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Research article

Terminal value problem for nonlinear parabolic equation with Gaussian white noise

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Abstract: In this paper, We are interested in studying the backward in time problem for nonlinear parabolic equation with time and space independent coefficients. The main purpose of this paper is to study the problem of determining the initial condition of nonlinear parabolic equations from noisy observations of the final condition. The final data are noisy by the process involving Gaussian white noise. We introduce a regularized method to establish an approximate solution. We establish an upper bound on the rate of convergence of the mean integrated squared error. This article is inspired by the article by Tuan and Nane [1].

Keywords: Quasi-reversibility method; backward problem; parabolic equation; Gaussian white noise regularization

1. Introduction

The forward problem for parabolic equations is finding the distribution at a later time when we know the initial distribution. In geophysical exploration, one is often faced with the problem of determining the temperature distribution in the object or any part of the Earth at a time $t_0 > 0$ from temperature measurements at a time $t_1 > t_0$. This is the backward in time parabolic problem. Backward parabolic problems arises in several practical areas such as image processing, mathematical finance, and physics (see [2,3]). Let T be a positive number and Ω be an open, bounded and connected domain in $\mathbb{R}^d, d \geq 1$ with a smooth boundary $\partial\Omega$. In this paper, we consider the question of finding the function $\mathbf{u}(x,t)$,

$(x,t) \in \Omega \times [0,T]$, satisfying the nonlinear problem

$$\begin{cases} \mathbf{u}_t - \nabla\Big(a(x,t)\nabla \mathbf{u}\Big) = F(x,t,\mathbf{u}(x,t)), & (x,t) \in \Omega \times (0,T), \\ \mathbf{u}|_{\partial\Omega} = 0, \quad t \in (0,T), \\ \mathbf{u}(x,T) = g(x), \quad (x,t) \in \Omega \times (0,T), \end{cases} \tag{1.1}$$

where the functions $a(x,t), g(x)$ are given and the source function F will be given later. Here the coefficient $a(x,t)$ is a C^1 smooth function and $0 < \overline{m} \le a(x,t) < M$ for all $(x,t) \in \Omega \times (0,T)$ for some finite constants $\overline{m},\ M$. The problem is well-known to be ill-posed in the sense of Hadamard. Hence, a solution corresponding to the data does not always exist, and in the case of existence, it does not depend continuously on the given data. In fact, from small noise contaminated physical measurements, the corresponding solutions will have large errors. Hence, one has to resort to a regularization. In the simple case of deterministic noise, Problem (1.1) with $a = 1$ and $F = 0$ was studied by many authors [4–6]. However, in the case of random noise, the analysis of regularization methods is still limited. The problem is to determine the initial temperature function f given a noisy version of the temperature distribution g at time T

$$g_\delta^{\mathrm{obs}}(x) = g(x) + \delta\xi(x) \tag{1.2}$$

where $\delta > 0$ is the amplitude of the noise and ξ is a Gaussian white noise. In practice, we only observe some finite errors as follows

$$\left\langle g_\delta^{\mathrm{obs}}, \phi_j \right\rangle = \left\langle g, \phi_j \right\rangle + \delta \left\langle \xi, \phi_j \right\rangle, \quad j = \overline{1,\mathbf{N}} = 1,2,3,\cdots,\mathbf{N}, \tag{1.3}$$

where the natural number $\mathbf{N}$ is the number of steps of discrete observations and ϕ_j is defined in section 2. The main goal is to find an approximate solution $\widehat{\mathbf{u}}_N(0)$ for $\mathbf{u}(0)$ and then investigate the rate of convergence $\mathbf{E}\|\widehat{\mathbf{u}}_N(0) - \mathbf{u}(0)\|$, which is called the mean integrated square error (MISE). Here $\mathbf{E}$ denotes the expectation w.r.t. the distribution of the data in the model (1.2).

There are two main approaches to considering inverse problem for noise modeling. The first approach is based on a formal technique if one is assuming that the noise is definite and small. The second approach is based on a statistical point of view and in this approach one does not need to assume smalll levels of noise. We consider in this paper a statistical point of view for the backward parabolic equation. Our aim is to reconstruct the initial function from the disturbance measurements of the final values in a statistical inverse problem framework. There are many different types of random noise, but we are interested in Gaussian noise here. The model (1.2) and (1.3) were considered in some recent papers; see [7–11]. In signal processing, Gaussian white noise is a random signal of equal intensity at different frequencies, giving it a constant power spectral density and this term is used in physics, acoustic engineering, telecommunications and statistical forecasting.

The inverse problem with random noise has a long history. The simple case of (1.1) is the homogeneous linear parabolic equation of finding the initial data $u_0 := u(x,0)$ that satisfies

$$\begin{cases} \mathbf{u}_t - \Delta u = 0, \quad (x,t) \in \Omega \times (0,T), \\ \mathbf{u}|_{\partial\Omega} = 0, \quad t \in (0,T), \\ \mathbf{u}(x,T) = g(x), \quad (x,t) \in \Omega \times (0,T). \end{cases} \tag{1.4}$$

This equation is a special form of statistical inverse problems and it can be transformed by a linear operator with random noise

$$g = Ku_0 + \text{"noise"}, \tag{1.5}$$

where K is a bounded linear operator that does not have a continuous inverse. Fomula (1.5) is interpreted as Ku_0 deviated from function g by a random error.

Problem (1.4) was studied by well-known methods including spectral cut-off (or called truncation method) [7, 9, 12, 13], the Tiknonov method [14], iterative regularization methods [15], the Bayes estimation method [16, 17], and the Lavrentiev regularization method [18]. In some parts of these works, the authors show that the error $\mathbf{E}\|\widehat{\mathbf{u}}_N(0) - \mathbf{u}(0)\|$ tend to zero when $\mathbf{N}$ is suitably chosen according to the value of δ and $\delta \to 0$. For more details, we refer the reader to [19].
To the best of our knowledge, there are no results for the backward problem for nonlinear parabolic equations with Gaussian white noise. There are two types of difficulty in solving our problem. The first difficulty occurs because the problem is nonlinear and nonlinear problems with random noise is more difficult since one cannot apply well known methods. The second is the random noise data, which makes the problem computationally complex. The problem of computation with random data requires some knowledge of the stochastic process,so one has to consider the expectation.

Very recently, in [20], the authors studied the discrete random model for backward nonlinear parabolic problems. However, the problem considered in [20] is in a rectangular domain which is limited in practice. The present paper uses another random model and also gives approximation of the solution in the case of more general bounded and smooth domains Ω. Our task in this paper is to show that the expectation between the solution and the approximate solution converges to zero when $\mathbf{N}$ tends to infinity.

This paper is organized as follows. In section 2, we give a couple of preliminary results. In section 3, we give an explanation for ill-posedness of the problem. To help the reader, we divide the problem into three cases under various assumptions on the coefficient a, and the source function F.
Case 1: $a := a(x, t)$ is a constant and F is a globally Lipschitz function. In section 4, we will study this case and give convergence rates in L^2 and H^p norms for $p > 0$. The method here is the well-known spectral method. The main idea is to approximate the final data g by the approximate data and use this function to establish a regularized problem by the truncation method.
Case 2: $a := a(x, t)$ depends on x and t and F is a locally Lipschitz function. This problem is more difficult. In most practical problems, the function F is often a locally Lipschitz function. The difficulty here is in the fact that the solution cannot be transformed into a Fourier series and therefore, we cannot apply well-known methods to find an approximate solution. In Section 5, we will study a new form of the quasi-reversibility method to construct a regularized solution and obtain the convergence rate. Our method is new and very different than the method of Lattes and Lions [21]. We approximate the locally Lipschitz function by a sequence of globally Lipschitz functions and use some new techniques to obtain the convergence rate.
Case 3 Various assumptions on F. In practice there are many functions that are not locally Lipschitz. Hence our analysis in section 4 cannot applied in section 6. Our method in section 6 is also the quasi-reversibility method and is very similar to the method in section 4. However in section 6, we do not approximate F as we do in section 4. This leads to a convergence rate that is better than the one in section 4. One difficulty that occurs in this section is showing the existence and uniqueness of the regularized solution. To prove the existence of the regularized solution, we do not follow previously mentioned methods. Instead, we use the Faedo – Galerkin method, and the compactness method introduced by Lions [22]. To the best of our knowledge, this is the first result where F is not necessarily a locally Lipschitz function. Finally, in section 7, we give some specific equations which can be applied by our method.

2. Preliminaries

To give some details on this random model (1.2), we give the following definitions (See [12, 19]):

Definition 2.1. *Let $\mathcal{H}$ be a Hilbert space. Let $g, g_\delta \in \mathcal{H}$ satisfy* (1.2). *We understand the equal relationship in fomula*

$$g_\delta^{obs}(x) = g(x) + \delta\xi(x)$$

as follows:

$$\langle g_\delta, \chi \rangle = \langle g, \chi \rangle + \delta \langle \xi, \chi \rangle, \quad \forall \chi \in \mathcal{H}, \tag{2.1}$$

here δ is the amplitude of the noise. We also assume that ξ is a zero-mean Gaussian random process indexed by $\mathcal{H}$ on a probability space. $\langle \xi, \chi \rangle \sim \mathcal{N}(0, \|\chi\|_{\mathcal{H}}^2)$. Moreover, given $\chi_1, \chi_2 \in H$ then

$$\mathbb{E}\Big(\langle \xi, \chi_1 \rangle \langle \xi, \chi_2 \rangle \Big) = \mathbb{E} \langle \chi_1, \chi_2 \rangle. \tag{2.2}$$

Definition 2.2. *The stochastic error is a Hilbert-space process, i.e., a bounded linear operator ξ : $\mathcal{H} \to L^2(\mathbf{\Omega}, \mathcal{A}, P)$ where $(\mathbf{\Omega}, \mathcal{A}, P)$ is the underlying probability space and $L^2(.,.)$ is the space of all square integrable measurable functions.*

Let us recall that the eigenvalue problem

$$\begin{cases} -\Delta\phi_j(x) = \lambda_j \phi_j(x), & x \in \Omega, \\ \phi_j(x) = 0, & x \in \partial\Omega, \end{cases} \tag{2.3}$$

admits a family of eigenvalues $0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le ... \le \lambda_j \le ...$ and eigenfunctions $\{\phi_j\}$ and $\lambda_j \to \infty$ as $j \to \infty$; see page 335 in [23].

Next, we introduce the abstract Gevrey class of functions of index $\sigma > 0$, see, *e.g.*, [24], defined by

$$\mathcal{W}_\sigma = \left\{ v \in L^2(\Omega) : \sum_{j=1}^{\infty} e^{2\sigma\lambda_j} \left| \langle v, \phi_j(x) \rangle_{L^2(\Omega)} \right|^2 < \infty \right\},$$

which is a Hilbert space equipped with the inner product

$$\langle v_1, v_2 \rangle_{\mathcal{W}_\sigma} := \left\langle e^{\sigma\sqrt{-\Delta}} v_1, e^{\sigma\sqrt{-\Delta}} v_2 \right\rangle_{L^2(\Omega)}, \quad \text{for all } v_1, v_2 \in \mathcal{W}_\sigma;$$

its corresponding norm is $\|v\|_{\mathcal{W}_\sigma} = \sqrt{\sum_{j=1}^{\infty} e^{2\sigma\lambda_j} \left| \langle v, \phi_j \rangle_{L^2(\Omega)} \right|^2} < \infty$.

3. The ill-posedness of the nonlinear parabolic equation with random noise

The ill-posedness of the backward heat equation is well known and has appeared in many previous articles. However, in the random case, we need to give an example to illustrate the ill-posedness. From the appearance of the expected component, the evaluation of the nonconformity of the random model is much more complicated than the deterministic model. Therefore, we have to choose a simple case to find a suitable example. In this section, for a special case of Eq (1.1), we show that the nonlinear parabolic equation with random noise is ill-posed in the sense of Hadamard.

Theorem 3.1. *Problem* (1.1) *is ill-posed in the special case when $a = 1, \Omega = (0, \pi)$.*

Proof. Let $\Omega = (0, \pi)$ and $a(x, t) = 1$, Then $\lambda_{\mathbf{N}} = \mathbf{N}^2$. Let us consider the following parabolic equation

$$\begin{cases} \dfrac{\partial \mathbf{V}_{\delta,\mathbf{N}(\delta)}}{\partial t} - \Delta \mathbf{V}_{\delta,\mathbf{N}(\delta)}(t) = F_0(\mathbf{V}_{\delta,\mathbf{N}(\delta)}(x,t)), \quad 0 < t < T, x \in (0, \pi) \\ \mathbf{V}_{\delta,\mathbf{N}(\delta)}(0,t) = \mathbf{V}_{\delta,\mathbf{N}(\delta)}(\pi,t) = 0, \\ \mathbf{V}_{\delta,\mathbf{N}(\delta)}(x,T) = G_{\delta,\mathbf{N}(\delta)}(x), \end{cases} \tag{3.1}$$

where F_0 is

$$F_0(v(x)) = \sum_{j=1}^{\infty} \frac{e^{-Tj^2}}{2T} \left\langle v, \phi_j(x) \right\rangle \phi_j(x) \tag{3.2}$$

for any $v \in L^2(\Omega)$, and $\phi_j(x) = \sqrt{\frac{2}{\pi}} \sin(jx)$. Let us choose $G_{\delta,\mathbf{N}(\delta)} \in \mathcal{L}^2(\Omega)$ such that

$$G_{\delta,\mathbf{N}(\delta)}(x) = \sum_{j=1}^{\mathbf{N}(\delta)} \left\langle g_\delta(x), \phi_j(x) \right\rangle \phi_j(x) \tag{3.3}$$

where g_δ is defined by

$$\left\langle g_\delta, \phi_j \right\rangle = \delta \left\langle \xi, \phi_j \right\rangle, \quad j = \overline{1, N} = \{j \in \mathbb{N}, 1 \le j \le N\}. \tag{3.4}$$

By the usual MISE decomposition which involves a variance term and a bias term, we get

$$\mathbf{E}\|G_{\delta,\mathbf{N}(\delta)}\|^2_{L^2(\Omega)} = \mathbf{E}\Big(\sum_{j=1}^{\mathbf{N}(\delta)} \left\langle G_{\delta,\mathbf{N}(\delta)}, \phi_j \right\rangle^2 \Big) = \delta^2 \mathbf{E}\Big(\sum_{j=1}^{\mathbf{N}(\delta)} \xi_j^2 \Big) = \delta^2 \mathbf{N}(\delta). \tag{3.5}$$

The solution of Problem (3.1) is given by the Fourier series (see [29])

$$\mathbf{V}_{\delta,\mathbf{N}(\delta)}(x,t) = \sum_{j=1}^{\infty} \left[e^{(T-t)\lambda_j} \left\langle G_{\delta,\mathbf{N}(\delta)}, \phi_j \right\rangle - \int_t^T e^{(s-t)\lambda_j} \langle F_0(\mathbf{V}_{\delta,\mathbf{N}(\delta)}(s)), \phi_j \rangle ds \right] \phi_j. \tag{3.6}$$

We show that Problem (3.6) has unique solution $\mathbf{V}_{\delta,\mathbf{N}(\delta)} \in C([0,T]; L^2(\Omega))$. Let us consider

$$\Phi v := \sum_{j=1}^{\infty} e^{(T-t)\lambda_j} \left\langle G_{\delta,\mathbf{N}(\delta)}, \phi_j \right\rangle - \sum_{j=1}^{\infty} \left[\int_t^T e^{(s-t)\lambda_j} \langle F_0(v(s)), \phi_j \rangle ds \right] \phi_j. \tag{3.7}$$

For any $v_1, v_2 \in C([0,T]; L^2(\Omega))$, using Hölder inequality, we have for all $t \in [0,T]$

$$\begin{aligned} \|\Phi v_1(t) - \Phi v_2(t)\|^2_{L^2(\Omega)} &= \sum_{j=1}^{\infty} \left[\int_t^T e^{(s-t)\lambda_j} \langle F_0(v_1(s)) - F_0(v_2(s)), \phi_j \rangle ds \right]^2 \\ &\le T \sum_{j=1}^{\infty} \int_t^T e^{2(s-t)\lambda_j} \langle F_0(v_1(s)) - F_0(v_2(s)), \phi_j \rangle^2 ds \\ &= \frac{T}{4T^2} \sum_{j=1}^{\infty} \int_t^T e^{2(s-t-T)\lambda_j} \langle v_1(s) - v_2(s), \phi_j \rangle^2 ds \\ &\le \frac{1}{4T} \sum_{j=1}^{\infty} \int_t^T \langle v_1(s) - v_2(s), \phi_j \rangle^2 ds \le \frac{1}{4} \|v_1 - v_2\|^2_{C([0,T];L^2(\Omega))}. \end{aligned} \tag{3.8}$$

Hence, we obtain that

$$\|\Phi v_1 - \Phi v_2\|_{C([0,T];L^2(\Omega))} \leq \frac{1}{2}\|v_1 - v_2\|_{C([0,T];L^2(\Omega))}. \tag{3.9}$$

Thus Φ is a contraction. Using the contraction principle and we conclude that the equation $\Phi(w) = w$ has a unique solution $\mathbf{V}_{\delta,\mathbf{N}(\delta)} \in C([0,T];L^2(\Omega))$. Using the inequality $a^2 + b^2 \geq \frac{1}{2}(a-b)^2,\ \ a,b \in \mathbb{R}$, we have the following estimate

$$\begin{aligned}\left\|\mathbf{V}_{\delta,\mathbf{N}(\delta)}\right\|^2_{L^2(\Omega)} \geq &\underbrace{\frac{1}{2}\Big\|\sum_{j=1}^{\infty} e^{(T-t)\lambda_j}\left\langle G_{\delta,\mathbf{N}(\delta)},\phi_j\right\rangle\phi_j\Big\|^2_{L^2(\Omega)}}_{I_1} \\ &- \underbrace{\Big\|\sum_{j=1}^{\infty}\left(\int_t^T e^{(s-t)\lambda_j}\langle F_0(\mathbf{V}_{\delta,\mathbf{N}(\delta)}(s)),\phi_j\rangle ds\right)\phi_j\Big\|^2_{L^2(\Omega)}}_{I_2}.\end{aligned} \tag{3.10}$$

First, using Hölder's inequality, we get

$$\begin{aligned} I_2 &\leq \sum_{j=1}^{\infty}\left(\int_t^T e^{(s-t)\lambda_j}\langle F_0(\mathbf{V}_{\delta,\mathbf{N}(\delta)}(s)),\phi_j\rangle ds\right)^2 \\ &\leq T\sum_{j=1}^{\infty}\int_t^T e^{2(s-t)\lambda_j}\langle F_0(\mathbf{V}_{\delta,\mathbf{N}(\delta)}(s)),\phi_j\rangle^2 ds \\ &\leq \frac{T}{4T^2}\int_t^T\sum_{j=1}^{\infty} e^{2(s-t-T)\lambda_j}\left\langle \mathbf{V}_{\delta,\mathbf{N}(\delta)}(t),\phi_j\right\rangle^2 ds \leq \frac{1}{4}\left\|\mathbf{V}_{\delta,\mathbf{N}(\delta)}\right\|^2_{C([0,T];L^2(\Omega))}.\end{aligned} \tag{3.11}$$

We have the lower bound for I_1:

$$\mathbf{E}I_1 = \frac{1}{2}\sum_{j=1}^{\infty} e^{2(T-t)\lambda_j}\mathbf{E}\left\langle G_{\delta,\mathbf{N}(\delta)},\phi_j\right\rangle^2 = \frac{1}{2}\sum_{j=1}^{\mathbf{N}}\delta^2 e^{2(T-t)\lambda_j} \geq \frac{1}{2}\delta^2 e^{2(T-t)\lambda_{\mathbf{N}(\delta)}}. \tag{3.12}$$

Combining (3.10), (3.11), (3.12), and we obtain

$$\mathbf{E}\left\|\mathbf{V}_{\delta,\mathbf{N}(\delta)}\right\|^2_{L^2(\Omega)} + \frac{1}{4}\mathbf{E}\left\|\mathbf{V}_{\delta,\mathbf{N}(\delta)}\right\|^2_{C([0,T];L^2(\Omega))} \geq \frac{1}{2}\delta^2 e^{2(T-t)\lambda_{\mathbf{N}(\delta)}}. \tag{3.13}$$

By taking supremum of both sides on $[0,T]$, we get

$$\mathbf{E}\left\|\mathbf{V}_{\delta,\mathbf{N}(\delta)}\right\|^2_{C([0,T];L^2(\Omega))} \geq \frac{2}{5}\sup_{0\leq t\leq T}\delta^2 e^{2(T-t)\lambda_{\mathbf{N}(\delta)}} = \frac{2}{5}\delta^2 e^{2T\lambda_{\mathbf{N}(\delta)}} = \frac{2}{5}\delta^2 e^{2T\mathbf{N}^2(\delta)}. \tag{3.14}$$

Choosing $\mathbf{N} := \mathbf{N}(\delta) = \sqrt{\frac{1}{2T}\ln(\frac{1}{\delta})}$, we obtain

$$\mathbf{E}\|G_{\delta,\mathbf{N}(\delta)}\|^2_{L^2(\Omega)} = \delta^2\mathbf{N}(\delta) = \delta^2\sqrt{\frac{1}{2T}\ln(\frac{1}{\delta})} \to 0,\ \text{when } \delta \to 0, \tag{3.15}$$

and

$$\mathbf{E}\left\|\mathbf{V}_{\delta,\mathbf{N}(\delta)}\right\|^2_{C([0,T];L^2(\Omega))} \geq \frac{2}{5}\delta^2 e^{2T\mathbf{N}^2(\delta)} = \frac{2}{5\delta} \to +\infty,\ \text{when } \delta \to 0. \tag{3.16}$$

From (3.15) and (3.16), we can conclude that Problem (1.1) is ill-posed. □

4. Regularization result with constant coefficient and globally Lipschitz source function

In this section, we consider the question of finding the function $\mathbf{u}(x,t)$, $(x,t) \in \Omega \times [0,T]$, that satisfies the problem

$$\begin{cases} \mathbf{u}_t - \Delta u = F(x,t,\mathbf{u}(x,t)), & (x,t) \in \Omega \times (0,T), \\ \mathbf{u}|_{\partial\Omega} = 0, & t \in (0,T), \\ \mathbf{u}(x,T) = g(x), & x \in \Omega. \end{cases} \tag{4.1}$$

In this section, we assume there exists a constant $K > 0$ with

$$|F(x,t;u) - F(x,t;v)| \leq K\,|u - v|,$$

where $(x,t) \in \Omega \times [0,T]$ and $u,\ v \in \mathbb{R}$.

