma = \partial P \cap \partial Q$ and $W = P \cup Q$. For $c \in \mathbb{R}$, $K_c = \{u \in E : I(u) = c, I'(u) = 0\}$ and $I^c = \{u \in E : I(u) \leq c\}$.

Definition 3.1. *[28] Suppose we have the following deformation properties: if $K_c \setminus W = \emptyset$, there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$, there exists $\sigma \in C(E, E)$ satisfying*
(1) $\sigma(\overline{P}) \subset \overline{P}$, $\sigma(\overline{Q}) \subset \overline{Q}$;
(2) $\sigma|_{I^{c-2\varepsilon}} = id$;
(3) $\sigma(I^{c+\varepsilon} \setminus W) \subset I^{c-\varepsilon}$.
Then, $\{P, Q\}$ is called an admissible family of invariant sets with respect to I at level c.

To obtain sign-changing solutions for Eq (2.4), the positive and negative cones as in many references such as [28, 31] are defined:

$$P^+ := \{u \in E : u \geq 0\} \text{ and } P^- := \{u \in E : u \leq 0\}.$$

For $\varepsilon > 0$, consider

$$P_\varepsilon^+ := \{u \in E : \mathrm{dist}(u, P^+) < \varepsilon\} \text{ and } P_\varepsilon^- := \{u \in E : \mathrm{dist}(u, P^-) < \varepsilon\}.$$

Now, we are ready to prove that there exists a sign-changing solution for the modified Eq (2.4), and for this we take $P = P_\varepsilon^+$, $Q = P_\varepsilon^-$ and $I = J_\lambda$.

Lemma 3.1. *Assume that $(f_1) - (f_3)$ and (V_0) hold. Then the Palais-Smale sequence of J_λ is bounded.*

Proof. Since $\{v_n\} \subset E$ is a Palais-Smale sequence, then

$$\begin{aligned} J_\lambda(v_n) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_n|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|H^{-1}(v_n)|^2 \mathrm{d}x - \lambda \int_{\mathbb{R}^N} \widetilde{F}(H^{-1}(v_n))\mathrm{d}x \\ &= d_\lambda + o_n(1) \end{aligned} \tag{3.1}$$

and for any $\varphi \in E$, $\langle J'_\lambda(v_n), \varphi\rangle = o_n(1)\|\varphi\|$, that is

$$\int_{\mathbb{R}^N} \left(\nabla v_n \nabla\varphi + V(x)\frac{H^{-1}(v_n)}{h(H^{-1}(v))}\varphi\right)\mathrm{d}x - \lambda\int_{\mathbb{R}^N} \frac{\widetilde{f}(H^{-1}(v_n))}{h(H^{-1}(v_n))}\varphi \mathrm{d}x = o_n(1)\|\varphi\|. \tag{3.2}$$

Fixing $\varphi = H^{-1}(v_n)h(H^{-1}(v_n))$, it follows from Lemma 2.2-(4) that

$$|\nabla(H^{-1}(v_n)h(H^{-1}(v_n)))| \leq \left(1 + \frac{H^{-1}(v_n)}{h(H^{-1}(v_n))}h'(H^{-1}(v_n))\right)|\nabla v_n| \leq |\nabla v_n|. \tag{3.3}$$

Notice that, Lemma 2.2-(3) implies that

$$|H^{-1}(v_n)h(H^{-1}(v_n))| \leq \sqrt{6}|v_n|. \tag{3.4}$$

Combining Eqs (3.3) and (3.4), we have

$$\|H^{-1}(v_n)h(H^{-1}(v_n))\| \leq \sqrt{6}\|v_n\|.$$

From $\langle J'_\lambda(v_n), H^{-1}(v_n)h(H^{-1}(v_n))\rangle = o_n(1)\|v_n\|$, we get

$$\begin{aligned} o_n(1)\|v_n\| &= \int_{\mathbb{R}^N}\left(1 + \frac{H^{-1}(v_n)}{h(H^{-1}(v_n))}h'(H^{-1}(v_n))\right)|\nabla v_n|^2 \mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N} V(x)|H^{-1}(v_n)|^2 \mathrm{d}x - \lambda\int_{\mathbb{R}^N}\widetilde{f}(H^{-1}(v_n))H^{-1}(v_n)\mathrm{d}x \\ &\leq \int_{\mathbb{R}^N}|\nabla v_n|^2\mathrm{d}x + \int_{\mathbb{R}^N} V(x)|H^{-1}(v_n)|^2\mathrm{d}x - \lambda\int_{\mathbb{R}^N}\widetilde{f}(H^{-1}(v_n))H^{-1}(v_n)\mathrm{d}x. \end{aligned} \tag{3.5}$$

Therefore, by Eqs (3.1), (3.2) and (3.5), Lemma 2.1-(5) and Lemma 2.2-(3), we have

$$\begin{aligned} \kappa d_\lambda + o_n(1) + o_n(1)\|v_n\| &= \kappa J_\lambda(v_n) - \langle J'_\lambda(v_n), H^{-1}(v_n)h(H^{-1}(v_n))\rangle \\ &\geq \frac{\kappa-2}{2}\int_{\mathbb{R}^N}|\nabla v_n|^2\mathrm{d}x + \frac{\kappa-2}{2}\int_{\mathbb{R}^N} V(x)|H^{-1}(v_n)|^2\mathrm{d}x \\ &\geq \frac{\kappa-2}{2}\min\{1, V_0\}\|v_n\|^2, \end{aligned}$$

which implies $\|v_n\| < +\infty$.

Lemma 3.2. *Up to subsequence, the Palais-Smale sequence $\{v_n\}$ converges to a critical point v_0 of J_λ with $J_\lambda(v_0) = c_0$.*

Proof. Since $\{v_n\} \subset E$ is bounded and the embedding $E \hookrightarrow L^{\alpha}(\mathbb{R}^N)$ is compact with $\alpha \in [2, 2^*)$, up to a subsequence, we get

$$v_n \rightharpoonup v_0 \text{ weakly in } E,\ v_n \to v_0 \text{ strongly in } L^{\alpha}(\mathbb{R}^N),\ v_n \to v_0 \ a.e. \text{ in } \mathbb{R}^N.$$

We rewrite

$$J_\lambda(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} V(x)v^2 \mathrm{d}x - \int_{\mathbb{R}^N} \overline{F}(x, v)\mathrm{d}x,$$

where

$$\overline{F}(x,t) = \frac{1}{2}V(x)\left(t^2 - |H^{-1}(t)|^2\right) + \lambda\widetilde{F}(H^{-1}(t)).$$

