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Research article

On global randomized block Kaczmarz method for image reconstruction

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Abstract: Image reconstruction represents an important technique applied in various fields such as medicine, biology, materials science, nondestructive testing, and so forth. In this paper, we transform the problem of image reconstruction into the problem of solving linear systems with multiple right-hand sides. Based on the idea of K-means clustering, we propose the global randomized block Kaczmarz method, so as to solve the problem of the linear systems with multiple right-hand sides effectively and use this method to image reconstruction. Theoretical analysis proves the convergence of this method, and the simulation results demonstrate the performance of this method in image reconstruction.

Keywords: linear systems with multiple right-hand sides; randomized iteration; block Kaczmarz method; convergence property; image accuracy

1. Introduction

Image reconstruction plays a vital role in magnetic resonance imaging [1], X-ray computed tomography [2] and radio astronomy imaging applications [3]. Reconstruction methods [4–7] are mainly divided into analytical reconstruction and iterative reconstruction, which are based on the Radon transform [8], the inverse Radon transform, and the projection slice theorem serving as mathematical foundations. The most commonly used analytical reconstruction methods are all in the form of the filtered back projection (FBP) method [9]. Iterative reconstruction methods formulate the final result as a solution of linear systems [10].

For example, image reconstruction, as a mathematical process, generates tomographic images from X-ray [11] projection data acquired at many different angles in computed tomography (CT). If there has a test image with $n \times n$ pixels, p is the number of X-ray beams, and t is the scanning angle, which ranges from 0° to 179°, with intervals of 1°. The X-ray beam X_s passing through the test image is shown in the following figure.

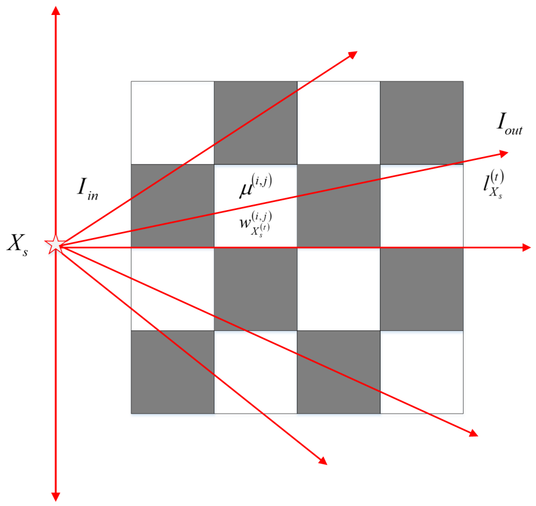


Figure 1. The X-ray beam X_s passing through the test image ($n = 4$).

In Figure 1, $w_{X_s^{(t)}}^{(i,j)}$ represents the length of the X-ray beam $X_s(s = 1, 2, \cdots, p)$ when it passes through the pixel $\{i, j\}$ along the line $l_{X_s}^{(t)}(t = 0, 2, \cdots, 179)$, and $\mu^{(i,j)}$ is the average attenuation coefficient for each pixel $\{i, j\}$. The data $g_{X_s^{(t)}}^{(i,j)}$ is measured when an X-ray beam $X_s^{(t)}$ crosses the pixel $\{i, j\}$ and intensity is absorbed. Thus, the image reconstruction problem amounts to solving the linear systems with multiple right-hand sides of the form