Lemma 4.1. *Let $\overline{G}_{\delta,\mathbf{N}(\delta)} \in L^2(\Omega)$ be such that*

$$\overline{G}_{\delta,\mathbf{N}(\delta)} = \sum_{j=1}^{\mathbf{N}(\delta)} \left\langle g_\delta^{obs}, \phi_j \right\rangle \phi_j. \tag{4.2}$$

Assume that $g \in H^{2\gamma}(\Omega)$. Then we have the following estimate

$$\mathbf{E}\|\overline{G}_{\delta,\mathbf{N}(\delta)} - g\|^2_{L^2(\Omega)} \leq \delta^2 \mathbf{N}(\delta) + \frac{1}{\lambda^{2\gamma}_{\mathbf{N}(\delta)}} \|g\|^2_{H^{2\gamma}(\Omega)} \tag{4.3}$$

for any $\gamma \geq 0$. Here $\mathbf{N}$ depends on δ and satisfies $\lim_{\delta\to 0} \mathbf{N}(\delta) = +\infty$ *and* $\lim_{\delta\to 0} \delta^2 \mathbf{N}(\delta) = 0$.

Remark 4.1. *Consider the right hand side of* (4.3)*. In order for the right-hand side of* (4.3) *to converge to zero we require* $\lim_{\delta\to 0} \delta^2 \mathbf{N}(\delta) = 0$ *and the condition*

$$\frac{1}{\lambda^{2\gamma}_{\mathbf{N}(\delta)}} \to 0, \quad \delta \to 0. \tag{4.4}$$

Since the fact that $\lambda_k \sim k^{2/d}$, we see that

$$\lambda^{2\gamma}_{\mathbf{N}(\delta)} \sim (\mathbf{N}(\delta))^{\frac{4\gamma}{\mathbf{d}}},$$

and to verify the condition (4.4) *we need the condition* $\lim_{\delta\to 0} \mathbf{N}(\delta) = +\infty$.

Proof. For the following proof, we consider the genuine model (1.3). By the usual MISE decomposition which involves a variance term and a bias term, we get

$$\begin{aligned} \mathbf{E}\|\overline{G}_{\delta,\mathbf{N}(\delta)} - g\|^2_{L^2(\Omega)} &= \mathbf{E}\Big(\sum_{j=1}^{\mathbf{N}(\delta)} \left\langle g_\delta^{\text{obs}} - g, \phi_j \right\rangle^2\Big) + \sum_{j \geq \mathbf{N}(\delta)+1} \left\langle g, \phi_j \right\rangle^2 \\ &= \delta^2 \mathbf{E}\Big(\sum_{j=1}^{\mathbf{N}(\delta)} \xi_j^2\Big) + \sum_{j \geq \mathbf{N}(\delta)+1} \lambda_j^{-2\gamma} \lambda_j^{2\gamma} \left\langle g, \phi_j \right\rangle^2. \end{aligned} \tag{4.5}$$

Since $\xi_j = \langle \xi, \phi_j \rangle \overset{iid}{\sim} N(0,1)$, it follows that $\mathbf{E}\xi_j^2 = 1$, so

$$\mathbf{E}\|\overline{G}_{\delta,\mathbf{N}(\delta)} - g\|^2_{L^2(\Omega)} \leq \delta^2 \mathbf{N}(\delta) + \frac{1}{\lambda^{2\gamma}_{\mathbf{N}(\delta)}} \|g\|_{H^{2\gamma}}. \tag{4.6}$$

□

Using the truncation method, we give a regularized problem for Problem (1.1) as follows

$$\begin{cases} \dfrac{\partial}{\partial t}\mathbf{u}^{\delta}_{\mathbf{N}(\delta)} - \Delta \mathbf{u}^{\delta}_{\mathbf{N}(\delta)} = \mathbf{J}_{\alpha_{\mathbf{N}(\delta)}} F(x,t,\mathbf{u}^{\delta}_{\mathbf{N}(\delta)}(x,t)), \quad (x,t)\in\Omega\times(0,T),\\ \mathbf{u}^{\delta}_{\mathbf{N}(\delta)}|_{\partial\Omega} = 0, \quad t\in(0,T),\\ \mathbf{u}^{\delta}_{\mathbf{N}(\delta)}(x,T) = \mathbf{J}_{\alpha_{\mathbf{N}(\delta)}}\overline{G}_{\delta,\mathbf{N}(\delta)}(x), \quad (x,t)\in\Omega\times(0,T), \end{cases} \tag{4.7}$$

where $\alpha_{\mathbf{N}(\delta)}$ is regularization parameter and $\mathbf{J}_{\alpha_{\mathbf{N}(\delta)}}$ is the following operator

$$\mathbf{J}_{\alpha_{\mathbf{N}(\delta)}} v := \sum_{\lambda_j \le \alpha_{\mathbf{N}(\delta)}} \left\langle v,\phi_j\right\rangle \phi_j, \text{ for all } v\in L^2(\Omega). \tag{4.8}$$

Our main result in this section is as follows

Theorem 4.1. *Problem* (4.7) *has a unique solution* $\mathbf{u}^{\delta}_{\mathbf{N}(\delta)} \in C([0,T];L^2(\Omega))$ *which satisfies*

$$\mathbf{u}^{\delta}_{\mathbf{N}(\delta)}(x,t) = \sum_{\lambda_j\le\alpha_{\mathbf{N}(\delta)}} \left[e^{(T-t)\lambda_j}\left\langle \overline{G}_{\delta,\mathbf{N}(\delta)},\phi_j\right\rangle - \int_t^T e^{(s-t)\lambda_j}\langle F(\mathbf{u}^{\delta}_{\mathbf{N}(\delta)}(s)),\phi_j\rangle ds\right]\phi_j. \tag{4.9}$$

Assume that problem (1.1) *has unique solution* $\mathbf{u}$ *such that*

$$\sum_{j=1}^{\infty} \lambda_j^{2\beta} e^{2t\lambda_j}\left\langle \mathbf{u}(.,t),\phi_j\right\rangle^2 < A', \quad t\in[0,T]. \tag{4.10}$$

Choose $\alpha_{\mathbf{N}(\delta)}$ *such that*

$$\lim_{\delta\to0}\alpha_{\mathbf{N}(\delta)} = +\infty,\ \lim_{\delta\to0}\frac{e^{kT\alpha_{\mathbf{N}(\delta)}}}{\lambda^{\gamma}_{\mathbf{N}(\delta)}} = 0,\ \lim_{\delta\to0} e^{KT\alpha_{\mathbf{N}(\delta)}}\sqrt{\mathbf{N}(\delta)}\delta = 0. \tag{4.11}$$

Then the following estimate holds

$$\mathbf{E}\|\mathbf{u}(.,t) - \mathbf{u}^{\delta}_{\mathbf{N}(\delta)}(.,t)\|^2_{L^2(\Omega)} \le 2e^{2K^2(T-t)}e^{-2t\alpha_{\mathbf{N}(\delta)}}\left[\delta^2\mathbf{N}(\delta)e^{2T\alpha_{\mathbf{N}(\delta)}} + \frac{e^{2T\alpha_{\mathbf{N}(\delta)}}}{\lambda^{2\gamma}_{\mathbf{N}(\delta)}}\|g\|_{H^{2\gamma}} + \alpha^{-2\beta}_{\mathbf{N}(\delta)}\right]. \tag{4.12}$$

Remark 4.2. *1. From the theorem above, it is easy to see that* $\mathbf{E}\left\|\mathbf{u}^{\delta}_{\mathbf{N}(\delta)}(x,t) - \mathbf{u}(x,t)\right\|^2_{L^2(\Omega)}$ *is of the order*

$$e^{-2t\alpha_{\mathbf{N}(\delta)}}\max\Big(\delta^2\mathbf{N}(\delta)e^{2T\alpha_{\mathbf{N}(\delta)}}, \frac{e^{2T\alpha_{\mathbf{N}(\delta)}}}{\lambda^{2\gamma}_{\mathbf{N}(\delta)}}, \alpha^{-2\beta}_{\mathbf{N}(\delta)}\Big). \tag{4.13}$$

2. Now, we give one example for the choice of $\mathbf{N}(\delta)$ *which satisfies condition* (4.11). *Since* $\lambda_{\mathbf{N}} \sim \mathbf{N}^{\frac{2}{d}}$, *see [25], we choose* $\alpha_{\mathbf{N}}$ *such that* $e^{kT\alpha_{\mathbf{N}(\delta)}} = |\mathbf{N}(\delta)|^a$ *for any* $0 < a < \frac{2\gamma}{d}$. *Then we have* $\alpha_{\mathbf{N}(\delta)} = \frac{a}{kT}\log(\mathbf{N}(\delta))$. *The number* $\mathbf{N}(\delta)$ *is chosen as*

$$\mathbf{N}(\delta) = \left(\frac{1}{\delta}\right)^{ba+\frac{b}{2}}$$

for $0 < b < 1$. *With* $\mathbf{N}(\delta)$ *chosen as above,* $\mathbf{E}\left\|\mathbf{u}^{\delta}_{\mathbf{N}(\delta)}(x,t) - \mathbf{u}(x,t)\right\|^2_{L^2(\Omega)}$ *is of the order* $\left(\frac{1}{\delta}\right)^{\frac{-(ba+\frac{b}{2})at}{KT}}$

3. The existence and uniqueness of the solution of Eq (1.1) *is an open problem, and we do not investigate this problem here. The case considered in Theorem 3.1 gives the existence of the solution of Problem* (1.1) *in a special case. The uniqueness of the backward parabolic problem has attracted the attention of many authors (see, for example, [26–28]) and this is also a challenging open problem.*

Proof of Theorem 4.1. We divide the proof into a number of parts.
Part 1. *Problem* (4.7) *has a unique solution* $\mathbf{u}^{\delta}_{\mathbf{N}(\delta)} \in C([0,T]; L^2(\Omega))$. The proof is similar to [29] (see Theorem 3.1, page 2975 [29]). Hence, we omit it here.
Part 2. Estimate the expectation of the error between the exact solution u and the regularized solution $\mathbf{u}^{\delta}_{\mathbf{N}(\delta)}$.
Let us consider the following integral equation

$$\mathbf{v}^{\delta}_{\mathbf{N}(\delta)}(x,t) = \sum_{\lambda_j \le \alpha_{\mathbf{N}(\delta)}} \left[e^{(T-t)\lambda_j} \left\langle g, \phi_j \right\rangle - \int_t^T e^{(s-t)\lambda_j} \langle F(\mathbf{v}^{\delta}_{\mathbf{N}(\delta)}(s)), \phi_j \rangle ds \right] \phi_j. \tag{4.14}$$

We have

$$\begin{aligned}
\|\mathbf{u}^{\delta}_{\mathbf{N}(\delta)}(.,t) - \mathbf{v}^{\delta}_{\mathbf{N}(\delta)}(.,t)\|^2_{L^2(\Omega)} &\le 2 \sum_{\lambda_j \le \alpha_N} e^{2(T-t)\lambda_j} \left\langle \overline{G}_{\delta,\mathbf{N}(\delta)} - g, \phi_j \right\rangle^2 \\
&\quad + 2 \sum_{\lambda_j \le \alpha_{\mathbf{N}(\delta)}} \left[\int_t^T e^{(s-t)\lambda_j} \Big(F_j(\mathbf{u}^{\delta}_{\mathbf{N}(\delta)})(s) - F_j(\mathbf{v}^{\delta}_{\mathbf{N}(\delta)})(s) \Big) ds \right]^2 \\
&\le 2e^{2(T-t)\alpha_{\mathbf{N}}} \sum_{\lambda_j \le \alpha_{\mathbf{N}(\delta)}} \left\langle \overline{G}_{\delta,\mathbf{N}(\delta)} - g, \phi_j \right\rangle^2 \\
&\quad + 2(T-t) \int_t^T e^{2(s-t)\alpha_{\mathbf{N}(\delta)}} \sum_{\lambda_j \le \alpha_{\mathbf{N}(\delta)}} \Big(F_j(\mathbf{u}^{\delta}_{\mathbf{N}(\delta)})(s) - F_j(\mathbf{v}^{\delta}_{\mathbf{N}(\delta)})(s) \Big)^2 ds \\
&\le 2e^{2(T-t)\alpha_{\mathbf{N}}} \|\overline{G}_{\delta,\mathbf{N}(\delta)} - g\|^2_{L^2(\Omega)} \\
&\quad + 2K^2 T \int_t^T e^{2(s-t)\alpha_{\mathbf{N}}} \|\mathbf{u}^{\delta}_{\mathbf{N}(\delta)}(.,s) - \mathbf{v}^{\delta}_{\mathbf{N}(\delta)}(.,s)\|^2_{L^2(\Omega)} ds.
\end{aligned} \tag{4.15}$$

Taking the expectation of both sides of the last inequality, we get

$$\begin{aligned}
\mathbf{E}\|\mathbf{u}^{\delta}_{\mathbf{N}(\delta)}(.,t) - \mathbf{v}^{\delta}_{\mathbf{N}(\delta)}(.,t)\|^2_{L^2(\Omega)} &\le 2e^{2(T-t)\alpha_{\mathbf{N}(\delta)}} \mathbf{E}\|\overline{G}_{\delta,\mathbf{N}(\delta)} - g\|^2_{L^2(\Omega)} \\
&\quad + 2K^2 T \int_t^T e^{2(s-t)\alpha_{\mathbf{N}}} \mathbf{E}\|\mathbf{u}^{\delta}_{\mathbf{N}(\delta)}(.,s) - \mathbf{v}^{\delta}_{\mathbf{N}(\delta)}(.,s)\|^2_{L^2(\Omega)} ds.
\end{aligned} \tag{4.16}$$

Multiplying both sides with $e^{2t\alpha_{\mathbf{N}}}$, we obtain

$$\begin{aligned}
e^{2t\alpha_{\mathbf{N}(\delta)}} \mathbf{E}\|\mathbf{u}^{\delta}_{\mathbf{N}(\delta)}(.,t) - \mathbf{v}^{\delta}_{\mathbf{N}(\delta)}(.,t)\|^2_{L^2(\Omega)} &\le 2e^{2T\alpha_{\mathbf{N}(\delta)}} \mathbf{E}\|\overline{G}_{\delta,\mathbf{N}(\delta)} - g\|^2_{L^2(\Omega)} \\
&\quad + 2K^2 T \int_t^T e^{2s\alpha_{\mathbf{N}(\delta)}} \mathbf{E}\|\mathbf{u}^{\delta}_{\mathbf{N}(\delta)}(.,s) - \mathbf{v}^{\delta}_{\mathbf{N}}(.,s)\|^2_{L^2(\Omega)} ds.
\end{aligned} \tag{4.17}$$

Applying Gronwall's inequality, we get

$$e^{2t\alpha_{\mathbf{N}(\delta)}} \mathbf{E}\|\mathbf{u}^{\delta}_{\mathbf{N}(\delta)}(.,t) - \mathbf{v}^{\delta}_{\mathbf{N}(\delta)}(.,t)\|^2_{L^2(\Omega)} \le 2e^{2T\alpha_{\mathbf{N}(\delta)}} e^{2K^2 T(T-t)} \mathbf{E}\|\overline{G}_{\delta,\mathbf{N}(\delta)} - g\|^2_{L^2(\Omega)}. \tag{4.18}$$

Hence, using Lemma 4.1, we deduce that

$$\begin{aligned}
\mathbf{E}\|\mathbf{u}^{\delta}_{\mathbf{N}(\delta)}(.,t) - \mathbf{v}^{\delta}_{\mathbf{N}(\delta)}(.,t)\|^2_{L^2(\Omega)} &\le 2e^{2K^2 T(T-t)} e^{2(T-t)\alpha_{\mathbf{N}(\delta)}} \mathbf{E}\|\overline{G}_{\delta,\mathbf{N}(\delta)} - g\|^2_{L^2(\Omega)} \\
&\le 2e^{2K^2 T(T-t)} e^{2(T-t)\alpha_{\mathbf{N}(\delta)}} \Big(\delta^2 \mathbf{N}(\delta) + \frac{1}{\lambda^{2\gamma}_{\mathbf{N}(\delta)}} \|g\|_{H^{2\gamma}} \Big).
\end{aligned} \tag{4.19}$$