Using Lemmas 2.1 and 2.2, we have that for all $x \in \mathbb{R}^N$

$$\lim_{t\to 0}\frac{\overline{f}(x,t)}{t} = 0 \text{ and } \lim_{t\to\infty}\frac{|\overline{f}(x,t)|}{|t|^{\alpha-1}} \le C,$$

where $\overline{f}(x,t) = \frac{d\overline{F}(x,t)}{dt}$. Thus, for all $\delta > 0$, there exists a constant C_δ, such that

$$|\overline{f}(x,t)| \le \delta|t| + C_\delta|t|^{\alpha-1}. \tag{3.6}$$

From Eq (3.6) and $v_n \to v_0$ strongly in $L^{\alpha}(\mathbb{R}^N)$, we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\left(\overline{f}(x,v_n) - \overline{f}(x,v_0)\right)(v_n - v_0)\mathrm{d}x = 0.$$

Thus,

$$\begin{aligned} o_n(1) =& \langle J'_\lambda(v_n) - J'_\lambda(v_0), v_n - v_0\rangle \\ =& \int_{\mathbb{R}^N}\left(|\nabla(v_n - v_0)|^2 + V(x)(v_n - v_0)^2\right)\mathrm{d}x \\ & - \int_{\mathbb{R}^N}\left(\overline{f}(x,v_n) - \overline{f}(x,v_0)\right)(v_n - v_0)\mathrm{d}x + o_n(1) \\ \ge& \|v_n - v_0\|^2 + o_n(1), \end{aligned}$$

which implies $v_n \to v_0$ in E and v_0 is critical point of J_λ.

3.1. Properties of operator $\mathcal{A}$

We now define an auxiliary operator $\mathcal{A}$ as follows: for any $v \in E$, assuming $w = \mathcal{A}(v) \in E$ is the unique solution to the following equation

$$-\Delta\omega + V(x)\omega = \overline{f}(x, v),\ \omega \in E, \tag{3.7}$$

where $\overline{f}(x,v) = \lambda\frac{\widetilde{f}(H^{-1}(v))}{h(H^{-1}(v))} - V(x)\frac{H^{-1}(v)}{h(H^{-1}(v))} + V(x)v$.

We can use the auxiliary operator $\mathcal{A}$ to construct a descending flow for the functional $J_\lambda(v)$. Actually, the following three statements are equivalent:

- v is a solution of Eq (2.4),
- v is a critical point of $J_\lambda(v)$,
- v is a fixed point of $\mathcal{A}$.

Lemma 3.3. *The operator $\mathcal{A}$ is well defined as well as continuous and compact.*

Proof. We firstly show that $\mathcal{A}$ is continuous. Assume that $v_n \to v$ in E. Up to a subsequence, suppose that $v_n \to v$ in $L^s(\mathbb{R}^N)$ with $s \in [2, 2^*]$. Set $\omega_n = \mathcal{A}(v_n)$ and $\omega = \mathcal{A}(v)$, we have

$$-\Delta\omega_n + V(x)\omega_n = \overline{f}(x, v_n), \tag{3.8}$$

and

$$-\Delta\omega + V(x)\omega = \overline{f}(x, v). \tag{3.9}$$

Testing with ω_n in Eq (3.8), by Eq (3.6) we have

$$\begin{aligned}\|\omega_n\|^2 &= \int_{\mathbb{R}^N} \overline{f}(x, v_n)\omega_n \mathrm{d}x \\ &\le \delta\|v_n\|\|\omega_n\| + C_\delta\|v_n\|^{\alpha-1}\|\omega_n\|.\end{aligned}$$

Then $\{\omega_n\}$ is bounded in E. After passing to subsequence, suppose $\omega_n \rightharpoonup \omega^*$ weakly in E, $\omega_n \to \omega^*$ in strongly $L^s(\mathbb{R}^N)$ with $s \in [2, 2^*)$. From $\omega_n \rightharpoonup \omega^*$ weakly in E, it is easy to see that ω^* is a solution of Eq (3.9) and then $\omega^* = \omega$ by the uniqueness. Moreover, testing with $\omega_n - \omega$ in Eqs (3.8) and (3.9), one has

$$\|\omega_n - \omega\|^2 = \int_{\mathbb{R}^N} \left(\overline{f}(x, v_n) - \overline{f}(x, v)\right)(\omega_n - \omega)\mathrm{d}x. \tag{3.10}$$

Next, we are ready to estimate the right term of Eq (3.10). Let $\phi \in C_0^\infty(\mathbb{R})$ be a cut-off function such that $\phi(t) \in [0, 1]$ for $t \in \mathbb{R}$, $\phi(t) = 1$ for $|t| \le 1$ and $\phi(t) = 0$ for $|t| \ge 2$. Setting

$$h_1(t) = \phi(t)\overline{f}(t),\ h_2(t) = \overline{f}(t) - h_1(t).$$

By Lemmas 2.1 and 2.2, there exists $C > 0$ such that $|h_1(t)| \le C|t|$ and $|h_2(t)| \le C|t|^{\alpha-1}$ for $t \in \mathbb{R}$. Then,

$$\begin{aligned}&\int_{\mathbb{R}^N} (h_1(v) - h_1(v_n))(\omega - \omega_n)\mathrm{d}x + \int_{\mathbb{R}^N} (h_2(v) - h_2(v_n))(\omega - \omega_n)\mathrm{d}x \\ &\le \left(\int_{\mathbb{R}^N} |h_1(v) - h_1(v_n)|^2 \mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} |\omega - \omega_n|^2 \mathrm{d}x\right)^{\frac{1}{2}} \\ &\quad + \left(\int_{\mathbb{R}^N} |h_2(v) - h_2(v_n)|^{\frac{\alpha}{\alpha-1}} \mathrm{d}x\right)^{\frac{\alpha-1}{\alpha}} \left(\int_{\mathbb{R}^N} |\omega - \omega_n|^\alpha \mathrm{d}x\right)^{\frac{1}{\alpha}} \\ &\le C\|\omega - \omega_n\| \left[\left(\int_{\mathbb{R}^N} |h_1(v) - h_1(v_n)|^2 \mathrm{d}x\right)^{\frac{1}{2}} + \left(\int_{\mathbb{R}^N} |h_2(v) - h_2(v_n)|^{\frac{\alpha}{\alpha-1}} \mathrm{d}x\right)^{\frac{\alpha-1}{\alpha}}\right].\end{aligned}$$

And it implies

$$\|\omega - \omega_n\| \le C\left[\left(\int_{\mathbb{R}^N} |h_1(v) - h_1(v_n)|^2 \mathrm{d}x\right)^{\frac{1}{2}} + \left(\int_{\mathbb{R}^N} |h_2(v) - h_2(v_n)|^{\frac{\alpha}{\alpha-1}} \mathrm{d}x\right)^{\frac{\alpha-1}{\alpha}}\right].$$

Therefore, we can conclude that $\|\omega - \omega_n\| \to 0$ as $n \to \infty$ by the dominated convergence theorem.

