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Research article

Existence of solutions for a coupled Schrödinger equations with critical exponent

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Abstract: In this paper we study the existence of multiple nontrivial solutions of the coupled Schrödinger system with external sources terms as perturbations. This type of the system arises from Bose-Einstein condensate. As these external sources terms are nonlinear functions and small in some sense, we use fibre map to divide the Nehari manifold into threes parts, and then prove the existence of a nontrivial ground state solution and a bound state solution.

Keywords: coupled Schrödinger equations; Sobolev critical exponent; variational method; Nehari Manifold

1. Introduction and main results

In this paper we consider solitary wave solutions of the time-dependent coupled nonlinear Schrödinger system with perturbation

$$\begin{cases} -i\frac{\partial \Phi_1}{\partial t} - \Delta\Phi_1 = \mu_1|\Phi_1|^2\Phi_1 + \beta|\Phi_2|^2\Phi_1 + f_1(x), & x \in \Omega,\ t > 0,\\ -i\frac{\partial \Phi_2}{\partial t} - \Delta\Phi_2 = \mu_2|\Phi_2|^2\Phi_2 + \beta|\Phi_1|^2\Phi_2 + f_2(x), & x \in \Omega,\ t > 0,\\ \Phi_1(t,x) = \Phi_2(t,x) = 0, \quad x \in \partial\Omega,\ t > 0,\ j = 1,2, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, i is the imaginary unit, $\mu_1, \mu_2 > 0$ and $\beta \neq 0$ is a coupling constant. When $N \leq 3$, the system (1.1) appears in many physical problems, especially in nonlinear optics. Physically, the solution j denotes the j-th component of the beam in Kerr-like photorefractive media (see [1]). The positive constant μ_j is for self-focusing in the j th component of the beam. The coupling constant β is the interaction between the two components of the beam. The problem (1.1) also arises in the Hartree-Fock theory for a double condensate, that is, a binary mixture of Bose-Einstein condensates in two different hyperfine states, for more information, see [2].

If we looking for the stationary solution of the system (1.1), i.e., the solution is independent of time t. Then the system (1.1) is reduced to the following elliptic system with perturbation

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1 |u|^2 u + \beta u v^2 + f_1(x), & x \in \Omega, \\ -\Delta v + \lambda_2 v = \mu_2 |v|^2 v + \beta u^2 v + f_2(x), & x \in \Omega, \\ u = v = 0, \quad x \in \partial\Omega. \end{cases} \tag{1.2}$$

In the case where $N \le 3$ and $f_1 = f_2 = 0$, then the nonlinearity and the coupling terms in (1.2) are subcritical, and the existence of solutions has recently received great interest, for instance, see [3–11] for the existence of a (least energy) solution, and [12–16] for semiclassical states or singularly perturbed settings, and [17–22] for the existence of multiple solutions.

In the present paper we consider the case when $N = 4$ and $p = 2^* = 4$ is the Sobolev critical exponent. If $f_1 = f_2 = 0$, the paper [23] proved the existence of positive least energy solution for negative β, positive small β and positive large β.Furthermore, for the case $\lambda_1 = \lambda_2$, they obtained the uniqueness of positive least energy solutions and they studied the limit behavior of the least energy solutions in the repulsive case $\beta \to -\infty$, and phase separation is obtained. Later, the paper [24] studied the high dimensional case $N \ge 5$. The paper [25] proved the existence of sign-changing solutions of (1.2). Recently, the paper [26] considered the system (1.2) with perturbation in dimension $N \le 3$. By using Nehari manifold methods, the authors proved the existence of a positive ground state solution and a positive bound state solution. To the best of our knowledge, the existence of multiple nontrivial solution to the system (1.2) with critical growth($N = 4$) is still unknown. In the present paper we shall fill this gap.

Another motivation to study the existence of multiple nontrivial solution of (1.2) is coming from studying of the scalar critical equation. In fact, the second-order semilinear and quasilinear problems have been object of intensive research in the last years. In the pioneering work [27], Brezis and Nirenberg have studied the existence of positive solutions of the scalar equation

$$\begin{cases} -\Delta u = u^p + f, & x \in \Omega, \\ u > 0, \quad x \in \Omega, \quad u = 0, \quad x \in \partial\Omega, \end{cases} \tag{1.3}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \ge 3$, $p = \frac{N+2}{N-2}$, $f(x, u)$ is a lower order perturbation of u^p. Particularly, when $f = \lambda u$, where $\lambda \in \mathbb{R}$ is a constant, they have discovered the following remarkable phenomenon: the qualitative behavior of the set of solutions of (1.7) is highly sensitive to N, the dimension of the space. Precisely, the paper [27] has shown that, in dimension $N \ge 4$, there exists a positive solution of (1.3), if and only if $\lambda \in (0, \lambda_1)$; while, in dimension $N = 3$ and when $\Omega = B_1$ is the unit ball, there exists a positive solution of (1.7), if and only if $\lambda \in (\lambda_1/4, \lambda_1)$, where $\lambda_1 > 0$ is the first eigenvalue of $-\Delta$ in Ω. The paper [28] proved the existence of both radial and nonradial solutions to the problem

$$\begin{cases} -\Delta u = b(r) u^p + f(r, u), & x \in \Omega, \quad r = |x|, \\ u > 0, \quad x \in \Omega, \quad u = 0, \quad x \in \partial\Omega \end{cases} \tag{1.4}$$

under some assumptions on $b(r)$ and $f(r, u)$, $p = \frac{N+2}{N-2}$, where $\Omega = B(0, 1)$ is the unit ball in $\mathbb{R}^N$. In the paper [29], G. Tarantello considered the critical case for (1.3). He proved that (1.3) has at least two solutions under some conditions of f: $f \neq 0$, $f \in H^{-1}$ and

$$\|f\|_{H^{-1}} < c_N S^{\frac{N}{4}}, c_N = \frac{4}{N-2}(\frac{N-2}{N+2})^{\frac{N+2}{4}}, \tag{1.5}$$

and

$$S = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{|\nabla u|_2^2}{|u|_4^2} \tag{1.6}$$

is the best Sobolev constant of the imbedding from $H_0^1(\Omega)$ to $L^p(\Omega)$. For more results on this direction we refer the readers to [30–35] and the references therein.

Motivated by the above works, in the present paper, we are interested in the critical coupled Schrödinger equations in (1.2) with $\lambda_1 = \lambda_2 = 0$

$$\begin{cases} -\Delta u = \mu_1 u^3 + \beta u v^2 + f_1, & x \in \Omega, \\ -\Delta v = \mu_2 v^3 + \beta u^2 v + f_2, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega. \end{cases} \tag{1.7}$$

where Ω is a smooth bounded domain in $\mathbb{R}^4$, Δ is the Laplace operator and $p = 2^* = 4$ is the Sobolev critical exponent, and $\mu_1 > 0, \mu_2 > 0, 0 < \beta \leq \min\{\mu_1, \mu_2\}$.

Obviously, the energy functional is denoted by

$$I(u,v) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + |\nabla v|^2)dx - \frac{1}{4}\int_\Omega (\mu_1|u|^4 + \mu_2|v|^4 + 2\beta u^2v^2)dx - \int_\Omega (f_1 u + f_2 v)dx \tag{1.8}$$

for $(u, v) \in H = H_0^1(\Omega) \times H_0^1(\Omega)$. So, the critical point of $I(u, v)$ is the solution of the system (1.7). We shall fill the gap and generalize the results of [26] to the critical case. Our main tool here is the Nehari manifold method which is similar to the fibering method of Pohozaev's, which was first used by Tarantello [29].

