Electronic Research Archive

http://www.aimspress.com/journal/era

ERA, 30(7): 2700–2718.
DOI: 10.3934/era.2022138
Received: 29 January 2022
Revised: 02 May 2022
Accepted: 06 May 2022
Published: 23 May 2022

Research article

Long-wavelength limit for the Green–Naghdi equations

Min Li*

Faculty of Applied Mathematics, Shanxi University of Finance and Economics, Taiyuan 030006, China

* **Correspondence:** Email: lmxx@sxufe.edu.cn.

Abstract: This paper studies the long-wavelength limit for the one-dimensional Green–Naghdi (GN) equations, which are often used to describe the propagation of fully nonlinear waves in coastal oceanography. We prove that, under the long-wavelength, small-amplitude approximation, the formal Korteweg–de Vries (KdV) equation for the GN equations is mathematically valid in the time interval for which the KdV dynamics survive. The main idea in the proof is to apply the Gardner–Morikawa transform, the reductive perturbation method, and some error energy estimates. The main novelties of this paper are the construction of valid approximate solutions of the GN equations with respect to the small wave amplitude parameter and global uniform energy estimates for the error system.

Keywords: Green–Naghdi equations; long-wavelength limit; Korteweg–de Vries equation

1. Introduction

In the present paper, we consider the limit of the Korteweg–de Vries (KdV) equation for the one-dimensional Green–Naghdi (GN) equations [1–4], which can be derived from the full water wave problem with shallow-water scaling [5]

$$\begin{cases} h_t + (hu)_x = 0, & \text{(1.1a)} \\ u_t + uu_x + gh_x - hh_x(u_{xt} + uu_{xx} - u_x^2) - \dfrac{1}{3}h^2(u_{xxt} - u_x u_{xx} + uu_{xxx}) = 0, & \text{(1.1b)} \end{cases}$$

where h and u are the elevation of the water surface above the bottom and the velocity in a channel, respectively. The parameter g represents gravity. Concerning nonlinear Galilean-invariant systems [2], one is usually interested in the dispersion (or dissipation) structure under the long-wavelength approximation. For such time and spatial scales, the dynamics can be obtained from the KdV equation [6–10] (or Burgers' equation) over sufficiently long time intervals.

This phenomenon has attracted considerable attention in recent years. For instance, many previous studies have examined the Euler–Poisson equations for ions [2, 11–15], whereby the solutions in one dimension [2, 12] and in two and three dimensions [13, 15] can be approximated by the KdV equation [16] under different scalings. In particular, the *rigorous mathematical justification* of such a limit has been established [12, 13] using the reductive perturbation method and uniform error energy estimates with respect to the amplitude of the initial disturbance. This result has been extended to the quantum Euler–Poisson equations from both *formal* [17] and *mathematical* [18] perspectives, where the electron fluid pressure is described by a Fermi–Dirac distribution. Moreover, the *formal* reduction to the KdV equation for the hydromagnetic waves in plasma and the incompressible two-dimensional water waves has been derived [19]. Numerical computations and well-posedness results for some KdV equations have also been presented [7, 8, 20–22]. As for irrotational and incompressible water waves [23, 24] in an infinitely long canal of fixed depth, Schneider and Wayne proved that the system can be reduced to two decoupled KdV equations (one moving to the right and the other moving to the left) under the long-wavelength approximation. For the one-dimensional Serre equations, a formal derivation has been reported [19], and Lannes [25] obtained a rigorous justification and derivation for many shallow-water asymptotic models. There are many other results for the water wave problems in the long-wave regime [9, 26–28].

Compared with previous work [2, 6, 19, 25], the main objective of this paper is to construct an explicit approximate solution for the GN equations via *an asymptotic expansion* with respect to the small dimensionless parameter ε. Moreover, the validity of such an asymptotic expansion is rigorously proved.

The basic plan is to first apply the singular perturbation method to obtain the formal derivation of the KdV equation, and then use some energy estimates to prove the validity of such an asymptotic expansion. One of the key mathematical difficulties lies in obtaining the uniform (in ε) energy bounds for the error system. Although the zeroth-, first-, and second-order energy estimates are no more difficult than those reported in previous work [12], the higher-order cases require a novel framework. To overcome this problem, we utilize the structure of Eq (2.16b) and then estimate the time dissipation for U_R in terms of the norm $\|(N_R, U_R)\|_{2,\varepsilon}$, which was not necessary in the previous study [12]. Further, we apply a new weighted energy norm, namely

$$\|(N_R, U_R)\|_{2,\varepsilon} \triangleq \|(N_R, U_R, \varepsilon^{\frac{1}{2}}\partial_\xi N_R, \varepsilon^{\frac{1}{2}}\partial_\xi U_R, \varepsilon\partial_{\xi\xi}N_R, \varepsilon\partial_{\xi\xi}U_R, \varepsilon^{\frac{3}{2}}\partial_{\xi\xi\xi}U_R)\|_{H^2}, \tag{1.2}$$

to close the *a priori* estimates of solutions to system (2.16).

We show that the solutions $((h-1)/\varepsilon, u/\varepsilon)$ of system (2.1) converge globally in time to those of the KdV equation (2.7) in $C([0, T_0], H^4 \times H^5)$ with a convergence rate of $O(\varepsilon^{m-1})$, where $m > 1$. Note that, compared with previous results [12, 13, 25], more accurate approximate solutions are constructed under the assumption that $N \geq m,\ m > 1$ because of the complex nonlinear structure of the system under consideration.

Here and in the following, α is an integer with $\alpha \geq 1$, and ∂^α is the spatial derivative. Moreover, we denote as H^s the Sobolev space with norm $\|f\|_{H^s} = \sum_{\alpha\leq s}\|\partial^\alpha f\|_{L^2}$ and as $\dot{H}^s$ the homogeneous Sobolev space with $\|f\|_{\dot{H}^s} = \|\partial^\alpha f\|_{L^2}$ $(\alpha = s)$. The commutator of A and B is denoted by $[A, B] = AB - BA$, and the commutator estimates are stated in Lemma 3.2.

The remainder of this paper is organized as follows. In Section 2, we present the formal asymptotic analysis and state the main result of this paper. Section 3 is devoted to uniform (in ε) energy estimates for the error system (2.16). Moreover, we complete the argument of the main theorem using the uniform (in ε) bounds and the continuity principle in Section 4.

2. Preliminaries

2.1. Formal KdV expansion

Letting $\tau = \varepsilon^{\frac{3}{2}} t, \xi = \varepsilon^{\frac{1}{2}}(x - Vt)$, we rewrite (1.1) as follows:

$$\begin{cases} \varepsilon\partial_\tau h + (u - V)\partial_\xi h + h\partial_\xi u = 0, & (2.1a) \\ \varepsilon\partial_\tau u + (u - V)\partial_\xi u + g\partial_\xi h - \dfrac{\varepsilon^2}{3} h^2 \partial_\tau \partial_{\xi\xi} u + \dfrac{\varepsilon}{3} h^2 (V\partial_{\xi\xi\xi} u + \partial_\xi u \partial_{\xi\xi} u - u\partial_{\xi\xi\xi} u) & \\ \qquad -h\partial_\xi h(\varepsilon^2 \partial_\tau \partial_\xi u - \varepsilon V \partial_{\xi\xi} u + \varepsilon u \partial_{\xi\xi} u - \varepsilon(\partial_\xi u)^2) = 0. & (2.1b) \end{cases}$$

Next, we introduce the formal expansion near the rest state $(1, 0)$ as

$$\begin{cases} h = 1 + \varepsilon h^1 + \ldots + \varepsilon^N h^N + \ldots, & (2.2a) \\ u = \varepsilon u^1 + \ldots + \varepsilon^N u^N + \ldots. & (2.2b) \end{cases}$$

Inserting the ansatz (2.2) to (2.1) and considering terms involving the same amplitude, we obtain a collection of equations.

At $O(\varepsilon)$, we have

$$\begin{cases} -V\partial_\xi h^1 + \partial_\xi u^1 = 0, & (2.3a) \\ -V\partial_\xi u^1 + g\partial_\xi h^1 = 0. & (2.3b) \end{cases}$$

This can be rewritten in matrix form as

$$\begin{pmatrix} -V, & 1 \\ g, & -V \end{pmatrix} \begin{pmatrix} \partial_\xi h^1 \\ \partial_\xi u^1 \end{pmatrix} = 0, \tag{2.4}$$

which implies $V = \pm\sqrt{g}$ to ensure a nontrivial solution. Therefore, we have

$$h^1 = \pm \frac{1}{\sqrt{g}} u^1 \tag{2.5}$$

under the zero Dirichlet boundary at infinity.

At $O(\varepsilon^2)$, we obtain

$$\begin{cases} \mp\sqrt{g}\partial_\xi h^2 + \partial_\xi u^2 = -\partial_\tau h^1 - \partial_\xi(h^1 u^1), & (2.6a) \\ \mp\sqrt{g}\partial_\xi u^2 + g\partial_\xi h^2 = -\partial_\tau u^1 \mp \dfrac{\sqrt{g}}{3}\partial_{\xi\xi\xi} u^1 - u^1 \partial_\xi u^1. & (2.6b) \end{cases}$$

Multiplying (2.6b) by $\pm\frac{1}{\sqrt{g}}$ and adding the resultant equation to (2.6a), we derive the following KdV equation:

$$\partial_\tau u^1 + \frac{3}{2} u^1 \partial_\xi u^1 \pm \frac{\sqrt{g}}{6} \partial_{\xi\xi\xi} u^1 = 0. \tag{2.7}$$

Note that (2.5) and (2.7) for (h^1, u^1) are self-consistent and independent of (h^j, u^j) for $j \geq 2$. This implies that the nonlinear waves of the GN equations can be formally approximated by the KdV equation, at least on time intervals of $O(\varepsilon^{-3/2})$. For the solvability of the KdV equation, we have the following theorem.

Theorem 2.1. *Let $\tilde{s} \geq 2$ be an integer. Then, there exists a constant $T_* > 0$ such that, for any given initial data $u_0^1 \in H^{\tilde{s}}$, problem* (2.7) *admits a unique solution u^1 that satisfies*

$$\sup_{\tau\in[0,T_*]} \|u^1(\tau)\|_{H^{\tilde{s}}} \leq C\|u_0^1\|_{H^{\tilde{s}}}, \tag{2.8}$$

where C is a generic constant independent of ε. Moreover, in view of the conservation laws of the KdV equation, we can extend the existence time to $[0, T_0]$ for any $T_0 > 0$.