Now, we continue to estimate $\|\mathbf{u}(.,t)-\mathbf{v}^{\delta}_{\mathbf{N}(\delta)}(.,t)\|_{L^2(\Omega)}$. Indeed, using Hölder's inequality and globally Lipschitz property of F, we get

$$\begin{aligned}
&\|\mathbf{u}(.,t)-\mathbf{v}^{\delta}_{\mathbf{N}(\delta)}(.,t)\|^2_{L^2(\Omega)}\\
&\le 2\sum_{\lambda_j\le\alpha_{\mathbf{N}(\delta)}}\left[\int_t^T e^{(s-t)\lambda_j}\Big(F_j(\mathbf{u})(s)-F_j(\mathbf{v}^{\delta}_{\mathbf{N}(\delta)})(s)\Big)ds\right]^2+2\sum_{\lambda_j>\alpha_N}\Big\langle\mathbf{u}(t),\phi_j\Big\rangle^2\\
&\le 2\sum_{\lambda_j>\alpha_N}\lambda_j^{-2\beta}e^{-2t\lambda_j}\lambda_j^{2\beta}e^{2t\lambda_j}\Big\langle\mathbf{u}(t),\phi_j\Big\rangle^2+2K^2\int_t^T e^{2(s-t)\lambda_N}\|\mathbf{u}(.,s)-\mathbf{v}^{\delta}_{\mathbf{N}(\delta)}(.,s)\|^2_{L^2(\Omega)}ds\\
&\le \alpha_N^{-2\beta}e^{-2t\alpha_N}\sum_{j=1}^{\infty}\lambda_j^{2\beta}e^{2t\lambda_j}\Big\langle\mathbf{u}(t),\phi_j\Big\rangle^2+2K^2\int_t^T e^{2(s-t)\alpha_{\mathbf{N}(\delta)}}\|\mathbf{u}(.,s)-\mathbf{v}^{\delta}_{\mathbf{N}(\delta)}(.,s)\|^2_{L^2(\Omega)}ds;
\end{aligned}$$

above, we have used the mild solution of u as follows

$$\mathbf{u}(x,t)=\sum_{j=1}^{\infty}\left[e^{(T-t)\lambda_j}\Big\langle g,\phi_j\Big\rangle-\int_t^T e^{(s-t)\lambda_j}\langle F(\mathbf{u}(s)),\phi_j\rangle ds\right]\phi_j.$$

Multiplying both sides with $e^{2t\alpha_{\mathbf{N}(\delta)}}$, we obtain

$$\begin{aligned}
e^{2t\alpha_{\mathbf{N}(\delta)}}\|\mathbf{u}(.,t)-\mathbf{v}^{\delta}_{\mathbf{N}(\delta)}(.,t)\|^2_{L^2(\Omega)}&\le\alpha_{\mathbf{N}(\delta)}^{-2\beta}\sum_{j=1}^{\infty}\lambda_j^{2\beta}e^{2t\lambda_j}\Big\langle\mathbf{u}(.,t),\phi_j\Big\rangle^2\\
&\quad+2K^2\int_t^T e^{2s\alpha_{\mathbf{N}}}\|\mathbf{u}(.,s)-\mathbf{v}^{\delta}_{\mathbf{N}(\delta)}(.,s)\|^2_{L^2(\Omega)}ds.
\end{aligned}\tag{4.20}$$

Gronwall's inequality implies that

$$e^{2t\alpha_{\mathbf{N}}}\|\mathbf{u}(.,t)-\mathbf{v}^{\delta}_{\mathbf{N}(\delta)}(.,t)\|^2_{L^2(\Omega)}\le e^{2K^2(T-t)}\alpha_{\mathbf{N}(\delta)}^{-2\beta}A'.\tag{4.21}$$

This together with the estimate (4.19) leads to

$$\begin{aligned}
\mathbf{E}\|\mathbf{u}(.,t)-\mathbf{u}^{\delta}_{\mathbf{N}(\delta)}(.,t)\|^2_{L^2(\Omega)}&\le 2\mathbf{E}\|\mathbf{u}^{\delta}_{\mathbf{N}}(.,t)-\mathbf{v}^{\delta}_{\mathbf{N}(\delta)}(.,t)\|^2_{L^2(\Omega)}+2\|\mathbf{u}(.,t)-\mathbf{v}^{\delta}_{\mathbf{N}(\delta)}(.,t)\|^2_{L^2(\Omega)}\\
&\le 2e^{2K^2(T-t)\alpha_{\mathbf{N}}}\Big(\delta^2\mathbf{N}(\delta)+\frac{1}{\lambda^{2\gamma}_{\mathbf{N}(\delta)}}\|g\|_{H^{2\gamma}}\Big)+2\alpha_{\mathbf{N}(\delta)}^{-2\beta}e^{-2t\alpha_{\mathbf{N}}}e^{2K^2(T-t)}A'
\end{aligned}\tag{4.22}$$

where A' is given in Eq (4.10). This completes our proof. □

The next theorem provides an error estimate in the Sobolev space $H^p(\Omega)$ which is equipped with a norm defined by

$$\|g\|^2_{H^p(\Omega)}=\sum_{j=1}^{\infty}\lambda_j^p\Big\langle g,\phi_j(x)\Big\rangle^2.\tag{4.23}$$

To estimate the error in H^p norm, we need a stronger assumption of the solution u.

Theorem 4.2. *Assume that problem* (1.1) *has unique solution* $\mathbf{u}$ *such that*

$$\sum_{j=1}^{\infty} e^{2(t+r)\lambda_j} \left\langle \mathbf{u}(.,t), \phi_j \right\rangle^2 < A", \quad t \in [0,T]. \tag{4.24}$$

for any $r > 0$*. Choose* $\alpha_{\mathbf{N}(\delta)}$ *such that*

$$\lim_{\delta\to 0} \alpha_{\mathbf{N}(\delta)} = +\infty, \ \lim_{\delta\to 0} \frac{e^{kT\alpha_{\mathbf{N}(\delta)}}}{\lambda^{\gamma}_{\mathbf{N}(\delta)}} = 0, \ \ \lim_{\delta\to 0} e^{kT\alpha_{\mathbf{N}(\delta)}} \sqrt{\mathbf{N}(\delta)}\delta = 0 \tag{4.25}$$

Then the following estimate holds

$$\mathbf{E}\|\mathbf{u}^{\delta}_{\mathbf{N}(\delta)}(.,t) - \mathbf{u}(.,t)\|^2_{H^p(\Omega)} \tag{4.26}$$

$$\leq 2e^{2k^2T(T-t)} e^{-2t\alpha_{\mathbf{N}}} |\alpha_{\mathbf{N}(\delta)}|^p \left[2\delta^2 \mathbf{N}(\delta) e^{2T\alpha_{\mathbf{N}(\delta)}} + 2\frac{e^{2T\alpha_{\mathbf{N}(\delta)}}}{\lambda^{2\gamma}_{\mathbf{N}(\delta)}} \|g\|_{H^{2\gamma}} + A" e^{-2r\alpha_{\mathbf{N}_\delta}} \right]$$

$$+ A"|\alpha_{\mathbf{N}(\delta)}|^p \exp\Big(-2(t+r)\alpha_{\mathbf{N}(\delta)} \Big). \tag{4.27}$$

Proof. First, we have

$$\begin{aligned}
\mathbf{E}\|\mathbf{u}^{\delta}_{\mathbf{N}(\delta)}(.,t) - \mathbf{J}_{\alpha_{\mathbf{N}(\delta)}}\mathbf{u}(.,t)\|^2_{H^p(\Omega)} &= \mathbf{E}\left(\sum_{\lambda_j \leq \alpha_{\mathbf{N}(\delta)}} \lambda_j^p \left\langle \mathbf{u}^{\delta}_{\mathbf{N}(\delta)}(x,t) - \mathbf{u}(x,t), \phi_j(x) \right\rangle^2 \right) \\
&\leq |\alpha_{\mathbf{N}(\delta)}|^p \mathbf{E}\left(\sum_{\lambda_j \leq \alpha_{\mathbf{N}(\delta)}} \left\langle \mathbf{u}^{\delta}_{\mathbf{N}(\delta)}(x,t) - \mathbf{u}(x,t), \phi_j(x) \right\rangle^2 \right) \\
&\leq |\alpha_{\mathbf{N}(\delta)}|^p \mathbf{E}\|\mathbf{u}^{\delta}_{\mathbf{N}(\delta)}(.,t) - \mathbf{u}(.,t)\|^2_{L^2(\Omega)}.
\end{aligned} \tag{4.28}$$

Next, we continue to estimate $\mathbf{E}\|\mathbf{u}^{\delta}_{\mathbf{N}(\delta)}(.,t) - \mathbf{u}(.,t)\|^2_{L^2(\Omega)}$ with assumption (4.24). Recall $\mathbf{v}^{\delta}_{\mathbf{N}(\delta)}$ from (4.14). The expectation of the error between $\mathbf{u}^{\delta}_{\mathbf{N}(\delta)}$ and $\mathbf{v}^{\delta}_{\mathbf{N}(\delta)}$ is given in the estimate (4.19) as

$$\mathbf{E}\|\mathbf{u}^{\delta}_{\mathbf{N}(\delta)}(.,t) - \mathbf{v}^{\delta}_{\mathbf{N}(\delta)}(.,t)\|^2_{L^2(\Omega)} \leq 2e^{2K^2T(T-t)} e^{2(T-t)\alpha_{\mathbf{N}(\delta)}} \Big(\delta^2 \mathbf{N}(\delta) + \frac{1}{\lambda^{2\gamma}_{\mathbf{N}(\delta)}} \|g\|_{H^{2\gamma}} \Big). \tag{4.29}$$

We only need to estimate $\|\mathbf{u}(.,t) - \mathbf{v}^{\delta}_{\mathbf{N}(\delta)}(.,t)\|_{L^2(\Omega)}$. Indeed, using Hölder's inequality and the globally Lipschitz property of F, we get

$$\begin{aligned}
&\|\mathbf{u}(.,t) - \mathbf{v}^{\delta}_{\mathbf{N}(\delta)}(.,t)\|^2_{L^2(\Omega)} \\
&\leq 2\sum_{\lambda_j > \alpha_N} \left\langle \mathbf{u}(t), \phi_j \right\rangle^2 + 2\sum_{\lambda_j \leq \alpha_{\mathbf{N}(\delta)}} \left[\int_t^T e^{(s-t)\lambda_j} \Big(F_j(\mathbf{u})(s) - F_j(\mathbf{v}^{\delta}_{\mathbf{N}(\delta)})(s) \Big) ds \right]^2 \\
&\leq 2\sum_{\lambda_j > \alpha_N} e^{-2(t+r)\lambda_j} e^{2(t+r)\lambda_j} \left\langle \mathbf{u}(t), \phi_j \right\rangle^2 + 2K^2T \int_t^T e^{-2(s-t)\alpha_{\mathbf{N}_\delta}} \|\mathbf{u}(.,s) - \mathbf{v}^{\delta}_{\mathbf{N}(\delta)}(.,s)\|^2_{L^2(\Omega)} ds \\
&\leq e^{-2(t+r)\alpha_{\mathbf{N}_\delta}} \sum_{j=1}^{\infty} e^{2(t+r)\lambda_j} \left\langle \mathbf{u}(t), \phi_j \right\rangle^2 + 2K^2T \int_t^T e^{2(s-t)\alpha_{\mathbf{N}(\delta)}} \|\mathbf{u}(.,s) - \mathbf{v}^{\delta}_{\mathbf{N}(\delta)}(.,s)\|^2_{L^2(\Omega)} ds.
\end{aligned}$$

Multiplying both sides with $e^{2t\alpha_{\mathbf{N}(\delta)}}$, we obtain

$$e^{2t\alpha_{\mathbf{N}(\delta)}} \|\mathbf{u}(.,t) - \mathbf{v}^{\delta}_{\mathbf{N}(\delta)}(.,t)\|^2_{L^2(\Omega)} \leq A" e^{-2r\alpha_{\mathbf{N}_\delta}}$$

$$+ 2K^2T\int_t^T e^{2s\alpha_{\mathbf{N}(\delta)}}\|\mathbf{u}(.,s)-\mathbf{v}^{\delta}_{\mathbf{N}(\delta)}(.,s)\|^2_{L^2(\Omega)}ds. \tag{4.30}$$

Gronwall's inequality implies that

$$e^{2t\alpha_{\mathbf{N}}}\|\mathbf{u}(.,t)-\mathbf{v}^{\delta}_{\mathbf{N}(\delta)}(.,t)\|^2_{L^2(\Omega)} \le e^{2K^2T(T-t)}A\text{"}e^{-2r\alpha_{\mathbf{N}_\delta}}. \tag{4.31}$$

This last estimate together with the estimate (4.29) leads to

$$\begin{aligned}
&\mathbf{E}\|\mathbf{u}(.,t)-\mathbf{u}^{\delta}_{\mathbf{N}(\delta)}(.,t)\|^2_{L^2(\Omega)}\\
&\le 2\mathbf{E}\|\mathbf{u}^{\delta}_{\mathbf{N}}(.,t)-\mathbf{v}^{\delta}_{\mathbf{N}(\delta)}(.,t)\|^2_{L^2(\Omega)} + 2\|\mathbf{u}(.,t)-\mathbf{v}^{\delta}_{\mathbf{N}(\delta)}(.,t)\|^2_{L^2(\Omega)}\\
&\le 4e^{2K^2T(T-t)}e^{2(T-t)\alpha_{\mathbf{N}(\delta)}}\Big(\delta^2\mathbf{N}(\delta)+\frac{1}{\lambda^{2\gamma}_{\mathbf{N}(\delta)}}\|g\|_{H^{2\gamma}}\Big) + 2e^{2K^2T(T-t)}A\text{"}e^{-2t\alpha_{\mathbf{N}}}e^{-2r\alpha_{\mathbf{N}_\delta}}\\
&= 2e^{2K^2T(T-t)}e^{-2t\alpha_{\mathbf{N}}}\left[2\delta^2\mathbf{N}(\delta)e^{2T\alpha_{\mathbf{N}(\delta)}} + 2\frac{e^{2T\alpha_{\mathbf{N}(\delta)}}}{\lambda^{2\gamma}_{\mathbf{N}(\delta)}}\|g\|_{H^{2\gamma}} + A\text{"}e^{-2r\alpha_{\mathbf{N}_\delta}}\right].
\end{aligned} \tag{4.32}$$

On the other hand, consider the function

$$G(\xi) = \xi^p e^{-D\xi}, \quad D>0. \tag{4.33}$$

The derivative of G is $G'(\xi) = \xi^{p-1}e^{-D\xi}(p-D\xi)$. Hence we know that G is strictly decreasing when $D\xi \ge p$. Since $\lim_{\delta\to 0}\alpha_{\mathbf{N}(\delta)} = +\infty$, we see that if δ is small enough then $2r\alpha_{\mathbf{N}(\delta)} \ge p$. Put $D = 2(t+r)$, $\xi = \alpha_{\mathbf{N}(\delta)}$ into (4.33), and we obtain for $\lambda_j > \alpha_{\mathbf{N}(\delta)}$

$$G(\lambda_j) = \lambda_j^p \exp\Big(-2(t+r)\lambda_j\Big) \le G(\alpha_{\mathbf{N}(\delta)}) = |\alpha_{\mathbf{N}(\delta)}|^p\exp\Big(-2(t+r)\alpha_{\mathbf{N}(\delta)}\Big).$$

The latter equality leads to

$$\begin{aligned}
\|\mathbf{u}(.,t)-\mathbf{J}_{\alpha_{\mathbf{N}(\delta)}}\mathbf{u}(.,t)\|^2_{H^p(\Omega)} &= \sum_{\lambda_j>\alpha_{\mathbf{N}(\delta)}}\lambda_j^p\Big\langle\mathbf{u}(x,t),\phi_j(x)\Big\rangle^2\\
&= \sum_{\lambda_j>\alpha_{\mathbf{N}(\delta)}}\lambda_j^p\exp\Big(-2(t+r)\lambda_j\Big)\exp\Big(2(t+r)\lambda_j\Big)\Big\langle\mathbf{u}(x,t),\phi_j(x)\Big\rangle^2\\
&\le |\alpha_{\mathbf{N}(\delta)}|^p\exp\Big(-2(t+r)\alpha_{\mathbf{N}(\delta)}\Big)\sum_{\lambda_j>\alpha_{\mathbf{N}(\delta)}}\exp\Big(2(t+r)\lambda_j\Big)\Big\langle\mathbf{u}(x,t),\phi_j(x)\Big\rangle^2\\
&\le A\text{"}|\alpha_{\mathbf{N}(\delta)}|^p\exp\Big(-2(t+r)\alpha_{\mathbf{N}(\delta)}\Big)
\end{aligned} \tag{4.34}$$

where we use assumption (4.24) for the last inequality. Combining (4.28), (4.32) and (4.34), and we deduce that

$$\begin{aligned}
&\mathbf{E}\|\mathbf{u}^{\delta}_{\mathbf{N}(\delta)}(.,t)-\mathbf{u}(.,t)\|^2_{H^p(\Omega)}\\
&\le \mathbf{E}\|\mathbf{u}^{\delta}_{\mathbf{N}(\delta)}(.,t)-\mathbf{J}_{\alpha_{\mathbf{N}(\delta)}}\mathbf{u}(.,t)\|^2_{H^p(\Omega)} + \|\mathbf{u}(.,t)-\mathbf{J}_{\alpha_{\mathbf{N}(\delta)}}\mathbf{u}(.,t)\|^2_{H^p(\Omega)}\\
&\le 2e^{2K^2T(T-t)}e^{-2t\alpha_{\mathbf{N}}}|\alpha_{\mathbf{N}(\delta)}|^p\left[2\delta^2\mathbf{N}(\delta)e^{2T\alpha_{\mathbf{N}(\delta)}} + 2\frac{e^{2T\alpha_{\mathbf{N}(\delta)}}}{\lambda^{2\gamma}_{\mathbf{N}(\delta)}}\|g\|_{H^{2\gamma}} + A\text{"}e^{-2r\alpha_{\mathbf{N}_\delta}}\right]\\
&+ A\text{"}|\alpha_{\mathbf{N}(\delta)}|^p\exp\Big(-2(t+r)\alpha_{\mathbf{N}(\delta)}\Big)
\end{aligned} \tag{4.35}$$

which completes the proof. □

Remark 4.3. *In the above Theorem, to obtain the error estimate, we require strong assumptions on* **u**. *This is a limitation of Theorem 4.1, because there are only certain types of functions* **u** *satisfying these conditions. To remove this limitation, we need to find a new estimator. The convergence rate in the case of weak assumptions of u is a difficult problem. Indeed, in the next Theorem, we give a regularization result in the case of a weaker assumption for u, i.e.,* $u \in C([0, T]; L^2(\Omega))$. *This is one of the first results in this case.*

4.1. The second regularized solution and the error estimate

To help the reader, we describe our analysis and methods in this subsection. To obtain the approximate solution when the solution u is in $C([0, T]; L^2(\Omega))$, we don't use a regularized solution as in Theorem 4.1. Since $\overline{G}_{\delta,\mathbf{N}(\delta)}$ is an approximation of G, we know that it is an observed data. It can also be called the "input data". Recall that K is the Lipschitz constant of F. We divide our results in Theorem 4.3 into two cases:
Case 1: $KT < 1$. By the way the input data $\overline{G}_{\delta,\mathbf{N}(\delta)}$ is defined, we construct a new regularized solution. Then we obtain the error between the new regularized solution and the sought solution **u**.
Case 2: $KT > 1$. In this case, the construction of the regularized solution is more difficult. To apply the known result in Case 1, we need to divide $[0, T]$ into a collection of sub intervals $[T_h, T_{h'}]$ where $K(T_{h'} - T_h) < 1$. From the given input data θ and appropriate parameter regularization ζ, we set the output function $\mathbf{Y}^{\zeta}_{T_h,T_{h'}}(f)(\mathbf{x}, t)$ satisfies the nonlinear integral equation (4.37). The existence of $\mathbf{Y}^{\zeta}_{T_h,T_{h'}}(f)$ in $C([T_h, T_{h'}]; L^2(\Omega))$ holds if $K(T_{h'} - T_h) < 1$. From (4.56), we have an important result: If ζ is suitably chosen and θ is an approximate function of $u(x, T_{h'})$ then the function $\mathbf{Y}^{\zeta}_{T_h,T_{h'}}(f)(x, t)$ is an approximate solution of the sought solution u in all intervals $[T_h, T_{h'}]$. Let s be a positive integer such that $s > KT$. Define a sequence of points $\{T_l\}$, $l = 0, 1, ...2s$ such that

$$T_0 = 0 < T_1 = \overline{h}T < T_2 = 2\overline{h}T < ... < T_{2s} = 2s\overline{h}T = T. \tag{4.36}$$

where $\overline{h} = \frac{1}{2s}$. In all the intervals $[T_i, T_{i+1}]$, $i = \overline{0, 2s - 1}$, we construct different regularized solutions and combine them into a final regularized solution. More details are as follows:

- In the first step, to construct an approximate solution on $[T_{2s-1}, T]$, we use the input data $\overline{G}_{\delta,\mathbf{N}(\delta)}$ and parameter regularization ζ_{2s} to establish a function $\mathbf{Y}^{\zeta_{2s}}_{T_{2s-2},T_{2s}}\left(\overline{G}_{\delta,\mathbf{N}(\delta)}\right)(x, t)$. Then we define a regularized solution $\mathcal{U}_\delta(x, t) = \mathbf{Y}^{\zeta_{2s}}_{T_{2s-2},T_{2s}}\left(\overline{G}_{\delta,\mathbf{N}(\delta)}\right)(x, t)$ for all $t \in [T_{2s-1}, T]$.
- In the second step, to construct an approximate solution on $[T_{2s-2}, T_{2s-1}]$, we use the input data $\mathcal{U}_n(x, T_{2s-1})$ (which is computed in the first step) and parameter regularization ζ_{2s-1} to establish a function $\mathbf{Y}^{\zeta_{2s}}_{T_{2s-2},T_{2s}}\left(\overline{G}_{\delta,\mathbf{N}(\delta)}\right)(x, t)$. Then we define a regularized solution

$$\mathcal{U}_\delta(x, t) = \mathbf{Y}^{\zeta_{2s-1}}_{T_{2s-3},T_{2s-1}}\left(\mathcal{U}_\delta(x, T_{2s-1})\right)(x, t)$$

  for all $t \in [T_{2s-2}, T_{2s-1}]$.
- We continue similarly for the remaining steps. Finally, we obtain the regularized solution in (4.61) and (4.62).

