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Research article

Non-singular solutions of p-Laplace problems, allowing multiple changes of sign in the nonlinearity

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Abstract: For the p-Laplace Dirichlet problem (where $\varphi(t) = t|t|^{p-2}$, $p > 1$)

$$\varphi(u'(x))' + f(u(x)) = 0 \quad \text{for } -1 < x < 1, \;\; u(-1) = u(1) = 0$$

assume that $f'(u) > (p-1)\frac{f(u)}{u} > 0$ for $u > \gamma > 0$, while $\int_u^\gamma f(t)\,dt < 0$ for all $u \in (0,\gamma)$. Then any positive solution, with $\max_{(-1,1)} u(x) = u(0) > \gamma$, is non-singular, no matter how many times $f(u)$ changes sign on $(0,\gamma)$. The uniqueness of solution follows.

Keywords: non-singular positive solutions; p-Laplace problems

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We consider positive solutions of

$$\varphi(u'(x))' + f(u(x)) = 0 \quad \text{for } -1 < x < 1, \;\; u(-1) = u(1) = 0\,, \tag{1}$$

where $\varphi(t) = t|t|^{p-2}$, $p > 1$, so that $\varphi'(t) = (p-1)|t|^{p-2}$. The linearized problem is

$$\begin{gathered}(\varphi'(u'(x))w'(x))' + f'(u(x))w(x) = 0 \quad \text{for } -1 < x < 1, \\ w(-1) = w(1) = 0\,.\end{gathered} \tag{2}$$

Recall that any positive solution of (1) is an even function $u(-x) = u(x)$, satisfying $xu'(x) < 0$ for $x \neq 0$ so that $\max_{(-1,1)} u(x) = u(0)$, and that any non-trivial solution of (2) is of one sign, so that we may assume that $w(x) > 0$ for $x \in (-1,1)$, see e.g., P. Korman [5], [6].

If $f'(u) > (p-1)\frac{f(u)}{u} > 0$ for $u > 0$, it is well known that any positive solution of (1) is *non-singular*, i.e., the problem (2) admits only the trivial solution $w(x) \equiv 0$. Now suppose that $f'(u) > (p-1)\frac{f(u)}{u} > 0$ holds only for $u > \gamma$, for some $\gamma > 0$. It turns out that positive solutions of (1), with maximum value greater than γ are still non-singular, provided that $\int_u^\gamma f(t)\,dt < 0$ for all $u \in (0,\gamma)$. The main result is stated next. It is customary to denote $F(u) = \int_0^u f(t)\,dt$.

Theorem 1. *Assume that $f(u) \in C^1(\bar{R}_+)$, and for some $\gamma > 0$ it satisfies*

$$f(\gamma) = 0, \quad \textit{and} \;\; f(u) > 0 \textit{ on } (\gamma, \infty), \tag{3}$$

$$f'(u) > (p-1)\frac{f(u)}{u}, \quad \textit{for } u > \gamma\,, \tag{4}$$

$$F(\gamma) - F(u) = \int_u^\gamma f(t)\,dt < 0, \;\; \textit{for } u \in (0, \gamma)\,. \tag{5}$$

Then any positive solution of (1), satisfying

$$u(0) > \gamma, \quad \textit{and} \;\; u'(1) < 0\,, \tag{6}$$

is non-singular, which means that the linearized problem (2) admits only the trivial solution.

In case $p = 2$ this result was proved in P. Korman [7], while for general $p > 1$ a weaker result, requiring that $f(u) < 0$ on $(0, \gamma)$, was given in J. Cheng [3] (and before that by R. Schaaf [10] for $p = 2$ case), see also P. Korman [5], [6] for a different proof, and a more detailed description of the solution curve. Other multiplicity results on p-Laplace equations include [1], [2], [4] and [9].