By (2.6), we have

$$h^2 = \pm\frac{1}{\sqrt{g}}u^2 + \frac{1}{g}\int^{\xi}(\partial_\tau u^1 - 2u^1\partial_\xi u^1). \tag{2.9}$$

Hence, to determine h^2, we need only determine u^2.

Similar to the above, at $O(\varepsilon^3)$, we have

$$\begin{cases} \mp\sqrt{g}\partial_\xi h^3 + \partial_\xi u^3 = -\partial_\tau h^2 - \partial_\xi(h^1u^2 + h^2u^1), & (2.10\text{a}) \\ \mp\sqrt{g}\partial_\xi u^3 + g\partial_\xi h^3 = -\partial_\tau u^2 \mp \frac{\sqrt{g}}{3}\partial_{\xi\xi\xi}u^2 + \frac{1}{3}\partial_\tau\partial_{\xi\xi}u^1 - \partial_\xi(u^1u^2) \\ \quad \mp\frac{2\sqrt{g}}{3}h^1\partial_{\xi\xi\xi}u^1 - \frac{1}{3}\partial_\xi u^1\partial_{\xi\xi}u^1 + \frac{1}{3}u^1\partial_{\xi\xi\xi}u^1 \mp \sqrt{g}\partial_\xi h^1\partial_{\xi\xi}u^1. & (2.10\text{b}) \end{cases}$$

Multiplying (2.10b) by $\pm\frac{1}{\sqrt{g}}$ and again adding the resultant equation to (2.10a), we derive

$$\partial_\tau u^2 + \frac{3}{2}\partial_\xi(u^1u^2) \pm \frac{\sqrt{g}}{6}\partial_{\xi\xi\xi}u^2 = G^1, \tag{2.11}$$

where G^1 depends only on the known function u^1. Likewise, (2.5) and (2.7) are self-consistent and do not depend on (h^j, u^j) for $j \geq 3$.

Generally, at $O(\varepsilon^k)$ $(k \geq 3)$, we have the evolution equation for (h^{k-1}, u^{k-1}), from which we can deduce the following relation:

$$h^k = \pm\frac{1}{\sqrt{g}}u^k + l^{k-1}, \tag{2.12}$$

where l^{k-1} depends on (h^j, u^j) for $1 \leq j \leq k-1$. Therefore, we can express h^k in terms of u^k. At $O(\varepsilon^{k+1})$, we obtain the evolution equation for (h^k, u^k). Similar to the derivation of (2.11), we deduce the equation satisfied by u^k to be

$$\partial_\tau u^k + \frac{3}{2}\partial_\xi(u^1u^k) \pm \frac{\sqrt{g}}{6}\partial_{\xi\xi\xi}u^k = G^{k-1}, \tag{2.13}$$

where G^{k-1} is known and has been determined in previous steps.

For the solvability of the linear KdV equation (2.13), we have the following theorem.

Theorem 2.2. *Let $k \geq 2$, $\tilde{s}_k \leq \tilde{s} - 3(k-1)$ be sufficiently large integers. Then, for any given initial data $u_0^k \in H^{\tilde{s}_k}$, problem* (2.13) *admits a unique solution u^k that satisfies*

$$\sup_{\tau\in[0,T_0]} \|u^k(\tau)\|_{H^{\tilde{s}_k}} \leq C\|u_0^k\|_{H^{\tilde{s}_k}} \tag{2.14}$$

for any $T_0 > 0$, where C is a generic constant independent of ε.

Based on (2.5), (2.12), and Theorems 2.1 and 2.2, we can assume (h^k, u^k) for $k \geq 1$ are as smooth as necessary.

2.2. Main results

To provide a rigorous procedure for studying the KdV limit for system (2.1), we introduce the perturbation expansion

$$\begin{cases} h = 1 + \varepsilon h^1 + \ldots + \varepsilon^N h^N + \varepsilon^m N_R \triangleq 1 + \varepsilon\bar{h} + \varepsilon^m N_R, & (2.15a)\\ u = \varepsilon u^1 + \ldots + \varepsilon^N u^N + \varepsilon^m U_R \triangleq \varepsilon\bar{u} + \varepsilon^m U_R. & (2.15b)\end{cases}$$

By careful computation, we derive the following equation for the remainders:

$$\begin{cases} \partial_\tau N_R + \dfrac{u-V}{\varepsilon}\partial_\xi N_R + \dfrac{h}{\varepsilon}\partial_\xi U_R + \partial_\xi\bar{h}U_R + \partial_\xi\bar{u}N_R + \varepsilon^{N-m}\mathfrak{R}_1 = 0, & (2.16a)\\ \partial_\tau U_R + \dfrac{(u-V)\partial_\xi U_R}{\varepsilon} + \dfrac{g}{\varepsilon}\partial_\xi N_R - \dfrac{\varepsilon}{3}h^2\partial_\tau\partial_{\xi\xi}U_R + \dfrac{(V-u)h^2}{3}\partial_{\xi\xi\xi}U_R + \dfrac{h^2}{3}\partial_\xi u\partial_{\xi\xi}U_R & \\ \quad -\varepsilon h\partial_\xi h\partial_\tau\partial_\xi U_R + (V-u)h\partial_\xi h\partial_{\xi\xi}U_R + h\partial_\xi h\partial_\xi u\partial_\xi U_R + Q + \varepsilon^{N-m}\mathfrak{R}_2 = 0, & (2.16b)\end{cases}$$

where $V = \pm\sqrt{g}$, $\mathfrak{R}_1$ and $\mathfrak{R}_2$ depend only on the known functions $(\bar{n},\bar{u})$, and

$$\begin{aligned} Q =& \partial_\xi\bar{u}U_R - \frac{\varepsilon^2 h\partial_\tau\partial_{\xi\xi}\bar{u}N_R}{3} - \frac{\varepsilon^2(1+\varepsilon\bar{h})\partial_\tau\partial_{\xi\xi}\bar{u}N_R}{3} + \frac{V\varepsilon(h\partial_{\xi\xi\xi}\bar{u} + (1+\varepsilon\bar{h})\partial_{\xi\xi\xi}\bar{u})N_R}{3}\\ &- \frac{\varepsilon(h^2U_R + \varepsilon h(1+\varepsilon\bar{h})N_R\bar{u} + \varepsilon(1+\varepsilon\bar{h})N_R\bar{u})\partial_{\xi\xi\xi}\bar{u}}{3}\\ &+ \frac{\varepsilon}{3}(h^2\partial_\xi U_R + \varepsilon hN_R\partial_\xi\bar{u} + \varepsilon N_R(1+\varepsilon\bar{h})\partial_\xi\bar{u})\partial_{\xi\xi}\bar{u} - \varepsilon^2(h\partial_\xi N_R + \varepsilon\partial_\xi\bar{h}N_R)\partial_\tau\partial_\xi\bar{u}\\ &- \varepsilon(h\partial_\xi hU_R + \varepsilon h\partial_\xi N_R\bar{u} + \varepsilon^2\partial_\xi\bar{h}\bar{u}N_R)\partial_{\xi\xi}\bar{u} + \varepsilon V(h\partial_\xi N_R + \varepsilon N_R\partial_\xi\bar{h})\partial_{\xi\xi}\bar{u}\\ &+ \varepsilon(h\partial_\xi h\partial_\xi u\partial_\xi U_R + \varepsilon h\partial_\xi N_R\partial_\xi\bar{u} + \varepsilon^2 N_R\partial_\xi\bar{h}\partial_\xi\bar{u})\partial_\xi\bar{u}.\end{aligned}$$

The main result of this paper can be stated as follows.

Theorem 2.3. *Let the integers $\tilde{s}, \tilde{s}_k$ in Theorems 2.1 and 2.2 be sufficiently large and the integers N, m in* (2.15) *satisfy $N \geq m,\ m > 1$. Let (h^1, u^1) be the solution to the KdV equation* (2.7) *with initial data (h_0^1, u_0^1) satisfying* (2.5)*, and let (h^k, u^k) $(k \geq 2)$ be the solution to the linear KdV equation* (2.13) *with initial data (h_0^k, u_0^k) satisfying* (2.12)*. Let (N_R, U_R) be the solution to the error system* (2.16) *with initial data (N_{R0}, U_{R0}). Assume that the initial data $(h_0, u_0) \in H^5$ of system* (2.1) *satisfy*

$$\begin{aligned} h_0 =& 1 + \varepsilon h_0^1 + \ldots + \varepsilon^N h_0^N + \varepsilon^m N_{R0},\\ u_0 =& \varepsilon u_0^1 + \ldots + \varepsilon^N u_0^N + \varepsilon^m U_{R0}.\end{aligned} \tag{2.17}$$

Then, for any $T_0 > 0$, there exists some ε_0 such that, for all $0 < \varepsilon < \varepsilon_0$, system (2.1) *with initial data (h_0, u_0) admits a strong solution that can be expressed as*

$$\begin{aligned} h =& 1 + \varepsilon h^1 + \ldots + \varepsilon^N h^N + \varepsilon^m N_R,\\ u =& \varepsilon u^1 + \ldots + \varepsilon^N u^N + \varepsilon^m U_R.\end{aligned}$$

Moreover, we have

$$\sup_{\tau\in[0,T_0]} |\!|\!|(h - 1 - \sum_{j=1}^N \varepsilon^j h^j, u - \sum_{j=1}^N \varepsilon^j u^j)(\tau)|\!|\!|_{2,\varepsilon} \leq C\varepsilon^m,$$

where C is a generic constant independent of ε.

Remark 2.4. *Under the conditions of Theorem 2.3, we have*

$$\sup_{t\in[0,T_0/\varepsilon^{\frac{3}{2}}]}\left\|\begin{pmatrix}(h-1)/\varepsilon\\ u/\varepsilon\end{pmatrix}-\psi_{KdV}\right\|_{H^s}\leq C\varepsilon, \tag{2.18}$$

where $\psi_{KdV} = \begin{pmatrix}\pm\frac{1}{\sqrt{g}}\\ 1\end{pmatrix}u^1$. *That is, the one-dimensional compressible GN equations can be approximated by the KdV equation in a time interval of* $O(\varepsilon^{-3/2})$ *when the initial data are well prepared, that is, when* (2.17) *holds initially.*

3. Rigorous justification

In this section, we prove the strong convergence of the solution (h, u) of system (2.1) to that of the KdV equation (2.7) in the time interval where the KdV dynamics survive. The main proposition can be stated as follows.