Finally, we give the proof of the compact of $\mathcal{A}$. Assume that $\{v_n\}$ is a bounded sequence, we can get the boundness of $\{\omega_n\} \subset E$ due to the continuous of $\mathcal{A}$. Passing to a subsequence, we may assume that $v_n \rightharpoonup v$ and $\omega_n \rightharpoonup \omega$ weakly in E and strongly in $L^s(\mathbb{R}^N)$ with $s \in [2, 2^*)$. From Eq (3.8), we have

$$\int_{\mathbb{R}^N} (\nabla\omega_n \nabla\varphi + V(x)\omega_n\varphi)\,\mathrm{d}x = \int_{\mathbb{R}^N} \overline{f}(x, v_n)\varphi \mathrm{d}x, \text{ for all } \varphi \in E. \tag{3.11}$$

Taking limit as $n \to \infty$ in Eq (3.11) yields

$$\int_{\mathbb{R}^N} (\nabla\omega\nabla\varphi + V(x)\omega\varphi)\,\mathrm{d}x = \int_{\mathbb{R}^N} \overline{f}(x, v)\varphi \mathrm{d}x.$$

This means $\omega = \mathcal{A}(v)$ and thus

$$\|\omega_n - \omega\|^2 = \int_{\mathbb{R}^N} \left(\overline{f}(x, v_n) - \overline{f}(x, v)\right)(\omega_n - \omega)\mathrm{d}x.$$

Using the similar method as before, we can get $\|\omega_n - \omega\| \to 0$, i.e., $\mathcal{A}(v_n) \to \mathcal{A}(v)$ in E as $n \to \infty$.

Lemma 3.4. *There exists $\varepsilon_0 > 0$ such that $\mathcal{A}(P_\varepsilon^\pm) \subset P_\varepsilon^\pm$, for all $\varepsilon \in (0, \varepsilon_0)$ and every nontrivial solution $v \in P_\varepsilon^-$ ($v \in P_\varepsilon^+$) is negative (positive).*

Proof. Due to the similarity of the above two conclusions, we only prove $v \in P_\varepsilon^-$. Let $v \in E$ and $\omega = \mathcal{A}(v)$, for all $q \in [2, 2^*]$, there exists $S_q > 0$ such that

$$\|v^\pm\|_q = \inf_{u\in P^\mp} \|v - u\|_q \le S_q \inf_{u\in P^\mp} \|v - u\| = S_q \mathrm{dist}(v, P^\mp).$$

Since $\mathrm{dist}(u, P^-) \le \|u^+\|$, we have

$$\begin{aligned}
\mathrm{dist}(\omega, P^-)\|\omega^+\| &\le \|\omega^+\|^2 \\
&= \langle \omega, \omega^+ \rangle \\
&= \int_{\mathbb{R}^N} \nabla\omega\nabla\omega^+ \mathrm{d}x + \int_{\mathbb{R}^N} V(x)\omega\omega^+ \mathrm{d}x \\
&= \int_{\mathbb{R}^N} \frac{\overline{f}(H^{-1}(v))}{h(H^{-1}(v))}\omega^+ \mathrm{d}x \\
&\le C \int_{\mathbb{R}^N} \left(\delta|v^+| + C_\delta |v^+|^{\alpha-1}\right)\omega^+ \mathrm{d}x \\
&\le \delta\|v^+\|_2\|\omega^+\|_2 + C_\delta\|v^+\|_\alpha^{\alpha-1}\|\omega^+\|_\alpha \\
&\le C(\delta \mathrm{dist}(v, P^-) + C_\delta(\mathrm{dist}(v, P^-))^{\alpha-1})\|\omega^+\|.
\end{aligned}$$

In consequence,

$$\mathrm{dist}(\mathcal{A}(v), P^-) \le C\left(\delta \mathrm{dist}(v, P^-) + C_\delta(\mathrm{dist}(v, P^-))^{\alpha-1}\right).$$

Therefore, if we choose δ small enough, there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$, it implies

$$\mathrm{dist}(\mathcal{A}(v), P^-) \le \frac{1}{2}\mathrm{dist}(v, P^-)$$

for any $v \in P_\varepsilon^-$. It implies that $\mathcal{A}(\partial P_\varepsilon^-) \subset P_\varepsilon^-$. And if $v \in P_\varepsilon^-$ with $\mathcal{A}(v) = v$, then $v \in P^-$.

Lemma 3.5. *(1) $\langle J'_\lambda(v), v - \mathcal{A}(v)\rangle \geq \|v - \mathcal{A}(v)\|^2$ for all $v \in E$;*
(2) $\|J'_\lambda(v)\| \leq C\|v - \mathcal{A}(v)\|$ for some $C > 0$ and all $v \in E$.

Proof. Because $\mathcal{A}(v)$ is the solution of Eq (3.7), we have that

$$\begin{aligned}
\langle J'_\lambda(v), v - \mathcal{A}(v)\rangle &= \int_{\mathbb{R}^N} \nabla v \nabla (v - \mathcal{A}(v))\mathrm{d}x + \int_{\mathbb{R}^N} V(x)\frac{H^{-1}(v)}{h(H^{-1}(v))}(v - A(v))\mathrm{d}x \\
&\quad - \lambda \int_{\mathbb{R}^N} \frac{\widetilde{f}(H^{-1}(v))}{h(H^{-1}(v))}(v - \mathcal{A}(v))\mathrm{d}x \\
&= \int_{\mathbb{R}^N} \nabla v \nabla (v - \mathcal{A}(v))\mathrm{d}x + \int_{\mathbb{R}^N} V(x)\frac{H^{-1}(v)}{h(H^{-1}(v))}(v - \mathcal{A}(v))\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^N} V(x)\frac{H^{-1}(v)}{h(H^{-1}(v))}(v - \mathcal{A}(v))\mathrm{d}x + \int_{\mathbb{R}^N} V(x)v(v - \mathcal{A}(v))\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^N} \nabla\mathcal{A}(v)\nabla(v - \mathcal{A}(v))\mathrm{d}x - \int_{\mathbb{R}^N} V(x)\mathcal{A}(v)(v - \mathcal{A}(v))\mathrm{d}x \\
&= \int_{\mathbb{R}^N} |\nabla(v - \mathcal{A}(v))|^2\mathrm{d}x + \int_{\mathbb{R}^N} V(x)(v - \mathcal{A}(v))^2\mathrm{d}x \\
&= \|v - \mathcal{A}(v)\|^2.
\end{aligned}$$