Proof: Assume, on the contrary, that the problem (2) admits a non-trivial solution $w(x) > 0$. Let $x_0 \in (0, 1)$ denote the point where $u(x_0) = \gamma$. Define

$$q(x) = (p-1)(1-x)\varphi(u'(x)) + \varphi'(u'(x))u(x)\,.$$

We claim that

$$q(x_0) < 0\,. \tag{7}$$

Rewrite (using that $(p-1)\varphi(t) = t\varphi'(t)$)

$$q(x) = \varphi'(u'(x))\left[(1-x)u'(x) + u(x)\right]\,.$$

Since $\varphi'(t) > 0$ for all $t \neq 0$, it suffices to show that the function $z(x) \equiv (1-x)u'(x) + u(x) < 0$ satisfies $z(x_0) < 0$. Indeed,

$$z(x_0) = \int_{x_0}^1 \left[u'(x_0) - u'(x)\right]\,dx < 0\,,$$

which implies the desired inequality (7), provided we can prove that

$$u'(x_0) - u'(x) < 0\,, \quad \text{for } x \in (x_0, 1)\,. \tag{8}$$

The "energy" function $E(x) = \frac{p-1}{p}|u'(x)|^p + F(u(x))$ is seen by differentiation to be a constant, so that $E(x) = E(x_0)$, or

$$\frac{p-1}{p}|u'(x)|^p + F(u(x)) = \frac{p-1}{p}|u'(x_0)|^p + F(\gamma)\,, \quad \text{for all } x\,.$$

By the assumption (5), it follows that

$$\frac{p-1}{p}\left[|u'(x)|^p - |u'(x_0)|^p\right] = F(\gamma) - F(u(x)) < 0\,, \quad \text{for } x \in (x_0, 1)\,,$$

justifying (8), and then giving (7).

Next, we claim that

$$(p-1)w(x_0)\varphi(u'(x_0)) - u(x_0)w'(x_0)\varphi'(u'(x_0)) > 0\,, \tag{9}$$

which implies, in particular, that

$$w'(x_0) < 0\,. \tag{10}$$

Indeed, by a direct computation, using (1) and (2),

$$[(p-1)w(x)\varphi(u'(x)) - u(x)w'(x)\varphi'(u'(x))]' = \left[f'(u) - (p-1)\frac{f(u)}{u}\right]uw\,.$$

The quantity on the right is positive on $(0, x_0)$, in view of our condition (4). Integration over $(0, x_0)$, gives (9).

We have for all $x \in [-1, 1]$

$$\varphi'(u')\,(u'w' - u''w) = constant = \varphi'(u'(1))u'(1)w'(1) > 0\,, \tag{11}$$

as follows by differentiation, and using the assumption $u'(1) < 0$. Hence

$$u'(x)w'(x) - u''(x)w(x) > 0\,, \quad \text{for } x \in (x_0, 1)\,. \tag{12}$$

Since $f(u(x_0)) = 0$, it follows from Eq (1) that $u''(x_0) = 0$. Then (11) implies

$$\begin{aligned} \varphi'(u'(1))u'(1)w'(1) &= \varphi'(u'(x_0))u'(x_0)w'(x_0) \\ &= (p-1)\varphi(u'(x_0))w'(x_0)\,. \end{aligned} \tag{13}$$

We need the following function, motivated by M. Tang [11] (which was introduced in P. Korman [5], and used in Y. An et al. [2])

$$T(x) = x\left[(p-1)\varphi(u'(x))w'(x) + f(u(x))w(x)\right] - (p-1)\varphi(u'(x))w(x)\,.$$