Proposition 3.1. *Let* (N_R, U_R) *be the solution of system* (2.16). *Then, there exists some constant* ε_0 *such that, for any* $0 < \varepsilon < \varepsilon_0$,

$$\begin{aligned}&\frac{d}{d\tau}|\!|(N_R,U_R)|\!|_{2,\varepsilon}^2+\|(\varepsilon^{\frac{3}{2}}\partial_\tau U_R,\varepsilon^2\partial_\tau\partial_\xi U_R,\varepsilon^{\frac{5}{2}}\partial_\tau\partial_{\xi\xi}U_R)\|_{H^2}^2\\&\leq C(1+\varepsilon^{4(m-1)}|\!|(N_R,U_R)|\!|_{2,\varepsilon}^4)|\!|(N_R,U_R)|\!|_{2,\varepsilon}^2+C\varepsilon^{2(N-m)+1},\end{aligned}\tag{3.1}$$

where the weighted norm is defined in (1.2).

Our next goal is to prove Proposition 3.1 using energy estimates and a deep analysis of the complex nonlinear structure of system (2.16). Indeed, Proposition 3.1 can be proved by a series of lemmas. First, the local well-posedness of (2.16) is known [4, 6]. Using this property of the system, we define

$$T_\varepsilon=\sup\{T\geq 0;\ \forall\tau\in[0,T],\ |\!|(N_R,U_R)(\tau)|\!|_{2,\varepsilon}\leq\tilde{C}\},\tag{3.2}$$

where $\tilde{C}$ is a constant depending on ε that will be determined later. Thus, by (2.15) and Lemma 3.1, we immediately obtain that there exists some sufficiently small positive constant $\varepsilon_0 = \varepsilon_0(\tilde{C})$ such that, on $[0, T_\varepsilon]$,

$$1/2 < h < 3/2,\ \ |u| < 1/2\tag{3.3}$$

for any $0 < \varepsilon < \varepsilon_0$. The key point for the proof of Theorem 2.3 is to obtain $T_\varepsilon > T_0$ for any $T_0 > 0$ as $\varepsilon \to 0$. For this, it suffices to obtain uniform energy estimates for the remainders with respect to ε in the Gardner–Morikawa transform.

Let $\alpha = 0, 1, 2$. Differentiating (2.16) with ∂^α, we obtain

$$\begin{cases}\partial_\tau\partial^\alpha N_R+\dfrac{u-V}{\varepsilon}\partial^\alpha\partial_\xi N_R+\dfrac{h}{\varepsilon}\partial^\alpha\partial_\xi U_R=-C_1+\partial^\alpha\mathcal{R}_1, & (3.4a)\\ \partial_\tau\partial^\alpha U_R+\dfrac{(u-V)\partial^\alpha\partial_\xi U_R}{\varepsilon}+\dfrac{g}{\varepsilon}\partial^\alpha\partial_\xi N_R-\dfrac{\varepsilon}{3}h^2\partial_\tau\partial^\alpha\partial_{\xi\xi}U_R+\dfrac{(V-u)h^2}{3}\partial^\alpha\partial_{\xi\xi\xi}U_R\\ \quad-\varepsilon h\partial_\xi h\partial_\tau\partial^\alpha\partial_\xi U_R=-C_2+\partial^\alpha\mathcal{R}_2, & (3.4b)\end{cases}$$

where

$$\begin{aligned} C_1 =& \frac{1}{\varepsilon}[\partial^\alpha, u]\partial_\xi N_R + \frac{1}{\varepsilon}[\partial^\alpha, h]\partial_\xi U_R, \\ C_2 =& \frac{1}{\varepsilon}[\partial^\alpha, u]\partial_\xi U_R - \frac{\varepsilon}{3}[\partial^\alpha, h^2]\partial_\tau\partial_{\xi\xi} U_R + \frac{1}{3}[\partial^\alpha, (V-u)h^2]\partial_{\xi\xi\xi} U_R - \varepsilon[\partial^\alpha, h\partial_\xi h]\partial_\tau\partial_\xi U_R \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} \mathcal{R}_1 =& \partial_\xi \bar{h} U_R + \partial_\xi \bar{u} N_R + \varepsilon^{N-m}\mathfrak{R}_1, \\ \mathcal{R}_2 =& \frac{h^2}{3}\partial_\xi u \partial_{\xi\xi} U_R + (V-u)h\partial_\xi h \partial_{\xi\xi} U_R + h\partial_\xi h \partial_\xi u \partial_\xi U_R + Q + \varepsilon^{N-m}\mathfrak{R}_2. \end{aligned} \tag{3.6}$$

3.1. Preliminary estimates

In this subsection, we list some elementary inequalities that will be used later in the paper. Specifically, we state the Gagliardo–Nirenberg inequality as follows.

Lemma 3.1. *Let p, q, r be any positive integers. Then, we have*

$$\|\nabla^\alpha f\|_{L^p} \le C\|\nabla^l f\|_{L^q}^c \|\nabla^m f\|_{L^r}^{1-c} \tag{3.7}$$

for any $f \in \mathcal{S}$ (the Schwartz class) and $0 \le \alpha, m \le l,\ 0 < c < 1$ such that

$$\alpha - \frac{1}{p} = (l - \frac{1}{q})c + (m - \frac{1}{r})(1-c).$$

Based on this and Hölder's inequality, one can deduce the following Moser-type inequality.

Lemma 3.2. *Assume that $f, g \in H^k \cap L^\infty$. Then, for any $p \ge 1$,*

$$\|\partial_\xi^\alpha (fg)\|_{L^p} \le C(\|f\|_{L^{p_1}}\|g\|_{\dot{W}^{k,p_2}} + \|f\|_{\dot{W}^{k,p_3}}\|g\|_{L^{p_4}}) \tag{3.8}$$

and

$$\|[\partial_\xi^\alpha, f]g\|_{L^p} \le C(\|\partial_\xi f\|_{L^{p_1}}\|g\|_{\dot{W}^{k-1,p_2}} + \|f\|_{\dot{W}^{k,p_3}}\|g\|_{L^{p_4}}), \tag{3.9}$$

where $\dot{W}$ is the homogeneous Sobolev space, $p_2, p_3 > 1$, and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$.

Using Lemma 3.2 and the Sobolev embedding $H^1 \hookrightarrow L^\infty$, we arrive at

$$\begin{aligned} \|\partial_\xi^\alpha (fg)\|_{L^2} \le& C(\|f\|_{L^\infty}\|g\|_{H^\alpha} + \|f\|_{H^\alpha}\|g\|_{L^\infty}) \\ \le& C\|f\|_{H^\alpha}\|g\|_{H^\alpha} \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} \|[\partial_\xi^\alpha, f]g\|_{L^2} \le& C(\|\partial_\xi f\|_{L^\infty}\|g\|_{H^{\alpha-1}} + \|f\|_{H^\alpha}\|g\|_{L^\infty}) \\ \le& C\|f\|_{H^\alpha}\|g\|_{H^{\alpha-1}} \end{aligned} \tag{3.11}$$

for $\alpha \ge 2$.

3.2. Estimates for the time dissipation

Different from previous work [12, 13], we need to bound the time dissipation estimate for U_R. For this, we derive the following lemma, which plays an important role in obtaining the closed Gronwall inequality.

Lemma 3.3. *For any $\tau \in [0, T_\varepsilon]$,*

$$\|(\varepsilon^{3/2}\partial_\tau U_R, \varepsilon^3\partial_\tau\partial_\xi U_R, \varepsilon^{5/2}\partial_\tau\partial_{\xi\xi}U_R)\|_{H^2}^2 \le C(1+\varepsilon^{2(m-1)}|\!|\!|(N_R, U_R)|\!|\!|_{2,\varepsilon}^2)|\!|\!|(N_R, U_R)|\!|\!|_{2,\varepsilon}^2 + C\varepsilon^{2(N-m)+1} \tag{3.12}$$

holds, where C is a generic constant that is independent of ε.

Proof. Multiplying (3.4b) by $\varepsilon^3\partial_\tau\partial^\alpha(U_R - \varepsilon\partial_{\xi\xi}U_R)$ and integrating the resulting expression, we obtain

$$\begin{aligned}
&\varepsilon^3\|\partial^\alpha\partial_\tau U_R\|_{L^2}^2 + \varepsilon^4\|\partial^\alpha\partial_\tau\partial_\xi U_R\|_{L^2}^2 + \frac{\varepsilon^4}{3}\int h^2|\partial^\alpha\partial_\tau\partial_\xi U_R|^2 + \frac{\varepsilon^5}{3}\int h^2|\partial^\alpha\partial_\tau\partial_{\xi\xi}U_R|^2 \\
&= -\varepsilon^2\int(u-V)\partial^\alpha\partial_\xi U_R\partial_\tau(\partial^\alpha U_R - \varepsilon\partial^\alpha\partial_{\xi\xi}U_R) - g\varepsilon^2\int\partial^\alpha\partial_\xi N_R\partial_\tau(\partial^\alpha U_R - \varepsilon\partial^\alpha\partial_{\xi\xi}U_R) \\
&\quad - \frac{\varepsilon^3}{3}\int(V-u)h^2\partial^\alpha\partial_{\xi\xi\xi}U_R\partial_\tau(\partial^\alpha U_R - \varepsilon\partial^\alpha\partial_{\xi\xi}U_R) - \varepsilon^5\int h\partial_\xi h\partial_\tau\partial^\alpha\partial_\xi U_R\partial_\tau\partial^\alpha\partial_{\xi\xi}U_R \\
&\quad - \frac{\varepsilon^4}{6}\int\partial_\xi(h\partial_\xi h)|\partial^\alpha\partial_\tau U_R|^2 + \varepsilon^3\int(-C_2 + \partial^\alpha\mathcal{R}_2)\partial_\tau(\partial^\alpha U_R - \varepsilon\partial^\alpha\partial_{\xi\xi}U_R),
\end{aligned} \tag{3.13}$$

where we have used the following fact:

$$\begin{aligned}
&\frac{2\varepsilon^4}{3}\int h\partial_\xi h\partial_\tau\partial^\alpha\partial_\xi U_R\partial^\alpha\partial_\tau U_R - \varepsilon^4\int h\partial_\xi h\partial_\tau\partial^\alpha\partial_\xi U_R\partial^\alpha\partial_\tau U_R \\
&= -\frac{\varepsilon^4}{3}\int h\partial_\xi h\partial^\alpha\partial_\tau\partial_\xi U_R\partial^\alpha\partial_\tau U_R \\
&= \frac{\varepsilon^4}{6}\int\partial_\xi(h\partial_\xi h)|\partial^\alpha\partial_\tau U_R|^2.
\end{aligned} \tag{3.14}$$