We define thee Nehari manifold

$$\mathscr{N} = \{(u,v) \in H | \; I'(u,v)(u,v) = 0\}. \tag{1.9}$$

It is clear that all critical points of I lie in the Nehari manifold, and it is usually effective to consider the existence of critical points in this smaller subset of the Sobolev space. For fixed $(u, v) \in H \setminus \{(0, 0)\}$, we set

$$g(t) = I(tu, tv) = \frac{A}{2}t^2 - \frac{B}{4}t^4 - Dt, \quad t > 0.$$

where

$$A = \int_\Omega (|\nabla u|^2 + |\nabla v|^2)dx, \; B = \int_\Omega (\mu_1|u|^4 + \mu_2|v|^4 + 2\beta u^2v^2)dx, \; D = \int_\Omega (f_1 u + f_2 v)dx. \tag{1.10}$$

The mapping $g(t)$ is called fibering map. Such maps are often used to investigate Nehari manifold for various semilinear problem. By using the relationship of I and $g(t)$, we can divide $\mathscr{N}$ into three parts as follow:

$$\mathscr{N}^+ = \{(u,v) \in \mathscr{N} \;\big|\; A - 3B > 0\},$$
$$\mathscr{N}^0 = \{(u,v) \in \mathscr{N} \;\big|\; A - 3B = 0\},$$
$$\mathscr{N}^- = \{(u,v) \in \mathscr{N} \;\big|\; A - 3B < 0\}.$$

In order to get our results, we assume that f_i satisfies

$$f_i \neq 0, f_i \in L^{\frac{4}{3}}(\Omega), |f_i|_{\frac{4}{3}} < \frac{S^{\frac{3}{2}}}{3\sqrt{3}K^{\frac{1}{2}}}, i = 1, 2, \tag{1.11}$$

where $K = \max\{\mu_1, \mu_2\}$, S is defined in (1.6). Then we have the following main results.

Theorem 1.1. *Assume that* $0 < \beta \leq \min\{\mu_1, \mu_2\}$*, and* f_1, f_2 *satisfies (1.11). Then*

$$\inf_{\mathcal{N}} I = \inf_{\mathcal{N}^+} I = c_0 \tag{1.12}$$

is achieved at a point $(u_0, v_0) \in \mathcal{N}$*. Furthermore,* (u_0, v_0) *is a critical point of I.*

Next we consider then following minimization problem

$$\inf_{\mathcal{N}^-} I = c_1. \tag{1.13}$$

Then we have the following result.

Theorem 1.2. *Assume that* $0 < \beta \leq \min\{\mu_1, \mu_2\}$*, and* f_1, f_2 *satisfies (1.11). Then* $c_1 > c_0$ *and the infimum in (1.13) is achieved at a point* $(u_1, v_1) \in \mathcal{N}^-$*, which is the second critical point of I.*

Remark 1.3. *We point out that to the best of our knowledge, the existence of multiple nontrivial solution to the system* (1.2) *with critical growth(*$N = 4$*) is still unknown. In the present paper we shall fill this gap and generalized the results of [26] to the critical case.*

2. Variational setting and preliminary results

Throughout the paper, we shall use the following notation.

- Let $(\cdot, \cdot)$ be the inner product of the usual Sobolev space $H_0^1(\Omega)$ defined by $(u, v) = \int_\Omega \nabla u \nabla v dx$, and the corresponding norm is $\|u\| = (u, u)^{\frac{1}{2}}$.
- Let $S = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \left(|\nabla u|_2^2 / |u|_4^2\right)$ be the best Sobolev constant of the imbedding from $H_0^1(\Omega)$ to $L^4(\Omega)$.
- $|u|_p$ is the norm of $L^p(\Omega)$ defined by $|u|_p = (\int_\Omega |u|^p dx)^{\frac{1}{p}}$, for $0 < p < \infty$.
- Let $\|(u, v)\|^2 = \int_\Omega (|\nabla u|^2 + |\nabla v|^2) dx$ be the norm in the space of $H = H_0^1(\Omega) \times H_0^1(\Omega)$.
- Let C or $C_i (i = 1, 2, ...)$ denote the different positive constants.

We shall use the variational methods to prove the main results. In this section we shall prove some basic results for the system (1.2). The next lemma states the purpose of the assuptions (1.11).

Lemma 2.1. *Assume that the conditions of Theorem 1.1 hold. Then for every* $(u, v) \in H \setminus \{(0, 0)\}$*, there exists a unique* $t_1 > 0$ *such that* $(t_1 u, t_1 v) \in \mathcal{N}^-$*. In particular, we have*

$$t_1 > t_0 := \left[\frac{A}{3B}\right]^{\frac{1}{2}} \tag{2.1}$$

and $g(t_1) = \max_{t \geq t_0} g(t)$*, where A and B are given in* (1.10)*. Moreover, if* $D > 0$*, then there exists a unique* $t_2 > 0$*, such that* $(t_2 u, t_2 v) \in \mathcal{N}^+$*, where* $D > 0$ *is given in* (1.10)*. In particular, one has*

$$t_2 < t_0 \quad and \quad I(t_2 u, t_2 v) \leq I(tu, tv), \quad \forall t \in [0, t_1]. \tag{2.2}$$

Proof. We first define the fibering map by

$$g(t) = \frac{A}{2}t^2 - \frac{B}{4}t^4 - Dt, \quad t > 0.$$

Then we have

$$g'(t) = At - Bt^3 - D = \Phi(t) - D.$$

We deduce from $\Phi'(t) = 0$ that

$$t = t_0 = \left[\frac{A}{3B}\right]^{\frac{1}{2}}.$$

If $0 < t < t_0$, we have $g''(t) = \Phi'(t) > 0$, and if $t > t_0$, one sees $g''(t) = \Phi'(t) < 0$. A direct computation shows that $\Phi(t)$ achieves its maximum at t_0 and $\Phi(t_0) = \frac{2}{3\sqrt{3}}\frac{A^{\frac{3}{2}}}{B^{\frac{1}{2}}}$.

From the assumption (1.11), Sobolev's and Hölder's inequalities, we infer that

$$\begin{aligned} D = \int_\Omega (f_1 u + f_2 v)dx &\leq |f_1|_{\frac{4}{3}}|u|_4 + |f_2|_{\frac{4}{3}}|v|_4 \leq \sqrt{(|f_1|^2_{\frac{4}{3}} + |f_2|^2_{\frac{4}{3}})(|u|_4^2 + |v|_4^2)} \\ &\leq \sqrt{2}\max\{|f_1|_{\frac{4}{3}}, |f_2|_{\frac{4}{3}}\}(|u|_4^2 + |v|_4^2)^{\frac{1}{2}} < \frac{\sqrt{2}S^{\frac{3}{2}}}{3\sqrt{3}K^{\frac{1}{2}}}(|u|_4^2 + |v|_4^2)^{\frac{1}{2}}. \end{aligned} \tag{2.3}$$