Now, we consider the following lemma.

Lemma 4.2. *Let* $0 \le T_h < T_{h'} \le T$. *For* $f \in C([T_h, T_{h'}]; L^2(\Omega))$, *we consider the following nonlinear* *integral equation*

$$\begin{aligned}
&\mathbf{Y}^{\zeta}_{T_h,T_{h'}}(f)(x,t)\\
&=\sum_{\lambda_j\le\zeta}\left[e^{(T_{h'}-t)\lambda_j}\left\langle f,\phi_j\right\rangle-\int_t^{T_{h'}}e^{(\tau-t)\lambda_j}\langle F(\mathbf{Y}^{\zeta}_{T_h,T_{h'}}(f)(\tau),\phi_j\rangle d\tau\right]\phi_j\\
&+\sum_{\lambda_j>\zeta}\left[\int_{T_h}^{t}e^{(\tau-t)\lambda_j}F_j\langle F(\mathbf{Y}^{\zeta}_{T_h,T_{h'}}(f)(\tau),\phi_j\rangle d\tau\right]\phi_j(x).
\end{aligned}\tag{4.37}$$

for $\zeta>0$*. Assume that* $K(T_{h'}-T_h)<1$*. Then Problem* (4.37) *has a unique solution* $\mathbf{Y}^{\zeta}_{T_h,T_{h'}}(f)\in C([T_h,T_{h'}];L^2(\Omega))$*. Moreover, we have the following estimate*

$$\begin{aligned}
&\mathbf{E}\|\mathbf{Y}^{\zeta}_{T_h,T_{h'}}(f)(.,t)-\mathbf{u}(.,t)\|^2_{L^2(\Omega)}\\
&\le\frac{2\left(1+\frac{1}{q_0}\right)e^{2(T_h-t)}}{1-(1+q_0)K^2(T_{h'}-T_h)^2}\left(e^{2(T_{h'}-T_h)\zeta}\mathbf{E}\left\|f-\mathbf{u}(.,T_{h'})\right\|^2_{L^2(\Omega)}+\|\mathbf{u}(.,T_{h'})\|^2_{L^2(\Omega)}\right).
\end{aligned}\tag{4.38}$$

for all $t\in[T_h,T_{h'}]$ *and* q_0 *satisfies* $0<q_0<\frac{1}{K^2(T_{h'}-T_h)^2}-1$.

Proof. **Part A.** We will begin by showing that Eq (4.37) has a unique solution in $C([T_h,T_{h'}];L^2(\Omega))$. Our analysis here is similar to the one in [29]. Define on $C([T_h,T_{h'}];L^2(\Omega))$ the following Bielecki norm

$$\|v\|_1=\sup_{T_h\le t\le T_{h'}}e^{(t-T_h)\zeta(\delta)}\|v(t)\|,\tag{4.39}$$

for all $v\in C([T_h,T_{h'}];L^2(\Omega))$. It is easy to check that $\|.\|_1$ is a norm of $C([T_h,T_{h'}];L^2(\Omega))$. Now, let f be in $L^2(\Omega)$. We want to show that the map given by

$$\begin{aligned}
\mathcal{I}(w(f))(x,t)&=\sum_{\lambda_j\le\zeta}\left[e^{(T_{h'}-t)\lambda_j}\left\langle f,\phi_j\right\rangle-\int_t^{T_{h'}}e^{(\tau-t)\lambda_j}\langle F(w(f)(\tau),\phi_j\rangle d\tau\right]\phi_j\\
&+\sum_{\lambda_j>\zeta}\left[\int_{T_h}^{t}e^{(\tau-t)\lambda_j}\langle F(w(f)(\tau),\phi_j\rangle d\tau\right]\phi_j(x),
\end{aligned}\tag{4.40}$$

for $w(f)\in C([T_h,T_{h'}];L^2(\Omega))$, is a contraction on $C([T_h,T_{h'}];L^2(\Omega))$ with the condition $K(T_{h'}-T_h)<1$. Indeed, we shall prove that, for every $w_1,w_2\in C([T_h,T_{h'}];L^2(\Omega))$,

$$\left\|\mathcal{I}(w_1(f))-\mathcal{I}(w_2(f))\right\|_1\le K(T_{h'}-T_h)\,\|w_1(f)-w_2(f)\|_1\,.\tag{4.41}$$

First, by using the Hölder inequality and the global Lipschitz property of F, we have the following estimates for all $t \in [T_{h_1}, T_{h_2}]$, namely

$$\sum_{\lambda_j \le \zeta} \left(\int_t^{T_{h'}} e^{(\tau-t)\lambda_j} \Big[F_j(w_1(f))(\tau) - F_j(w_2(f))(\tau) \Big] d\tau \right)^2$$

$$\le (T_{h'} - t) \sum_{\lambda_j \le \zeta} \int_t^{T_{h'}} \left| e^{(\tau-t)\lambda_j} \Big[F_j(w_1(f))(\tau) - F_j(w_2)(f)(\tau) \Big] \right|^2 d\tau$$

$$\le (T_{h'} - t) \sum_{\lambda_j \le \zeta} \int_t^{T_{h'}} e^{2(\tau-t)\zeta} \Big[F_j(w_1(f))(\tau) - F_j(w_2)(f)(\tau) \Big]^2 d\tau$$

$$\le K^2 (T_{h'} - t) \int_t^{T_{h'}} e^{2(\tau-t)\zeta} \|w_1(f)(\tau) - w_2(f)(\tau)\|^2 d\tau$$

$$\le e^{-2(t-T_h)\zeta} K^2 (T_{h'} - t)^2 \sup_{T_h \le \tau \le T_{h'}} e^{2(\tau-T_h)\zeta} \|w_1(f)(\tau) - w_2(f)(\tau)\|^2$$

$$= e^{-2(t-T_h)\zeta} K^2 (T_{h'} - t)^2 \|w_1(f) - w_2(f)\|_1^2.$$

Noting that if $\lambda_j > \zeta$ then $e^{(\tau-t)\lambda_j} \le e^{(\tau-t)\zeta}$ for $T_h \le \tau \le t$, it follows that

$$\sum_{\lambda_j > \zeta} \left(\int_{T_h}^{t} e^{(\tau-t)\lambda_j} j \Big[F_j(w_1(f))(\tau) - F_j(w_2(f))(\tau) \Big] d\tau \right)^2$$

$$\le (t - T_h) \sum_{\lambda_j > \zeta} \int_{T_h}^{t} \left| e^{(\tau-t)\zeta} \Big[F_j(w_1(f))(\tau) - F_{\mathbf{p}}(w_2(f))(\tau) \Big] \right|^2 d\tau$$

$$\le (t - T_h) \sum_{\lambda_j > \zeta} \int_{T_h}^{t} e^{2(\tau-t)\zeta} \Big| F_j(w_1(f))(\tau) - F_j(w_2(f))(\tau) \Big|^2 d\tau$$

$$\le K^2 (t - T_h) \int_{T_h}^{t} e^{2(\tau-t)\zeta} \|w_1(f)(\tau) - w_2(f)(\tau)\|^2 d\tau$$

$$\le e^{-2(t-T_h)\zeta} K^2 (t - T_h)^2 \sup_{0 \le \tau \le T} e^{2(\tau-T_h)\zeta} \|w_1(f)(\tau) - w_2(f)(\tau)\|^2$$

$$= e^{-2(t-T_h)\zeta} K^2 (t - T_h)^2 \|w_1(f) - w_2(f)\|_1^2.$$

From the definition of $\mathcal{I}$ in (4.40), we have

$$\mathcal{I}(w_1(f))(x,t) - \mathcal{I}(w_2(f))(x,t)$$
$$= \sum_{\lambda_j \le \zeta} \left(\int_t^{T_{h'}} e^{(\tau-t)\lambda_j} \Big[F_j(w_1(f))(\tau) - F_j(w_2(f))(\tau) \Big] d\tau \right) \phi_j(x)$$
$$+ \sum_{\lambda_j > \zeta} \left(\int_{T_h}^{t} e^{(\tau-t)\lambda_j} \Big[F_j(w_1(f))(\tau) - F_j(w_2(f))(\tau) \Big] d\tau \right) \phi_j(x).$$

Combining (4.42), (4.42), (4.42) and using the inequality $(a+b)^2 \le (1+\theta_0)a^2 + \left(1 + \frac{1}{\theta_0}\right) b^2$ for any real numbers a, b and $\theta_0 > 0$, we get the following estimate for all $t \in (T_h, T_{h'})$

$$\|\mathcal{I}(w_1(f))(.,t) - \mathcal{I}(w_2(f))(.,t)\|^2$$
$$\le e^{-2(t-T_h)\zeta} K^2 (t - T_h)^2 (1 + \theta_0) \|w_1(f) - w_2(f)\|_1^2$$
$$+ e^{-2(t-T_h)\zeta} \left(1 + \frac{1}{\theta_0} \right) K^2 (T_{h'} - t)^2 \|w_1(f) - w_2(f)\|_1^2.$$

By choosing $\theta_0 = \frac{T_{h'}-t}{t}$, we obtain for all $t \in (T_h, T_{h'})$

$$e^{2(t-T_h)\zeta}\|\mathcal{I}(w_1(f))(.,t) - \mathcal{I}(w_2(f))(.,t)\|^2 \leq K^2(T_{h'} - T_h)^2\|w_1(f) - w_2(f)\|_1^2. \tag{4.42}$$

On the other hand, letting $t = T_{h'}$ in (4.42), we get

$$e^{2(T_{h'}-T_h)\zeta}\|\mathcal{I}(w_1(f))(.,T_h) - \mathcal{I}(w_2(f))(.,T_{h'})\|^2 \leq K^2(T_{h'} - T_h)^2\|w_1(f) - w_2(f)\|_1^2. \tag{4.43}$$

By letting $t = T_h$ in (4.42), we obtain

$$\|\mathcal{I}(w_1(f))(.,T_h) - \mathcal{I}(w_2(f))(.,T_h)\|^2 \leq K^2(T_{h'} - T_h)^2\|w_1(f) - w_2(f)\|_1^2. \tag{4.44}$$

Combining (4.42), (4.43) and (4.44), we deduce that for all $T_h \leq t \leq T_{h'}$

$$e^{2(t-T_h)\zeta}\|\mathcal{I}(w_1(f))(t) - \mathcal{I}(w_2(f))(t)\| \leq K(T_{h'} - T_h)\|w_1(f) - w_2(f)\|_1, \tag{4.45}$$

which leads to (4.41). Since $K(T_{h'} - T_h) < 1$, it follows that $\mathcal{I}$ is a well-defined contraction on $C([T_h, T_{h'}]; L^2(\Omega))$. By the Banach fixed point theorem, it therefore has a unique fixed point, i.e., the equation $\mathcal{I}(w) = w$ has a unique solution which we denote by $\mathbf{Y}^{\zeta}_{T_h,T_{h'}}(f) \in C([T_h, T_{h'}]; L^2(\Omega))$.

Part B. The error estimate $\mathbf{E}\|\mathbf{Y}^{\zeta}_{T_h,T_{h'}}(f)(.,t) - \mathbf{u}(.,t)\|^2_{L^2(\Omega)}$.

By a similar technique as in the proof of Theorem 4.1, we obtain

$$\mathbf{u}_j(T_h) = e^{(T_{h'}-T_h)\lambda_j}\mathbf{u}_j(T_{h'}) - \int_{T_h}^{T_{h'}} e^{(\tau-t)\lambda_j}F_j(\mathbf{u})(\tau)d\tau. \tag{4.46}$$

This leads to

$$e^{-(t-T_h)\lambda_j}\mathbf{u}_j(T_h) = e^{(T_{h'}-t)\lambda_j}\left[\mathbf{u}_j(T_{h'}) - \int_{T_h}^{T_{h'}} e^{(s-T_{h'})\lambda_j}F_j(\mathbf{u})(\tau)d\tau\right]. \tag{4.47}$$

The last equality implies that after some simple transformation

$$\begin{aligned}\sum_{\lambda_j>\zeta} & e^{(T_{h'}-t)\lambda_j}\left[\mathbf{u}_j(T_{h'}) - \int_t^{T_{h'}} e^{(s-T_{h'})\lambda_j}F_j(\mathbf{u})(\tau)d\tau\right]\phi_j(x) \\ &= \sum_{\lambda_j>\zeta}\left[\int_{T_h}^t e^{(\tau-t)\lambda_j}F_j(\mathbf{u})(\tau)d\tau\right]\phi_j(x) + \sum_{\lambda_j>\zeta} e^{-(t-T_h)\lambda_j}\mathbf{u}_j(T_h)\phi_j(x).\end{aligned} \tag{4.48}$$

Using the last equality and (4.47), we get

$$\begin{aligned}\mathbf{u}(x,t) &= \sum_{\lambda_j\leq\zeta}\left[e^{(T_{h'}-t)\lambda_j}\mathbf{u}_j(T_{h'}) - \int_t^{T_{h'}} e^{(\tau-t)\lambda_j}F_j(\mathbf{u})(\tau)d\tau\right]\phi_j(x) \\ &\quad + \sum_{\lambda_j\leq\zeta}\left[\int_{T_h}^t e^{(\tau-t)\lambda_j}F_j(\mathbf{u})(\tau)d\tau\right]\phi_j(x) + \sum_{\lambda_j>\zeta} e^{-(t-T_h)\lambda_j}\mathbf{u}_j(T_h)\phi_j(x).\end{aligned} \tag{4.49}$$

We have

$$\mathbf{Y}^{\zeta}_{T_h,T_{h'}}(f)(x,t) - \mathbf{u}(x,t) = \sum_{\lambda_j\leq\zeta}\left[e^{(T_{h'}-t)\zeta}\left(f_j - \mathbf{u}_j(T_{h'})\right)\right]\phi_j(x)$$

$$
\begin{aligned}
&-\sum_{\lambda_j\le\zeta}\left[\int_t^{T_{h'}} e^{(\tau-t)\lambda_j}\Big(F_j\left(\mathbf{Y}^{\zeta}_{T_h,T_{h'}}(f)\right)(\tau)-F_j(\mathbf{u})(\tau)\Big)d\tau\right]\phi_j(x)\\
&+\sum_{\lambda_j>\zeta}\left[\int_{T_h}^{t} e^{(\tau-t)\lambda_j}\Big(F_j\left(\mathbf{Y}^{\zeta}_{T_h,T_{h'}}(f)\right)(\tau)-F_j(\mathbf{u})(\tau)\Big)d\tau\right]\phi_j(x)\\
&-\sum_{\lambda_j>\zeta} e^{-(t-T_h)\lambda_j}\mathbf{u}_j(T_h)\phi_j(x).
\end{aligned}\tag{4.50}
$$

This implies that

$$
\begin{aligned}
&\left|\left\langle \mathbf{Y}^{\zeta}_{T_h,T_{h'}}(f)(.,t)-\mathbf{u}(.,t),\phi_j(.)\right\rangle\right|\\
&\le e^{(T_{h'}-t)\zeta}\left|\Big(f_j-\mathbf{u}_j(T_{h'})\Big)\right|+\int_{T_h}^{T_{h'}} e^{(\tau-t)\zeta}\left|F_j\left(\mathbf{Y}^{\zeta}_{T_h,T_{h'}}(f)\right)(\tau)-F_j(\mathbf{u})(\tau)\right|d\tau\\
&+e^{-(t-T_h)\zeta}\left|\mathbf{u}_j(T_h)\right|.
\end{aligned}\tag{4.51}
$$

Hence, using Parseval's identity and the inequality

$$
(c_1+c_2+c_3)^2\le 2\left(1+\frac{1}{q_0}\right)c_1^2+2\left(1+\frac{1}{q_0}\right)c_2^2+(1+q_0)c_3^2
$$

for any real numbers c_1, c_2, c_3 and $q_0>0$ we have

$$
\begin{aligned}
&\mathbf{E}\|\mathbf{Y}^{\zeta}_{T_h,T_{h'}}(f)(.,t)-\mathbf{u}(.,t)\|^2_{L^2(\Omega)}=\mathbf{E}\left(\sum_{j=1}^{\infty}\left|\left\langle \mathbf{Y}^{\zeta}_{T_h,T_{h'}}(f)(.,t)-\mathbf{u}(.,t),\phi_j(x)\right\rangle\right|^2\right)\\
&\le 2\left(1+\frac{1}{q_0}\right)\mathbf{E}\left(\sum_{j=1}^{\infty} e^{(T_{h'}-t)\zeta}\left|\Big(f_j-\mathbf{u}_j(T_{h'})\Big)\right|^2\right)\\
&+(1+q_0)\mathbf{E}\left((T_{h'}-T_h)\sum_{j=1}^{\infty}\int_{T_h}^{T_{h'}} e^{2(\tau-t)\zeta}\left|F_j\left(\mathbf{Y}^{\zeta}_{T_h,T_{h'}}(f)\right)(\tau)-F_j(\mathbf{u})(\tau)\right|d\tau\right)\\
&+2\left(1+\frac{1}{q_0}\right)\sum_{j=1}^{\infty} e^{-2(t-T_h)\zeta}\left\|\mathbf{u}_j(T_h)\right\|^2\\
&\le 2\left(1+\frac{1}{q_0}\right)e^{2(T_{h'}-t)\zeta}\mathbf{E}\left\|f-\mathbf{u}(.,T_{h'})\right\|^2_{L^2(\Omega)}\\
&+2\left(1+\frac{1}{q_0}\right)e^{-2(t-T_h)\zeta}\|\mathbf{u}(.,T_h)\|^2_{L^2(\Omega)}\\
&+(1+q_0)(T_{h'}-T_h)\int_{T_h}^{T_{h'}} e^{2(\tau-t)\zeta}\mathbf{E}\left(\left\|F\left(\mathbf{Y}^{\zeta}_{T_h,T_{h'}}(f)\right)(\tau)-F(\mathbf{u})(\tau)\right\|^2_{L^2(\Omega)}\right)d\tau.
\end{aligned}
$$