By (3.3), Hölder's inequality, the Sobolev embedding $H^1 \hookrightarrow L^\infty$, and Lemmas 3.1 and 3.2, the right-hand side of (3.13) can be bounded by

$$\begin{aligned}
&\|(\varepsilon^{3/2}\partial^\alpha\partial_\tau U_R, \varepsilon^2\partial^\alpha\partial_\tau\partial_\xi U_R, \varepsilon^{5/2}\partial^\alpha\partial_\tau\partial_{\xi\xi}U_R)\|_{L^2}^2 \\
&\le C\varepsilon^3\|(-C_2 + \partial^\alpha\mathcal{R}_2)\|_{L^2}^2 + \delta\|(\varepsilon^{3/2}\partial_\tau U_R, \varepsilon^{5/2}\partial_\tau\partial_{\xi\xi}U_R)\|_{H^2}^2 \\
&\quad + C\varepsilon^{1/2}(1+\varepsilon^m\|(N_R, \varepsilon^{\frac{1}{2}}\partial_\xi N_R)\|_{H^2}^2)\|(\varepsilon^{1/2}\partial_\tau U_R, \varepsilon^2\partial_\tau\partial_\xi U_R, \varepsilon^{5/2}\partial_\tau\partial_{\xi\xi}U_R)\|_{H^2}^2 \\
&\quad + C(1+\varepsilon^{2(m-1)}|\!|\!|(N_R, U_R)|\!|\!|_{2,\varepsilon}^2)\|(\varepsilon^{\frac{1}{2}}\partial_\xi N_R, \varepsilon^{\frac{1}{2}}\partial_\xi U_R, \varepsilon\partial_{\xi\xi}U_R, \varepsilon^{\frac{3}{2}}\partial_{\xi\xi\xi}U_R)\|_{H^2}^2 + C\varepsilon^{2N-2m+1},
\end{aligned}$$

where δ is a sufficiently small positive constant. Recalling (3.3), (3.11), and Lemmas 3.1 and 3.2, we have

$$\begin{aligned}
\varepsilon\|C_2\|_{L^2} \le& C(\|\partial_\xi u\|_{L^\infty}\|\partial_\xi U_R\|_{\dot H^1} + \|u\|_{\dot H^2}\|\partial_\xi U_R\|_{L^\infty}) + C\varepsilon^2(\|h\partial_\xi h\|_{L^\infty}\|\partial_\tau\partial_\xi U_R\|_{\dot H^2} \\
&+ \|h^2\|_{\dot H^1}\|\partial_\tau\partial_{\xi\xi}U_R\|_{L^\infty}) + C\varepsilon(\|\partial_\xi((V-u)h^2)\|_{L^\infty}\|\partial_{\xi\xi}U_R\|_{\dot H^2} \\
&+ \|(V-u)h^2\|_{\dot H^2}\|\partial_{\xi\xi\xi}U_R\|_{L^\infty}) + C\varepsilon^2(\|\partial_\xi(h\partial_\xi h)\|_{L^\infty}\|\partial_\tau U_R\|_{\dot H^2} + \|h\partial_\xi h\|_{\dot H^2}\|\partial_\tau\partial_\xi U_R\|_{L^\infty}) \\
\le& C(1+\varepsilon^{2m}\|(N_R, U_R, \varepsilon^{1/2}\partial_\xi N_R)\|_{H^2}^2)\|(U_R, \varepsilon\partial_{\xi\xi}U_R, \varepsilon^{3/2}\partial_\tau U_R, \varepsilon^2\partial_\tau\partial_\xi U_R)\|_{H^2}
\end{aligned} \tag{3.15}$$

and

$$
\begin{aligned}
\varepsilon\|\partial^\alpha \mathcal{R}_2\|_{L^2} \le & \|h^2\partial_\xi u\|_{H^2}\|\varepsilon\partial_{\xi\xi}U_R\|_{H^2} + \|(V-u)h\partial_\xi h\|_{H^2}\|\varepsilon\partial_{\xi\xi}U_R\|_{H^2} \\
& + \|h\partial_\xi h\partial_\xi u\|_{H^2}\|\varepsilon^{1/2}\partial_\xi U_R\|_{H^2} + \|(N_R,U_R)\|_{H^2} + C\varepsilon^{N-m+1} \\
\le & C(1+\varepsilon^{2(m-1)}|\!|\!|(N_R,U_R)|\!|\!|_{2,\varepsilon}^2)\|(N_R,U_R,\varepsilon^{1/2}\partial_\xi U_R,\varepsilon\partial_{\xi\xi}U_R)\|_{H^2} + C\varepsilon^{N-m+1}.
\end{aligned}
\tag{3.16}
$$

Combining the above estimates and taking ε,δ to be sufficiently small, we have completed the proof of Lemma 3.3.

In the following, we derive uniform (in ε) energy estimates on the lower-order derivatives of (N_R, U_R).

3.3. Estimate for the lower-order derivatives

Lemma 3.4. *For any $\tau \in [0, T_\varepsilon]$,*

$$
\begin{aligned}
\frac{d}{d\tau}\|(N_R,U_R,\varepsilon^{1/2}\partial_\xi U_R)\|_{H^2}^2 \le & \delta\|(\varepsilon^{3/2}\partial_\tau U_R,\varepsilon^2\partial_\tau\partial_\xi U_R)\|_{H^2}^2 + C\varepsilon^{2(N-m+1)} \\
& + C(1+\varepsilon^{2(m-1)}|\!|\!|(N_R,U_R)|\!|\!|_{2,\varepsilon}^2)|\!|\!|(N_R,U_R)|\!|\!|_{2,\varepsilon}^2
\end{aligned}
\tag{3.17}
$$

holds, where C is a generic constant that is independent of ε.

Proof. Multiplying system (3.4) by $(\frac{g}{h}\partial^\alpha N_R, \partial^\alpha U_R)$ and using integration by parts, we derive

$$
\begin{aligned}
& \frac{g}{2}\frac{d}{dt}\int\frac{|\partial^\alpha N_R|^2}{h} + \frac{1}{2}\frac{d}{dt}\int|\partial^\alpha U_R|^2 + \frac{g}{\varepsilon}\int\partial^\alpha\partial_\xi U_R\partial^\alpha N_R + \frac{g}{\varepsilon}\int\partial^\alpha\partial_\xi N_R\partial^\alpha U_R \\
= & \frac{g}{2}\int\partial_\tau\Big(\frac{1}{h}\Big)|\partial^\alpha N_R|^2 - \frac{g}{\varepsilon}\int\frac{u-V}{h}\partial^\alpha\partial_\xi N_R\partial^\alpha N_R - \frac{1}{\varepsilon}\int(u-V)\partial^\alpha\partial_\xi U_R\partial^\alpha U_R \\
& + \frac{\varepsilon}{3}\int h^2\partial^\alpha\partial_\tau\partial_{\xi\xi}U_R\partial^\alpha U_R - \int\frac{(V-u)h^2}{3}\partial^\alpha\partial_{\xi\xi\xi}U_R\partial^\alpha U_R + \varepsilon\int h\partial_\xi h\partial^\alpha\partial_\tau\partial_\xi U_R\partial^\alpha U_R \\
& + g\int\frac{(\partial^\alpha\mathcal{R}_1 - C_1)\partial^\alpha N_R}{h} + \int(\partial^\alpha\mathcal{R}_2 - C_2)\partial^\alpha U_R \\
\triangleq & \sum_{i=1}^{8} I_i^{(\alpha)}.
\end{aligned}
\tag{3.18}
$$

It is easy to see that

$$
\frac{g}{\varepsilon}\int\partial^\alpha\partial_\xi U_R\partial^\alpha N_R + \frac{g}{\varepsilon}\int\partial^\alpha\partial_\xi N_R\partial^\alpha U_R = \frac{g}{\varepsilon}\int\partial_\xi(\partial^\alpha N_R\partial^\alpha U_R) = 0.
$$

We now derive estimates for the right-hand side of (3.18). From (2.15a), (2.16a), and (3.3), we can deduce that

$$
\begin{aligned}
\|\partial_\tau h\|_{L^\infty} = & \varepsilon\|\partial_\tau\bar{h}\|_{L^\infty} + \varepsilon^{m-1}\|\varepsilon\partial_\tau N_R\|_{L^\infty} \\
\le & C\varepsilon + \varepsilon^{m-1}\|(\partial_\xi N_R,\partial_\xi U_R)\|_{L^\infty},
\end{aligned}
\tag{3.19}
$$

where we have used the fact that $\bar{h}$ is a known smooth function according to Theorems 2.1 and 2.2. Directly applying (3.19), Hölder's inequality, and the Sobolev embedding, we find that $I_1^{(\alpha)}$ can be bounded by

$$\begin{aligned} I_1^{(\alpha)} \le& C\|\partial_\tau h\|_{L^\infty}\|N_R\|_{H^2}^2 \\ \le& C(1+\varepsilon^{m-1}\|(N_R,U_R)\|_{H^2})\|N_R\|_{H^2}^2. \end{aligned}$$

For the second term $I_2^{(\alpha)}$, we apply integration by parts, the Sobolev embedding $H^1 \hookrightarrow L^\infty$, and Young's inequality to show that

$$\begin{aligned} I_2^{(\alpha)} =& \frac{g}{2\varepsilon}\int \partial_\xi\Big(\frac{u-V}{h}\Big)|\partial^\alpha N_R|^2 \\ \le& C(1+\varepsilon^{m-1}\|(N_R,U_R)\|_{H^2})\|N_R\|_{H^2}^2. \end{aligned}$$

Similarly,

$$I_3^{(\alpha)} \le C(1+\varepsilon^{m-1}\|U_R\|_{H^2})\|U_R\|_{H^2}^2.$$

For the fourth term $I_4^{(\alpha)}$, we apply integration by parts, Young's inequality, and (3.19) to arrive at

$$\begin{aligned} I_4^{(\alpha)} =& -\frac{\varepsilon}{3}\int h^2\partial^\alpha\partial_\tau\partial_\xi U_R\partial^\alpha\partial_\xi U_R - \frac{2\varepsilon}{3}\int h\partial_\xi h\partial^\alpha\partial_\tau\partial_\xi U_R\partial^\alpha U_R \\ =& -\frac{\varepsilon}{6}\frac{d}{dt}\int h^2|\partial^\alpha\partial_\xi U_R|^2 + \frac{\varepsilon}{3}\int h\partial_\tau h|\partial^\alpha\partial_\xi U_R|^2 - \frac{2\varepsilon}{3}\int h\partial_\xi h\partial^\alpha\partial_\tau\partial_\xi U_R\partial^\alpha U_R \\ \le& -\frac{\varepsilon}{6}\frac{d}{dt}\int h^2|\partial^\alpha\partial_\xi U_R|^2 + \delta\|\varepsilon^2\partial_\tau\partial_\xi U_R\|_{H^2}^2 + C(1+\varepsilon^{2(m-1)}\|(N_R,U_R)\|_{H^2}^2)\|(U_R,\varepsilon^{1/2}\partial_\xi U_R\|_{H^2}^2, \end{aligned}$$

where δ is a sufficiently small positive constant.