On the other hand, since $0 < \beta \leq \min\{\mu_1, \mu_2\}$, it follows that

$$\begin{aligned} \Phi(t_0) = \frac{2}{3\sqrt{3}}\frac{A^{\frac{3}{2}}}{B^{\frac{1}{2}}} &= \frac{2}{3\sqrt{3}}\frac{(\int_\Omega(|\nabla u|^2 + |\nabla v|^2)dx)^{\frac{3}{2}}}{(\int_\Omega(\mu_1|u|^4 + \mu_2|v|^4 + 2\beta u^2v^2)dx)^{\frac{1}{2}}} \\ &\geq \frac{2}{3\sqrt{3}}\frac{S^{\frac{3}{2}}(|u|_4^2 + |v|_4^2)^{\frac{3}{2}}}{K^{\frac{1}{2}}(\int_\Omega(|u|^4 + |v|^4 + 2\beta u^2v^2)dx)^{\frac{1}{2}}} \\ &\geq \frac{2S^{\frac{3}{2}}}{3\sqrt{3}K^{\frac{1}{2}}}\frac{(|u|_4^2 + |v|_4^2)^{\frac{3}{2}}}{\sqrt{2}(|u|_4^4 + |v|_4^4)^{\frac{1}{2}}} \\ &\geq \frac{\sqrt{2}S^{\frac{3}{2}}}{3\sqrt{3}K^{\frac{1}{2}}}\frac{(|u|_4^2 + |v|_4^2)^{\frac{3}{2}}}{|u|_4^2 + |v|_4^2} = \frac{\sqrt{2}S^{\frac{3}{2}}}{3\sqrt{3}K^{\frac{1}{2}}}(|u|_4^2 + |v|_4^2)^{\frac{1}{2}}, \end{aligned}$$

where $K = \max\{\mu_1, \mu_2\}$. Hence we get

$$g'(t_0) = \Phi(t_0) - D > 0 \quad \text{and} \quad g'(t) \to -\infty, \ as \quad t \to +\infty. \tag{2.4}$$

Thus, there exists an unique $t_1 > t_0$ such that $g'(t_1) = 0$. We infer from the monotonicity of $\Phi(t)$ that for $t_1 > t_0$

$$g''(t_1) = \Phi'(t_1) < 0, \ \ t_1^2\Phi'(t_1) = t_1^2(A - 3Bt_1^2) < 0.$$

This implies that $(t_1u, t_1v) \in \mathcal{N}^-$. If $D > 0$, then we have $g'(0) = \Phi(0) - D = -D < 0$. Furthermore, there exists an unique $t_2 \in [0, t_0]$ such that $g'(t_2) = 0$ and $\Phi(t_2) = D$. A direct computation shows that $(t_2u, t_2v) \in \mathcal{N}^+$ and $I(t_2u, t_2v) \leq I(tu, tv), \ \forall t \in [0, t_1]$. □

Next we study the structure of $\mathcal{N}^0$.

Lemma 2.2. *Let $f_i \neq 0 (i = 1, 2)$ satisfy (1.11). Then for every $(u, v) \in \mathcal{N} \setminus \{(0, 0)\}$, we have*

$$\int_\Omega (|\nabla u|^2 + |\nabla v|^2)dx - 3\int_\Omega (\mu_1|u|^4 + \mu_2|v|^4 + 2\beta u^2v^2)dx \neq 0. \tag{2.5}$$

Hence we can get the conclusion that $\mathcal{N}^0 = \{(0, 0)\}$.

Proof. In order to prove that $\mathcal{N}^0 = \{(0, 0)\}$, we only need to show that for $(u, v) \in H \setminus \{(0, 0)\}$, $g(t)$ has no critical point that is a turning point. We use contradiction argument. Assume that there exists $\exists(u, v) \neq (0, 0)$ such that $(t_0u, t_0v) \in \mathcal{N}^0$ and $t_0 > 0$. Thus, we get

$$g'(t_0) = At_0 - Bt_0^3 - D = 0 \quad \text{and} \quad g''(t_0) = A - 3Bt_0^2 = 0.$$

Then we have $t_0 = \left[\frac{A}{3B}\right]^{\frac{1}{2}}$. This contradicts (2.4). This finishes the proof. □

In the next lemma, we shall prove the properties of Nehari manifolds $\mathcal{N}$.

Lemma 2.3. *Let $f_i \neq 0 (i = 1, 2)$ satisfy (1.11). For $(u, v) \in \mathcal{N} \setminus \{(0, 0)\}$, then there exist $\varepsilon > 0$ and a differentiable function $t = t(w, z) > 0$, $(w, z) \in H$, $\|(w, z)\| < \varepsilon$, and satisfying the following conditions*

$$t(0, 0) = 1, \quad t(w, z)((u, v) - (w, z)) \in \mathcal{N}, \quad \forall \|(w, z)\| < \varepsilon,$$

and

$$< t'(0, 0), (w, z) >= \frac{2\int_\Omega(\nabla u\nabla w + \nabla v\nabla z)dx - 4\int_\Omega[\mu_1|u|^2uw + \mu_2|v|^2vz + \beta(uv^2w + u^2vz)]dx - \int_\Omega(f_1w + f_2z)dx}{\int_\Omega(|\nabla u|^2 + |\nabla v|^2)dx - 3\int_\Omega(\mu_1|u|^4 + \mu_2|v|^4 + 2\beta u^2v^2)dx}.$$

Proof. We define $F : \mathbb{R} \times H \to \mathbb{R}$ by

$$\begin{aligned} F(t, (w, z)) = t\|\nabla(u - w)\|_2^2 + t\|\nabla(v - z)\|_2^2 - t^3\int_\Omega(\mu_1|u - w|^4 \\ + \mu_2|v - z|^4 + 2\beta(u - w)^2(v - z)^2)dx - \int_\Omega(f_1(u - w) + f_2(v - z))dx. \end{aligned}$$

We deduce from Lemma 2.2 and $(u, v) \in \mathcal{N}$ that $F(1, (0, 0)) = 0$. Moreover, one has

$$F_t(1, (0, 0)) = \int_\Omega(|\nabla u|^2 + |\nabla v|^2)dx - 3\int_\Omega(\mu_1|u|^4 + \mu_2|v|^4 + 2\beta u^2v^2)dx \neq 0.$$

By applying the implicit function theorem at point (1,(0,0)), we can obtain the results. □

3. The Proof of Theorem 1.1

In this section we are devoted to proving Theorem 1.1. We begin the following lemma for the property of inf I.

Lemma 3.1. *Let*

$$c_0 = \inf_{\mathcal{N}} I = \inf_{\mathcal{N}^+} I.$$

Hence I is bounded from below in $\mathcal{N}$ and $c_0 < 0$.

Proof. For $(u, v) \in \mathcal{N}$, we have $\langle I'(u, v), (u, v)\rangle = 0$. We infer from (1.10) that $A - B - D = 0$. Thus, one deduces from (2.3) and Hölder inequality that

$$D < C(|u|_4^2 + |v|_4^2)^{\frac{1}{2}} \leq C_1(|\nabla u|_2^2 + |\nabla v|_2^2)^{\frac{1}{2}} = C_1 A^{\frac{1}{2}}.$$

Hence, one deduces that

$$I(u, v) = \frac{A}{2} - \frac{B}{4} - D = \frac{A}{4} - \frac{3D}{4} > \frac{A}{4} - C_2 A^{\frac{1}{2}}.$$

Thus, the infimum c_0 in $\mathcal{N}^+$ is bounded from below. Next we prove the upper bound for c_0. Let $w_i \in H_0^1(\Omega)(i = 1, 2)$ be the solution for $-\Delta w = f_i, (i = 1, 2)$. So, for $f_i \neq 0$ one sees that

$$\int_\Omega (f_1 w_1 + f_2 w_2) dx = |\nabla w_1|_2^2 + |\nabla w_2|_2^2 > 0.$$

We let $t_2 = t_2(u, v) > 0$ as defined by Lemma 2.1. Thus, we infer that $(t_2 w_1, t_2 w_2) \in \mathcal{N}^+$ and

$$t_2^2 \int_\Omega (|\nabla w_1|^2 + |\nabla w_2|^2) dx - t_2^4 \int_\Omega (\mu_1 |w_1|^4 + \mu_2 |w_2|^4 + 2\beta w_1^2 w_2^2) dx - t_2 \int_\Omega (f_1 w_1 + f_2 w_2) dx = 0.$$

Furthermore, it follows from (2.2) that

$$c_0 = \inf_{(u,v)\in\mathcal{N}^+} I(u, v) \leq I(t_2 w_1, t_2 w_2)) < I(0, 0) = 0.$$

This completes the proof. □

The next lemma studies the properties of the infimum c_0.