$$AX = B, \tag{1.1}$$

where

$$A = \begin{bmatrix} w_{X_1^{(0)}}^{(1,1)} & \cdots & w_{X_1^{(0)}}^{(1,n)} \\ \vdots & \ddots & \vdots \\ w_{X_1^{(0)}}^{(n,1)} & \cdots & w_{X_1^{(0)}}^{(n,n)} \\ \vdots & \ddots & \vdots \\ w_{X_p^{(179)}}^{(1,1)} & \cdots & w_{X_p^{(179)}}^{(1,n)} \\ \vdots & \ddots & \vdots \\ w_{X_p^{(179)}}^{(n,1)} & \cdots & w_{X_p^{(179)}}^{(n,n)} \end{bmatrix}, X = \begin{bmatrix} \mu^{(1,1)} & \cdots & \mu^{(1,n)} \\ \vdots & \ddots & \vdots \\ \mu^{(n,1)} & \cdots & \mu^{(n,n)} \end{bmatrix}, B = \begin{bmatrix} g_{X_1^{(0)}}^{(1,1)} & \cdots & g_{X_1^{(0)}}^{(1,n)} \\ \vdots & \ddots & \vdots \\ g_{X_1^{(0)}}^{(n,1)} & \cdots & g_{X_1^{(0)}}^{(n,n)} \\ \vdots & \ddots & \vdots \\ g_{X_p^{(179)}}^{(1,1)} & \cdots & g_{X_p^{(179)}}^{(1,n)} \\ \vdots & \ddots & \vdots \\ g_{X_p^{(179)}}^{(n,1)} & \cdots & g_{X_p^{(179)}}^{(n,n)} \end{bmatrix}.$$

The linear systems with multiple right-hand sides (1.1) can be written as

$$Ax^{(l)} = b^{(l)}, \tag{1.2}$$

where $l = 1, 2, \cdots, n$. The numerical methods [12–15] for solving linear systems (1.2) have seen a significant maturation. Among them the Kaczmarz method [16] is a classic and effective row-action method. Out of its simplicity, the Kaczmarz method has been a preferable solution tool in many fields, such as image reconstruction [15, 17], computerized tomography [9, 18] and signal processing [19, 20]. Based on the Kaczmarz method, a series of row iteration methods are proposed [21–23].

In order to reduce the computational costs, Censor proposed the block Kaczmarz method [18], Niu [24] presented the greedy block Kaczmarz method, Chen [25] declared the randomized double block Kaczmarz method and raised the upper bound of the error estimate in expectation. The approximate solution $x_k^{(l)}(l = 1, \cdots, n)$ can be obtained by solving the linear systems (1.2) with one of the above-mentioned solving methods, thereby getting the approximate solution $X_k = \left[x_k^{(1)}, \cdots, x_k^{(n)}\right]$ of the linear systems with multiple right-hand sides. In the above solving process, the calculation and storage requirements increase with the iteration.

Instead of solving each of the n linear systems independently by using some iterative methods, it is more efficient to solve the linear systems with multiple right-hand sides globally. A great deal of research has been finished, including the block conjugate gradient method [26], the block generalized minimal residual method [27] and the GMRES seed projection method [28], see also [29–34] and the references therein.

In regard to the problem of image reconstruction, we transform it into solving a linear systems with multiple right-hand sides and propose the idea of global iteration. Based on the idea of K-means clustering, we put forward the global randomized block Kaczmarz method, and use this method to solve the linear systems with multiple right-hand sides effectively, so as to image reconstruction. Theoretical analysis proves the convergence of the global randomized block Kaczmarz method. Simulation results show that the reconstructed image quality of this method is superior to that of the filtered back projection method.

In this paper, we adopt the following notations. Let A^* be the conjugate transpose of matrix A, $A^\dagger$ the Moore-Penrose pseudoinverse, $\lambda_{\min}(A)$ the smallest nonzero eigenvalue, $\|A\|_2$ the spectral norm and $\|A\|_F$ the Frobenius norm respectively. $B^{(i)}$ represents the ith column of matrix B. Let $[m] := \{1, 2, \cdots, m\}$, the index set $J = \left\{J_1, J_2, \cdots, J_q\right\} \subseteq [m]$ satisfies $J_i \cap J_j = \emptyset$ and $\cup_{i=1}^q J_i = [m]$. A_{J_i} is the row submatrix indexed by J_i. $X_\star = A^\dagger B$ denotes the least Euclidean-norm solution of the linear systems with multiple right-hand sides (1.1). $\mathbb{E}_k$ indicates the expected value conditional on the first k iterations, that is,

$$\mathbb{E}_k[\cdot] = \mathbb{E}[\cdot \mid J_o, J_1, \cdots, J_{k-1}],$$

where $J_l\,(l = 0, 1, \cdots, k-1)$ is the J_lth submatrix chosen at the lth iteration, and then we can obtain that $\mathbb{E}[\mathbb{E}_k[\cdot]] = \mathbb{E}[\cdot]$.