For any $\varphi \in E$, we get

$$\begin{aligned}
\langle J'_\lambda(v), \varphi\rangle &= \int_{\mathbb{R}^N} \nabla v \nabla \varphi\mathrm{d}x + \int_{\mathbb{R}^N} V(x)\frac{H^{-1}(v)}{h(H^{-1}(v))}\varphi\mathrm{d}x \\
&\quad - \lambda \int_{\mathbb{R}^N} \frac{\widetilde{f}(H^{-1}(v))}{h(H^{-1}(v))}\varphi\mathrm{d}x \\
&= \int_{\mathbb{R}^N} \nabla v \nabla \varphi\mathrm{d}x + \int_{\mathbb{R}^N} V(x)\frac{H^{-1}(v)}{h(H^{-1}(v))}\varphi\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^N} V(x)\frac{H^{-1}(v)}{h(H^{-1}(v))}\varphi\mathrm{d}x + \int_{\mathbb{R}^N} V(x)v\varphi\mathrm{d}x - \langle\mathcal{A}(v), \varphi\rangle \\
&= \langle v - \mathcal{A}(v), \varphi\rangle \\
&\leq \|v - \mathcal{A}(v)\|\|\varphi\|.
\end{aligned}$$

Lemma 3.6. *For $v \in E$, $a < b$ and $\alpha > 0$, if $J_\lambda(v) \in [a, b]$ and $\|J'_\lambda(v)\| \geq \alpha$, then there exists $\beta > 0$ such that $\|v - \mathcal{A}(v)\| \geq \beta$.*

Proof. Otherwise, there exists a sequence $\{v_n\} \subset E$ such that

$$J_\lambda(v_n) \in [a, b],\ \|J'_\lambda(v_n)\| \geq \alpha,\ \text{and } \|v_n - \mathcal{A}(v_n)\| \to 0.$$

But, by Lemma 3.5-(2), we have a contradiction.

Following from [29] and [30], we can construct a locally Lipschitz continuous operator $\mathcal{B}$ on $E_0 := E \setminus K$ which inherits the main properties of $\mathcal{A}$.

Lemma 3.7. *The locally Lipschitz continuous operator $\mathcal{B} : E_0 \to E$ satisfying*
(1) $\mathcal{B}(\partial P_\varepsilon^+) \subset P_\varepsilon^+$ and $\mathcal{B}(\partial P_\varepsilon^-) \subset P_\varepsilon^-$ for $\varepsilon \in (0, \varepsilon_0)$;
(2) $\frac{1}{2}\|v - \mathcal{B}(v)\| \le \|v - \mathcal{A}(v)\| \le 2\|v - \mathcal{B}(v)\|$ for all $v \in E_0$;
(3) $\langle J_\lambda'(v), v - \mathcal{A}(v)\rangle \ge \frac{1}{2}\|v - \mathcal{A}(v)\|^2$ for all $v \in E_0$.

By the proof of Lemma 3.5 in [28] and Lemma 3.7, we have

Lemma 3.8. $\{P_\varepsilon^+, P_\varepsilon^-\}$ *is an admissible family of invariant sets of the functional J_λ at any level $c \in \mathbb{R}$.*

3.2. Existence of one sign-changing solution

Next, we are ready to construct φ_0 satisfying the hypotheses in the Theorem 2.4 in [28]. For $(t, s) \in \Delta$, $v_1, v_2 \in C_0^\infty(\mathbb{R}^N)$ with $\mathrm{supp}(v_1) \cap \mathrm{supp}(v_2) = \emptyset$ and $v_1 \le 0, v_2 \ge 0$, define

$$\varphi_0(t, s) := R(tv_1 + sv_2),$$

here R is a positive constant which be determined later. Actually, for $t, s \in [0, 1]$, $\varphi_0(0, s) = Rsv_2 \in P_\varepsilon^+$ and $\varphi_0(t, 0) = Rsv_1 \in P_\varepsilon^-$.

Lemma 3.9. *Assume that* (V_0)*,* (V_1)*,* (f_1)*,* (f_2) *and* (f_3) *hold. Then, for* $\lambda \ge 1$*, problem Eq (2.4) has a sign-changing solution.*

Proof. We shall prove two claims as follows, which will be useful for us to prove Lemma 3.9.
Claim 1. For $q \in [2, 2^*]$, there exists $S_q > 0$ independence of ε such that $\|v\|_q \le 2S_q\varepsilon$ for $v \in M = P_\varepsilon^+ \cap P_\varepsilon^-$.

In order to prove this claim, we consider

$$\|v^\pm\|_q = \inf_{w \in P^\mp} \|v - w\|_q \le S_q \inf_{w \in P^\mp} \|v - w\| = S_q \mathrm{dist}(v, P^\mp) \le S_q\varepsilon.$$

Claim 2. If $\varepsilon > 0$ is small enough then $J_\lambda(v) \ge \frac{\varepsilon^2}{2}$ for $v \in \Sigma = \partial P_\varepsilon^+ \cap \partial P_\varepsilon^-$.

For $v \in \partial P_\varepsilon^+ \cap \partial P_\varepsilon^-$, then

$$\|v^\pm\| \ge \mathrm{dist}(v, P^\mp) = \varepsilon.$$

Since $\|v^\pm\|_q \le S_q\varepsilon$ and $\|v\|^2 = \|v^+\|^2 + \|v^-\|^2$, for $\varepsilon > 0$ small enough, we have

$$\begin{aligned} J_\lambda(v) &\ge \frac{1}{2}\|v\|^2 - \delta C\|v\|_2^2 - C_\delta\|v\|_{2^*}^{2^*} \\ &\ge 2\varepsilon^2(\frac{1}{2} - \delta C) - CC_\delta S_{2^*}^{2^*}\varepsilon^{2^*} \\ &\ge \frac{1}{2}\varepsilon^2. \end{aligned}$$

Next, we are ready to verify the conditions (2) and (3) in Theorem 2.4 in [28]. Notice that $\rho = \min\{\|tv_1 + (1 - t)v_2\|_2 : 0 \le t \le 1\} > 0$. Then, from the above Claim 1, we have $\varphi_0(\partial_0\Delta) \cap M = \emptyset$. In fact, for R large enough, if $v \in \varphi_0(\partial_0\Delta)$, we have $\|v\|_2 > \rho R$.