Lemma 3.2. *(1) The level c_0 can be attained. That is, there exists $(u_0, v_0) \in \mathcal{N}^+$ such that $I(u_0, v_0) = c_0$.*

(2) (u_0, v_0) is a local minimum for I in H.

Proof. From Lemma 3.1, we can apply Ekeland's variational principle to the minimization problem, which gives a minimizing sequence $\{(u_n, v_n)\} \subset \mathcal{N}$ such that

(i) $I(u_n, v_n) < c_0 + \frac{1}{n}$,

(ii) $I(w, z) \geq I(u_n, v_n) - \frac{1}{n}(|\nabla(w - u_n)|_2 + |\nabla(z - v_n)|_2), \quad \forall (w, z) \in \mathcal{N}$.

For n large enough, by Lemma 3.1 and (i)-(ii) of the above, we can get

$$\exists C_1 > 0, C_2 > 0, \ \ 0 < C_1 \leq |\nabla u_n|_2^2 + |\nabla v_n|_2^2 \leq C_2.$$

In the following we shall prove that $\|I'(u_n, v_n)\| \to 0$ as $n \to \infty$. In fact, we can apply Lemma 2.3 with $(u, v) = (u_n, v_n)$ and $(w, z) = \delta \frac{(I_u(u_n,v_n), I_v(u_n,v_n))}{\|I'(u_n,v_n)\|} (\delta > 0)$. Then can find $t_n(\delta)$ such that

$$(w_\delta, z_\delta) = t_n(\delta)\left[(u_n, v_n) - \delta \frac{(I_u(u_n, v_n), I_v(u_n, v_n))}{\|I'(u_n, v_n)\|}\right] \in \mathcal{N}.$$

Thus, we infer from the condition (ii) that

$$I(u_n, v_n) - I(w_\delta, z_\delta) \leq \frac{1}{n} (|\nabla(w_\delta - u_n)|_2 + |\nabla(z_\delta - v_n)|_2). \tag{3.1}$$

On the other hand, by using Taylor expansion we have that

$$I(u_n,v_n)-I(w_\delta,z_\delta)=(1-t_n(\delta))(I'(w_\delta,z_\delta),(u_n,v_n))+\delta t_n(\delta)\left(I'(w_\delta,z_\delta),\frac{I'(u_n,v_n)}{\|I'(u_n,v_n)\|}\right)+o(\delta).$$

Dividing by $\delta>0$ and letting $\delta\to 0$, we get

$$\frac{1}{n}\left(2+t_n'(0)(|\nabla u_n|_2+|\nabla v_n|_2)\right)\geq -t_n'(0)(I'(u_n,v_n),(u_n,v_n))+\|I'(u_n,v_n)\|=\|I'(u_n,v_n)\|. \tag{3.2}$$

Combining (3.1) and (3.2) we conclude that

$$\|I'(u_n,v_n)\|\leq \frac{C}{n}(2+t_n'(0)).$$

We infer from Lemma 2.3 and $(u_n,v_n)\subset\mathscr{N}$ that $t_n'(0)$ is bounded. That is,

$$|t_n'(0)|\leq C.$$

Hence we obtain that

$$\|I'(u_n,v_n)\|\to 0\ \ as\ \ n\to\infty. \tag{3.3}$$

Therefore, by choosing a subsequence if necessary, we have that

$$(u_n,v_n)\rightharpoonup(u_0,v_0)\ \text{ in }\ H\ \text{ and }\ I'(u_0,v_0)=0,$$

and

$$\begin{aligned}c_0\leq I(u_0,v_0)&=\frac{1}{4}(|\nabla u_0|_2^2+|\nabla v_0|_2^2)-\int_\Omega(f_1u_0+f_2v_0)dx\\&\leq\lim_{n\to\infty}I(u_n,v_n)=c_0.\end{aligned}$$

Consequently, we infer that

$$(u_n,v_n)\to(u_0,v_0)\ \ in\ \ H,\ I(u_0,v_0)=c_0=\inf_{\mathscr{N}}I.$$

From Lemma 2.1 and (3.3), we deduce that $(u_0,v_0)\in\mathscr{N}^+$.

(2) In order to get the conclusion, it suffices to prove that $\forall(w,z)\in H,\exists\varepsilon>0$, if $\|(w,z)\|<\varepsilon$, then $I(u_0-w,v_0-z)\geq I(u_0,v_0)$. In fact, notice that for every $(w,z)\in H$ with $\int_\Omega(f_1u+f_2v)dx>0$, we infer from Lemma 2.1 that

$$I(su,sv)\geq I(t_1u,t_1v),\ \forall\ s\in[0,t_0].$$

In particular, for $(u_0,v_0)\in\mathscr{N}^+$, we have

$$t_2=1<t_0=\left[\frac{\int_\Omega(|\nabla u_0|^2+|\nabla v_0|^2)dx}{3\int_\Omega(\mu_1|u_0|^4+\mu_2|v_0|^4+2\beta u_0^2v_0^2)dx}\right]^{\frac{1}{2}}.$$

Let $\varepsilon>0$ sufficiently small. Then we infer that for $\|(w,z)\|<\varepsilon$

$$1<\left[\frac{\int_\Omega(|\nabla(u_0-w)|^2+|\nabla(v_0-z)|^2)dx}{3\int_\Omega(\mu_1|u_0-w|^4+\mu_2|v_0-z|^4+2\beta(u_0-w)^2(v_0-z)^2)dx}\right]^{\frac{1}{2}}=\tilde{t}_0. \tag{3.4}$$

From Lemma 2.3, let $t(w,z) > 0$ satisfy $t(w,z)(u_0 - w, v_0 - z) \in \mathcal{N}$ for every $\|(w,z)\| < \varepsilon$. Since $t(w,z) \to 1$ as $\|(w,z)\| \to 0$, we can assume that

$$t(w,z) < \tilde{t}_0, \quad \forall \, \|(w,z)\| < \varepsilon.$$

Hence we obtain that $t(w,z)(u_0 - w, v_0 - z) \in \mathcal{N}^+$ and

$$I(s(u_0 - w), s(v_0 - z)) \geq I(t(w,z)(u_0 - w),\ t(w,z)(v_0 - z)) \geq I(u_0, v_0), \ \forall \, 0 < s < \tilde{t}_0.$$

From (3.4) we can take $s = 1$ and conclude

$$I(u_0 - w, v_0 - z) \geq I(u_0, v_0), \quad \forall (w,z) \in H, \ \ \|(w,z)\| < \varepsilon.$$

This finishes the proof. □

Proof of Theorem 1.1. From Lemma 3.2, we know that (u_0, v_0) is the critical point of I. □

4. Proof of Theorem 1.2

In this section we focus on the proof of Theorem 1.2. The main difficulty here is the lack of compactness(due to the embedding $H \hookrightarrow L^4(\Omega) \times L^4(\Omega)$ is noncompact). Motivated by previous works of [27, 29, 37], we shall seek the local compactness. Then by using the Mountain pass principle to find the second nontrivial solution of equation (1.7). The pioneering paper [29] has used this methods to find the second solution of the scalar Schrödinger equation. To this purpose, we first begin with the following lemma to find the threshold to recover the compactness.