2. Global randomized block Kaczmarz method

K-means clustering [35] is the simplest unsupervised learning method as well as the most widely used to solve clustering problems. Recently, several Kaczmarz-type methods that embed the clustering methods appeared, see e.g., [36, 37], and the complete process of the K-means method is given in Method 2.1.

Method 2.1. The K-means method

Input: A date set X containing m data items, and the number of clusters q.

Output: ρ_i, C_i.

1: Randomly choose q items from X as the initial cluster centers;

2: Repeat Steps 3 and 4 until the termination;

3: Assign each data item to the cluster whose centroid is nearest in terms of $\sum_{i=1}^{q} \sum_{x_k \in C_i} \left| 1 - \frac{x_k^\top \rho_i}{\|x_k\|_2 \|\rho_i\|_2} \right|$;

4: Based on the mean value of the data objects in the cluster, update the cluster centers ρ_i;

5: Return ρ_i and C_i with ρ_i being the cluster center and C_i the clusters for $i = 1, 2, \cdots, q$.

Using this method to divide the rows of coefficient matrix of linear systems with multiple right-hand sides (1.1), we can obtain

$$A = \begin{bmatrix} A_{J_1} \\ \vdots \\ A_{J_q} \end{bmatrix}, B = \begin{bmatrix} B_{J_1} \\ \vdots \\ B_{J_q} \end{bmatrix}, \tag{2.1}$$

where $A_{J_i} \in \mathbb{C}^{m_{J_i} \times n}$, $B_{J_i} \in \mathbb{C}^{m_{J_i}}$ and $\sum_{i=1}^{q} m_{J_i} = m$. Starting from an initial guess X_0, the current iterate X_k is orthogonally projected to the hyperplane $A_{J_{i_k}} X_k = B_{J_{i_k}}$ under the probability criterion

$$\Pr(row = J_{i_k}) = \frac{\left\|A_{J_{i_k}}\right\|_F^2}{\|A\|_F^2}.$$

The global randomized block Kaczmarz (MGRBK) method for solving linear systems with multiple right-hand sides can be formulated as

$$X_{k+1} = X_k + A_{J_{i_k}}^{\dagger} \left(B_{J_{i_k}} - A_{J_{i_k}} X_k \right), \tag{2.2}$$

where the index of working submatrix J_{i_k} is chosen from the set $J = \{J_1, J_2, \cdots, J_q\}$ at random, with probability proportional to $\left\|A_{J_{i_k}}\right\|_F^2$. The form of $A_{J_{i_k}}^{\dagger}$ in our simulation experiments can be seen in the reference [38]. Then the global randomized block Kaczmarz method can be described in the following.

Method 2.2. The global randomized block Kaczmarz method

Input: A, B, l, q and X_0.

Output: X_l.

1: The blocks A_{J_i}, B_{J_i} $(i = 1, 2, \cdots, q)$ are obtained by K-means method;

2: **for** $k = 0, 1, 2, \cdots, l-1$ **do**

3: Select $J_{i_k} \in J$ with probability $\Pr(\text{row} = J_{i_k}) = \frac{1}{q}$;

4: Compute $X_{k+1} = X_k + A_{J_{i_k}}^{\dagger} \left(B_{J_{i_k}} - A_{J_{i_k}} X_k \right)$;

5: **end for**

Note that the index set J is nonempty for all iteration index k. According to a large number of numerical experiments, we find that the number of blocks q fluctuates around $\frac{m}{2000}$, where m is the number of rows of coefficient matrix A. Hence, we have a rough estimate is that $q \approx \frac{m}{2000}$. For the convergence property of the global randomized block Kaczmarz method, we establish the following theorem.