By the definition of $\widetilde{F}$, for any $v \in \varphi_0(\partial_0\Delta)$, denote $A = \{x : |v| \ge 2\delta\}$, $B = \{x : |v| < 2\delta\}$, and let $v_A = v|_A$, $v_B = v|_B$. Then, we have

$$\widetilde{F}(v_B) \ge F(v_B) \ge C|v_B|^\beta, \tag{3.12}$$

$$\widetilde{F}(v_A) = F_\infty(v_A) = C|v_A|^\alpha. \tag{3.13}$$

And then

$$\begin{aligned}
J_\lambda(v) &\le \frac{1}{2}\|v_A\|^2 + \frac{1}{2}\|v_B\|^2 - C\lambda\|v_A\|^\alpha - C\lambda\|v_B\|^\beta \\
&\le \frac{1}{2}\|v_A\|^2 + \frac{1}{2}\|v_B\|^2 - C\|v_A\|^\alpha - C\|v_B\|^\beta,
\end{aligned}$$

which togethers with the above Claim 2. One has for R large enough and ε small enough,

$$\sup_{v\in\varphi_0(\partial_0\Delta)} J_\lambda(v) < 0 < c_*.$$

Finally, from the Theorem 2.4 in [28], there exists $v \in E \setminus (P_\varepsilon^+ \cup P_\varepsilon^-)$, which is a sign-changing solution of Eq (2.4).

We observe that the weak solutions of Eq (2.4) with L^∞-norm less than $\min\{\sqrt{1/6}, \delta\}$ are equivalent to the weak solutions of Eq (1.4). Next, we turn to study the L^∞ estimates of the critical points of J_λ .

Lemma 3.10. *If $v \in E$ is a weak solution of problem Eq (2.4), then $v \in L^\infty(\mathbb{R}^N)$. Moreover,*

$$|v|_\infty \le C\lambda^{\frac{1}{2^*-\alpha}}\|v\|^{\frac{2^*-2}{2^*-\alpha}}, \tag{3.14}$$

where $C > 0$ only depends on α, N.

Proof. Let $v \in E$ be a weak solution of $-\Delta v + V(x)\frac{H^{-1}(v)}{h(H^{-1}(v))} = \lambda\frac{\widetilde{f}(H^{-1}(v))}{h(H^{-1}(v))}$, i.e.,

$$\int_{\mathbb{R}^N} \nabla v\nabla\varphi \mathrm{d}x + \int_{\mathbb{R}^N} V(x)\frac{H^{-1}(v)}{h(H^{-1}(v))}\varphi \mathrm{d}x = \int_{\mathbb{R}^N} \lambda\frac{\widetilde{f}(H^{-1}(v))}{h(H^{-1}(v))}\varphi \mathrm{d}x, \text{ for all } \varphi \in E. \tag{3.15}$$

Let $T > 0$, and define

$$v_T = \begin{cases} -T, \text{ if } v \le -T, \\ v, \text{ if } 0 < |v| \le T, \\ T, \text{ if } v \ge T. \end{cases}$$

Choosing $\varphi = v_T^{2(\eta-1)}v$ in Eq (3.15), where $\eta > 1$ will be determined later, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla v|^2 v_T^{2(\eta-1)}\mathrm{d}x + 2(\eta-1)\int_{\{x||v(x)|<T\}} v_T^{2(\eta-1)-1}v|\nabla v|^2\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N} V(x)\frac{H^{-1}(v)}{h(H^{-1}(v))}v_T^{2(\eta-1)}v\mathrm{d}x = \lambda\int_{\mathbb{R}^N} \frac{\widetilde{f}(H^{-1}(v))}{h(H^{-1}(v))}v_T^{2(\eta-1)}v\mathrm{d}x.
\end{aligned}$$

It follows from $\int_{\{x||v(x)|<T\}} v_T^{2(\eta-1)-1}v|\nabla v|^2\mathrm{d}x \ge 0$, $\int_{\mathbb{R}^N} V(x)\frac{H^{-1}(v)}{h(H^{-1}(v))}v_T^{2(\eta-1)}v\mathrm{d}x \ge 0$ and Lemma 2.1-(3), that

$$\begin{aligned}
\int_{\mathbb{R}^N} |\nabla v|^2 v_T^{2(\eta-1)}\mathrm{d}x &\le \lambda\int_{\mathbb{R}^N} \frac{\widetilde{f}(H^{-1}(v))}{h(H^{-1}(v))}v_T^{2(\eta-1)}v\mathrm{d}x \\
&\le \lambda C\int_{\mathbb{R}^N} \frac{|H^{-1}(v)|^{\alpha-1}}{h(H^{-1}(v))}v_T^{2(\eta-1)}v\mathrm{d}x \\
&\le \lambda C\int_{\mathbb{R}^N} |v|^\alpha v_T^{2(\eta-1)}\mathrm{d}x.
\end{aligned} \tag{3.16}$$

On the other hand, due to the Sobolev inequality, it implies

$$\begin{aligned}\left(\int_{\mathbb{R}^N}(v|v_T|^{\eta-2}v_T)^{2^*}\mathrm{d}x\right)^{\frac{2}{2^*}} &\leq C\int_{\mathbb{R}^N}|\nabla(vv_T^{\eta-1})|^2\mathrm{d}x\\ &\leq C\int_{\mathbb{R}^N}|\nabla v|^2v_T^{2(\eta-1)}\mathrm{d}x + C(\eta-1)^2\int_{\mathbb{R}^N}|\nabla v|^2v_T^{2(\eta-1)}\mathrm{d}x\\ &\leq C\eta^2\int_{\mathbb{R}^N}|\nabla v|^2v_T^{2(\eta-1)}\mathrm{d}x,\end{aligned}$$

where we used that $(a+b)^2 \leq 2(a^2+b^2)$ and $\eta^2 \geq (\eta-1)^2+1$.

From Eq (3.16), the Sobolev embedding theorem and the Hölder inequality, it implies

$$\begin{aligned}\left(\int_{\mathbb{R}^N}(v|v_T|^{\eta-2}v_T)^{2^*}\mathrm{d}x\right)^{\frac{2}{2^*}} &\leq \lambda C\eta^2\int_{\mathbb{R}^N}|v|^{\alpha-2}v^2v_T^{2(\eta-1)}\mathrm{d}x\\ &\leq \lambda C\eta^2\left(\int_{\mathbb{R}^N}|v|^{2^*}\mathrm{d}x\right)^{\frac{\alpha-2}{2^*}}\left(\int_{\mathbb{R}^N}(|v||v_T|^{\eta-1})^{\frac{22^*}{2^*-\alpha+2}}\mathrm{d}x\right)^{\frac{2^*-\alpha+2}{2^*}}\\ &\leq \lambda C\eta^2\|v\|^{\alpha-2}\left(\int_{\mathbb{R}^N}|v|^{\frac{\eta 22^*}{2^*-\alpha+2}}\mathrm{d}x\right)^{\frac{2^*-\alpha+2}{2^*}}.\end{aligned}$$