The fifth term $I_5^{(\alpha)}$ involves third-order derivatives of U_R, which are not closed in terms of the weighted norm (1.2). To overcome the difficulty, we apply integration by parts twice to decompose this term into

$$\begin{aligned} I_5^{(\alpha)} =& \frac{1}{3}\int (V-u)h^2\partial^\alpha\partial_{\xi\xi}U_R\partial^\alpha\partial_\xi U_R + \frac{1}{3}\int \partial_\xi((V-u)h^2)\partial^\alpha\partial_{\xi\xi}U_R\partial^\alpha U_R \\ =& -\frac{1}{6}\int \partial_\xi((V-u)h^2)|\partial^\alpha\partial_{\xi\xi}U_R|^2 + \frac{1}{3}\int \partial_\xi((V-u)h^2)\partial^\alpha\partial_{\xi\xi}U_R\partial^\alpha U_R \\ \le& C(1+\varepsilon^{m-1}|\!|\!|(N_R,U_R)|\!|\!|_{2,\varepsilon})\|(U_R,\varepsilon\partial_{\xi\xi}U_R)\|_{H^2}^2. \end{aligned}$$

For the sixth term $I_6^{(\alpha)}$, Young's inequality yields

$$I_6^{(\alpha)} \le \delta\|\varepsilon^2\partial_\tau\partial_\xi U_R)\|_{H^2}^2 + C(1+\varepsilon^{2(m-1)}\|N_R\|_{H^2}^2)\|U_R\|_{H^2}^2.$$

For the seventh term $I_7^{(\alpha)}$, by applying Lemma 3.2, Hölder's inequality, and the Sobolev embedding $H^1 \hookrightarrow L^\infty$, we have that

$$\begin{aligned} &-\frac{1}{3}\int \partial^\alpha(h^2\partial_\xi u\partial_{\xi\xi}U_R)\partial^\alpha U_R \\ &\le C(\|h^2\partial_\xi u\|_{L^\infty}\|\partial_{\xi\xi}U_R\|_{H^2} + (\|h\|_{L^\infty}^2\|\partial_\xi u\|_{H^2} + \|h\|_{L^\infty}\|h\|_{H^2}\|\partial_\xi u\|_{L^\infty})\|\partial_{\xi\xi}U_R\|_{L^\infty})\|U_R\|_{H^2} \\ &\le C(1+\varepsilon^{2(m-1)}\|(N_R,U_R)\|_{H^2}^2)\|(U_R,\varepsilon\partial_{\xi\xi}U_R)\|_{H^2}^2 \end{aligned}$$

and

$$
\begin{aligned}
&\frac{\varepsilon}{3}\int [\partial^\alpha, h^2]\partial_\tau\partial_{\xi\xi}U_R\partial^\alpha U_R - \frac{1}{3}\int [\partial^\alpha, (V-u)h^2]\partial_{\xi\xi\xi}U_R\partial^\alpha U_R \\
&\le C\varepsilon(\|h\partial_\xi h\|_{L^\infty}\|\partial_\tau\partial_{\xi\xi}U_R\|_{\dot H^1} + \|h\partial_\xi h\|_{\dot H^1}\|\partial_\tau\partial_{\xi\xi}U_R\|_{L^\infty})\|U_R\|_{H^2} \\
&\quad + C(\|\partial_\xi((V-u)h^2)\|_{L^\infty}\|\partial_{\xi\xi\xi}U_R\|_{\dot H^1} + \|\partial_\xi((V-u)h^2)\|_{\dot H^1}\|\partial_{\xi\xi\xi}U_R\|_{L^\infty})\|U_R\|_{H^2} \\
&\le \delta\|\varepsilon^2\partial_\tau\partial_\xi U_R\|_{H^2}^2 + C(1+\varepsilon^{2(m-1)}\|(N_R, U_R)\|_{H^2}^2)\|(U_R, \varepsilon\partial_{\xi\xi}U_R)\|_{H^2}^2.
\end{aligned}
$$

The other terms in $I_7^{(\alpha)}$ and $I_8^{(\alpha)}$ can be similarly bounded by

$$
\delta\|\varepsilon^{3/2}\partial_\tau U_R\|_{H^2}^2 + C(1+\varepsilon^{2(m-1)}|\!|\!|(N_R, U_R)|\!|\!|_{2,\varepsilon}^2)|\!|\!|(N_R, U_R)|\!|\!|_{2,\varepsilon}^2 + C\varepsilon^{2(N-m)},
$$

where Q, $C_{1,2}$, and $\mathcal{R}_{1,2}$ are defined in (2.17), (3.5), and (3.6), respectively. Moreover, $\partial^\alpha Q$ depends only on the H^2-norm of (N_R, U_R) and the known functions $(\bar h, \bar u)$, and is therefore integrable.

This completes the proof of Lemma 3.4.

It is obvious that the H^2-norm of the solution is not closed because the right-hand side of inequality (Eq 3.17) cannot be controlled by the terms on the left, which leads to higher-order energy estimates (see the next subsection). The strategy is no more difficult than that of Lemma 3.4, but the argument for the higher-order case is much more delicate.

3.4. Estimates for the weighted third-order derivatives of (N_R, U_R)

Lemma 3.5. *For any* $\tau \in [0, T_\varepsilon]$,

$$
\begin{aligned}
&\frac{d}{dt}\|(\varepsilon^{1/2}\partial_\xi N_R, \varepsilon^{1/2}\partial_\xi U_R, \varepsilon\partial_{\xi\xi}N_R, \varepsilon\partial_{\xi\xi}U_R, \varepsilon^{3/2}\partial_{\xi\xi\xi}U_R)\|_{H^2}^2 \\
&\le \delta\|(\varepsilon^{3/2}\partial_\tau U_R, \varepsilon^2\partial_\tau\partial_\xi U_R, \varepsilon^{5/2}\partial_\tau\partial_{\xi\xi}U_R)\|_{H^2}^2 + C\varepsilon^{2(N-m)+1} \\
&\quad + C(1+\varepsilon^{4(m-1)}|\!|\!|(N_R, U_R)|\!|\!|_{2,\varepsilon}^4)|\!|\!|(N_R, U_R)|\!|\!|_{2,\varepsilon}^2
\end{aligned}
\tag{3.20}
$$

holds.

Proof. Applying the operator $\partial_\xi\partial^\alpha$ to (2.16) and multiplying by $(\frac{g\varepsilon}{h}\partial^\alpha\partial_\xi N_R, \varepsilon\partial^\alpha\partial_\xi U_R)$ on both sides, integration by parts yields

$$
\begin{aligned}
&\frac{g\varepsilon}{2}\frac{d}{dt}\int \frac{|\partial^\alpha\partial_\xi N_R|^2}{h} + \frac{\varepsilon}{2}\frac{d}{dt}\int |\partial^\alpha\partial_\xi U_R|^2 \\
&= \frac{g\varepsilon}{2}\int \partial_\tau\Big(\frac{1}{h}\Big)|\partial^\alpha\partial_\xi N_R|^2 - g\int \frac{\partial^\alpha\partial_\xi((u-V)\partial_\xi N_R)\partial^\alpha\partial_\xi N_R}{h} \\
&\quad - \int \partial^\alpha\partial_\xi((u-V)\partial_\xi U_R)\partial^\alpha\partial_\xi U_R + \frac{\varepsilon^2}{3}\int \partial^\alpha\partial_\xi(h^2\partial_\tau\partial_{\xi\xi}U_R)\partial^\alpha\partial_\xi U_R \\
&\quad - \frac{\varepsilon}{3}\int \partial^\alpha\partial_\xi((V-u)h^2\partial_{\xi\xi\xi}U_R)\partial^\alpha\partial_\xi U_R - g\int \frac{[\partial^\alpha\partial_\xi, h]\partial_\xi U_R\partial^\alpha\partial_\xi N_R}{h} \\
&\quad - \varepsilon\int \partial^\alpha\partial_\xi((V-u)h\partial_\xi h\partial_{\xi\xi}U_R)\partial^\alpha\partial_\xi U_R \\
&\quad + \varepsilon^2\int \partial^\alpha\partial_\xi(h\partial_\xi h\partial_\tau\partial_\xi U_R)\partial^\alpha\partial_\xi U_R - \frac{\varepsilon}{3}\int \partial^\alpha\partial_\xi(h^2\partial_\xi u\partial_{\xi\xi}U_R)\partial^\alpha\partial_\xi U_R
\end{aligned}
$$

$$
\begin{aligned}
&- g\varepsilon \int \frac{\partial^\alpha \partial_\xi (\partial_\xi \bar{h} U_R + \partial_\xi \bar{u} N_R + \varepsilon^{N-m} \mathfrak{R}_1) \partial^\alpha \partial_\xi N_R}{h} \\
&- \varepsilon \int \partial^\alpha \partial_\xi (h \partial_\xi h \partial_\xi u \partial_\xi U_R + Q + \varepsilon^{N-m} \mathfrak{R}_2) \partial^\alpha \partial_\xi U_R \triangleq \sum_{i=1}^{11} II_i^{(\alpha)},
\end{aligned} \tag{3.21}
$$

where we have used

$$
\begin{aligned}
& g \int \frac{\partial^\alpha \partial_\xi (h \partial_\xi U_R) \partial^\alpha \partial_\xi N_R}{h} + g \int \partial^\alpha \partial_{\xi\xi} N_R \partial^\alpha \partial_\xi U_R \\
&= g \int \frac{[\partial^\alpha \partial_\xi, h] \partial_\xi U_R \partial^\alpha \partial_\xi N_R}{h} + g \int \partial^\alpha \partial_{\xi\xi} U_R \partial^\alpha \partial_\xi N_R + g \int \partial^\alpha \partial_{\xi\xi} N_R \partial^\alpha \partial_\xi U_R \\
&= g \int \frac{[\partial^\alpha \partial_\xi, h] \partial_\xi U_R \partial^\alpha \partial_\xi N_R}{h},
\end{aligned}
$$

which comes from integration by parts and the commutator.

We now derive estimates for the right-hand side of (3.21). For $II_1^{(\alpha)}$, using (3.19), the Sobolev embedding, and Young's inequality leads to

$$
\begin{aligned}
II_1^{(\alpha)} \leq & C\varepsilon \|\partial_\tau h\|_{L^\infty} \|\partial_\xi N_R\|_{H^2}^2 \\
\leq & C(1 + \varepsilon^{m-1} \|(N_R, U_R)\|_{H^2}) \|\varepsilon^{1/2} \partial_\xi N_R\|_{H^2}^2,
\end{aligned}
$$

where δ is a sufficiently small positive constant.