Multiplying both sides of the last inequality by $e^{2(t-T_h)\zeta}$, and using the global Lipschitz property of F, we obtain

$$
\begin{aligned}
&e^{2(t-T_h)\zeta}\mathbf{E}\|\mathbf{Y}^{\zeta}_{T_h,T_{h'}}(f)(.,t)-\mathbf{u}(.,t)\|^2_{L^2(\Omega)}\\
&\le 2\left(1+\frac{1}{q_0}\right)e^{2(T_{h'}-T_h)\zeta}\mathbf{E}\left\|f-\mathbf{u}(.,T_{h'})\right\|^2_{L^2(\Omega)}+2\left(1+\frac{1}{q_0}\right)\|\mathbf{u}(.,T_h)\|^2_{L^2(\Omega)}
\end{aligned}
$$

$$+ (1+q_0)K^2(T_{h'}-T_h)\int_{T_h}^{T_{h'}} e^{2(\tau-T_h)\zeta}\mathbf{E}\|\mathbf{Y}^{\zeta}_{T_h,T_{h'}}(f)(.,s)-\mathbf{u}(.,s)\|^2_{L^2(\Omega)}ds. \tag{4.52}$$

Since $\mathbf{Y}^{\zeta}_{T_h,T_{h'}}(f),\ \mathbf{u}\in C([T_h,T_{h'}];L^2(\Omega))$ we obtain that the function

$$e^{2(t-T_h)\zeta}\mathbf{E}\|\mathbf{Y}^{\zeta}_{T_h,T_{h'}}(f)(.,t)-\mathbf{u}(.,t)\|^2_{L^2(\Omega)}$$

is continuous on $[T_h,T_{h'}]$. Therefore, the following is a finite positive constant

$$\widetilde{A}=\sup_{T_h\le t\le T_{h'}} e^{2(t-T_h)\zeta}\mathbf{E}\|\mathbf{Y}^{\zeta}_{T_h,T_{h'}}(f)(.,t)-\mathbf{u}(.,t)\|^2_{L^2(\Omega)}.$$

This implies that

$$\begin{aligned}\widetilde{A}&\le 2\left(1+\frac{1}{q_0}\right)e^{2(T_{h'}-T_h)\zeta}\mathbf{E}\Big\|f-\mathbf{u}(.,T_{h'})\Big\|^2_{L^2(\Omega)}+2\left(1+\frac{1}{q_0}\right)\|\mathbf{u}(.,T_h)\|^2_{L^2(\Omega)}\\&\quad+(1+q_0)K^2(T_{h'}-T_h)^2\widetilde{A}\end{aligned} \tag{4.53}$$

Hence

$$\begin{aligned}\Big(1-(1+q_0)K^2(T_{h'}-T_h)^2\Big)\widetilde{A}&\le 2\left(1+\frac{1}{q_0}\right)e^{2(T_{h'}-T_h)\mathbf{M}(\zeta)}\mathbf{E}\Big\|f-\mathbf{u}(.,T_{h'})\Big\|^2_{L^2(\Omega)}\\&\quad+2\left(1+\frac{1}{q_0}\right)\|\mathbf{u}(.,T_h)\|^2_{L^2(\Omega)}.\end{aligned} \tag{4.54}$$

Since by assumption $0<q_0<\frac{1}{K^2(T_{h'}-T_h)^2}-1$, it follows that the term on the left hand-side that is in parenthesis is positive. This implies that for all $t\in[T_h,T_{h'}]$

$$\begin{aligned}&e^{2(t-T_h)\zeta}\mathbf{E}\|\mathbf{Y}^{\zeta}_{T_h,T_{h'}}(f)(.,t)-\mathbf{u}(.,t)\|^2_{L^2(\Omega)}\\&\le\frac{2\left(1+\frac{1}{q_0}\right)e^{2(T_{h'}-T_h)\zeta}\mathbf{E}\Big\|f-\mathbf{u}(.,T_{h'})\Big\|^2_{L^2(\Omega)}+2\left(1+\frac{1}{q_0}\right)\|\mathbf{u}(.,T_h)\|^2_{L^2(\Omega)}}{1-(1+q_0)K^2(T_{h'}-T_h)^2}.\end{aligned} \tag{4.55}$$

Hence for all $t\in[T_h,T_{h'}]$ we conclude that

$$\begin{aligned}&\mathbf{E}\|\mathbf{Y}^{\zeta}_{T_h,T_{h'}}(f)(.,t)-\mathbf{u}(.,t)\|^2_{L^2(\Omega)}\\&\le\frac{2\left(1+\frac{1}{q_0}\right)}{1-(1+q_0)K^2(T_{h'}-T_h)^2}\left(e^{2(T_{h'}-T_h)\zeta}\mathbf{E}\Big\|f-\mathbf{u}(.,T_{h'})\Big\|^2_{L^2(\Omega)}+\|\mathbf{u}(.,T_h)\|^2_{L^2(\Omega)}\right)e^{2(T_h-t)\zeta}.\end{aligned} \tag{4.56}$$

□

Our main result in this subsection is as follows.

Theorem 4.3. *Let g be as in Theorem 4.1. Assume that* $\mathbf{u}$ *is the unique solution of Problem* (1.1).

(a) *Assume that* $KT<1$, *where K is the Lipschitz constant of F. A new regularized solution is given as follows*

$$\begin{aligned}\widehat{U}_\delta(x,t)&=\sum_{\lambda_j\le\zeta(\delta)}\Big[e^{(T-t)\lambda_j}\overline{G}_{\delta,\mathbf{N}(\delta)}-\int_t^T e^{(\tau-t)\lambda_j}F_j(\widehat{U}_\delta)(\tau)d\tau\Big]\phi_j(x)\\&\quad+\sum_{\lambda_j>\zeta(\delta)}\Big[\int_0^t e^{(\tau-t)\lambda_j}F_j(\widehat{U}_\delta)(\tau)d\tau\Big]\phi_j(x).\end{aligned} \tag{4.57}$$

Let us choose $\zeta(\delta)$ *such that*

$$\lim_{\delta\to 0}\zeta(\delta)=+\infty,\ \lim_{\delta\to 0}\frac{e^{kT\zeta(\delta)}}{\lambda^{\gamma}_{\mathbf{N}(\delta)}}=0,\quad \lim_{\delta\to 0}e^{kT\zeta(\delta)}\sqrt{\mathbf{N}(\delta)}\delta=0. \tag{4.58}$$

If $\mathbf{u}\in C([0,T];L^2(\Omega))$ *then as* $|\delta|\to 0$

$$\mathbf{E}\|\widehat{U}_\delta(.,t)-\mathbf{u}(.,t)\|^2_{L^2(\Omega)}\quad \text{is of order}\quad e^{-2t\zeta(\delta)}. \tag{4.59}$$

(b) *Suppose that* $KT>1$ *and let us assume that* $\mathbf{u}\in C([0,T];L^2(\Omega))$. *Let*

$$\begin{aligned}\zeta_1(\delta):&=\frac{s}{T2^{2s-1}}\log\left(\frac{1}{\xi(\delta)}\right)\\ \zeta_k(\delta):&=\frac{s}{T2^{2s-k}}\log\left(\frac{1}{\xi(\delta)}\right),\ k=\overline{2,2s}.\end{aligned} \tag{4.60}$$

We construct a regularized solution $\widehat{U}_\delta$ *as follows*

$$\widehat{U}_\delta(x,t)=\mathbf{Y}^{\zeta_{2s-i}(\delta)}_{T_{2s-i-2},T_{2s-i}}\Big(\widehat{U}_\delta(x,T_{2s-i})\Big)(x,t),\quad \text{if}\quad T_{2s-i-1}\le t\le T_{2s-i},\quad i=\overline{0,2s-2} \tag{4.61}$$

and

$$\widehat{U}_\delta(x,t)=\mathbf{Y}^{\zeta_1(\delta)}_{T_0,T_1}\Big(\widehat{U}_\delta(x,T_1)\Big)(x,t),\quad \text{if}\quad 0\le t\le T_1. \tag{4.62}$$

where $\mathbf{Y}^{\zeta(\delta)}_{T_{h_1},T_{h_2}}(\theta)(x,t)$ *is defined in* (4.37). *Then we have*

- *If* $t\in[T_k,T_{k+1}]$ *and* $k=\overline{1,2s-1}$ *then*

$$\mathbf{E}\|\widehat{U}_\delta(.,t)-\mathbf{u}(.,t)\|^2_{L^2(\Omega)}\quad \text{is of order}\quad \Big(\xi(\delta)\Big)^{\frac{1}{2^{2s-k}}}. \tag{4.63}$$

- *If* $t\in[0,T_1]$ *then*

$$\mathbf{E}\|\widehat{U}_\delta(.,t)-\mathbf{u}(.,t)\|^2_{L^2(\Omega)}\quad \text{is of order}\quad \Big(\xi(\delta)\Big)^{\frac{st}{2^{2s-1}}}. \tag{4.64}$$

Remark 4.4. *In [29], we only need the regularization result for* $0<KT<1$. *Our Theorem 4.3 extends this result for any* $K>0$.

Proof of part (a) of Theorem 4.3. By setting $T_h=0$ and $T_{h'}=T$ $f=\overline{G}_{\delta,\mathbf{N}(\delta)}$ then $\mathbf{Y}^{\zeta}_{T_h,T_{h'}}(f)$ given by (4.37) in Lemma 4.2 is equal to $\widehat{U}_\delta$ given by (4.57). Then apply the result from (4.38). Since $KT<1$, applying Lemma 4.2 , we obtain

$$\begin{aligned}&\mathbf{E}\|\widehat{U}_\delta(.,t)-\mathbf{u}(.,t)\|^2_{L^2(\Omega)}\\ &\le\frac{2\left(1+\frac{1}{q_0}\right)}{1-(1+q_0)K^2T^2}\left(e^{2T\zeta(\delta)}\mathbf{E}\Big\|\overline{G}_{\delta,\mathbf{N}(\delta)}-g\Big\|^2_{L^2(\Omega)}+\|g\|^2_{L^2(\Omega)}\right)e^{-2t\zeta(\delta)}\\ &\le\frac{2\left(1+\frac{1}{q_0}\right)}{1-(1+q_0)K^2T^2}e^{2(T-t)\zeta(\delta)}\delta^2\mathbf{N}(\delta)+4\frac{2\left(1+\frac{1}{q_0}\right)}{1-(1+q_0)K^2T^2}e^{2(T-t)\zeta(\delta)}\frac{1}{\lambda^{2\gamma}_{\mathbf{N}(\delta)}}\|g\|_{H^{2\gamma}}\\ &+\frac{2\left(1+\frac{1}{q_0}\right)}{1-(1+q_0)K^2T^2}\|g\|^2_{L^2(\Omega)}e^{-2t\zeta(\delta)}.\end{aligned}$$

This completes the proof of part (a).

Proof of part (b) of Theorem 4.3

By Theorem 3.1, we have $\mathbf{E}\|\overline{G}_{\delta,\mathbf{N}(\delta)}-g\|^2_{L^2(\Omega)} \le \widetilde{C}\xi^2(\delta)$. where $\widetilde{C}=1+\|g\|^2_{H^{2\gamma}}$. We will estimate the error for time variable in interval $[T_l, T_{l+1}]$ for $l=\overline{0,2s}$.

Case 1. Let $t\in[T_{2s-1},T]$. Since $\zeta_{2s}(\delta)=\frac{s}{T}\log\left(\frac{1}{\xi(\delta)}\right)$, by Lemma 4.2 we get

$$\begin{aligned}
\mathbf{E}\|\widehat{U}_\delta(.,t)-\mathbf{u}(.,t)\|^2_{L^2(\Omega)} &= \mathbf{E}\left\|\mathbf{Y}^{\zeta_{2s}(\delta)}_{T_{2s-2},T_{2s}}(\overline{G}_{\delta,\mathbf{N}(\delta)})(.,t)-\mathbf{u}(.,t)\right\|^2_{L^2(\Omega)}\\
&\le \frac{2s^2\left(1+\frac{1}{q_0}\right)}{s^2-T^2K^2(1+q_0)}\Big[e^{2(T_{2s}-T_{2s-2})\zeta_{2s}(\delta)}\mathbf{E}\|\overline{G}_{\delta,\mathbf{N}(\delta)}-g\|^2_{L^2(\Omega)}\Big]e^{2(T_{2s-2}-t)\zeta_{2s}(\delta)}\\
&+\frac{2s^2\left(1+\frac{1}{q_0}\right)}{s^2-T^2K^2(1+q_0)}\Big[\|\mathbf{u}(.,T_{2s-2})\|^2_{L^2(\Omega)}\Big]e^{2(T_{2s-2}-t)\zeta_{2s}(\delta)}\\
&\le \frac{2s^2\left(1+\frac{1}{q_0}\right)}{s^2-T^2K^2(1+q_0)}\Big(\widetilde{C}+\|\mathbf{u}\|^2_{L^\infty(0,T;L^2(\Omega))}\Big)\xi(\delta)\\
&=\chi(s,K,q_0)\Big(\widetilde{C}+\|\mathbf{u}\|^2_{L^\infty(0,T;L^2(\Omega))}\Big)\xi(\delta),
\end{aligned}\tag{4.65}$$

which we note that $e^{2(T_{2s-2}-t)\zeta_{2s}(\delta)}\le e^{2(T_{2s-2}-T_{2s-1})\zeta_{2s}(\delta)}=\xi(\delta)$ and

$$\chi(s,K,q_0)=\max\left\{1,\frac{2s^2\left(1+\frac{1}{q_0}\right)}{s^2-T^2K^2(1+q_0)}\right\},\ \text{then}\ \chi(s,K,q_0)\ge 1.$$

Case 2. Let $t\in[T_{2s-2},T_{2s-1}]$. Since $\zeta_{2s-1}(\delta)=\frac{s}{2T}\log\left(\frac{1}{\xi(\delta)}\right)$, by Lemma 4.2 we get

$$\begin{aligned}
&\mathbf{E}\|\widehat{U}_\delta(.,t)-\mathbf{u}(.,t)\|^2_{L^2(\Omega)}\\
&=\mathbf{E}\left\|\mathbf{Y}^{\zeta_{2s-1}(\delta)}_{T_{2s-3},T_{2s-1}}\left(\widehat{\mathcal{U}}_\delta(.,T_{2s-1})\right)(.,t)-\mathbf{u}(.,t)\right\|^2_{L^2(\Omega)}\\
&\le\chi(s,K,q_0)\exp\Big(2(T_{2s-3}-t)\zeta_{2s-1}(\delta)\Big)\exp\Big(2(T_{2s-1}-T_{2s-3})\zeta_{2s-1}(\delta)\Big)\mathbf{E}\|\widehat{U}_\delta(.,T_{2s-1})-\mathbf{u}(.,T_{2s-1}\|^2_{L^2(\Omega)}\\
&+\chi(s,K,q_0)\exp\Big(2(T_{2s-3}-t)\zeta_{2s-1}(\delta)\Big)\|\mathbf{u}(.,T_{2s-3})\|^2_{L^2(\Omega)}\\
&\le\chi(s,K,q_0)\bigg(\chi(s,K,q_0)\Big(\widetilde{C}+\|\mathbf{u}\|_{L^\infty(0,T;L^2(\Omega))}\Big)+\|\mathbf{u}\|_{L^\infty(0,T;L^2(\Omega))}\bigg)\Big(\xi(\delta)\Big)^{\frac{1}{2}}\\
&\le 2\chi^2(s,K,q_0)\Big(\widetilde{C}+\|\mathbf{u}\|_{L^\infty(0,T;L^2(\Omega))}\Big)\Big(\xi(\delta)\Big)^{\frac{1}{2}},
\end{aligned}$$

where we used the following result from (4.65):

$$\mathbf{E}\left\|\widehat{U}_\delta(.,T_{2s-1})-\mathbf{u}(.,T_{2s-1}\right\|^2_{L^2(\Omega)}\le\chi(s,K,q_0)\Big(\widetilde{C}+\|\mathbf{u}\|^2_{L^\infty(0,T;L^2(\Omega))}\Big)\xi(\delta).$$

Therefore, repeating the argument as in the above cases and using the induction method, we can prove the following estimate

$$\begin{aligned}
&\mathbf{E}\|\widehat{U}_\delta(.,t)-u(.,t)\|^2_{L^2(\Omega)}\\
&\le(2s-k)|\chi^2(s,K,q_0)|^{2s-k}\Big(\widetilde{C}+\|\mathbf{u}\|^2_{L^\infty(0,T;L^2(\Omega))}\Big)\Big(\xi(\delta)\Big)^{\frac{1}{2^{2s-k-1}}},
\end{aligned}$$

for all $t \in [T_k, T_{k+1}]$ and $k = \overline{1, 2s-1}$.
If $t \in [0, T_1]$, then by a similar technique as above, we obtain the error estimate

$$\begin{aligned}&\mathbf{E}\|\widehat{U}_\delta(.,t) - u(.,t)\|^2_{L^2(\Omega)}\\ &\leq s|\chi^2(s,K,q_0)|^{2s}\Big(\widetilde{C} + \|u\|^2_{L^\infty(0,T;L^2(\Omega))}\Big)\Big(\xi(\delta)\Big)^{\frac{st}{2^{2s-2}}}.\end{aligned}$$

□

5. Regularization result with locally Lipschitz source

Section 4 addressed a problem in which F is a global Lipschitz function. In this section we extend the analysis to a locally Lipschitz function F. Results for the locally Lipschitz case are difficult. Here, we have to find another regularization method to study the problem with a locally Lipschitz source. Assume that a is noisy by the observation data $a_\delta^{\text{obs}} : \Omega \times [0,T] \to \mathbb{R}$ as follows

$$\mathbf{a}_\delta^{\text{obs}}(x,t) = a(x,t) + \delta\psi(t) \tag{5.1}$$

where $\delta > 0$ and $\psi \in L^\infty(0,T)$ such that

$$\|\psi\|_{L^\infty(0,T)} = \sup_{0\leq t\leq T} |\psi(t)| \leq \overline{M}, \tag{5.2}$$

where $\overline{M} > 0$. In the case when a is not disturbed, we can use the method in the previous sections (the case when a is not disturbed is simpler than the case a is noisy). If a is disturbed by random data, it is difficult to use the old method and we need a new approach, as outlined below.