Next, taking $\zeta = \frac{22^*}{2^*-\alpha+2}$, we obtain

$$\left(\int_{\mathbb{R}^N}(v|v_T|^{\eta-2}v_T)^{2^*}\mathrm{d}x\right)^{\frac{2}{2^*}} \leq \lambda C\eta^2\|v\|^{\alpha-2}\|v\|_{\eta\zeta}^{2\eta}.$$

From the Fatou's lemma, it follows that

$$\|v\|_{\eta 2^*} \leq (\lambda C\eta^2\|v\|^{\alpha-2})^{\frac{1}{2\eta}}\|v\|_{\eta\zeta}. \tag{3.17}$$

Let us define $\eta_{n+1}\zeta = 2^*\eta_n$ where $n = 0, 1, 2, ...$ and $\eta_0 = \frac{2^*+2-\alpha}{2}$. By Eq (3.17) we have

$$\|v\|_{\eta_1 2^*} \leq (\lambda C\eta_1^2\|v\|^{\alpha-2})^{\frac{1}{2\eta_1}}\|v\|_{2^*\eta_0} \leq (\lambda C\|v\|^{\alpha-2})^{\frac{1}{2\eta_1}+\frac{1}{2\eta_0}}\eta_0^{\frac{1}{\eta_0}}\eta_1^{\frac{1}{\eta_1}}\|v\|_{2^*}.$$

It follows from Moser's iteration method that

$$\|v\|_{\eta_n 2^*} \leq (\lambda C\|v\|^{\alpha-2})^{\frac{1}{2\eta_0}\sum_{i=0}^n(\frac{\zeta}{2^*})^i}(\eta_0)^{\frac{1}{\eta_0}\sum_{i=0}^n(\frac{\zeta}{2^*})^i}(\frac{2^*}{\zeta})^{\frac{1}{\eta_0}\sum_{i=0}^n i(\frac{\zeta}{2^*})^i}\|v\|_{2^*}.$$

Thus, we have

$$|v|_\infty \leq C\lambda^{\frac{1}{2^*-\alpha}}\|v\|^{\frac{2^*-2}{2^*-\alpha}}.$$

Lemma 3.11. *Assume that $(f_1)-(f_3)$ and (V_0) hold. Let v_λ be a critical point of J_λ with $J_\lambda(v_\lambda) = d_\lambda$. Then there exists $C > 0$ (independent of λ) such that*

$$\|v_\lambda\|^2 \leq Cd_\lambda. \tag{3.18}$$

Proof. From Lemma 2.1-(5) and Eq (3.4), we obtain

$$
\begin{aligned}
\kappa d_\lambda &= \kappa J_\lambda(v_\lambda) - \langle J'_\lambda(v_\lambda), H^{-1}(v_\lambda)h(H^{-1}(v_\lambda))\rangle \\
&= \frac{\kappa}{2}\int_{\mathbb{R}^N} |\nabla v_\lambda|^2 \mathrm{d}x + \frac{\kappa}{2}\int_{\mathbb{R}^N} V(x)|H^{-1}(v_\lambda)|^2 \mathrm{d}x - \lambda\kappa \int_{\mathbb{R}^N} \widetilde{F}(H^{-1}(v_\lambda))\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^N} \nabla v_\lambda \nabla\left(H^{-1}(v_\lambda)h(H^{-1}(v_\lambda))\right)\mathrm{d}x - \int_{\mathbb{R}^N} V(x)|H^{-1}(v_\lambda)|^2 \mathrm{d}x \\
&\quad + \lambda\int_{\mathbb{R}^N} \widetilde{f}(H^{-1}(v_\lambda))H^{-1}(v_\lambda)\mathrm{d}x \\
&\geq \frac{\kappa-2}{2}\int_{\mathbb{R}^N} |\nabla v_\lambda|^2 \mathrm{d}x + \frac{\kappa-2}{2}\int_{\mathbb{R}^N} V(x)|H^{-1}(v_\lambda)|^2 \mathrm{d}x \\
&\geq \frac{\kappa-2}{2}\min\{1, V_0\}\|v_\lambda\|^2.
\end{aligned}
$$

It implies that $\|v_\lambda\|^2 \leq Cd_\lambda$.

Proof of Theorem 1.1. Let $v_1, v_2 \in C_0^\infty(\mathbb{R}^N)$, $v_1 \leq 0, v_2 \geq 0$ with $\mathrm{supp}(v_1) \cap \mathrm{supp}(v_2) = \emptyset$ and $R > 0$ are large enough. Let $\varphi_0(t,s) := tRv_1 + sRv_2$ for $(t,s) \in \Delta$. Define

$$
d_\lambda = \inf_{\varphi\in\Gamma} \sup_{u\in\varphi(\Delta)\setminus W} J_\lambda(v),
$$

where $\Gamma := \{\varphi \in C(\Delta, E) : \varphi(\partial_1\Delta) \subset P,\ \varphi(\partial_2\Delta) \subset Q,\ \varphi|_{\partial_0\Delta} = \varphi_0|_{\partial_0\Delta}\}$.

By Lemma 3.9, J_λ has a sign-changing critical point v_λ and $J_\lambda(v_\lambda) = d_\lambda$. Furthermore, from Eqs (3.12) and (3.13), we obtain

$$
\begin{aligned}
d_\lambda &\leq \max_{(t,s)\in\Delta} J_\lambda(tRv_1 + sRv_2) \\
&\leq \max_{t\in[0,1]}\left(\frac{t^2}{2}\int_{\mathbb{R}^N} (|\nabla Rv_1|^2 + 6V(x)Rv_1^2)\mathrm{d}x - \lambda\int_{\mathbb{R}^N} \widetilde{F}(H^{-1}(tRv_1))\mathrm{d}x\right) \\
&\quad + \max_{s\in[0,1]}\left(\frac{s^2}{2}\int_{\mathbb{R}^N} (|\nabla Rv_2|^2 + 6V(x)Rv_2^2)\mathrm{d}x - \lambda\int_{\mathbb{R}^N} \widetilde{F}(H^{-1}(sRv_2))\mathrm{d}x\right) \\
&\leq C\lambda^{-\frac{2}{\alpha-2}} + C\lambda^{-\frac{2}{\beta-2}}.
\end{aligned}
\tag{3.19}
$$

By Eqs (3.14), (3.18) and (3.19), we have

$$
|v_\lambda|_\infty \leq C(\lambda^{\frac{1}{2-\alpha}} + \lambda^{\frac{2^*-\beta}{(2^*-\alpha)(2-\beta)}}).
$$

Hence, there exists $\lambda_1 > 0$ such that for all $\lambda > \lambda_1$

$$
|u_\lambda|_\infty = |H^{-1}(v_\lambda)|_\infty \leq \sqrt{6}|v_\lambda|_\infty < \min\{\sqrt{1/6}, \delta\},
$$

where δ is fixed in Eq (2.1). Thus, for $\lambda > \lambda_1$, $u_\lambda = H^{-1}(v_\lambda)$ is a sign-changing solution of the original Eq (1.4).