For $II_2^{(\alpha)}$, applying integration by parts again and using the commutator estimates, we arrive at

$$
\begin{aligned}
II_2^{(\alpha)} = & - g \int \frac{\partial^\alpha \partial_\xi ((u - V) \partial_\xi N_R) \partial^\alpha \partial_\xi N_R}{h} \\
= & - g \int \frac{[\partial^\alpha \partial_\xi, u] \partial_\xi N_R \partial^\alpha \partial_\xi N_R}{h} + g \int \frac{(u - V) \partial^\alpha \partial_{\xi\xi} N_R \partial^\alpha \partial_\xi N_R}{h} \\
= & - g \int \frac{[\partial^\alpha \partial_\xi, u] \partial_\xi N_R \partial^\alpha \partial_\xi N_R}{h} - \frac{g}{2} \int \partial_\xi \Big(\frac{u - V}{h} \Big) |\partial^\alpha \partial_\xi N_R|^2 \\
\leq & \frac{C}{\varepsilon} (\|\partial_\xi u\|_{L^\infty} \|\varepsilon^{1/2} \partial_\xi N_R\|_{H^2} + \|\varepsilon^{1/2} \partial_\xi u\|_{H^2} \|\partial_\xi N_R\|_{L^\infty}) \|\varepsilon^{1/2} \partial_\xi N_R\|_{H^2} \\
& + \frac{C}{\varepsilon} \|(\partial_\xi u, \partial_\xi h)\|_{L^\infty} \|\varepsilon^{1/2} \partial_\xi N_R\|_{H^2}^2 \\
\leq & C(1 + \varepsilon^{m-1} \|(N_R, U_R)\|_{H^2}) \|(\varepsilon^{1/2} \partial_\xi N_R, \varepsilon^{1/2} \partial_\xi U_R)\|_{H^2}^2.
\end{aligned}
$$

Similar to $II_2^{(\alpha)}$, $II_3^{(\alpha)}$ can be estimated as

$$
II_3^{(\alpha)} \leq C(1 + \varepsilon^{m-1} \|U_R\|_{H^2}) \|\varepsilon^{1/2} \partial_\xi U_R\|_{H^2}^2.
$$

For the fourth term $II_4^{(\alpha)}$, integration by parts, Hölder's inequality, and the Gagliardo–Nirenberg inequality lead to

$$
\begin{aligned}
II_4^{(\alpha)} =& \frac{\varepsilon^2}{3}\int \partial^\alpha\partial_\xi(h^2\partial_\tau\partial_{\xi\xi}U_R)\partial^\alpha\partial_\xi U_R\\
=& -\frac{\varepsilon^2}{3}\int \partial^\alpha(h^2\partial_\tau\partial_{\xi\xi}U_R)\partial^\alpha\partial_{\xi\xi}U_R\\
=& -\frac{\varepsilon^2}{6}\frac{d}{dt}\int h^2|\partial^\alpha\partial_{\xi\xi}U_R|^2 + \frac{\varepsilon^2}{3}\int h\partial_\tau h|\partial^\alpha\partial_{\xi\xi}U_R|^2 - \frac{\varepsilon^2}{3}\int [\partial^\alpha, h^2]\partial_\tau\partial_{\xi\xi}U_R\partial^\alpha\partial_{\xi\xi}U_R\\
\le& -\frac{\varepsilon^2}{6}\frac{d}{dt}\int h^2|\partial^\alpha\partial_{\xi\xi}U_R|^2 + \delta\|\varepsilon^2\partial_\tau\partial_\xi U_R\|_{H^2}^2 + C(1+\varepsilon^{2(m-1)}|\!|\!|(N_R, U_R)|\!|\!|_{2,\varepsilon}^2)\|\varepsilon\partial_{\xi\xi}U_R\|_{H^2}^2,
\end{aligned}
$$

where δ is a sufficiently small positive constant.

For $II_5^{(\alpha)}$, using integration by parts again, we divide this term into

$$
\begin{aligned}
II_5^{(\alpha)} =& -\frac{\varepsilon}{3}\int \partial^\alpha\partial_\xi((V-u)h^2\partial_{\xi\xi\xi}U_R)\partial^\alpha\partial_\xi U_R\\
=& -\frac{\varepsilon}{3}\int [\partial^\alpha\partial_\xi, (V-u)h^2]\partial_{\xi\xi\xi}U_R\partial^\alpha\partial_\xi U_R - \frac{\varepsilon}{3}\int (V-u)h^2\partial^\alpha\partial_{\xi\xi\xi\xi}U_R\partial^\alpha\partial_\xi U_R\\
=& -\frac{\varepsilon}{3}\int [\partial^\alpha\partial_\xi, (V-u)h^2]\partial_{\xi\xi\xi}U_R\partial^\alpha\partial_\xi U_R + \frac{\varepsilon}{3}\int \partial_\xi((V-u)h^2)\partial^\alpha\partial_{\xi\xi\xi}U_R\partial^\alpha\partial_\xi U_R\\
&+ \frac{\varepsilon}{3}\int (V-u)h^2\partial^\alpha\partial_{\xi\xi\xi}U_R\partial^\alpha\partial_{\xi\xi}U_R\\
=& -\frac{\varepsilon}{3}\int [\partial^\alpha\partial_\xi, (V-u)h^2]\partial_{\xi\xi\xi}U_R\partial^\alpha\partial_\xi U_R - \frac{\varepsilon}{3}\int \partial_{\xi\xi}((V-u)h^2)\partial^\alpha\partial_{\xi\xi}U_R\partial^\alpha\partial_\xi U_R\\
&- \frac{\varepsilon}{2}\int \partial_\xi((V-u)h^2)|\partial^\alpha\partial_{\xi\xi}U_R|^2\\
\le& C(1+\varepsilon^{2(m-1)}|\!|\!|(N_R, U_R)|\!|\!|_{2,\varepsilon}^2)\|(\varepsilon^{1/2}\partial_\xi U_R, \varepsilon\partial_{\xi\xi}U_R, \varepsilon^{3/2}\partial_{\xi\xi\xi}U_R)\|_{H^2}^2.
\end{aligned}
$$

For $II_6^{(\alpha)}$, a Moser-type inequality yields

$$
\begin{aligned}
II_6^{(\alpha)} =& -g\int \frac{[\partial^\alpha\partial_\xi, h]\partial_\xi U_R\partial^\alpha\partial_\xi N_R}{h}\\
\le& \frac{C}{\varepsilon}(\|\partial_\xi h\|_{L^\infty}\|\varepsilon^{1/2}\partial_\xi U_R\|_{\dot H^2} + \|\varepsilon^{1/2}\partial_\xi h\|_{\dot H^2}\|\partial_\xi U_R\|_{L^\infty})\|\varepsilon^{1/2}\partial_\xi N_R\|_{H^2}\\
\le& C(1+\varepsilon^{2(m-1)}|\!|\!|(N_R, U_R)|\!|\!|_{2,\varepsilon}^2)\|(\varepsilon^{1/2}\partial_\xi U_R, \varepsilon^{1/2}\partial_\xi N_R)\|_{H^2}^2.
\end{aligned}
$$

For $II_7^{(\alpha)}$, we apply Lemma 3.2 again to obtain

$$
\begin{aligned}
II_7^{(\alpha)} =& -\varepsilon\int (V-u)\partial^\alpha\partial_\xi((V-u)h\partial_\xi h\partial_{\xi\xi}U_R)\partial_\xi\partial^\alpha U_R\\
\le& C\|\varepsilon^{1/2}\partial_\xi U_R)\|_{H^2}(\|(V-u)h\partial_\xi h\|_{L^\infty})\|\varepsilon^{3/2}\partial_{\xi\xi\xi}U_R\|_{\dot H^2} + \|(V-u)h\partial_\xi h\|_{\dot H^3}\|\varepsilon^{1/2}\partial_{\xi\xi}U_R\|_{L^\infty})\\
\le& C(1+\varepsilon^{2(m-1)}\|(N_R, \varepsilon^{1/2}\partial_\xi N_R)\|_{H^2}^2)\|(\varepsilon^{1/2}\partial_\xi U_R, \varepsilon^{3/2}\partial_{\xi\xi\xi}U_R)\|_{H^2}^2.
\end{aligned}
$$

The other four terms in (3.21) can be dealt with in a similar manner to $II_7^{(\alpha)}$:

$$
\begin{aligned}
II_{8\sim 11}^{(\alpha)} \le& \delta\|(\varepsilon^{3/2}\partial_\tau U_R, \varepsilon^2\partial_\tau\partial_\xi U_R)\|_{H^2}^2 + C(1+\varepsilon^{2(m-1)}\|(N_R, U_R, \varepsilon^{1/2}\partial_\xi U_R)\|_{H^2}^2)|\!|\!|(N_R, U_R)|\!|\!|_{2,\varepsilon}^2\\
&+ C\varepsilon^{2N-2m+1}.
\end{aligned}
$$

In summary, we conclude that

$$\frac{g\varepsilon}{2}\frac{d}{dt}\int\frac{|\partial^\alpha\partial_\xi N_R|^2}{h}+\frac{\varepsilon}{2}\frac{d}{dt}\int|\partial^\alpha\partial_\xi U_R|^2-\varepsilon^2\int(V-u)h\partial_\xi h\partial^\alpha\partial_\tau\partial_{\xi\xi}U_R\partial^\alpha\partial_\xi U_R$$
$$\leq\delta\|(\varepsilon^{3/2}\partial_\tau U_R,\varepsilon^2\partial_\tau\partial_\xi U_R)\|_{H^2}^2+C(1+\varepsilon^{2(m-1)}|\!|(N_R,U_R)|\!|_{2,\varepsilon}^2)|\!|(N_R,U_R)|\!|_{2,\varepsilon}^2+C\varepsilon^{2N-2m+1}.$$