Assume that for each $\mathcal{R} > 0$, there exists $K_\mathcal{R} > 0$ such that

$$|F(x,t;u) - F(x,t;v)| \leq K_\mathcal{R}|u-v|, \text{ if } \max\{|u|,|v|\} \leq \mathcal{R}, \tag{5.3}$$

where $(x,t) \in \Omega \times [0,T]$ and

$$K_\mathcal{R} := \sup\left\{\left|\frac{F(x,t;u) - F(x,t;v)}{u-v}\right| : \max\{|u|,|v|\} \leq \mathcal{R}, u \neq v, (x,t) \in \Omega \times [0,T]\right\} < +\infty.$$

We note that $K_\mathcal{R}$ is increasing and $\lim_{\mathcal{R}\to+\infty} K_\mathcal{R} = +\infty$. Now, we outline our idea to construct a regularization for problem (1.1). For all $\mathcal{R} > 0$, we approximate F by $\mathcal{F}_\mathcal{R}$ defined by

$$\mathcal{F}_\mathcal{R}(x,t;w) := \begin{cases} F(x,t;-\mathcal{R}), & w \in (-\infty, -\mathcal{R}) \\ F(x,t;u), & w \in [-\mathcal{R}, \mathcal{R}] \\ F(x,t;\mathcal{R}), & w \in (\mathcal{R}, +\infty). \end{cases} \tag{5.4}$$

For each $\delta > 0$, we consider a parameter $\mathcal{R}(\delta) \to +\infty$ as $\delta \to 0^+$. Let us denote the operator $\mathbb{P} = M\Delta$, where M is a positive number such that $M > \mathbf{a}_\delta^{\text{obs}}(x,t)$ for all $(x,t) \in \Omega \times (0,T)$. Define the following operator

$$\mathbf{P}^\delta_{\beta_{\mathbf{N}(\delta)}} = \mathbb{P} + \mathbf{Q}^\delta_{\beta_{\mathbf{N}(\delta)}},$$

where

$$\mathbf{Q}^\delta_{\beta_{\mathbf{N}(\delta)}} v(x) = \frac{1}{T}\sum_{j=1}^{\infty} \ln\left(1 + \beta_{\mathbf{N}(\delta)} e^{MT\lambda_j}\right)\langle v(x), \phi_j(x)\rangle_{L^2(\Omega)}\phi_j(x), \tag{5.5}$$

for any function $v \in L^2(\Omega)$. Here $\mathbf{N}(\delta)$ is defined in Lemma (4.1).

We introduce the main idea to solve problem (1.1) with a generalized case of source term defined by (5.4), and we consider the problem:

$$\begin{cases} \dfrac{\partial \mathbf{u}^{\delta}_{\mathbf{N}(\delta)}}{\partial t} - \nabla\Big(\mathbf{a}^{\text{obs}}_{\delta}(x,t)\nabla \mathbf{u}^{\delta}_{\mathbf{N}(\delta)}\Big) - \mathbf{Q}^{\delta}_{\beta_{\mathbf{N}(\delta)}}(\mathbf{u}^{\delta}_{\mathbf{N}(\delta)})(x,t) \\ \qquad\qquad\qquad = \mathcal{F}_{R_\delta}\left(x,t,\mathbf{u}^{\delta}_{\mathbf{N}(\delta)}(x,t)\right), \quad (x,t) \in \Omega \times (0,T), \\ \mathbf{u}^{\delta}_{\mathbf{N}(\delta)}|_{\partial\Omega} = 0, \quad t \in (0,T), \\ \mathbf{u}^{\delta}_{\mathbf{N}}(x,T) = \overline{G}_{\delta,\mathbf{N}(\delta)}(x), \quad (x,t) \in \Omega \times (0,T), \end{cases} \tag{5.6}$$

Here $\overline{G}_{\delta,\mathbf{N}(\delta)}(x)$ is defined in Eq (4.2). Now, we introduce some Lemmas which will be useful for our main results. First, we recall the abstract Gevrey class of functions of index $\sigma > 0$, see, *e.g.*, [24], defined by

$$\mathcal{W}_\sigma = \left\{ v \in L^2(\Omega) : \sum_{n=1}^{\infty} e^{2\sigma\lambda_n} \left|\langle v, \phi_n(x)\rangle_{L^2(\Omega)}\right|^2 < \infty \right\},$$

which is a Hilbert space equipped with the inner product

$$\langle v_1, v_2\rangle_{\mathcal{W}_\sigma} := \left\langle e^{\sigma\sqrt{-\Delta}} v_1, e^{\sigma\sqrt{-\Delta}} v_2 \right\rangle_{L^2(\Omega)}, \quad \text{for all } v_1, v_2 \in \mathcal{W}_\sigma;$$

and the corresponding norm is $\|v\|_{\mathcal{W}_\sigma} = \sqrt{\sum_{n=1}^{\infty} e^{2\sigma\lambda_n}\left|\langle v, \phi_n\rangle_{L^2(\Omega)}\right|^2} < \infty$.

Lemma 5.1. *For $\mathcal{F}_{\mathcal{R}} \in L^\infty(\Omega \times [0,T] \times \mathbb{R})$, we have*

$$|\mathcal{F}_{\mathcal{R}}(x,t;u) - \mathcal{F}_{\mathcal{R}}(x,t;v)| \le K_{\mathcal{R}}|u - v|, \quad \forall (x,t) \in \Omega \times [0,T],\ u,v \in \mathbb{R}.$$

Proof. See the proof of Lemma 2.4 in [35]. □

Lemma 5.2. *1. Let $M, T > 0$. For any $v \in \mathcal{W}_{MT}(\Omega)$, we have*

$$\|\mathbf{Q}^{\delta}_{\beta_{\mathbf{N}(\delta)}}(v)\|_{L^2(\Omega)} \le \frac{\beta_{\mathbf{N}(\delta)}}{T} \|v\|_{\mathcal{W}_{MT}(\Omega)}. \tag{5.7}$$

2. Let $\beta_{\mathbf{N}(\delta)} < 1 - e^{-MT\lambda_1}$. For any $v \in L^2(\Omega)$, we have

$$\left\|\mathbf{P}^{\delta}_{\beta_{\mathbf{N}(\delta)}} v\right\|_{L^2(\Omega)} \le \frac{1}{T}\ln\left(\frac{1}{\beta_{\mathbf{N}(\delta)}}\right)\|v\|_{L^2(\Omega)}. \tag{5.8}$$

Proof. Using the inequality $\ln(1 + a) \le a$, $\forall a > 0$, we have

$$\begin{aligned} \left\|\mathbf{Q}^{\delta}_{\beta_{\mathbf{N}(\delta)}}(v)\right\|^2_{L^2(\Omega)} &= \frac{1}{T^2}\sum_{j=1}^{\infty} \ln^2\left(1 + \beta_{\mathbf{N}(\delta)} e^{MT\lambda_j}\right)\left|\langle v, \phi_j\rangle_{L^2(\Omega)}\right|^2 \\ &\le \frac{\beta^2_{\mathbf{N}(\delta)}}{T^2}\sum_{j=1}^{\infty} e^{2MT\lambda_j}\left|\langle v, \phi_j\rangle_{L^2(\Omega)}\right|^2 \le \frac{\beta^2_{\mathbf{N}(\delta)}}{T^2}\|v\|^2_{\mathcal{W}_{MT}}. \end{aligned} \tag{5.9}$$

Since $\beta_{\mathbf{N}(\delta)} < 1 - e^{-MT\lambda_1}$, we know that $\beta_{\mathbf{N}(\delta)} + e^{-MT\lambda_j} < 1$. Using Parseval's equality, we easily get

$$\left\|\mathbf{P}^{\delta}_{\beta_{\mathbf{N}(\delta)}}(v)\right\|^2_{L^2(\Omega)} = \frac{1}{T^2}\sum_{j=1}^{\infty}\ln^2\left(\frac{1}{\beta_{\mathbf{N}(\delta)} + e^{-MT\lambda_j}}\right)\left|\langle v, \phi_j\rangle_{L^2(\Omega)}\right|^2$$

$$\leq \frac{1}{T^2}\ln^2\left(\frac{1}{\beta_{\mathbf{N}(\delta)}}\right)\sum_{j=1}^{\infty}\left|\langle v,\phi_j\rangle_{L^2(\Omega)}\right|^2 \leq \frac{1}{T^2}\ln^2\left(\frac{1}{\beta_{\mathbf{N}(\delta)}}\right)\|v\|^2_{L^2(\Omega)}.$$

□

Theorem 5.1. *Problem* (5.6) *has a unique solution* $\mathbf{u}^{\delta}_{\mathbf{N}(\delta)} \in C\left([0,T];L^2(\Omega)\right)$. *Assume that the problem* (1.1) *has a unique solution* $\mathbf{u}$ *satisfying* $\mathbf{u}(\cdot,t) \in \mathcal{W}_{MT}$. *Choose* $\beta_{\mathbf{N}(\delta)}$ *such that*

$$\lim_{\delta\to 0}\delta\sqrt{\mathbf{N}(\delta)}\beta^{-1}_{\mathbf{N}(\delta)} = \lim_{\delta\to 0}\beta^{-1}_{\mathbf{N}(\delta)}\lambda^{-\gamma}_{\mathbf{N}(\delta)} = \lim_{\delta\to 0}\beta_{\mathbf{N}(\delta)} = 0. \tag{5.10}$$

Choose $\mathcal{R}_\delta$ *such that*

$$\lim_{\delta\to 0}\beta^{\frac{2t}{T}}_{\mathbf{N}(\delta)}e^{2K\mathcal{R}_\delta T} = 0, \quad t>0. \tag{5.11}$$

Then we have the following estimate

$$\mathbf{E}\left\|\mathbf{u}^{\delta}_{\mathbf{N}(\delta)}(x,t)-\boldsymbol{u}(x,t)\right\|^2_{L^2(\Omega)} \leq \beta^{\frac{2t}{T}}_{\mathbf{N}(\delta)}e^{(2K(\mathcal{R}_\delta))+1)T}\widetilde{C}(\delta). \tag{5.12}$$

Here $\widetilde{C}(\delta)$ *is*

$$\widetilde{C}(\delta) = \delta^2\mathbf{N}(\delta)\beta^{-2}_{\mathbf{N}_\delta} + \frac{1}{\lambda^{2\gamma}_{\mathbf{N}(\delta)}\beta^2_{\mathbf{N}_\delta}}\|g\|_{H^{2\gamma}(\Omega)} + \|\boldsymbol{u}\|^2_{C([0,T];\mathcal{W}_{MT}(\Omega))} + \frac{\delta^2T^3}{b_0\beta^2_{\mathbf{N}_\delta}}\|\boldsymbol{u}\|^2_{L^\infty(0,T;\mathcal{H}^1_0(\Omega))}.$$

and assume that Ω *is one dimensional domain.*

Remark 5.1. *1. Under asumption* (5.11)*, the right hand side of Eq* (5.12) *converges to zero when* $t>0$.
2. Choose $\beta_{\mathbf{N}(\delta)} = \mathbf{N}(\delta)^{-c}$ *for any* $0<c<\min(\frac{1}{2},\frac{2\gamma}{d})$*, and* $\mathbf{N}(\delta)$ *is chosen as*

$$\mathbf{N}(\delta) = \left(\frac{1}{\delta}\right)^{m(\frac{1}{2}-c)}, \quad 0<m<1. \tag{5.13}$$

Choose $\mathcal{R}_\delta$ *such that*

$$K(\mathcal{R}_\delta) \leq \frac{1}{kT}\ln\Big(\ln(\mathbf{N}(\delta))\Big) = \frac{1}{kT}\ln\Big(m(\frac{1}{2}-c)\ln\left(\frac{1}{\delta}\right)\Big).$$

Then $\mathbf{E}\left\|\mathbf{u}^{\delta}_{\mathbf{N}(\delta)}(x,t)-\boldsymbol{u}(x,t)\right\|^2_{L^2(\Omega)}$ *is of the order* $\delta^{mc(\frac{1}{2}-c)\frac{t}{T}}\ln(\frac{1}{\delta})$.

Proof of Theorem 5.1. The proof is divided into two Steps.
Step 1. The existence and uniqueness of the solution to the regularized problem (5.6)**.**
Let $b(x,t)$ be defined by $b(x,t) = M - a(x,t)$. It is clear that $0<b(x,t)<M$. Then from (5.6), we obtain

$$\frac{\partial \mathbf{u}^{\delta}_{\mathbf{N}(\delta)}}{\partial t} + \nabla\Big(b(x,t)\nabla\mathbf{u}^{\delta}_{\mathbf{N}(\delta)}\Big) = F\left(x,t,\mathbf{u}^{\delta}_{\mathbf{N}(\delta)}(x,t)\right) - \frac{1}{T}\sum_{j=1}^{\infty}\ln\left(\frac{1}{\beta_{\mathbf{N}(\delta)}+e^{-MT\lambda_j}}\right)\langle\mathbf{u}^{\delta}_{\mathbf{N}(\delta)}(\cdot,t),\phi_j\rangle\phi_j(x), \tag{5.14}$$

for $(x,t) \in \Omega \times (0,T)$.

Let $\mathbf{v}^{\delta}_{\mathbf{N}(\delta)}$ be the function defined by $\mathbf{v}^{\delta}_{\mathbf{N}(\delta)}(x,t) = \mathbf{u}^{\delta}_{\mathbf{N}(\delta)}(x, T-t)$. Then we have

$$\frac{\partial \mathbf{v}^{\delta}_{\mathbf{N}(\delta)}}{\partial t}(x,t) = -\frac{\partial \mathbf{u}^{\delta}_{\mathbf{N}(\delta)}}{\partial t}(x,T-t),\ \nabla\Big(b(x,t)\nabla \mathbf{v}^{\delta}_{\mathbf{N}(\delta)}\Big)(x,t) = \nabla\Big(b(x,t)\nabla \mathbf{u}^{\delta}_{\mathbf{N}(\delta)}\Big)(x,T-t)$$

and

$$\begin{aligned}&\frac{1}{T}\sum_{j=1}^{\infty} \ln\Big(\beta_{\mathbf{N}(\delta)} + e^{-MT\lambda_j}\Big)\langle \mathbf{v}^{\delta}_{\mathbf{N}(\delta)}(x,t), \phi_j(x)\rangle \phi_j(x)\\ &\qquad = \frac{1}{T}\sum_{j=1}^{\infty} \ln\Big(\beta_{\mathbf{N}(\delta)} + e^{-MT\lambda_j}\Big)\langle \mathbf{u}^{\delta}_{\mathbf{N}(\delta)}(x,T-t), \phi_j(x)\rangle \phi_j(x).\end{aligned}$$

This implies that $\mathbf{v}^{\delta}_{\mathbf{N}(\delta)}$ satisfies the problem

$$\begin{cases}\dfrac{\partial \mathbf{v}^{\delta}_{\mathbf{N}(\delta)}}{\partial t} - \nabla\Big(b(x,t)\nabla \mathbf{v}^{\delta}_{\mathbf{N}(\delta)}\Big) = \mathcal{G}(x,t,\mathbf{v}^{\delta}_{\mathbf{N}}(x,t)), \quad (x,t) \in \Omega\times(0,T),\\ \mathbf{v}^{\delta}_{\mathbf{N}(\delta)}|_{\partial\Omega} = 0, \quad t\in(0,T),\\ \mathbf{v}^{\delta}_{\mathbf{N}(\delta)}(x,0) = \overline{G}_{\delta,\mathbf{N}(\delta)}(x), \quad (x,t)\in\Omega\times(0,T),\end{cases} \tag{5.15}$$

where $\mathcal{G}$ is defined by

$$\begin{aligned}\mathcal{G}(x,t,v(x,t)) &= -F(x,t,v(x,t))\\ &\quad + \frac{1}{T}\sum_{j=1}^{\infty} \ln\left(\frac{1}{\beta_{\mathbf{N}(\delta)} + e^{-MT\lambda_j}}\right)\langle v(\cdot,t), \phi_j\rangle_{L^2(\Omega)}\phi_j(x),\end{aligned} \tag{5.16}$$

for any $v \in C\Big([0,T]; L^2(\Omega)\Big)$.

Since

$$\beta_{\mathbf{N}(\delta)} \in \Big(0, 1-e^{-MT\lambda_1}\Big), \qquad 0 < \ln\left(\frac{1}{\beta_{\mathbf{N}(\delta)} + e^{-MT\lambda_n}}\right) < \ln\left(\frac{1}{\beta_{\mathbf{N}(\delta)}}\right)$$

and using Parseval's identity, we obtain for any $v_1, v_2 \in L^2(\Omega)$,

$$\begin{aligned}&\|\mathcal{G}(\cdot,t,v_1(\cdot,t)) - \mathcal{G}(\cdot,t,v_2(\cdot,t))\|_{L^2(\Omega)}\\ &\quad\le \|F(\cdot,t,v_1(\cdot,t)) - F(\cdot,t,v_2(\cdot,t))\|_{L^2(\Omega)}\\ &\quad + \left\|\frac{1}{T}\sum_{j=1}^{\infty}\ln\left(\frac{1}{\beta_{\mathbf{N}(\delta)} + e^{-MT\lambda_j}}\right)\langle v_1(x,t) - v_2(x,t), \phi_j(x)\rangle_{L^2(\Omega)}\phi_j(x)\right\|_{L^2(\Omega)}\\ &\quad\le K\|v_1(\cdot,t) - v_2(\cdot,t)\|_{L^2(\Omega)}\\ &\quad + \frac{1}{T}\sqrt{\sum_{j=1}^{\infty}\ln^2\left(\frac{1}{\beta_{\mathbf{N}(\delta)} + e^{-MT\lambda_j}}\right)\Big|\langle v_1(\cdot,t) - v_2(\cdot,t), \phi_n\rangle_{L^2(\Omega)}\Big|^2}\\ &\quad\le \left[K + \frac{1}{T}\ln\left(\frac{1}{\beta_{\mathbf{N}(\delta)}}\right)\right]\ \|v_1(\cdot,t) - v_2(\cdot,t)\|_{L^2(\Omega)}.\end{aligned} \tag{5.17}$$

Thus $\mathcal{G}$ is a Lipschitz function. Using the results of Theorem 12.2 in [32], we complete the proof of Step 1.

Step 2. Error estimate

We consider the error estimate between the regularized solution of problem (5.6) and the exact solution of problem (1.1).