3.3. Multiplicity of sign-changing solutions

To prove Theorem 1.2, we make on further assumption, $G : E \to E$ is an isometric involution, i.e., $G^2 = id$ and $d(Gu, Gv) = d(u, v)$ for $u, v \in E$. We assume I is G-invariant on E in the sense that $I(Gu) = I(u)$ for any $u \in E$. We also assume $Q = GP$. If for any $u \in F$, $Gu \in F$, then the subset $F \subset E$ is said to by symmetric. $\gamma(F)$ can be called the genus of a closed symmetric subset F of $E \setminus \{0\}$.

Definition 3.2. *If the following deformation property holds: there exist $\varepsilon_0 > 0$ and a symmetric open neighborhood N of $K_c \setminus W$ with $\gamma(\overline{N}) < +\infty$, such that for $\varepsilon \in (0, \varepsilon_0)$, there exists $\sigma \in C(E, E)$ meet the following four conditions:*
(1) $\sigma(\overline{P}) \subset \overline{P}$, $\sigma(\overline{Q}) \subset \overline{Q}$;
(2) $\sigma|_{I^{c-2\varepsilon}} = id$;
(3) $\sigma \circ G = G \circ \sigma$;
(4) $\sigma(I^{c+\varepsilon} \setminus (N \cup W)) \subset I^{c-\varepsilon}$.
Then, we call P is a G-admissible invariant set with respect to I at level c.

We now assume that f is odd and we turn to prove the existence of infinitely many sign-changing solutions to Eq (1.4). We plan to apply the Theorem 2.6 in [28], for this we take $G = -id$, $P = P_\varepsilon^+$, $Q = P_\varepsilon^-$ and $I = J_\lambda$. Next, lemma is used to prove P is a G-admissible invariant set with respect to $J_\lambda \in C^1(E, \mathbb{R})$ at any level c.

Lemma 3.12. *P_ε^+ is a G-admissible invariant set for the functional J_λ at any level c.*

Proof. The proof is similar to Lemma 3.8. Since J_λ is even, thus σ is odd in u. Here, we omit the details.

Proof of Theorem 1.2. Firstly, we shall use the Theorem 2.6 in [28] to get solutions for Eq (2.4) first. Making use of estimates on the critical values, for any fixed $n \in \mathbb{N}$ we shall show Eq (1.4) has $n - 1$ pairs of sign-changing solutions for large λ.

For any $n \in \mathbb{N}$, let $\{v_i\}_{i=1}^n \subset C_0^\infty(\mathbb{R}^N) \setminus \{0\}$ be such that $\text{supp}(v_i) \cap \text{supp}(v_j) = \emptyset$ for $i \neq j$. Define $\varphi_n \in C(B_n, E)$ as

$$\varphi_n(t) = R_n \sum_{i=1}^n t_i v_i(\cdot), \quad t = (t_1, t_2, \cdots, t_n) \in B_n,$$

where $R_n > 0$ will be determined later. Actually, $\varphi_n(0) = 0 \in P_\varepsilon^+ \cap P_\varepsilon^-$ and $\varphi_n(-t) = -\varphi_n(t)$ for $t \in B_n$.

Observe that

$$\rho_n = \min\{\|t_1 v_1 + t_2 v_2 + \cdots + t_n v_n\|_2 : \sum_{i=1}^n t_i^2 = 1\} > 0,$$

then $\|v\|_2^2 \geq \rho_n^2 R_n^2$ for $v \in \varphi_n(\partial B_n)$ and it follows from Claims 1 and 2 in Lemma 3.9 that $\varphi_n(\partial B_n) \cap (P_\varepsilon^+ \cap P_\varepsilon^-) = \emptyset$. Similar to the proof of Theorem 1.1 (existence part), for large enough $R_n > 0$ independent on λ we also have

$$\sup_{v \in \varphi_n(\partial B_n)} J_\lambda(v) < 0 < \inf_{v \in \Sigma} J_\lambda(v).$$

For $j = 2, 3, \cdots, n$, let

$$c_{j,\lambda} = \inf_{B \in \Gamma_j} \sup_{v \in B \setminus W} J_\lambda(v),$$

where

$$\Gamma_j = \{B: \ B = \varphi(B_n \setminus Y) \text{ for some } \varphi \in H_n, Y \subset B_n, n \geq j \text{ with } -Y = Y, \ \gamma(\overline{Y}) \leq n - j\}$$

and

$$H_n = \{\varphi: \ \varphi \in C(B_n, E), \varphi(-t) = -\varphi(t) \text{ for } t \in B_n, \varphi(0) \in M \text{ and } \varphi|_{\partial B_n} = \varphi_n|_{\partial B_n}\}.$$

Then, by the Theorem 2.6 in [28], we have that $0 < c_{2,\lambda} \leq c_{3,\lambda} \leq \cdots \leq c_{n,\lambda}$ are all critical values of J_λ and there are at least $(n-1)$ pairs of sign-changing critical points at these critical values. Since $\varphi_n \in H_n$, we have

$$c_{n,\lambda} \leq b_{n,\lambda} := \sup_{v \in \varphi_n(B_n)} J_\lambda(v).$$

Due to $\mathrm{supp}(v_i) \cap \mathrm{supp}(v_j) = \emptyset$ for $i \neq j$, similar with Eq (3.19), we have

$$\sup_{v \in \varphi_n(B_n)} J_\lambda(v) \leq C\lambda^{-\frac{2}{\alpha-2}} + C\lambda^{-\frac{2}{\beta-2}}.$$

Therefore, it follows from Lemmas 3.10 and 3.11, for λ large that these $(n-1)$ pairs of sign-changing critical points of J_λ are also solutions of the original Eq (1.4).

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Conflict of interest

The authors declare there is no conflicts of interest.

References

1. A. Borovskii, A. Galkin, Dynamical modulation of an ultrashort high-intensity laser pulse in matter, *J. Exp. Theor. Phys.*, **77** (1983), 562–573.
2. H. Brandi, C. Manus, G. Mainfray, T. Lehner, G. Bonnaud, Relativistic and ponderomotive self-focusing of a laser beam in a radially inhomogeneous plasma, *Phys. Fluids. B Plasma Phys.*, **5** (1993), 3539–3550. https://doi.org/10.1063/1.860828
3. L. Brizhik, A. Eremko, B. Piette, W. J. Zakrzewski, Static solutions of a D-dimensional modified nonlinear Schrödinger equation, *Nonlinearity*, **16** (2003), 1481–1497. https://doi.org/10.1088/0951-7715/16/4/317
4. W. Krolikowski, O. Bang, J. Rasmussen, J. Wyller, Modulational instability in nonlocal nonlinear Kerr media, *Phys. Rev. E*, **64** (2001), 016612. https://doi.org/10.1103/PhysRevE.64.016612
5. S. Kurihura, Large-amplitude quasi-solitons in superfluids films, *J. Phys. Soc. Japan*, **50** (1981), 3262–3267. https://doi.org/10.1143/JPSJ.50.3262
6. H. Berestycki, P. L. Lions, Nonlinear scalar field equations, I existence of a ground state, *Arch. Ration. Mech. Anal.*, **82** (1983), 313–346.