Theorem 2.1. *Let the linear systems with multiple right-hand sides (1.1), with the coefficient matrix $A \in \mathbb{C}^{m\times n}$ and the right-hand side $B \in \mathbb{C}^{m\times n}$, be consistent. The iteration sequence $\{X_k\}_{k=0}^{\infty}$ generated by the global randomized block Kaczmarz method starting from any initial approximation $X_0 \in \mathcal{R}(A^*)$, converges to the unique least-norm solution $X_\star = A^\dagger B$ in expectation. Moreover, the solution error in expectation for the iteration sequence $\{X_k\}_{k=0}^{\infty}$ obeys*

$$\mathbb{E}\,\|X_k - X_\star\|_F^2 \leq \left(1 - \frac{\lambda_{\min}(A^*A)}{\|A\|_F^2}\right)^k \|X_0 - X_\star\|_F^2 \,. \tag{2.3}$$

Proof. From the iteration scheme of the global randomized block Kaczmarz method, we can straightforwardly obtain

$$X_{k+1} - X_k = A_{J_{i_k}}^\dagger \left(B_{J_{i_k}} - A_{J_{i_k}} X_k\right).$$

Since $B_{J_{i_k}} = A_{J_{i_k}} X_\star$, it holds that

$$X_k - X_{k+1} = A_{J_{i_k}}^\dagger A_{J_{i_k}} (X_k - X_\star)\,, \tag{2.4}$$

$$X_{k+1} - X_\star = \left(I_n - A_{J_{i_k}}^\dagger A_{J_{i_k}}\right)(X_k - X_\star)\,. \tag{2.5}$$

From the properties of the Moore-Penrose pseudoinverse, we have

$$\begin{aligned}
&(X_k - X_\star)^*\left(A_{J_{i_k}}^\dagger A_{J_{i_k}}\right)^* \left(I_n - A_{J_{i_k}}^\dagger A_{J_{i_k}}\right)(X_k - X_\star) \\
&= (X_k - X_\star)^* \left(A_{J_{i_k}}^\dagger A_{J_{i_k}} - A_{J_{i_k}}^\dagger A_{J_{i_k}}\right)(X_k - X_\star) \\
&= 0.
\end{aligned}$$

Therefore, we know that matrix $X_k - X_{k+1}$ is orthogonal to $X_{k+1} - X_\star$ for any $k \geq 0$, in other words, $(X_k - X_{k+1})^* (X_{k+1} - X_\star) = 0$. It follows that

$$\|X_k - X_\star\|_F^2 = \|X_k - X_{k+1}\|_F^2 + \|X_{k+1} - X_\star\|_F^2 \,. \tag{2.6}$$

By Eq (2.4), Eq (2.6) can be rewritten as

$$\|X_{k+1} - X_\star\|_F^2 = \|X_k - X_\star\|_F^2 - \left\|A_{J_{i_k}}^\dagger A_{J_{i_k}} (X_k - X_\star)\right\|_F^2 \,. \tag{2.7}$$

With this, we obtain

$$\begin{aligned}
&\mathbb{E}_k \|X_{k+1} - X_\star\|_F^2 = \|X_k - X_\star\|_F^2 - \mathbb{E}_k \left\|A_{J_{i_k}}^\dagger A_{J_{i_k}} (X_k - X_\star)\right\|_F^2 \\
&= \|X_k - X_\star\|_F^2 - \sum_{i=1}^{q} \left(\frac{\left\|A_{J_{i_k}}\right\|_F^2}{\|A\|_F^2} \left\|A_{J_{i_k}}^\dagger A_{J_{i_k}} (X_k - X_\star)\right\|_F^2 \right) \\
&= \|X_k - X_\star\|_F^2 - \frac{1}{\|A\|_F^2} \sum_{i=1}^{q} \left\|A_{J_{i_k}} (X_k - X_\star)\right\|_F^2 \\
&= \|X_k - X_\star\|_F^2 - \frac{1}{\|A\|_F^2} \|A (X_k - X_\star)\|_F^2 \\
&\leq \|X_k - X_\star\|_F^2 - \frac{1}{\|A\|_F^2} \lambda_{\min}(A^*A) \|X_k - X_\star\|_F^2 \,.
\end{aligned}$$