Now, applying the operator $\partial^\alpha\partial_{\xi\xi}$ to (2.16) and multiplying the resultant equations by $(\frac{g\varepsilon^2}{h}\partial^\alpha\partial_{\xi\xi}N_R,\varepsilon^2\partial^\alpha\partial_{\xi\xi}U_R)$ on both sides, we derive

$$\begin{aligned}&\frac{g\varepsilon^2}{2}\frac{d}{dt}\int\frac{|\partial^\alpha\partial_{\xi\xi}N_R|^2}{h}+\frac{\varepsilon^2}{2}\frac{d}{dt}\int|\partial^\alpha\partial_{\xi\xi}U_R|^2+\mathcal{B}_1+\mathcal{B}_2\\&=\varepsilon^3\int h\partial_\xi h\partial^\alpha\partial_\tau\partial_{\xi\xi\xi}U_R\partial^\alpha\partial_{\xi\xi}U_R+\frac{\varepsilon^3}{3}\int\partial^\alpha\partial_{\xi\xi}(h^2\partial_\tau\partial_{\xi\xi}U_R)\partial^\alpha\partial_{\xi\xi}U_R\\&-\frac{\varepsilon^2}{3}\int\partial^\alpha\partial_{\xi\xi}((V-u)h^2\partial_{\xi\xi\xi}U_R)\partial^\alpha\partial_{\xi\xi}U_R\triangleq\sum_{i=1}^3 III_i^{(\alpha)},\end{aligned}\tag{3.22}$$

where

$$\begin{aligned}\mathcal{B}_1=&\varepsilon^2\int\partial^\alpha\partial_{\xi\xi}(\frac{h^2}{3}\partial_\xi u\partial_{\xi\xi}U_R+(V-u)h\partial_\xi h\partial_{\xi\xi}U_R)\partial^\alpha\partial_{\xi\xi}U_R\\&+\varepsilon^2\int\partial^\alpha\partial_{\xi\xi}(h\partial_\xi h\partial_\xi u\partial_\xi U_R+Q+\varepsilon^{N-m}\mathfrak{R}_2)\partial_{\xi\xi}\partial^\alpha U_R\\&+g\varepsilon^2\int\frac{\partial^\alpha\partial_{\xi\xi}(\partial_\xi\bar{h}U_R+\partial_\xi\bar{u}N_R+\varepsilon^{N-m}\mathfrak{R}_1)\partial_{\xi\xi}\partial^\alpha N_R}{h}\\&-\frac{g\varepsilon^2}{2}\int\partial_\tau\Big(\frac{1}{h}\Big)|\partial^\alpha\partial_{\xi\xi}N_R|^2-\frac{\varepsilon}{2}\int\partial_\xi u|\partial^\alpha\partial_{\xi\xi}N_R|^2-\frac{\varepsilon}{2}\int\partial_\xi u|\partial_{\xi\xi}\partial^\alpha U_R|^2\end{aligned}\tag{3.23}$$

and

$$\begin{aligned}\mathcal{B}_2=&g\varepsilon\int\frac{[\partial^{\alpha+2},u]\partial_\xi N_R\partial^\alpha\partial_{\xi\xi}N_R}{h}+g\varepsilon\int\frac{[\partial^{\alpha+2},h]\partial_\xi U_R\partial^\alpha\partial_{\xi\xi}N_R}{h}\\&+\varepsilon\int[\partial^{\alpha+2},u]\partial_\xi U_R\partial^\alpha\partial_{\xi\xi}U_R-\varepsilon^3\int[\partial^{\alpha+2},h\partial_\xi h]\partial_\tau\partial_\xi U_R\partial^\alpha\partial_{\xi\xi}U_R,\end{aligned}\tag{3.24}$$

using integration by parts.

We now derive estimates for the left-hand side of (3.22). For the second term $\mathcal{B}_1$, we deal with the most difficult part as follows. From integration by parts and Lemmas 3.1 and 3.2, we have that

$$\begin{aligned}&\varepsilon^2\int\partial^\alpha\partial_{\xi\xi}(\frac{h^2}{3}\partial_\xi u\partial_{\xi\xi}U_R+(V-u)h\partial_\xi h\partial_{\xi\xi}U_R)\partial^\alpha\partial_{\xi\xi}U_R\\&=-\varepsilon^2\int\partial^\alpha\partial_\xi(\frac{h^2}{3}\partial_\xi u\partial_{\xi\xi}U_R+(V-u)h\partial_\xi h\partial_{\xi\xi}U_R)\partial^\alpha\partial_{\xi\xi\xi}U_R\\&\leq\frac{C}{\varepsilon}(\|(h^2\partial_\xi u,(V-u)h\partial_\xi h)\|_{L^\infty}\|\varepsilon^{3/2}\partial_{\xi\xi\xi}U_R\|_{\dot{H}^2}\\&\quad+\|(\varepsilon h^2\partial_\xi u,\varepsilon(V-u)h\partial_\xi h)\|_{\dot{H}^3}\|\varepsilon^{1/2}\partial_{\xi\xi}U_R\|_{L^\infty})\|\varepsilon^{3/2}\partial_{\xi\xi\xi}U_R\|_{H^2}\\&\leq C(1+\varepsilon^{3(m-1)}|\!|(N_R,U_R)|\!|_{2,\varepsilon}^3)\|(\varepsilon^{1/2}\partial_\xi U_R,\varepsilon\partial_{\xi\xi}U_R,\varepsilon^{3/2}\partial_{\xi\xi\xi}U_R)\|_{H^2}^2.\end{aligned}$$

The other terms can be treated much more easily, and hence we have

$$\mathcal{B}_1 \leq \delta \|\varepsilon^{5/2}\partial_\tau\partial_{\xi\xi}U_R\|_{H^2}^2 + C(1+\varepsilon^{3(m-1)}|\!|(N_R,U_R)|\!|_{2,\varepsilon}^3)|\!|(N_R,U_R)|\!|_{2,\varepsilon}^2 + C\varepsilon^{2(N-m)+1}. \tag{3.25}$$

Similar to $II_{5\sim 6}^{(\alpha)}$, the standard commutator estimate yields

$$\mathcal{B}_2 \leq \delta \|(\varepsilon^{3/2}\partial_\tau U_R, \varepsilon^2\partial_\tau\partial_\xi U_R, \varepsilon^{5/2}\partial_\tau\partial_{\xi\xi}U_R)\|_{H^2}^2 + C(1+\varepsilon^{4(m-1)}|\!|(N_R,U_R)|\!|_{2,\varepsilon}^4)|\!|(N_R,U_R)|\!|_{2,\varepsilon}^2. \tag{3.26}$$

We now derive estimates for the right-hand side of (3.22). For $III_1^{(\alpha)}$, applying integration by parts gives

$$\begin{aligned}
III_1^{(\alpha)} =& \varepsilon^3 \int h\partial_\xi h \partial^\alpha\partial_\tau\partial_{\xi\xi\xi}U_R\partial^\alpha\partial_{\xi\xi}U_R \\
=& -\varepsilon^3 \int \partial_\xi(h\partial_\xi h)\partial^\alpha\partial_\tau\partial_{\xi\xi}U_R\partial^\alpha\partial_{\xi\xi}U_R - \varepsilon^3\int h\partial_\xi h\partial^\alpha\partial_\tau\partial_{\xi\xi}U_R\partial^\alpha\partial_{\xi\xi\xi}U_R \\
\leq& \delta\|\varepsilon^{5/2}\partial_\tau\partial_{\xi\xi}U_R\|_{H^2}^2 + C(1+\varepsilon^{4(m-1)}|\!|(N_R,U_R)|\!|_{2,\varepsilon}^4)|\!|(N_R,U_R)|\!|_{2,\varepsilon}^2.
\end{aligned}$$

For $III_2^{(\alpha)}$, we investigate the H^5-norm of $\partial_\tau U_R$, which cannot be handled using the previous lemmas and hence requires more effort. Fortunately, a useful term appears after integration by parts, which provides the possibility of closing the proof later. Specifically, using integration by parts and Lemma 3.2, the term $III_2^{(\alpha)}$ can be decomposed into

$$\begin{aligned}
III_2^{(\alpha)} =& \frac{\varepsilon^3}{3}\int \partial^\alpha\partial_{\xi\xi}(h^2\partial_\tau\partial_{\xi\xi}U_R)\partial^\alpha\partial_{\xi\xi}U_R \\
=& -\frac{\varepsilon^3}{3}\int h^2\partial^\alpha\partial_\tau\partial_{\xi\xi\xi}U_R\partial^\alpha\partial_{\xi\xi\xi}U_R - \frac{\varepsilon^3}{3}\int [\partial^{\alpha+1},h^2]\partial_\tau\partial_{\xi\xi}U_R\partial^\alpha\partial_{\xi\xi\xi}U_R \\
=& -\frac{\varepsilon^3}{6}\frac{d}{d\tau}\int h^2|\partial^\alpha\partial_{\xi\xi\xi}U_R|^2 + \frac{\varepsilon^3}{3}\int h\partial_\tau h|\partial^\alpha\partial_{\xi\xi\xi}U_R|^2 \\
& - \frac{\varepsilon^3}{3}\int [\partial^{\alpha+1},h^2]\partial_\tau\partial_{\xi\xi}U_R\partial^\alpha\partial_{\xi\xi\xi}U_R \\
\leq& -\frac{\varepsilon^3}{6}\frac{d}{d\tau}\int h^2|\partial^\alpha\partial_{\xi\xi\xi}U_R|^2 + C\|\partial_\tau h\|_{L^\infty}\|\varepsilon^{3/2}\partial_{\xi\xi\xi}U_R\|_{H^2}^2 \\
& + \frac{C}{\varepsilon}(\|h\partial_\xi h\|_{L^\infty}\|\varepsilon^{5/2}\partial_\tau\partial_{\xi\xi}U_R\|_{H^2} + \|\varepsilon^{1/2}h^2\|_{\dot{H}^3}\|\varepsilon^2\partial_\tau\partial_{\xi\xi}U_R\|_{L^\infty}) \\
\leq& -\frac{\varepsilon^3}{6}\frac{d}{d\tau}\int h^2|\partial^\alpha\partial_{\xi\xi\xi}U_R|^2 + \delta\|\varepsilon^{5/2}\partial_\tau\partial_{\xi\xi}U_R\|_{H^2}^2 \\
& + C(1+\varepsilon^{4(m-1)}|\!|(N_R,U_R)|\!|_{2,\varepsilon}^4)\|\varepsilon^{3/2}\partial_{\xi\xi\xi}U_R\|_{H^2}^2.
\end{aligned} \tag{3.27}$$

For $III_3^{(\alpha)}$, using integration by parts and Lemma 3.2 again, we obtain

$$
\begin{aligned}
III_3^{(\alpha)} =& -\frac{\varepsilon^2}{3}\int \partial^\alpha\partial_{\xi\xi}((V-u)h^2\partial_{\xi\xi\xi}U_R)\partial^\alpha\partial_{\xi\xi}U_R\\
=&\frac{\varepsilon^2}{3}\int \partial^\alpha\partial_{\xi}((V-u)h^2\partial_{\xi\xi\xi}U_R)\partial^\alpha\partial_{\xi\xi\xi}U_R\\
=&\frac{\varepsilon^2}{3}\int h^2(V-u)\partial^\alpha\partial_{\xi\xi\xi\xi}U_R\partial^\alpha\partial_{\xi\xi\xi}U_R+\frac{\varepsilon^2}{3}\int[\partial^{\alpha+1},(V-u)h^2]\partial_{\xi\xi\xi}U_R\partial_{\xi\xi\xi}U_R\\
=&-\frac{\varepsilon^2}{6}\int \partial_\xi(h^2(V-u))|\partial^\alpha\partial_{\xi\xi\xi}U_R|^2+\frac{\varepsilon^2}{3}\int[\partial^{\alpha+1},(V-u)h^2]\partial_{\xi\xi\xi}U_R\partial_{\xi\xi\xi}U_R\\
\le& C(1+\varepsilon^{2(m-1)}|\!|\!|(N_R,U_R)|\!|\!|_{2,\varepsilon}^2)\|(\varepsilon\partial_{\xi\xi}U_R,\varepsilon^{3/2}\partial_{\xi\xi\xi}U_R)\|_{H^2}^2.
\end{aligned}
$$

Adding the above estimates together and taking δ to be suitably small, we have completed the proof of Lemma 3.5.