Lemma 4.1. *For every sequence $(u_n, v_n) \in H$ satisfying*

(*i*) $I(u_n, v_n) \to c$ *with* $c < c_0 + \frac{1}{4}\min\left\{\frac{S^2}{\mu_1}, \frac{S^2}{\mu_2}\right\}$, *where c_0 is defined in (1.12), S is the best Sobolev constant of the imbedding from $H_0^1(\Omega)$ to $L^4(\Omega)$,*
(*ii*) $\|I'(u_n, v_n)\| \to 0$ *as* $n \to \infty$.

Then $\{(u_n, v_n)\}$ has a convergent subsequence. This means that the $(PS)_c$ condition holds for all level $c < c_0 + \frac{1}{4}\min\left\{\frac{S^2}{\mu_1}, \frac{S^2}{\mu_2}\right\}$.

Proof. From condition (*i*) and (*ii*), it is easy to verify that $\|(u_n, v_n)\|$ is bounded. So, for the subsequence $\{(u_n, v_n)\}$(which we still call $\{(u_n, v_n)\}$, we can find a $(w_0, z_0) \in H$ such that $(u_n, v_n) \rightharpoonup (w_0, z_0)$ in H. Then from the condition (*ii*), we obtain that

$$(I'(w_0, z_0), (w, z)) = 0, \quad \forall (w,z) \in H.$$

That is, (w_0, z_0) is a solution in H. Moreover, $(w_0, z_0) \in \mathcal{N}$ and $I(w_0, z_0) \geq c_0$. Let

$$(u_n, v_n) = (w_0 + w_n, z_0 + z_n).$$

Then $(w_n, z_n) \rightharpoonup (0,0)$ in H. Then it suffices to prove that

$$(w_n, z_n) \to (0,0) \ \text{ in } \ H. \tag{4.1}$$

We use the indirect argument. Assume that (4.1) does not hold. Then we divide the following three cases to find the contradiction.

Case 1: $w_n \to 0$ and $z_n \nrightarrow 0$ in H. Since $\|(u_n, v_n)\|$ is bounded, it follows that

$$\int_\Omega w_n^2 z_n^2 dx = o(1).$$

by (1.7), we can get

$$\begin{aligned} c_0 + \frac{1}{4}\min\left\{\frac{S^2}{\mu_1}, \frac{S^2}{\mu_2}\right\} &> I(u_n, v_n) = I(w_0 + w_n, z_0 + z_n) \\ &= I(w_0, z_0) + \frac{1}{2}\int_\Omega |\nabla z_n|^2 dx - \frac{\mu_2}{4}\int_\Omega |z_n|^4 dx + o(1) \\ &\geq c_0 + \frac{1}{2}|\nabla z_n|_2^2 - \frac{\mu_2}{4}|z_n|_4^4 + o(1), \end{aligned}$$

and then

$$\frac{1}{2}|\nabla z_n|_2^2 - \frac{\mu_2}{4}|z_n|_4^4 < \frac{S^2}{4\mu_2}. \tag{4.2}$$

We infer from the condition (*ii*) that

$$o(1) = (I'(u_n, v_n), (u_n, v_n)) = (I'(w_0, z_0), (w_0, z_0)) + |\nabla z_n|_2^2 - \mu_2 |z_n|_4^4 + o(1).$$

That is, we get

$$|\nabla z_n|_2^2 = \mu_2 |z_n|_4^4 + o(1).$$

By using the embedding from $H_0^1(\Omega)$ to $L^4(\Omega)$, we get

$$\mu_2 |z_n|_4^4 = |\nabla z_n|_2^2 \geq S|z_n|_4^2 + o(1).$$

Since $z_n \nrightarrow 0$, we infer that $|z_n|_4^2 \geq S/\mu_2 + +o(1)$. That is,

$$|z_n|_4^4 \geq \frac{S^2}{\mu_2^2} + o(1).$$

Hence we get

$$\frac{1}{2}|\nabla z_n|_2^2 - \frac{\mu_2}{4}|z_n|_4^4 = \frac{\mu_2}{4}|z_n|_4^4 + o(1) \geq \frac{1}{4}\frac{S^2}{\mu_2}. \tag{4.3}$$

This contradicts with the fact (4.2).

Case 2: $w_n \nrightarrow 0$ and $z_n \to 0$ in H. This can be accomplished by using same argument as in the proof of the Case 1.

Case 3: $w_n \nrightarrow 0$ and $z_n \nrightarrow 0$ in H. Similar to the Case 1, we infer from condition (*ii*) that

$$\begin{aligned} o(1) = (I'(u_n, v_n), (u_n, 0)) &= |\nabla u_n|^2 - \mu_1 |u_n|_4^4 - \beta\int_\Omega u_n^2 v_n^2 dx - \int_\Omega f_1 u_n dx \\ &= (I'(w_0, z_0), (w_0, 0)) + |\nabla w_n|_2^2 - \mu_1 |w_n|_4^4 - \beta\int_\Omega w_n^2 z_n^2 dx + o(1). \end{aligned}$$

Then we have

$$|\nabla w_n|_2^2 = \mu_1|w_n|_4^4 + \beta \int_\Omega w_n^2 z_n^2 dx + o(1). \tag{4.4}$$

One infers from Hölder and Sobolev inequality that

$$S|w_n|_4^2 \leq |\nabla w_n|_2^2 = \mu_1|w_n|_4^4 + \beta \int_\Omega w_n^2 z_n^2 dx + o(1) \leq \mu_1|w_n|_4^4 + \beta|w_n|_4^2|z_n|_4^2 + o(1). \tag{4.5}$$

Since $w_n \nrightarrow 0$, we have

$$S \leq \mu_1|w_n|_4^2 + \beta|z_n|_4^2 + o(1) \leq \mu_1(|w_n|_4^2 + |z_n|_4^2) + o(1). \tag{4.6}$$

Similarly, we obtain that

$$S \leq \mu_2|z_n|_4^2 + \beta|w_n|_4^2 + o(1) \leq \mu_2(|w_n|_4^2 + |z_n|_4^2) + o(1). \tag{4.7}$$

Thus, we conclude that

$$|w_n|_4^2 + |z_n|_4^2 \geq \max\left\{\frac{S}{\mu_1}, \frac{S}{\mu_2}\right\} + o(1). \tag{4.8}$$

On the other hand, we infer from the condition (*ii*) that

$$\begin{aligned} o(1) &= (I'(u_n, v_n), (u_n, v_n)) \\ &= |\nabla u_n|^2 + |\nabla v_n|^2 - \mu_1|u_n|_4^4 - \mu_2|v_n|_4^4 - 2\beta \int_\Omega u_n^2 v_n^2 dx - \int_\Omega f_1 u_n dx - \int_\Omega f_2 v_n dx \\ &= (I'(w_0, z_0), (w_0, z_0)) + |\nabla w_n|_2^2 + |\nabla z_n|_2^2 - \mu_1|w_n|_4^4 - \mu_2|z_n|_4^4 - 2\beta \int_\Omega w_n^2 z_n^2 dx + o(1). \end{aligned} \tag{4.9}$$