For $(x,t) \in \Omega \times (0,T)$, we begin by establishing that the functions $b(x,t), \mathbf{b}_\delta^{\text{obs}}(x,t)$ satisfy

$$0 < b(x,t) \le M, \; 0 < b_0 \le \mathbf{b}_\delta^{\text{obs}}(x,t) \le M$$

and

$$\begin{pmatrix} a(x,t) \\ \mathbf{a}_\delta^{\text{obs}}(x,t) \end{pmatrix} = \begin{pmatrix} M \\ M \end{pmatrix} - \begin{pmatrix} b(x,t) \\ \mathbf{b}_\delta^{\text{obs}}(x,t) \end{pmatrix}, \; \forall (x,t) \in \Omega \times (0,T). \tag{5.18}$$

The functions $\mathbf{u}_{\mathbf{N}(\delta)}^\delta(x,t)$ and $\mathbf{u}(x,t)$ solve the following equations

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + \nabla\left(\mathbf{b}_\delta^{\text{obs}}(x,t)\nabla\mathbf{u}\right) &= F(x,t;\mathbf{u}(x,t)) \\ &\quad + \nabla\Big((\mathbf{b}_\delta^{\text{obs}}(x,t) - b(x,t))\nabla\mathbf{u}\Big) + \mathbb{P}\mathbf{u} \end{aligned} \tag{5.19}$$

and

$$\frac{\partial \mathbf{u}_{\mathbf{N}(\delta)}^\delta}{\partial t} + \nabla\Big(\mathbf{b}_\delta^{\text{obs}}(x,t)\nabla\mathbf{u}_{\mathbf{N}(\delta)}^\delta\Big) = \mathcal{F}_{R_\delta}\left(x,t,\mathbf{u}_{\mathbf{N}(\delta)}^\delta(x,t)\right) + \mathbf{P}_{\beta_{\mathbf{N}(\delta)}}^\delta \mathbf{u}_{\mathbf{N}(\delta)}^\delta. \tag{5.20}$$

For $\rho_\delta > 0$, we put $\mathbf{V}_{\mathbf{N}(\delta)}^\delta(x,t) = e^{\rho_\delta(t-T)}\left[\mathbf{u}_{\mathbf{N}(\delta)}^\delta(x,t) - \mathbf{u}(x,t)\right]$. Then for $(x,t) \in \Omega \times (0,T)$

$$\begin{aligned} &\frac{\partial \mathbf{V}_{\mathbf{N}(\delta)}^\delta}{\partial t} + \nabla\Big(\mathbf{b}_\delta^{\text{obs}}(x,t)\nabla\mathbf{V}_{\mathbf{N}(\delta)}^\delta\Big) - \rho_\delta \mathbf{V}_{\mathbf{N}(\delta)}^\delta \\ &\quad = \mathbf{P}_{\beta_{\mathbf{N}(\delta)}}^\delta \mathbf{V}_{\mathbf{N}(\delta)}^\delta + e^{\rho_\delta(t-T)}\mathbf{Q}_{\beta_{\mathbf{N}(\delta)}}^\delta \mathbf{u} - e^{\rho_\delta(t-T)}\nabla\left((\mathbf{b}_\delta^{\text{obs}}(x,t) - b(x,t))\nabla\mathbf{u}\right) \\ &\qquad + e^{\rho_\delta(t-T)}\left[\mathcal{F}_{R_\delta}\left(x,t,\mathbf{u}_{\mathbf{N}(\delta)}^\delta(x,t)\right) - F(x,t;\mathbf{u}(x,t))\right], \end{aligned} \tag{5.21}$$

and

$$\mathbf{V}_{\mathbf{N}(\delta)}^\delta|_{\partial\Omega} = 0, \; \mathbf{V}_{\mathbf{N}(\delta)}^\delta(x,T) = \overline{G}_{\delta,\mathbf{N}(\delta)}(x) - g(x).$$

By taking the inner product on both sides of Eq (5.21) with $\mathbf{V}_{\mathbf{N}(\delta)}^\delta$ and noting the equality

$$\int_\Omega \nabla\Big(\mathbf{b}_\delta^{\text{obs}}(x,t)\nabla\mathbf{V}_{\mathbf{N}(\delta)}^\delta\Big)\mathbf{V}_{\mathbf{N}(\delta)}^\delta dx = -\int_\Omega \mathbf{b}_\delta^{\text{obs}}(x,t)|\nabla\mathbf{V}_{\mathbf{N}(\delta)}^\delta|^2 dx,$$

we obtain

$$\begin{aligned} &\|\mathbf{V}_{\mathbf{N}(\delta)}^\delta(\cdot,T)\|_{L^2(\Omega)}^2 - \|\mathbf{V}_{\mathbf{N}(\delta)}^\delta(\cdot,t)\|_{L^2(\Omega)}^2 \\ &\quad - 2\int_t^T \int_\Omega \mathbf{b}_\delta^{\text{obs}}(x,s)|\nabla\mathbf{V}_{\mathbf{N}(\delta)}^\delta|^2 dxds - 2\rho_\delta \int_t^T \|\mathbf{V}_{\mathbf{N}(\delta)}^\delta(\cdot,s)\|_{L^2(\Omega)}^2 ds \\ &\quad = \underbrace{2\int_t^T \left\langle \mathbf{P}_{\beta_{\mathbf{N}(\delta)}}^\delta \mathbf{V}_{\mathbf{N}(\delta)}^\delta, \mathbf{V}_{\mathbf{N}(\delta)}^\delta \right\rangle_{L^2(\Omega)} ds}_{=:\widetilde{A_4}} + \underbrace{2\int_t^T \left\langle e^{\rho_\delta(t-T)}\mathbf{Q}_{\beta_{\mathbf{N}(\delta)}}^\delta \mathbf{u}, \mathbf{V}_{\mathbf{N}(\delta)}^\delta \right\rangle_{L^2(\Omega)} ds}_{=:\widetilde{A_5}} \\ &\quad + \underbrace{2\int_t^T \left\langle -e^{\rho_\delta(t-T)}\nabla\Big((\mathbf{b}_\delta^{\text{obs}}(x,t) - b(x,t))\nabla\mathbf{u}\Big), \mathbf{V}_{\mathbf{N}(\delta)}^\delta \right\rangle_{L^2(\Omega)} ds}_{=:\widetilde{A_6}} \end{aligned}$$

$$+\underbrace{2\int_t^T \left\langle e^{\rho_\delta(t-T)}\left[\mathcal{F}_{R_\delta}\left(x,t,\mathbf{u}^\delta_{\mathbf{N}(\delta)}(x,t)\right)-F(x,t;\mathbf{u}(x,t))\right],\mathbf{V}^\delta_{\mathbf{N}(\delta)}\right\rangle_{L^2(\Omega)}ds}_{=:\widetilde{A_7}}. \tag{5.22}$$

First, thanks to inequality (5.8), the expectation of $\widetilde{A_4}$ is estimated as follows:

$$\mathbf{E}\left|\widetilde{A_4}\right| \le \frac{2}{T}\ln\left(\frac{1}{\beta_{\mathbf{N}_\delta}}\right)\int_t^T \mathbf{E}\|\mathbf{V}^\delta_{\mathbf{N}(\delta)}(\cdot,s)\|^2_{L^2(\Omega)}ds, \tag{5.23}$$

Next, using the inequality (5.7) and the Hölder inequality, we have

$$\begin{aligned}\mathbf{E}\left|\widetilde{A_5}\right| &\le \int_t^T e^{2\rho_\beta(s-T)}\frac{\beta_{\mathbf{N}_\delta}}{T}\|\mathbf{u}\|^2_{C([0,T];\mathcal{W}_{MT})}ds+\int_t^T \mathbf{E}\|\mathbf{V}^\delta_{\mathbf{N}(\delta)}(\cdot,s)\|^2_{L^2(\Omega)}ds\\ &\le \frac{\beta_{\mathbf{N}_\delta}}{T}\|\mathbf{u}\|^2_{C([0,T];\mathcal{W}_{MT})}+\int_t^T \mathbf{E}\|\mathbf{V}^\delta_{\mathbf{N}(\delta)}(\cdot,s)\|^2_{L^2(\Omega)}ds. \end{aligned}\tag{5.24}$$

For estimating the expectation of $\left|\widetilde{A_6}\right|$, we use the Green's formula to get the equality

$$\left\langle\nabla\left((\mathbf{b}^{\text{obs}}_\delta(x,t)-b(x,t))\nabla\mathbf{u}\right),\mathbf{V}^\delta_{\mathbf{N}(\delta)}\right\rangle_{L^2(\Omega)}=\left\langle\left((\mathbf{b}^{\text{obs}}_\delta(x,t)-b(x,t)\right)\nabla\mathbf{u},\nabla\mathbf{V}^\delta_{\mathbf{N}(\delta)}\right\rangle_{L^2(\Omega)}$$

then using Hölder's inequality and noting the fact that

$$\int_\Omega|\nabla\mathbf{u}(.,s)|^2dx\le\|\mathbf{u}\|^2_{L^\infty\left(0,T;\mathcal{H}^1_0(\Omega)\right)}=\sup_{0\le s\le T}\int_\Omega|\nabla\mathbf{u}(.,s)|^2dx,$$

we obtain

$$\begin{aligned}\mathbf{E}\left|\widetilde{A_6}\right| &= 2\mathbf{E}\left|\int_t^T\left\langle e^{\rho_\delta(s-T)}\left((\mathbf{b}^{\text{obs}}_\delta(x,t)-b(x,t)\right)\nabla\mathbf{u},\nabla\mathbf{V}^\delta_{\mathbf{N}(\delta)}\right\rangle_{L^2(\Omega)}ds\right|\\ &\le \mathbf{E}\int_t^T\frac{e^{2\rho_\delta(s-T)}}{b_0}\int_\Omega\left((\mathbf{b}^{\text{obs}}_\delta(x,t)-b(x,t)\right)^2|\nabla\mathbf{u}(x,t)|^2\,dxds+\mathbf{E}\int_t^T\int_\Omega b_0\left|\nabla\mathbf{V}^\delta_{\mathbf{N}(\delta)}\right|^2dxds\\ &= \frac{\delta^2\int_t^T|\psi(s)|^2ds\int_\Omega|\nabla\mathbf{u}(.,s)|^2dx}{b_0}+\mathbf{E}\int_t^T\int_\Omega b_0\left|\nabla\mathbf{V}^\delta_{\mathbf{N}(\delta)}\right|^2dxds\\ &\le \frac{\overline{M}^2\delta^2T^2}{2b_0}\|\mathbf{u}\|^2_{L^\infty\left(0,T;\mathcal{H}^1_0(\Omega)\right)}+\mathbf{E}\int_t^T\int_\Omega b_0\left|\nabla\mathbf{V}^\delta_{\mathbf{N}(\delta)}\right|^2dxds; \end{aligned}\tag{5.25}$$

here in the last inequality, we have used the fact that $\mathbf{E}|\psi(s)|^2=s$ since ψ is Brownian motion. Finally, since $\lim_{\delta\to0^+}\mathcal{R}_\delta=+\infty$, for a sufficiently small $\delta>0$, there is an $\mathcal{R}_\delta>0$ such that $\mathcal{R}_\delta\ge\|\mathbf{u}\|_{L^\infty([0,T];L^2(\Omega))}$. For this value of $\mathcal{R}_\delta$ we have

$$\mathcal{F}_{\mathcal{R}_\delta}(x,t;\mathbf{u}(x,t))=F(x,t;\mathbf{u}(x,t)).$$

Using the global Lipschitz property of $\mathcal{F}_\mathcal{R}$ (see Lemma 5.1), one obtains similarly the estimate

$$\begin{aligned}\mathbf{E}\left|\widetilde{A_7}\right| &= 2\mathbf{E}\left|\int_t^T\left\langle e^{\rho_\delta(t-T)}\left[\mathcal{F}_{R_\delta}\left(x,t,\mathbf{u}^\delta_{\mathbf{N}(\delta)}(x,t)\right)-F(x,t;\mathbf{u}(x,t))\right],\mathbf{V}^\delta_{\mathbf{N}(\delta)}\right\rangle_{L^2(\Omega)}ds\right|\\ &\le 2\mathbf{E}\int_t^T\left\|e^{\rho_\delta(t-T)}\left[\mathcal{F}_{R_\delta}\left(x,s,\mathbf{u}^\delta_{\mathbf{N}(\delta)}(x,s)\right)-F(x,s;\mathbf{u}(x,s))\right]\right\|_{L^2(\Omega)}\|\mathbf{V}^\delta_{\mathbf{N}(\delta)}(\cdot,s)\|_{L^2(\Omega)}ds\end{aligned}$$

$$\leq 2K(\mathcal{R}_\delta)\int_t^T \mathbf{E}\|\mathbf{V}^\delta_{\mathbf{N}(\delta)}(\cdot,s)\|^2_{L^2(\Omega)}ds. \tag{5.26}$$

Combining (5.22), (5.23), (5.24),(5.25) and (5.26), and we obtain

$$\begin{aligned}
\mathbf{E}\|\mathbf{V}^\delta_{\mathbf{N}(\delta)}(\cdot,T)\|^2_{L^2(\Omega)} &- \mathbf{E}\|\mathbf{V}^\delta_{\mathbf{N}(\delta)}(\cdot,t)\|^2_{L^2(\Omega)} \\
&+ \int_t^T \left(\frac{\beta_{\mathbf{N}_\delta}}{T}\|\mathbf{u}\|^2_{C([0,T];\mathcal{W}_{MT})} + \frac{\delta^2 T^2}{2b_0}\|\mathbf{u}\|^2_{L^\infty\left(0,T;\mathcal{H}^1_0(\Omega)\right)}\right)ds \\
&\geq 2\mathbf{E}\int_t^T\int_\Omega \left(\mathbf{b}^{\mathrm{obs}}_\delta(x,s)-b_0\right)|\nabla \mathbf{V}^\delta_{\mathbf{N}(\delta)}|^2 dxds \\
&\quad + \mathbf{E}\int_t^T \left(2\rho_\delta - \frac{2}{T}\ln\left(\frac{1}{\beta_{\mathbf{N}_\delta}}\right) - 2K(\mathcal{R}_\delta) - 1\right)\|\mathbf{V}^\delta_{\mathbf{N}(\delta)}(\cdot,s)\|^2_{L^2(\Omega)}ds \\
&\geq \mathbf{E}\int_t^T \left(2\rho_\delta - \frac{2}{T}\ln\left(\frac{1}{\beta_{\mathbf{N}_\delta}}\right) - 2K(\mathcal{R}_\delta) - 1\right)\|\mathbf{V}^\delta_{\mathbf{N}(\delta)}(\cdot,s)\|^2_{L^2(\Omega)}ds.
\end{aligned} \tag{5.27}$$

Thus,

$$\begin{aligned}
\mathbf{E}\|\mathbf{V}^\delta_{\mathbf{N}(\delta)}(\cdot,t)\|^2_{L^2(\Omega)} \leq \mathbf{E}\|\overline{G}_{\delta,\mathbf{N}(\delta)} - g\|^2_{L^2(\Omega)} + \beta_{\mathbf{N}_\delta}\|\mathbf{u}\|^2_{C([0,T];\mathcal{W}_{MT}(\Omega))} + \frac{\delta^2 T^3}{b_0}\|\mathbf{u}\|^2_{L^\infty(0,T;\mathcal{H}^1_0(\Omega))} \\
+ \mathbf{E}\int_t^T \left(-2\rho_\delta + \frac{2}{T}\ln\left(\frac{1}{\beta_{\mathbf{N}_\delta}}\right) + 2K(\mathcal{R}_\delta) + 1\right)\|\mathbf{V}^\delta_{\mathbf{N}(\delta)}(\cdot,s)\|^2_{L^2(\Omega)}ds.
\end{aligned} \tag{5.28}$$

Since $\mathbf{V}^\delta_{\mathbf{N}(\delta)}(x,t) = e^{\rho_\delta(t-T)}\left(\mathbf{u}^\delta_{\mathbf{N}(\delta)}(x,t) - \mathbf{u}(x,t)\right)$ and applying Lemma 4.1, we observe that

$$\begin{aligned}
e^{2\rho_\delta(t-T)}\mathbf{E}\left\|\mathbf{u}^\delta_{\mathbf{N}(\delta)}(\cdot,t) - \mathbf{u}(\cdot,t)\right\|^2_{L^2(\Omega)} &\leq \delta^2\mathbf{N}(\delta) + \frac{1}{\lambda^{2\gamma}_{\mathbf{N}(\delta)}}\|g\|_{H^{2\gamma}(\Omega)} \\
&\quad + \beta_{\mathbf{N}_\delta}\|\mathbf{u}\|^2_{C([0,T];\mathcal{W}_{MT}(\Omega))} + \frac{\overline{M}^2\delta^2 T^3}{b_0}\|\mathbf{u}\|^2_{L^\infty(0,T;\mathcal{H}^1_0(\Omega))} \\
&\quad + (2K(\mathcal{R}_\delta)+1)\int_t^T e^{2\rho_\delta(s-T)}\mathbf{E}\left\|\mathbf{u}^\delta_{\mathbf{N}(\delta)}(\cdot,s) - \mathbf{u}(\cdot,s)\right\|^2_{L^2(\Omega)}ds.
\end{aligned} \tag{5.29}$$

Gronwall's lemma allows us to obtain

$$\begin{aligned}
&e^{2\rho_\delta(t-T)}\mathbf{E}\left\|\mathbf{u}^\delta_{\mathbf{N}(\delta)}(x,t) - \mathbf{u}(x,t)\right\|^2_{L^2(\Omega)} \\
&\leq \left[\delta^2\mathbf{N}(\delta) + \frac{1}{\lambda^{2\gamma}_{\mathbf{N}(\delta)}}\|g\|_{H^{2\gamma}(\Omega)} + \beta_{\mathbf{N}_\delta}\|\mathbf{u}\|^2_{C([0,T];\mathcal{W}_{MT}(\Omega))} + \frac{\overline{M}^2\delta^2T^3}{b_0}\|\mathbf{u}\|^2_{L^\infty(0,T;\mathcal{H}^1_0(\Omega))}\right]e^{(2K(\mathcal{R}_\delta))+1)(T-t)}.
\end{aligned} \tag{5.30}$$

By choosing $\rho_\delta = \frac{1}{T}\ln\left(\frac{1}{\beta_{\mathbf{N}_\delta}}\right) > 0$ we have

$$\mathbf{E}\left\|\mathbf{u}^\delta_{\mathbf{N}(\delta)}(\cdot,t) - \mathbf{u}(\cdot,t)\right\|^2_{L^2(\Omega)} \leq \beta^{\frac{2t}{T}}_{\mathbf{N}(\delta)} e^{(2K(\mathcal{R}_\delta))+1)T}\widetilde{C}(\delta). \tag{5.31}$$

The proof of Theorem 5.1 is complete. □

6. Regularization result with more general source term

In most previous works on backward nonlinear problems the assumption, that the source is global or locally Lipschitz, is required. To the best of our knowledge, this section is the first result when the source term F is not necessarily a locally Lipschitz source. We will solve the problem (1.1) with a special generalized case of source term defined by (5.4). Our regularized problem is different to the one in section 4 because we do not approximate the source function F. Indeed, we have the following regularized problem

$$\begin{cases} \dfrac{\partial \mathbf{u}^{\delta}_{\mathbf{N}(\delta)}}{\partial t} - \nabla\Big(\mathbf{a}^{\text{obs}}_{\delta}(x,t)\nabla \mathbf{u}^{\delta}_{\mathbf{N}(\delta)}\Big) - \mathbf{Q}^{\delta}_{\beta_{\mathbf{N}(\delta)}}(\mathbf{u}^{\delta}_{\mathbf{N}(\delta)})(x,t) \\ \qquad\qquad\qquad\qquad = F\left(x,t,\mathbf{u}^{\delta}_{\mathbf{N}(\delta)}(x,t)\right), \quad (x,t)\in\Omega\times(0,T), \\ \mathbf{u}^{\delta}_{\mathbf{N}(\delta)}|_{\partial\Omega} = 0, \quad t\in(0,T), \\ \mathbf{u}^{\delta}_{\mathbf{N}}(x,T) = \overline{G}_{\delta,\mathbf{N}(\delta)}(x), \quad (x,t)\in\Omega\times(0,T), \end{cases} \tag{6.1}$$

We make the following assumptions on $F\in C^0(\mathbb{R})$ in the following: There exists C_1 and C_1', C_2 and $p>1$ and $\overline{\gamma}$ such that

$$zF(x,t,z) \geq C_1|z|^p - C_1' \tag{6.2}$$

$$|F(x,t,z)| \leq C_2(1+|z|^{p-1}) \tag{6.3}$$

$$(z_1-z_2)\left(F(x,t,z_1)-F(x,t,z_2)\right) \geq -\overline{\gamma}|z_1-z_2|^2. \tag{6.4}$$

It is easy to check that the function $F(x,t,z)=z^{\frac{1}{3}}$ satisfies conditions (6.2), (6.3) and (6.4). Note here that this function is not locally Lipschitz.

Now we have the following result

Theorem 6.1. *Let us assume that F satisfies* (6.2), (6.3) *and* (6.4). *Then, there exists a unique weak solution* $\mathbf{u}^{\delta}_{\mathbf{N}(\delta)}$ *of problem* (6.1) *such that*

$$\mathbf{u}^{\delta}_{\mathbf{N}(\delta)} \in L^2(0,T;H^1)\cap L^\infty(0,T;L^2).$$

Assume that the problem (1.1) *has a unique solution* $\mathbf{u}$ *satisfying* $\mathbf{u}(\cdot,t)\in\mathcal{W}_{MT}$. *Choose* $\beta_{\mathbf{N}_\delta}$ *as in Theorem 5.1. Then we have the following estimate*

$$\mathbf{E}\left\|\mathbf{u}^{\delta}_{\mathbf{N}(\delta)}(x,t) - \boldsymbol{u}(x,t)\right\|^2_{L^2(\Omega)} \leq \beta^{\frac{2t}{T}}_{\mathbf{N}(\delta)} e^{(2\overline{\gamma}+1)T}\widetilde{C}(\delta). \tag{6.5}$$

where $\widetilde{C}(\delta)$ *is defined in* (6.51).