7. B. Li, Y. Ma, A firewall effect during the rogue wave and breather interactions to the Manakov system, *Nonlinear Dyn.*, **2022** (2022). https://doi.org/10.1007/s11071-022-07878-6

8. B. Li, Y. Ma, Interaction properties between rogue wave and breathers to the manakov system arising from stationary self-focusing electromagnetic systems, *Chaos Solitons Fractals*, **156** (2022), 111832. https://doi.org/10.1016/j.chaos.2022.111832

9. B. Li, Y. Ma, Extended generalized Darboux transformation to hybrid rogue wave and breather solutions for a nonlinear Schrödinger equation, *Appl. Math. Comput.*, **386** (2020), 125469. https://doi.org/10.1016/j.amc.2020.125469

10. P. L. Lions, The concentration compactness principle in the calculus of variations. The locally compact case. Part I, *Ann. Inst. Henri Poincaré C Anal. non Lineairé*, **1** (1984), 109–145. https://doi.org/10.1016/S0294-1449(16)30428-0

11. P. L. Lions, The concentration compactness principle in the calculus of variations. The locally compact case. Part II, *Ann. Inst. Henri Poincaré Anal. non Lineairé*, **1** (1984), 223–283. https://doi.org/10.1016/S0294-1449(16)30422-X

12. M. Poppenberg, K. Schmitt, Z. Q. Wang, On the existence of soliton solutions to quasilinear Schrödinger equations, *Calc. Var. Partial Differ. Equations*, **14** (2002), 329–344. https://doi.org/10.1007/s005260100105

13. J. Liu, Y. Wang, Z. Q. Wang, Soliton solutions for quasilinear Schrödinger equations II, *J. Differ. Equations*, **187** (2003), 473–493. https://doi.org/10.1016/S0022-0396(02)00064-5

14. M. Colin, L. Jeanjean, Solutions for a quasilinear Schrödinger equations: a dual approach, *Nonlinear Anal. Theory Methods Appl.*, **56** (2004), 213–226. https://doi.org/10.1016/j.na.2003.09.008

15. S. Adachi, T. Watanabe, G-invariant positive solutions for a quasilinear Schrödinger equation, *Adv. Differ. Equations*, **16** (2011), 289–324.

16. X. Fang, J. Zhang, Multiplicity of positive solutions for quasilinear elliptic equations involving critical nonlinearity, *Adv. Nonlinear Anal.*, **9** (2020), 1420–1436. https://doi.org/10.1515/anona-2020-0058

17. H. Liu, Positive solution for a quasilinear elliptic equation involving critical or supercritical exponent, *J. Math. Phys.*, **57** (2016), 159–180. https://doi.org/10.1063/1.4947109

18. J. Liu, Y. Wang, Z. Q. Wang, Solutions for quasilinear Schrödinger equations via the Nehari Method, *Commun. Partial Differ. Equation*, **29** (2004), 879–901. https://doi.org/10.1081/PDE-120037335

19. E. Medeiros, U. Severo, On the existence of signed solution for a quasilinear elliptic problem in $\mathbb{R}^N$, *Mat. Contemp.*, **32** (2007), 193–205.

20. E. Silva, G. Vieira, Quasilinear asymptotically periodic Schrödinger equations with critical growth, *Calc. Var. Partial Differ. Equations*, **39** (2010), 1–33. https://doi.org/10.1007/s00526-009-0299-1

21. Y. Wang, Z. Li, Existence of solutions to quasilinear Schrödinger equations involving critical Sobolev exponent, *Taiwanese J. Math.*, **22** (2018), 401–420. https://doi.org/10.11650/tjm/8150

22. H. Zhang, F. Meng, J. Zhang, Nodal solutions for quasilinear schrödinger equations with asymptotically 3-Linear nonlinearity, *J. Geom. Anal.*, **32** (2022). https://doi.org/10.1007/s12220-022-01043-6

23. C. Alves, Y. Wang, Y. Shen, Soliton solutions for a class of quasilinear Schrödinger equations with a parameter, *J. Differ. Equations*, **259** (2015), 318–343. https://doi.org/10.1016/j.jde.2015.02.030

24. C. Huang, G. Jia, Existence of positive solutions for supercritical quasilinear Schrödinger elliptic equations, *J. Math. Anal. Appl.*, **472** (2019), 705–727. https://doi.org/10.1016/j.jmaa.2018.11.048

25. C. Huang, G. Jia, Multiple solutions for a class of quasilinear Schrödinger equations, *Complex Var. Elliptic Equations*, **66** (2021), 347–359. https://doi.org/10.1080/17476933.2020.1727899

26. D. Costa, Z. Q. Wang, Multiplicity results for a class of superlinear elliptic problems, *Pro. Amer. Math. Soc.*, **133** (2005), 787–794. https://doi.org/10.1090/S0002-9939-04-07635-X

27. T. Bartsch, A. Pankov, Z. Q. Wang, Nonlinear Schrödinger equations with steep potential well, *Commun. Contemp. Math.*, **3** (2001), 549–569. https://doi.org/10.1142/S0219199701000494

28. J. Liu, X. Liu, Z. Q. Wang, Multiple mixed states of nodal solutions for nonlinear Schrödinger systems, *Calc. Var. Partial Differ. Equations*, **52** (2015), 565–586. https://doi.org/10.1007/s00526-014-0724-y

29. T. Bartsch, Z. Liu, On a superlinear elliptic p-Laplacian equation, *J. Differ. Equations*, **198** (2004), 149–175. https://doi.org/10.1016/j.jde.2003.08.001

30. T. Bartsch, Z. Liu, T. Weth, Nodal solutions of a p-Laplacian equation, *Proc. London Math. Soc.*, **91** (2005), 129–152. https://doi.org/10.1112/S0024611504015187

31. Z. Liu, Z. Wang, J. Zhang, Infinitely many sign-changing solutions for the nonlinear Schrödinger-Poisson system, *Ann. Mat. Pura Appl.*, **195** (2016), 775–794. https://doi.org/10.1007/s10231-015-0489-8

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