Here the last inequality is achieved with the use of the estimate

$$\|AC\|_F^2 = tr\,(C^*A^*AC) \geq \lambda_{\min}\,(A^*A)\,tr\,(C^*C) = \lambda_{\min}\,(A^*A)\,\|C\|_F^2\,,$$

where $C \in \mathbb{C}^{n\times c}$. Thus, we can get the estimate

$$\mathbb{E}_k\,\|X_{k+1} - X_\star\|_F^2 \leq \left(1 - \frac{1}{\|A\|_F^2}\lambda_{\min}\,(A^*A)\right)\|X_k - X_\star\|_F^2\,, \quad k = 0, 1, 2, \cdots. \tag{2.8}$$

Finally, by taking the full expectation on both sides of (2.8), we obtain

$$\mathbb{E}\,\|X_{k+1} - X_\star\|_F^2 \leq \left(1 - \frac{1}{\|A\|_F^2}\lambda_{\min}\,(A^*A)\right)\mathbb{E}\,\|X_k - X_\star\|_F^2\,, \quad k = 0, 1, 2, \cdots.$$

By induction on the iteration index k, we have

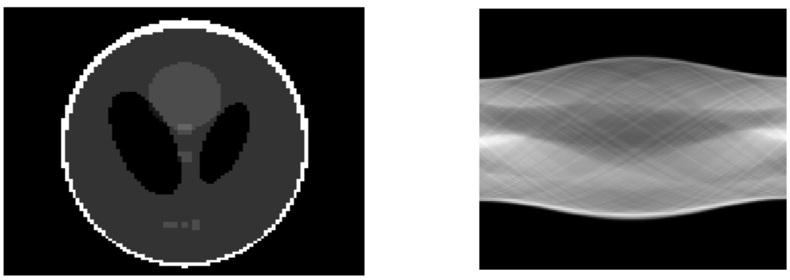
$$\mathbb{E}\,\|X_k - X_\star\|_F^2 \leq \left(1 - \frac{1}{\|A\|_F^2}\lambda_{\min}\,(A^*A)\right)^k \|X_0 - X_\star\|_F^2\,.$$

Consequently, the convergence property of the global randomized block Kaczmarz method is proved.

3. Simulation results

In this section, we investigate the performance of the *global randomized block Kaczmarz* (MGRBK) method in image reconstruction through practical experiments. We also compare the image reconstruction results of the MGRBK method and filtered back projection (FBP) method, in which the FBP method is realized by using the MATLAB function **iradon**. All computations are started from the initial matrix $X_0 = 0$, and the row vectors of coefficient matrix A are divided into q blocks.

Example 3.1. The MGRBK method is used to solve the linear systems with multiple right-hand sides to reconstruct the test image. The test image is obtained through the image processing toolbox function **phantom** in MATLAB. The Shepp-Logan phantom is shown in the following Figure.



(a) Shepp-Logan phantom **(b)** Radon transform

Figure 2. Shepp-Logan phantom and its Radon transform.

For the Shepp-Logan phantom with 100×100 pixels, $p = 4$, the corresponding linear systems with multiple right-hand sides is $AX = B$, where the dimension of coefficient matrix A is 72000×100. Then we solve this linear systems with multiple right-hand sides by using the MGRBK method, $q = 60$ and the number of iterations is 30. The computational process of image reconstruction is shown in the following figure.