One verifies that

$$T'(x) = pf(u(x))w(x)\,. \tag{14}$$

Integrating (14) over $(x_0, 1)$, and using (5) and (12), obtain

$$\begin{aligned} T(1) - T(x_0) &= p\int_{x_0}^1 f(u(x))w(x)\,dx \\ &= p\int_{x_0}^1 [F(u(x)) - F(\gamma)]'\,\frac{w(x)}{u'(x)}\,dx \\ &= -p\int_{x_0}^1 [F(u(x)) - F(\gamma)]\,\frac{w'(x)u'(x) - w(x)u''(x)}{u'^2(x)}\,dx < 0\,, \end{aligned}$$

which implies that

$$L \equiv (p-1)\varphi(u'(1))w'(1) - (p-1)x_0\varphi(u'(x_0))w'(x_0) + (p-1)\varphi(u'(x_0))w(x_0) < 0\,.$$

On the other hand, using (13), then (9), followed by (10) and (7), we estimate the same quantity as follows

$$\begin{aligned} L &> (p-1)\varphi(u'(x_0))w'(x_0) - (p-1)x_0\varphi(u'(x_0))w'(x_0) + u(x_0)w'(x_0)\varphi'(u'(x_0)) \\ &= w'(x_0)q(x_0) > 0\,, \end{aligned}$$

a contradiction. ◇

We remark that in case $f(0) < 0$ it is possible to have a singular positive solution with $u'(1) = 0$, so that the assumption $u'(1) < 0$ is necessary.

We now consider the problem (where $\varphi(t) = t|t|^{p-2}$, $p > 1$)

$$\varphi(u'(x))' + \lambda f(u(x)) = 0 \quad \text{for } -1 < x < 1, \;\; u(-1) = u(1) = 0\,, \tag{15}$$

depending on a positive parameter λ. The following result follows the same way as the Theorem 3.1 in [5].

Theorem 2. *Assume that $f(u) \in C^1(\bar{R}_+)$, and the conditions (3), (4) and (5) hold. Then there exists $0 < \lambda_0 \le \infty$ so that the problem (15) has a unique positive solution for $0 < \lambda < \lambda_0$. All positive solutions, satisfying $u(0) > \gamma$, lie on a continuous solution curve that is decreasing in the $(\lambda, u(0))$ plane (see Figure 1). In case $f(0) < 0$, one has $\lambda_0 < \infty$, and at $\lambda = \lambda_0$ a positive solution with $u'(\pm 1) = 0$ exists, and no positive solutions exist for $\lambda > \lambda_0$. In case $f(0) = 0$ and $f'(0) < 0$, we have $\lambda_0 = \infty$.*

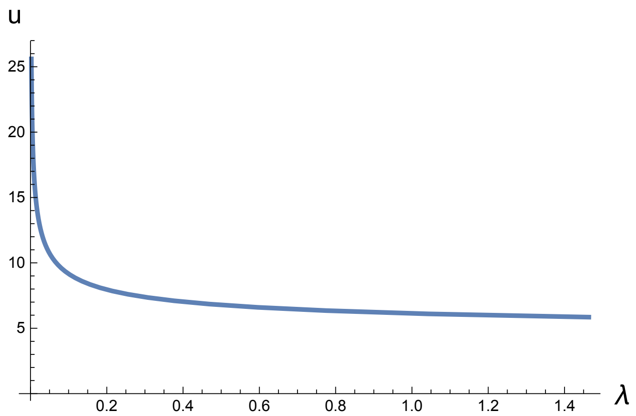

Figure 1. The curve of positive solutions for the problem (15), in case $p = 3$ and $f(u) = u(u-1)(u-2)(u-4)$.

Example In Figure 1 we present the solution curve of the problem (15) in case $p = 3$ and $f(u) = u(u-1)(u-2)(u-4)$. Here $\gamma = 4$, and one verifies that the Theorem 2 applies. The *Mathematica* program to perform numerical computations for this problem is explained in detail in [8] (it uses the shoot-and-scale method). The solution curve in Figure 1 exhausts the set of all positive solutions (since $\int_0^2 f(u)\,du < 0$, there are no solutions with $u(0) = \max_{(-1,1)} u(x) \in (1,2)$).

Conflict of interest

The author declares there is no conflicts of interest.

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