From (4.6)-(4.9), we deduce that

$$\begin{aligned} c_0 + \frac{1}{4}\min\left\{\frac{S^2}{\mu_1}, \frac{S^2}{\mu_2}\right\} &> I(u_n, v_n) = I(w_0 + w_n, z_0 + z_n) \\ &= I(w_0, z_0) + \frac{1}{2}|\nabla w_n|_2^2 + \frac{1}{2}|\nabla z_n|_2^2 - \frac{1}{4}(\mu_1|w_n|_4^4 + \mu_2|z_n|_4^4 \\ &\quad + 2\beta \int_\Omega w_n^2 z_n^2 dx) + o(1) \\ &\geq c_0 + \frac{1}{4}(\mu_1|w_n|_4^4 + \mu_2|z_n|_4^4 + 2\beta \int_\Omega w_n^2 z_n^2 dx) + o(1) \\ &\geq c_0 + \frac{S}{4}(|w_n|_4^2 + |z_n|_4^2) + o(1) \\ &\geq c_0 + \frac{1}{4}\max\{\frac{S^2}{\mu_1}, \frac{S^2}{\mu_2}\} + o(1). \end{aligned}$$

This is a contradiction. □

In order to applying Lemma 4.1 to get the compactness, we need to prove the following inequality

$$c_1 = \inf_{\mathcal{N}^-} I < c_0 + \frac{1}{4}\min\{\frac{S^2}{\mu_1}, \frac{S^2}{\mu_2}\}.$$

Let

$$u_\varepsilon(x) = \frac{\varepsilon}{\varepsilon^2 + |x|^2} \quad \varepsilon > 0,\ x \in \mathbb{R}^4$$

be an extremal function for the Sobolev inequality in $\mathbb{R}^4$. Let $u_{\varepsilon,a} = u(x-a)$ for $x \in \Omega$ and the cut-off function $\xi_a \in C_0^\infty(\Omega)$ with $\xi_a \geq 0$ and $\xi_a = 1$ near a. We set

$$U_{\varepsilon,a}(x) = \xi_a(x) u_{\varepsilon,a}(x), \quad x \in \mathbb{R}^4.$$

Following [37], we let $\Omega_1 \subset \Omega$ be a positive measure set such that $u_0 > 0,\ v_0 > 0$, where $c_0 = I(u_0, v_0)$ is given in Theorem 1.1. Then we have the following conclusion.

Lemma 4.2. *For every $R > 0$, and a.e. $a \in \Omega_1$, there exists $\varepsilon_0 = \varepsilon_0(R, a) > 0$, such that*

$$\min\{I(u_0 + RU_{\varepsilon,a}, v_0), I(u_0, v_0 + RU_{\varepsilon,a})\} < c_0 + \frac{1}{4}\min\left\{\frac{S^2}{\mu_1}, \frac{S^2}{\mu_2}\right\} \tag{4.10}$$

for every $0 < \varepsilon < \varepsilon_0$.

Proof. As in [37], a direct computation shows that

$$\begin{aligned} I(u_0 + RU_{\varepsilon,a}, v_0) = & \frac{1}{2}|\nabla u_0|_2^2 + R\int_\Omega \nabla u_0 \nabla U_{\varepsilon,a} dx + \frac{R^2}{2}|\nabla U_{\varepsilon,a}|_2^2 + \frac{1}{2}|\nabla v_0|_2^2 \\ & - \frac{\mu_1}{4}(|u_0|_4^4 + R^4|U_{\varepsilon,a}|_4^4 + 4R\int_\Omega u_0^3 U_{\varepsilon,a} dx + 4R^3\int_\Omega U_{\varepsilon,a}^3 u_0 dx) \\ & - \frac{\mu_2}{4}|v_0|_4^4 - \frac{\beta}{2}\int_\Omega (u_0^2 v_0^2 + 2Ru_0 v_0^2 U_{\varepsilon,a} + R^2 U_{\varepsilon,a}^2 v_0^2) dx \\ & - \int_\Omega (f_1 u_0 + f_2 v_0) dx - R\int_\Omega f U_{\varepsilon,a} dx + o(\varepsilon). \end{aligned} \tag{4.11}$$

We infer from [27] that

$$|\nabla U_{\varepsilon,a}|_2^2 = F + O(\varepsilon^2) \quad \text{and} \quad |U_{\varepsilon,a}|_4^4 = G + O(\varepsilon^4), \tag{4.12}$$

where

$$F = \int_{\mathbb{R}^4} |\nabla u_1(x)|^2 dx, \ \ G = \int_{\mathbb{R}^4} \frac{dx}{(1+|x|^2)^4}, \ \ S = \frac{F}{G^{\frac{1}{2}}}$$

If we let $u_0 = 0$ outside Ω, then

$$\int_\Omega U_{\varepsilon,a}^3 u_0 dx = \int_{\mathbb{R}^4} u_0 \xi_a(x) \frac{\varepsilon^3}{(\varepsilon^2 + |x-a|^2)^3} dx = \varepsilon \int_{\mathbb{R}^4} u_0 \xi_a(x) \frac{1}{\varepsilon^4} \varphi(\frac{x}{\varepsilon}) dx$$

where

$$\varphi(x) = \frac{1}{(1+|x|^2)^3} \in L^1(\mathbb{R}^4).$$

Set

$$E = \int_{\mathbb{R}^4} \frac{1}{(1+|x|^2)^3} dx.$$

Then we can derive

$$\int_{\mathbb{R}^4} u_0 \xi_a(x) \frac{1}{\varepsilon^4} \varphi(\frac{x}{\varepsilon}) dx \to u_0(a) E.$$

Since (u_0, v_0) is the critical point of I, it follows that

$$\int_\Omega (|\nabla u_0|^2 + |\nabla v_0|^2) dx - \int_\Omega (\mu_1 u_0^4 + \mu_2 v_0^4 + 2\beta u_0^2 v_0^2) dx - \int_\Omega (f_1 u_0 + f_2 v_0) dx = 0. \tag{4.13}$$

We infer from (4.11)-(4.13) that

$$\begin{aligned} I(u_0 + RU_{\varepsilon,a}, v_0) &= I(u_0, v_0) + \frac{R^2}{2} F - \frac{R^4}{4} \mu_1 G - \mu_1 R^3 \int_\Omega U_{\varepsilon,a}^3 u_0 dx - \frac{\beta R^2}{2} \int_\Omega U_{\varepsilon,a}^2 v_0^2 dx + o(\varepsilon) \\ &\le c_0 + \frac{R^2}{2} F - \frac{R^4}{4} \mu_1 G - \mu_1 \varepsilon R^3 E u_0(a) + o(\varepsilon). \end{aligned} \tag{4.14}$$

In order to get the upper bound of (4.14), we define

$$q_1(s) = \frac{F}{2} s^2 - \frac{\mu_1 G}{4} s^4 - k\varepsilon s^3, \quad k = \mu_1 E u_0(a) > 0,$$

and

$$q_2(s) = \frac{F}{2} s^2 - \frac{\mu_1 G}{4} s^4.$$

It is easy to get the maximum of $q_2(s)$ is achieved at $s_0 = (\frac{F}{\mu_1 G})^{\frac{1}{2}}$. Let the maximum of $q_1(s)$ is achieved at s_ε, so we can let $s_\varepsilon = (1 - \delta_\varepsilon) s_0$, and get $\delta_\varepsilon \to 0 (\varepsilon \to 0)$. Substituting $s_\varepsilon = (1 - \delta_\varepsilon) s_0$ into $q_1'(s) = 0$, we can get

$$F - F(1 - \delta_\varepsilon)^2 = 3 s_0 (1 - \delta_\varepsilon) k\varepsilon.$$