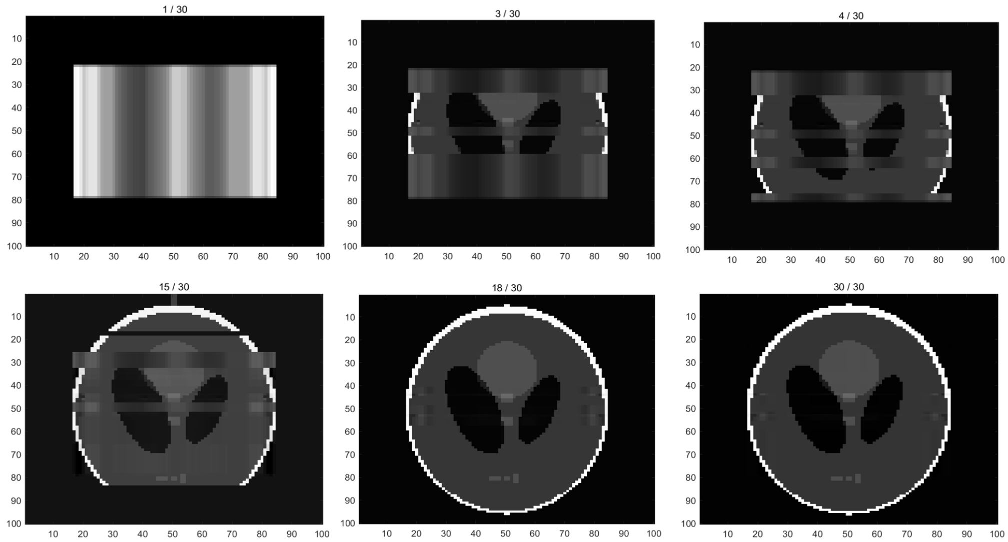
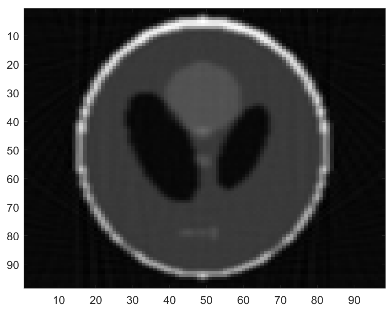
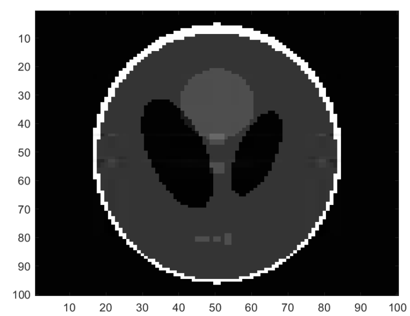


Figure 3. The process of image reconstruction.

In Figure 3, we can see that at the 18th iteration, the test image can be reconstructed with the global randomized block Kaczmarz method. Through observation, the reconstructed image can retain the details of the original image.



(a) The FBP method



(b) The MGRBK method

Figure 4. Comparison of reconstructed images.

In Figure 4, the image reconstruction is realized by using the global randomized block Kaczmarz method and the filtered back-projection method respectively, and the reconstruction results are compared. From the visual inspection of the reconstructed image, the MGRBK method outperforms the FBP method.

Example 3.2. Select the test image from the photo gallery in MATLAB, and then solve the linear systems with multiple right-hand sides through the MGRBK method to reconstruct the test image. The test image is shown in the following figure.

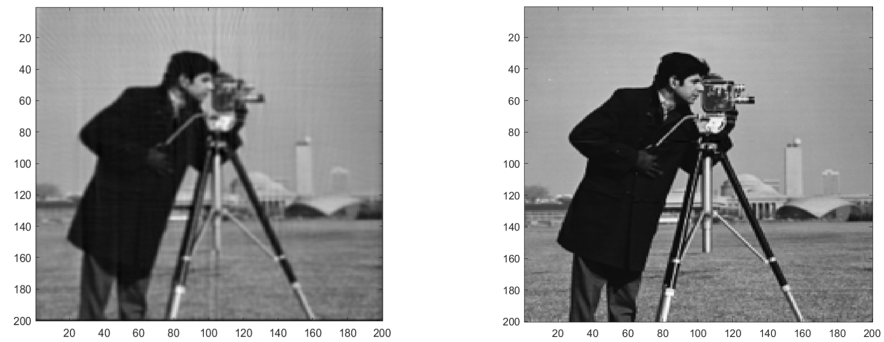
(a) Cameraman image **(b)** Radon transform

Figure 5. Cameraman image and its Radon transform.