4. Proof of the main theorem

Integrating (3.1) over $[0,\tau]$, and recalling definition (1.2) and the prior estimate (3.2), we conclude that

$$
\begin{aligned}
|\!|\!|(N_R,U_R)(\tau)|\!|\!|_{2,\varepsilon}^2 \le& C|\!|\!|(N_R,U_R)(0)|\!|\!|_{2,\varepsilon}^2+\int_0^\tau C(1+\varepsilon^{4(m-1)}|\!|\!|(N_R,U_R)|\!|\!|_{2,\varepsilon}^4)|\!|\!|(N_R,U_R)|\!|\!|_{2,\varepsilon}^2ds\\
&+C\varepsilon^{2(N-m)+1}\\
\le& C|\!|\!|(N_R,U_R)(0)|\!|\!|_{2,\varepsilon}^2+\int_0^\tau C(1+\varepsilon\tilde{C})|\!|\!|(N_R,U_R)|\!|\!|_{2,\varepsilon}^2ds+C\varepsilon^{2(N-m)+1}.
\end{aligned}
$$

There exists a suitably small constant ε_0 such that, for any $0<\varepsilon<\varepsilon_0$, $\varepsilon\tilde{C}<1$. Therefore, we obtain

$$
|\!|\!|(N_R,U_R)(\tau)|\!|\!|_{2,\varepsilon}^2 \le C|\!|\!|(N_R,U_R)(0)|\!|\!|_{2,\varepsilon}^2+\int_0^\tau 2C|\!|\!|(N_R,U_R)|\!|\!|_{2,\varepsilon}^2ds+C\varepsilon^{2(N-m)+1}.
$$

Using the Gronwall inequality and choosing $\tilde{C}$ in (3.2) to be sufficiently large that $\tilde{C}>2(1+C_0)(1+2CT_0e^{2CT_0})$, we then have

$$
\sup_{\tau\in[0,T]}|\!|\!|(N_R,U_R)(\tau)|\!|\!|_{2,\varepsilon}^2\le(1+2CTe^{2CT})(1+C_0)<\tilde{C}/2,
$$

where C_0 is a constant that depends on the initial data. Here, we need the assumption that the integers N,m satisfy $N\ge m,\ m>1$.

In view of the continuity principle, we can extend the existence time $T_\varepsilon>T_0$ as $\varepsilon\to 0$ for any $T_0>0$. Recalling (2.15) and Theorems 2.1–2.2 completes the proof of Theorem 2.3.

5. Conclusions

The convergence of the strong solution for the one-dimensional GN equations to that for the KdV equation has been rigorously proved for the small-amplitude, long-wavelength case. We have established a valid asymptotic expansion with respect to the small wave amplitude parameter ε, which is different from previous studies [6, 25]. In future work, it will be interesting to consider a similar derivation to the Camassa–Holm equation from the GN equations.

Acknowledgments

The work is supported by the Applied Basic Research Program of Shanxi Province under grant number 20210302124380, and the Scientific and Technological Innovation Programs of Higher Education Institutions in Shanxi under grant number 2020L0257.

Conflict of interest

The author declares no conflicts of interest in this paper.

References

1. A. Green, P. Naghdi, A derivation of equations for wave propagation in water of variable depth, *J. Fluid Mech.*, **78** (1976), 237–246. https://doi.org/10.1017/s0022112076002425
2. C. Su, C. Gardner, Korteweg-de Vries equation and generalizations III: Derivation of the Korteweg-de Vries equation and Burgers equation, *J. Math. Phys.*, **10** (1969), 536–539. https://doi.org/10.1063/1.1664873
3. G. Wei, J. T. Kirby, S. T. Grilli, R. Subramanya, A fully nonlinear Boussinesq model for surface waves, I. Highly nonlinear unsteady waves, *J. Fluid Mech.*, **294** (1995), 71–92. https://doi.org/10.1017/S0022112095002813
4. D. Lannes, B. Alvarez-Samaniego, A Nash-Moser theorem for singular evolution equations, application to the Serre and Green-Naghdi equations, *Indiana Univ. Math. J.*, **57** (2008), 97–132. https://doi.org/10.1512/iumj.2008.57.3200
5. Y. A. Li, A shallow-water approximation to the full water wave problem, *Commun. Pure Appl. Math.*, **59** (2006), 1225–1285. https://doi.org/10.1002/cpa.20148
6. B. Alvarez-Samaniego, D. Lannes, Large time existence for 3D water-waves and asymptotics, *Invent. Math.*, **171** (2008), 485–541. https://doi.org/10.1007/s00222-007-0088-4
7. Y. Benia, A. Scapellato, Existence of solution to Korteweg-de Vries equation in a non-parabolic domain, *Nonlinear Anal.*, **195** (2020), 111758. https://doi.org/10.1016/j.na.2020.111758
8. Y. Benia, B. K. Sadallah, Existence of solution to Korteweg-de Vries equation in domains that can be transformed into rectangles, *Math. Methods Appl. Sci.*, **41** (2018), 2684–2698. https://doi.org/10.1002/mma.4773
9. K. Kano, T. Nishida, A mathematical justification for Korteweg-de Vries equation and Boussinesq equation of water surface waves, *Osaka J. Math.*, **23** (1986), 389–413. https://doi.org/10.1016/0362-546X(86)90145-8
10. D. J. Korteweg, G. de Vries, On the change of form of long waves advancing in a rectangular channel, and on a new type of long stationary waves, *Philos. Mag.*, **39** (1895), 422–442. https://doi.org/10.1080/14786449508620739
11. J. Bae, B. Kwon, Small amplitude limit of solitary waves for the Euler-Poisson system, *J. Differ. Equations*, **266** (2019), 3450–3478. https://doi.org/10.1016/j.jde.2018.09.006

12. Y. Guo, X. Pu, KdV limit of the Euler-Poisson system, *Arch. Ration. Mech. Anal.*, **11** (2014), 673–710. https://doi.org/10.1007/s00205-013-0683-z
13. X. Pu, Dispersive limit of the Euler-Poisson system in higher dimensions, *SIAM J. Math. Anal.*, **45** (2013), 834–878. https://doi.org/10.1137/120875648
14. X. Pu, M. Li, KdV limit of the hydromagnetic waves in cold plasma, *Z. Angew. Math. Phys.*, **70** (2019), 81. https://doi.org/10.1007/s00033-019-1076-4
15. D. Lannes, F. Linares, J. Saut, The Cauchy problem for the Euler–Poisson system and derivation of the Zakharov–Kuznetsov equation, in *Studies in Phase Space Analysis with Applications to PDEs*, Birkhäuser, New York, NY, **84** (2013), 181–213. https://doi.org/10.1007/978-1-4614-6348-1_10
16. R. Miura, The Korteweg-de Vries equation: A survey of results, *SIAM Rev.*, **18** (1976), 412–459. https://doi.org/10.2307/2028638
17. F. Haas, L. Garcia, J. Goedert, G. Manfredi, Quantum ion-acoustic waves, *Phys. Plasmas*, **10** (2003), 3858–3866. https://doi.org/10.1063/1.1609446
18. H. Liu, X. Pu, Long wavelength limit for the quantum Euler-Poisson equation, *SIAM J. Math. Anal.*, **48** (2016), 2345–2381. https://doi.org/10.1137/15M1046587
19. C. Gardner, G. Morikawa, Similarity in the asymptotic behavior of collision-free hydromagnetic waves and water waves, *Report MF-2, NYO-9080*, Courant Institute of Mathematical Sciences, New York University, 1960.
20. I. Ahmad, H. Ahmad, M. Inc, H. Rezazadeh, M. A. Akbar, M. M. A. Khater, et al., Solution of fractional-order Korteweg-de Vries and Burgers' equations utilizing local meshless method, *J. Ocean Eng. Sci.*, 2021. https://doi.org/10.1016/j.joes.2021.08.014
21. L. Akinyemi, O. S. Iyiola, A reliable technique to study nonlinear time-fractional coupled Korteweg-de Vries equations, *Adv. Differ. Equations*, (2020), 169. https://doi.org/10.1186/s13662-020-02625-w
22. L. Akinyemi, P. Veeresha, S. O. Ajibola, Numerical simulation for coupled nonlinear Schrödinger-Korteweg-de Vries and Maccari systems of equations, *Mod. Phys. Lett. B*, **35** (2021), 2150339. https://doi.org/10.1142/S0217984921503395
23. G. Schneider, C. Wayne, The long-wave limit for the water wave problem I. The case of zero surface tension, *Commun. Math. Phys.*, **53** (2000), 1475–1535. https://doi.org/10.1002/1097-0312(200012)53:12<1475::AID-CPA1>3.0.CO;2-V
24. G. Schneider, C. Wayne, Corrigendum: The long-wave limit for the water wave problem I. The case of zero surface tension, *Commun. Pure Appl. Math.*, **65** (2012), 587–591. https://doi.org/10.1002/cpa.21391
25. D. Lannes, *The water waves problem: mathematical analysis and asymptotics*, AMS Ebooks Program, 2013. https://doi.org/10.1090/surv/188
26. J. L. Bona, T. Colin, D. Lannes, Long wave approximations for water waves, *Arch. Ration. Mech. Anal.*, **178** (2005), 373–410. https://doi.org/10.1007/s00205-005-0378-1
27. W. Craig, An existence theory for water waves and the Boussinesq and Korteweg-de Vries scaling limits, *Commun. Partial Differ. Equations*, **10** (1985), 787–1003. https://doi.org/10.1080/03605308508820396

28. T. Iguchi, A long wave approximation for capillary-gravity waves and an effect of the bottom, *Commun. Partial Differ. Equations*, **32** (2007), 37–85. https://doi.org/10.1080/03605300601088708