As in [29], we infer that

$$\delta_\varepsilon \sim \varepsilon, \quad \varepsilon \to 0.$$

Then we can get the upper bound estimation of $I(u_0 + RU_{\varepsilon,a}, v_0)$:

$$\begin{aligned} I(u_0 + RU_{\varepsilon,a}, v_0) &\le c_0 + \frac{R^2}{2} F - \frac{R^4}{4} \mu_1 G - k\varepsilon R^3 + o(\varepsilon) \\ &\le c_0 + \frac{[(1 - \delta_\varepsilon) s_0]^2}{2} F - \frac{[(1 - \delta_\varepsilon) s_0]^4}{4} \mu_1 G - k\varepsilon [(1 - \delta_\varepsilon) s_0]^3 + o(\varepsilon) \\ &= c_0 + (\frac{s_0^2}{2} F - \frac{s_0^4}{4} \mu_1 G) + (s_0^4 \mu_1 G - s_0^2 F) \delta_\varepsilon - k\varepsilon s_0^3 + o(\varepsilon) \\ &< c_0 + \frac{S^2}{4\mu_1} + o(\varepsilon). \end{aligned}$$

Thus, for $\varepsilon_0 > 0$ small, we get

$$I(u_0 + RU_{\varepsilon,a}, v_0) < c_0 + \frac{S^2}{4\mu_1}.$$

Similarly, we obtain that

$$I(u_0, v_0 + RU_{\varepsilon,a}) < c_0 + \frac{S^2}{4\mu_2}.$$

So, we prove

$$\min\{I(u_0+RU_{\varepsilon,a},v_0), I(u_0,v_0+RU_{\varepsilon,a})\} < c_0 + \frac{1}{4}\min\{\frac{S^2}{\mu_1},\frac{S^2}{\mu_2}\}, \quad \forall\, 0<\varepsilon<\varepsilon_0.$$

This finishes the proof. □

Without loss of generality, from above Lemma 4.2 we can assume

$$I(u_0+RU_{\varepsilon,a},v_0) < c_0 + \frac{S^2}{4\mu_1}, \quad R>0,\ \ \forall\ \ 0<\varepsilon<\varepsilon_0.$$

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. It is clear that there exists an uniqueness of $t_1>0$ such that

$$(t_1u,t_1v)\in\mathcal{N}^- \quad \text{and} \quad I(t_1u,t_1v)=\max_{t\geq t_0} I(t_1u,t_1v),\ \forall (u,v)\in H,\ \|(u,v)\|=1.$$

Moreover, $t_1(u,v)$ is a continuous function of (u,v), and $\mathcal{N}^-$ divides H into two components H_1 and H_2, which are disconnect with each other. Let

$$H_1=\left\{(u,v)=(0,0)\ \ or\ \ (u,v):\|(u,v)\|<t_1\Big(\frac{(u,v)}{\|(u,v)\|}\Big)\right\} \quad \text{and}$$
$$H_2=\left\{(u,v):\|(u,v)\|>t_1\Big(\frac{(u,v)}{\|(u,v)\|}\Big)\right\}.$$

Obviously, we have $H\setminus\mathcal{N}^-=H_1\cup H_2$. Furthermore, we obtain that $\mathcal{N}^+\subset H_1$ for $(u_0,v_0)\in H_1$. We choose a constant C_0 such that

$$0<t_1(u,v)\leq C_0,\ \ \forall\|(u,v)\|=1.$$

In the following we deduce that

$$(w,z)=(u_0+R_0U_{\varepsilon,a},v_0)\in H_2, \tag{4.15}$$

where $R_0=\left(\frac{1}{F}|C_0^2-\|(u_0,v_0)\|^2\right)^{\frac{1}{2}}+1$. Since

$$\begin{aligned}\|(w,z)\|^2&=\|(u_0,v_0)\|^2+R_0^2|\nabla U_{\varepsilon,a}|^2+2R_0\int_\Omega|\nabla u_0\|\nabla U_{\varepsilon,a}|dx\\&=\|(u_0,v_0)\|^2+R_0^2F+o(1)>C_0^2\geq\left[t_1\Big(\frac{(w,z)}{\|(w,z)\|}\Big)\right]^2\end{aligned}$$

for $\varepsilon>0$ small enough. We fix $\varepsilon>0$ small to make both (4.10) and (4.15) hold by the choice of R_0 and $a\in\Omega_1$. Set

$$\Gamma_1=\{\gamma\in C([0,1],H):\gamma(0)=(u_0,v_0),\gamma(1)=(u_0+R_0U_{\varepsilon,a},v_0)\}.$$

We take $h(t) = (u_0 + tR_0U_{\varepsilon,a}, v_0)$. Then $h(t) \in \Gamma_1$. From Lemma 4.1, we conclude that

$$c' = \inf_{h\in\Gamma_1}\max_{t\in[0,1]} I(h(t)) < c_0 + \frac{S^2}{4\mu_1}.$$

Since the range of every $h \in \Gamma_1$ intersect $\mathcal{N}^-$, we have

$$c_1 = \inf_{\mathcal{N}^-} I \leq c' < c_0 + \frac{S^2}{4\mu_1}. \tag{4.16}$$

Set

$$\Gamma_2 = \{\gamma \in C([0,1], H) : \gamma(0) = (u_0, v_0), \gamma(1) = (u_0, v_0 + R_0U_{\varepsilon,a})\}.$$

By using the same argument, we can get similar results

$$c'' = \inf_{h\in\Gamma_2}\max_{t\in[0,1]} I(h(t)) < c_0 + \frac{S^2}{4\mu_2}.$$

Moreover, since the range of every $h \in \Gamma_2$ intersect $\mathcal{N}^-$, we have

$$c_1 = \inf_{\mathcal{N}^-} I \leq c'' < c_0 + \frac{S^2}{4\mu_2}. \tag{4.17}$$

Combining (4.16) and (4.17), we obtain that

$$c_1 < c_0 + \frac{1}{4}\min\left\{\frac{S^2}{\mu_1}, \frac{S^2}{\mu_2}\right\}.$$

Next by using Mountain-Pass lemma(see [36]) to obtain that there exist $\{(u_n, v_n)\} \subset \mathcal{N}^-$ such that

$$I(u_n) \to c_1, \quad \|I'((u_n, v_n))\| \to 0.$$

From Lemma 4.1, we can obtain a subsequence (still denote $\{(u_n, v_n)\}$) of $\{(u_n, v_n)\}$, and $(u_1, v_1) \in H$ such that

$$(u_n, v_n) \to (u_1, v_1) \quad in\ H.$$

Hence, we get (u_1, v_1) is a critical point for $I, (u_1, v_1) \in \mathcal{N}^-$ and $I(u_1, v_1) = c_1$. □

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Conflict of interest

The authors declare there is no conflicts of interest.