For the cameraman image with 200×200 pixels, $p = 5$, the corresponding linear systems with multiple right-hand sides is $AX = B$, where the dimension of coefficient matrix A is 180000×200. Then we solve this linear systems with multiple right-hand sides by using the MGRBK method. Let $q = 70$ and the number of iterations is 40. The computational process of image reconstruction is shown in the following figure.

Figure 6. The process of image reconstruction.

In Figure 6, we can see that test image can be reconstructed with the global randomized block Kaczmarz method as reach the 38th iteration. Through observation, the reconstructed image can retain the details of the original image.

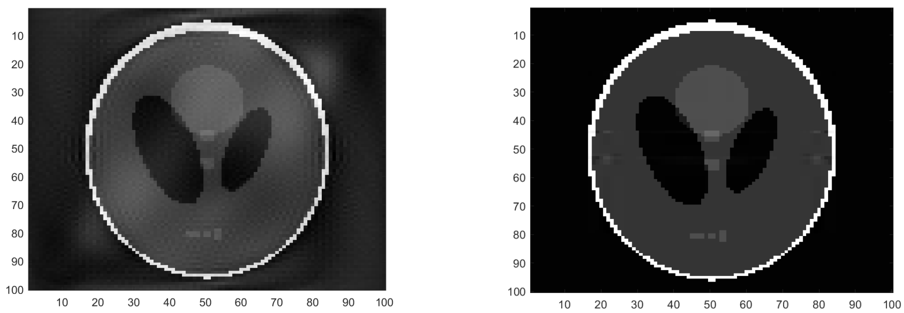


(a) The FBP method **(b)** The MGRBK method

Figure 7. Comparison of reconstructed images.

In Figure 7, the reconstruction results of the MGRBK method and the FBP method are compared. From the visual inspection of the reconstructed image, the former method is superior to the latter method.

Example 3.3. The test image is obtained through the image processing toolbox function **phantom** in MATLAB, which is the same as that in Example 3.1. We compare the reconstructed image of the global randomized block Kaczmarz (MGRBK) method with that of a typical iterative method, that is Kaczmarz method. The reconstructed images of these two iterative methods are shown in the following figure.



(a) The Kaczmarz method **(b)** The MGRBK method

Figure 8. Comparison of reconstructed images.

From the visual inspection, the reconstructed image obtained by using the MGRBK method has higher resolution. In the MGRBK method, the relative residual (Rres) is defined by

$$\text{Rres} = \frac{||B - AX_k||_F^2}{||B||_F^2},$$

and Rres of the MGRBK method is 1.6049e-04. In [39], the relative residual of Kaczmarz method is

0.0015. This means that the approximate solution of MGRBK method is approach to exact solution. Hence, the reconstructed image quality of the MGRBK method is much better.

4. Conclusions

Iterative reconstruction refers to iterative methods used to reconstruct 2D and 3D images in certain imaging techniques. A variety of reconstruction methods have been investigated to improve the quality of reconstructed images. In this paper, we transform the problem of image reconstruction into solving a linear systems with multiple right-hand sides. In order to solve the linear systems with multiple right-hand sides effectively, we propose the global randomized block Kaczmarz method based on the idea of K-means clustering, and use this method to image reconstruction. Theoretical analysis proves the convergence of this method. Simulation results show that the reconstructed image quality of this method is better than that of the FBP method.

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Conflict of interest

The authors declare there is no conflicts of interest.

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