References

1. N. Akhmediev, A. Ankiewicz, Partially coherent solitons on a finite background, *Phys. Rev. Lett.*, **82** (1999), 2661–2664. https://doi.org/10.1103/PhysRevLett.82.2661
2. B. Esry, C. Greene, J. Burke, J. Bohn, Hartree-Fock theory for double condensates, *Phys. Rev. Lett.*, **78** (1997), 3594–3597. https://doi.org/10.1103/PhysRevLett.78.3594
3. A. Ambrosetti, E. Colorado, Bound and ground states of coupled nonlinear Schrödinger equations, *C. R. Math. Acad. Sci. Paris*, **342** (2006), 453–458. https://doi.org/10.1016/j.crma.2006.01.024
4. A. Ambrosetti, E. Colorado, Standing waves of some coupled nonlinear Schr̈dinger equations, *J. Lond. Math. Soc.*, **75** (2007), 67–82.
5. T. Bartsch, Z.-Q. Wang, Note on ground states of nonlinear Schrödinger systems, *J. Partial Differ. Equ.*, **19** (2006), 2000–2007.
6. T. Bartsch, Z.-Q. Wang, J.-C. Wei, Bound states for a coupled Schrödinger system, *J. Fixed Point Theory Appl.*, **2** (2007), 353–367. https://doi.org/10.1007/s11784-007-0033-6
7. Z.-J. Chen, W. Zou, An optimal constant for the existence of least energy solutions of a coupled Schrödinger system, *Calc. Var. Partial Differ. Equ.*, **48** (2013), 695–711. https://doi.org/10.1007/s00526-012-0568-2
8. Z.-J. Chen, C-S. Lin, Asymptotic Behavior of Least Energy Solutions for a Critical Elliptic System, *Int. Math. Res. Not.*, **21** (2015), 11045–11082. https://doi.org/10.1093/imrn/rnv016
9. T. Lin, J. Wei, Ground state of N coupled nonlinear Schrödinger equations in $\mathbb{R}^n$, $n \leq 3$, *Commun. Math. Phys.*, **255** (2005), 629–653.
10. L. Maia, E. Montefusco, B. Pellacci, Positive solutions for a weakly coupled nonlinear Schrödinger systems, *J. Differ. Equ.*, **229** (2006), 743–767. https://doi.org/10.1016/j.jde.2006.07.002
11. B. Sirakov, Least energy solitary waves for a system of nonlinear Schrödinger equations in $\mathbb{R}^n$, *Commun. Math. Phys.*, **271** (2007), 199–221.
12. T. Lin, J. Wei, Spikes in two coupled nonlinear Schrödinger equations, *Ann. Inst. H. Poincar é Anal. Non Linéire*, **22** (2005), 403–439.
13. T. Lin, J. Wei, Spikes in two-component systems of nonlinear Schr̈dinger equations with trapping potentials, *J. Differ. Equ.*, **229** (2006), 538–569.
14. L. Maia, B. Pellacci, M. Squassina, Semiclassical states for weakly coupled nonlinear Schrödinger systems, *J. Eur. Math. Soc.*, **10** (2007), 47–71.
15. A. Pomponio, Coupled nonlinear Schrödinger systems with potentials, *J. Differ. Equ.*, **227** (2006), 258–281. https://doi.org/10.1016/j.jde.2005.09.002
16. J. Wang, J.-P. Shi, Standing waves of a weakly coupled Schr̈dinger system with distinct potential functions, *J. Differ. Equ.*, **260** (2016), 1830–1864.
17. T. Bartsch, N. Dancer, Z.-Q. Wang, A Liouville theorem, a priori bounds, and bifurcating branches of positive solutions for a nonlinear elliptic system, *Calc. Var. PDE*, **37** (2010), 345–361.
18. N. Dancer, J. Wei, T. Weth, A priori bounds versus multiple existence of positive solutions for a nonlinear Schrödinger systems, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **27** (2010), 953–969.

19. Z. Liu, Z.-Q. Wang, Multiple bound states of nonlinear Schrödinger systems, *Commun. Math. Phys.*, **282** (2008), 721–731. https://doi.org/10.1007/s00220-008-0546-x

20. B. Noris, M. Ramos, Existence and bounds of positive solutions for a nonlinear Schrödinger system, *Proc. Am. Math. Soc.*, **138** (2010), 1681–1692. https://doi.org/10.1090/S0002-9939-10-10231-7

21. J. Wei, T. Weth, Nonradial symmetric bound states for a system of two coupled Schrödinger equations, *Rend. Lincei Mat. Appl.*, **18** (2007), 279–293.

22. J. Wei, T. Weth, Radial solutions and phase separation in a system of two coupled Schrödinger equations, *Arch. Ration. Mech. Anal.*, **190** (2008), 83–106. https://doi.org/10.1007/s00205-008-0121-9

23. Z.-J. Chen, W.-M. Zou, Positive Least Energy Solutions and Phase Separation for Coupled Schrödinger Equations with Critical Exponent, *Arch. Ration. Mech. Anal.*, **205** (2012), 515–551. https://doi.org/10.1007/s00205-012-0513-8

24. Z.-J. Chen, W.-M. Zou, Positive least energy solutions and phase separation for coupled Schrödinger equations with critical exponent: higher dimensional case, *Calc. Var. Partial Differ. Equ.*, **52** (2015), 423–467. https://doi.org/10.1007/s00526-014-0717-x

25. Z.-J. Chen, C-S. Lin, W. Zou, Sign-changing solutions and phase separation for an elliptic system with critical exponent, *Commun. Partial Differ. Equ.*, **39** (2014), 1827–1859. https://doi.org/10.1080/03605302.2014.908391

26. Z.-X. Qi, Z.-T. Zhang, Existence of multiple solutions to a class of nonlinear Schrödinger system with external sources terms, *J. Math. Anal. Appl.*, **420** (2014), 972–986. https://doi.org/10.1016/j.jmaa.2014.06.038

27. H. Brezis, L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, *Commun. Pure Appl. Math.*, **36** (1983), 437–477. https://doi.org/10.1002/cpa.3160360405

28. X.-J. Wang, Existence of positive solutions to nonlinear elliptic equations involving critical Sobolev exponents, *Acta Math. Sin.*, **8** (1992), 273–291.

29. G. Tarantello, On nonhomogeneous elliptic equations involving critical Sobolev exponent, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **9** (1992), 281–304.

30. V. Benci, G. Cerami, Existence of positive solutions of the Equation $-\Delta + a(x)u = u^{\frac{N+2}{N-2}}$ in $\mathbb{R}^N$, *J. Funct. Anal.*, **88** (1990), 91–117.

31. M. Cuesta, C. De Coster, Superlinear critical resonant problems with small forcing term, *Calc. Var. Partial Differ. Equ.*, **54** (2015), 349–363. https://doi.org/10.1007/s00526-014-0788-8

32. P. Drabek, Y. Huang, Multiplicity of Positive Solutions for Some Quasilinear Elliptic Equation in RNwith Critical Sobolev Exponent, *J. Differ. Equ.*, **140** (1997), 106–132. https://doi.org/10.1006/jdeq.1997.3306

33. F. Faraci, C. Farkas, A quasilinear elliptic problem involving critical Sobolev exponents, *Collect. Math.*, **66** (2015), 243–259. https://doi.org/10.1007/s13348-014-0125-8

34. O. Rey, Concentration of solutions to elliptic equations with critical nonlinearity, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **9** (1992), 201–218. https://doi.org/10.1016/s0294-1449(16)30245-1

35. S. Wang, The existence of a positive solution of semilinear elliptic equations with limiting Sobolev exponent, *Proc. of the Royal Soc. Edinburgh*, **117A** (1991), 75–88. https://doi.org/10.1017/S030821050002761X

36. A. Ambrosetti, P. Rabinowitz, Dual Variational Methods in Critical Point Theory and Applications, *J. Funct. Anal.*, **11** (1973), 349–381. https://doi.org/10.1016/0022-1236(73)90051-7

37. H. Brezis, L. Nirenberg, A minimization problem with critical exponent and nonzero data, *Symmetry in Nature, Scuola Norm. Sup. Pisa*, **1** (1989), 129–140.

38. M. Willem. *Minimax theorems*.Progress in Nonlinear Differential Equations and their Applications, 24.Birkhauser Boston, MA, 1996.