Remark 6.1. *Our method in this theorem give the convergence rate* (6.5) *which is better than the error rate in* (5.12). *Indeed, since* $\lim_{\delta\to 0} K(\mathcal{R}_\delta)=+\infty$, *we have*

$$\frac{\textit{The right hand side of } (5.12)}{\textit{The right hand side of } (6.5)} = \frac{\beta^{\frac{2t}{T}}_{\mathbf{N}(\delta)} e^{(2K(\mathcal{R}_\delta))+1)T}\widetilde{C}(\delta)}{\beta^{\frac{2t}{T}}_{\mathbf{N}(\delta)} e^{(2\overline{\gamma}+1)T}\widetilde{C}(\delta)} \to +\infty \tag{6.6}$$

when $\delta\to 0$.

6.1. Proof of Theorem 6.1

6.1.1. Proof of the existence of solution of Problem (6.1)

First, by changing variable $\mathbf{v}^{\delta}_{\mathbf{N}(\delta)}(x,t) = \mathbf{u}^{\delta}_{\mathbf{N}(\delta)}(x, T-t)$, we transform Problem (6.1) into the initial value problem

$$\begin{cases} \dfrac{\partial \mathbf{v}^{\delta}_{\mathbf{N}(\delta)}}{\partial t} - \nabla\Big(\mathbf{b}^{\text{obs}}_{\delta}(x,t)\nabla \mathbf{v}^{\delta}_{\mathbf{N}(\delta)}\Big) = -F(x,t,\mathbf{v}^{\delta}_{\mathbf{N}}(x,t)) + \mathbf{P}^{\delta}_{\beta_{\mathbf{N}(\delta)}}(\mathbf{v}^{\delta}_{\mathbf{N}(\delta)}(x,t)), \quad (x,t)\in\Omega\times(0,T), \\ \mathbf{v}^{\delta}_{\mathbf{N}(\delta)}|_{\partial\Omega} = 0, \quad t\in(0,T), \\ \mathbf{v}^{\delta}_{\mathbf{N}(\delta)}(x,0) = \overline{G}_{\delta,\mathbf{N}(\delta)}(x), \quad (x,t)\in\Omega\times(0,T). \end{cases} \tag{6.7}$$

where $\mathbf{b}^{\text{obs}}_{\delta}(x,t) = M - \mathbf{a}^{\text{obs}}_{\delta}(x,t)$.
The weak formulation of the initial boundary value problem (6.7) can then be given in the following manner: Find $\mathbf{v}^{\delta}_{\mathbf{N}(\delta)}(t)$ defined in the open set $(0,T)$ such that $\mathbf{v}^{\delta}_{\mathbf{N}(\delta)}$ satisfies the following variational problem

$$\begin{aligned} \int_{\Omega}\frac{d}{dt}\mathbf{v}^{\delta}_{\mathbf{N}(\delta),m}\varphi dx + \int_{\Omega}\mathbf{b}^{\text{obs}}_{\delta}(x,t)\nabla\mathbf{v}^{\delta}_{\mathbf{N}(\delta),m}\nabla\varphi dx + \int_{\Omega}F(\mathbf{v}^{\delta}_{\mathbf{N}(\delta),m}(t))\varphi dx \\ = \int_{\Omega}\mathbf{P}^{\delta}_{\beta_{\mathbf{N}(\delta)}}(\mathbf{v}^{\delta}_{\mathbf{N}(\delta),m}(t))\varphi dx \end{aligned} \tag{6.8}$$

for all $\varphi\in H^1$, and the initial condition

$$\mathbf{v}^{\delta}_{\mathbf{N}(\delta)}(0) = \overline{G}_{\delta,\mathbf{N}(\delta)}. \tag{6.9}$$

Proof of the existence of solution of Problem (6.1) . The main technique of this proof is learned from the article [34]. The proof consists of several steps.

Step 1: *The Faedo – Galerkin approximation* (introduced by Lions [22]).
In the space $H^1(\Omega)$, we take a basis $\{e_j\}_{j=1}^{\infty}$ and define the finite dimensional subspace

$$V_m = \mathrm{s}pan\{e_1, e_2, ...e_m\}.$$

Let $\overline{G}_{\delta,\mathbf{N}(\delta),m}$ be an element of V_m such that

$$\overline{G}_{\delta,\mathbf{N}(\delta),m} = \textstyle\sum_{j=1}^{m} d^{\delta}_{mj}e_j \to \overline{G}_{\delta,\mathbf{N}(\delta)} \text{ strongly in } L^2 \tag{6.10}$$

as $m\to+\infty$. We can express the approximate solution of the problem (6.7) in the form

$$\mathbf{v}^{\delta}_{\mathbf{N}(\delta),\mathbf{m}}(t) = \sum_{j=1}^{m} c^{\delta}_{mj}(t)e_j, \tag{6.11}$$

where the coefficients c^{δ}_{mj} satisfy the system of linear differential equations

$$\begin{aligned} \int_{\Omega}\frac{d}{dt}\mathbf{v}^{\delta}_{\mathbf{N}(\delta),m}e_i dx + \int_{\Omega}\mathbf{b}^{\text{obs}}_{\delta}(x,t)\nabla\mathbf{v}^{\delta}_{\mathbf{N}(\delta),m}\nabla e_i dx + \int_{\Omega}F(\mathbf{v}^{\delta}_{\mathbf{N}(\delta),m}(t))e_i dx \\ = \int_{\Omega}\mathbf{P}^{\delta}_{\beta_{\mathbf{N}(\delta)}}(\mathbf{v}^{\delta}_{\mathbf{N}(\delta),m}(t))e_i dx \end{aligned} \tag{6.12}$$

with $i = \overline{1, m}$ and the initial conditions

$$c_{mj}^{\delta}(0) = d_{mj}^{\delta}, \ j = \overline{1, m}. \tag{6.13}$$

The existence of a local solution of system (6.12)–(6.13) is guaranteed by Peano's theorem on the existence of solutions. For each m there exists a solution $\mathbf{v}_{\mathbf{N}(\delta),\mathbf{m}}^{\delta}(t)$ in the form (6.11) which satisfies (6.12) and (6.13) almost everywhere on $0 \le t \le T_m$ for some T_m, $0 < T_m \le T$. The following estimates allow one to take $T_m = T$ for all m.

Step 2. A priori estimates.

a) **The first estimate**. Multiplying the i^{th} equation of (6.12) by $c_{mi}^{\delta}(t)$ and summing up with respect to i, afterwards, integrating by parts with respect to the time variable from 0 to t, we get after some rearrangements

$$\begin{aligned} &\left\|\mathbf{v}_{\mathbf{N}(\delta),\mathbf{m}}^{\delta}(t)\right\|_{L^2(\Omega)}^2 + 2\int_0^t \int_\Omega \mathbf{b}_\delta^{\text{obs}}(x,t)|\nabla \mathbf{v}_{\mathbf{N}(\delta),m}^{\delta}(s)|^2 dxds + 2\int_0^t \int_\Omega F(\mathbf{v}_{\mathbf{N}(\delta),m}^{\delta}(s))\mathbf{v}_{\mathbf{N}(\delta),m}^{\delta}(s)dxds \\ &= \left\|\overline{G}_{\delta,\mathbf{N}(\delta),m}\right\|^2 + 2\int_0^t \int_\Omega \mathbf{P}_{\beta_{\mathbf{N}(\delta)}}^{\delta}(\mathbf{v}_{\mathbf{N}(\delta),m}^{\delta}(s))\mathbf{v}_{\mathbf{N}(\delta),m}^{\delta}(s)dxds \end{aligned} \tag{6.14}$$

From (6.10), we have

$$\left\|\overline{G}_{\delta,\mathbf{N}(\delta),m}\right\|^2 \le B_0(\delta), \ \text{for all } \ m, \tag{3.8}$$

where $B_0(\delta)$ depends on $\overline{G}_{\delta,\mathbf{N}(\delta)}$ and is independent of m.
Using the lower bound of $\mathbf{b}_\delta^{\text{obs}}(x,t)$, we have the following estimate

$$2\int_0^t \int_\Omega \mathbf{b}_\delta^{\text{obs}}(x,t)|\nabla \mathbf{v}_{\mathbf{N}(\delta),m}^{\delta}(s)|^2 dxds \ge 2b_0 \int_0^t \|\mathbf{v}_{\mathbf{N}(\delta),m}^{\delta}(s)\|_{H^1(\Omega)} ds. \tag{6.15}$$

Using the assumption on F, we have

$$2\int_0^t \int_\Omega F(\mathbf{v}_{\mathbf{N}(\delta),m}^{\delta}(s))\mathbf{v}_{\mathbf{N}(\delta),m}^{\delta}(s)dxds \ge 2C_1 \int_0^t \left\|\mathbf{v}_{\mathbf{N}(\delta),m}^{\delta}(s)\right\|_{L^p(\Omega)}^p ds - 2TC_1' \tag{6.16}$$

and

$$2\int_0^t \int_\Omega \mathbf{P}_{\beta_{\mathbf{N}(\delta)}}^{\delta}(\mathbf{v}_{\mathbf{N}(\delta),m}^{\delta}(s))\mathbf{v}_{\mathbf{N}(\delta),m}^{\delta}(s)dxds \le \frac{2}{T}\ln\left(\frac{1}{\beta_{\mathbf{N}(\delta)}}\right)\int_0^t \left\|\mathbf{v}_{\mathbf{N}(\delta),m}^{\delta}(s)\right\|_{L^2(\Omega)}^2 ds. \tag{6.17}$$

Hence, it follows from (6.15)–(6.17) that

$$\begin{aligned} &\left\|\mathbf{v}_{\mathbf{N}(\delta),\mathbf{m}}^{\delta}(t)\right\|_{L^2(\Omega)}^2 + 2b_0 \int_0^t \|\mathbf{v}_{\mathbf{N}(\delta),m}^{\delta}(s)\|_{H^1(\Omega)} ds + 2C_1 \int_0^t \left\|\mathbf{v}_{\mathbf{N}(\delta),m}^{\delta}(s)\right\|_{L^p(\Omega)}^p ds \\ &\le B_0(\delta) + 2TC_1' + \frac{1}{T}\ln\left(\frac{1}{\beta_{\mathbf{N}(\delta)}}\right)\int_0^t \left\|\mathbf{v}_{\mathbf{N}(\delta),m}^{\delta}(s)\right\|_{L^2(\Omega)}^2 ds. \end{aligned} \tag{6.18}$$

Let

$$\mathbf{S}_m^{\delta}(t) = \left\|\mathbf{v}_{\mathbf{N}(\delta),\mathbf{m}}^{\delta}(t)\right\|_{L^2(\Omega)}^2 + 2b_0 \int_0^t \|\mathbf{v}_{\mathbf{N}(\delta),m}^{\delta}(s)\|_{H^1(\Omega)} ds + 2C_1 \int_0^t \left\|\mathbf{v}_{\mathbf{N}(\delta),m}^{\delta}(s)\right\|_{L^p(\Omega)}^p ds. \tag{6.19}$$

Using the fact that $\int_0^t \left\|\mathbf{v}_{\mathbf{N}(\delta),m}^{\delta}(s)\right\|_{L^2(\Omega)}^2 ds \le \int_0^t \mathbf{S}_m^{\delta}(s)ds$, we know from (6.18) that

$$\mathbf{S}_m^{\delta}(t) \le B_0(\delta) + 2TC_1' + \frac{1}{T}\ln\left(\frac{1}{\beta_{\mathbf{N}(\delta)}}\right)\int_0^t \mathbf{S}_m^{\delta}(s)ds \tag{6.20}$$

Applying Gronwall's lemma, and we obtain

$$\mathbf{S}_m^\delta(t) \le [B_0(\delta) + 2TC_1'] \exp\left(\frac{t}{T}\ln\left(\frac{1}{\beta_{\mathbf{N}(\delta)}}\right)\right) \le [B_0(\delta) + 2TC_1'] \exp\left(\ln\left(\frac{1}{\beta_{\mathbf{N}(\delta)}}\right)\right) = B_1(\delta, T), \quad (6.21)$$

for all $m \in \mathbb{N}$, for all t, $0 \le t \le T_m \le T$, i.e., $T_m = T$, where C_T always indicates a bound depending on T.

b) **The second estimate**. Multiplying the i^{th} equation of (6.12) by $t^2 \frac{d}{dt} c_{mi}^\delta(t)$ and summing up with respect to i, we have

$$\begin{aligned}\left\| t\frac{d}{dt}\mathbf{v}_{\mathbf{N}(\delta),m}^\delta(t)\right\|_{L^2(\Omega)}^2 &+ t^2\int_\Omega \mathbf{b}_\delta^{\text{obs}}(x,t)\nabla \mathbf{v}_{\mathbf{N}(\delta),m}^\delta(t)\nabla\left(\frac{d}{dt}\nabla\mathbf{v}_{\mathbf{N}(\delta),m}^\delta(t)\right)dx \\ &+ \int_\Omega t^2 F\left(\mathbf{v}_{\mathbf{N}(\delta),m}^\delta(t)\right)\frac{d}{dt}\mathbf{v}_{\mathbf{N}(\delta),m}^\delta(t)dx \\ &= \int_\Omega t^2 \mathbf{P}_{\beta_{\mathbf{N}(\delta)}}^\delta\left(\mathbf{v}_{\mathbf{N}(\delta),m}^\delta(t)\right)\frac{d}{dt}\mathbf{v}_{\mathbf{N}(\delta),m}^\delta(t)dx. \end{aligned} \quad (6.22)$$

It is easy to check that for any $u \in H^1(\Omega)$

$$\frac{d}{dt}\left[\int_\Omega \mathbf{b}_\delta^{\text{obs}}(x,t)|\nabla u(t)|^2dx\right] = 2\int_\Omega \mathbf{b}_\delta^{\text{obs}}(x,t)\nabla u(t)\nabla u'(t)dx + \int_\Omega \frac{\partial}{\partial t}\mathbf{b}_\delta^{\text{obs}}(x,t)|\nabla u(t)|^2dx. \quad (6.23)$$

The equality (6.22) is equivalent to

$$\begin{aligned} 2\left\| t\frac{d}{dt}\mathbf{v}_{\mathbf{N}(\delta),m}^\delta(t)\right\|_{L^2(\Omega)}^2 &+ \frac{d}{dt}\left[t^2\int_\Omega \mathbf{b}_\delta^{\text{obs}}(x,t)|\mathbf{v}_{\mathbf{N}(\delta),m}^\delta(t)|^2dx\right] + 2\int_\Omega t^2F\left(\mathbf{v}_{\mathbf{N}(\delta),m}^\delta(t)\right)\frac{d}{dt}\mathbf{v}_{\mathbf{N}(\delta),m}^\delta(t)dx \\ &= 2t\int_\Omega \mathbf{b}_\delta^{\text{obs}}(x,t)|\nabla\mathbf{v}_{\mathbf{N}(\delta),m}^\delta(t)|^2dx + t^2\int_\Omega \frac{\partial}{\partial t}\mathbf{b}_\delta^{\text{obs}}(x,t)|\nabla\mathbf{v}_{\mathbf{N}(\delta),m}^\delta(s)|^2dx \\ &+ \int_\Omega t^2\mathbf{P}_{\beta_{\mathbf{N}(\delta)}}^\delta\left(\mathbf{v}_{\mathbf{N}(\delta),m}^\delta(t)\right)\frac{d}{dt}\mathbf{v}_{\mathbf{N}(\delta),m}^\delta(t)dx. \end{aligned} \quad (6.24)$$

By integrating the last equality from 0 to t, we get

$$\begin{aligned} &2\int_0^t \left\| s\frac{d}{ds}\mathbf{v}_{\mathbf{N}(\delta),m}^\delta(s)\right\|_{L^2(\Omega)}^2 ds + \underbrace{t^2\int_\Omega \mathbf{b}_\delta^{\text{obs}}(x,t)|\mathbf{v}_{\mathbf{N}(\delta),m}^\delta(t)|^2dx}_{I_1} \\ &+ \underbrace{2\int_0^t\int_\Omega s^2F\left(\mathbf{v}_{\mathbf{N}(\delta),m}^\delta(s)\right)\frac{d}{ds}\mathbf{v}_{\mathbf{N}(\delta),m}^\delta(s)dxds}_{I_2} \\ &= \underbrace{2\int_0^t\int_\Omega s\mathbf{b}_\delta^{\text{obs}}(x,s)|\nabla\mathbf{v}_{\mathbf{N}(\delta),m}^\delta(s)|^2dxds}_{I_3} + \underbrace{\int_0^t\int_\Omega s^2\frac{\partial}{\partial s}\mathbf{b}_\delta^{\text{obs}}(x,s)|\nabla\mathbf{v}_{\mathbf{N}(\delta),m}^\delta(s)|^2dxds}_{I_4} \\ &+ \underbrace{\int_0^t\int_\Omega s^2\mathbf{P}_{\beta_{\mathbf{N}(\delta)}}^\delta\left(\mathbf{v}_{\mathbf{N}(\delta),m}^\delta(s)\right)\frac{d}{ds}\mathbf{v}_{\mathbf{N}(\delta),m}^\delta(s)dxds}_{I_5}. \end{aligned} \quad (6.25)$$

Estimate I_1. Since the assumption $\mathbf{b}_\delta^{\text{obs}}(x,t) \ge b_0$, we know that

$$I_1 = t^2\int_\Omega \mathbf{b}_\delta^{\text{obs}}(x,t)|\mathbf{v}_{\mathbf{N}(\delta),m}^\delta(t)|^2dx \ge b_0\left\|t\mathbf{v}_{\mathbf{N}(\delta),m}^\delta(t)\right\|_{H^1}^2. \quad (6.26)$$

Estimate I_2. To estimate I_2, we need the following Lemma

Lemma 6.1. *Let* $\mu_0 = \left(\frac{C_1'}{C_1}\right)^{1/p}$, $\overline{m} = \int_{-\mu_0}^{+\mu_0} |F(\xi)| d\xi$, $\widetilde{F}(z) = \int_0^z F(y)dy$, $z \in \mathbb{R}$. *Then we get*

$$-\overline{m} \leq \widetilde{F}(z) \leq C_2 \left(|z| + \frac{1}{p}|z|^p\right), \quad z \in \mathbb{R}. \tag{6.27}$$

The proof of Lemma 6.1 is easy and we omit it here. Now we return to estimate I_2. By a simple computation and then using Lemma 6.1, we have

$$\begin{aligned}
I_2 &= 2\int_0^t s^2 ds \frac{d}{ds}\left[\int_\Omega dx \int_0^{\mathbf{v}^\delta_{\mathbf{N}(\delta),m}(x,s)} F(y)dy\right] = 2\int_0^t s^2 ds \frac{d}{ds}\left[\int_\Omega dx \int_0^{\mathbf{v}^\delta_{\mathbf{N}(\delta),\mathbf{m}}(x,s)} F(y)dy\right] \\
&= 2\int_0^t \left[\frac{d}{ds}\left(s^2 \int_\Omega \widetilde{F}\left(\mathbf{v}^\delta_{\mathbf{N}(\delta),\mathbf{m}}(x,s)\right)dx\right) - 2s\int_\Omega \widetilde{F}\left(\mathbf{v}^\delta_{\mathbf{N}(\delta),\mathbf{m}}(x,s)\right)dx\right] \\
&= 2t^2 \int_\Omega \widetilde{F}\left(\mathbf{v}^\delta_{\mathbf{N}(\delta),\mathbf{m}}(x,t)\right)dx - 4\int_0^t sds \int_\Omega \widetilde{F}\left(\mathbf{v}^\delta_{\mathbf{N}(\delta),\mathbf{m}}(x,s)\right)dx \\
&\geq -2T^2\overline{m}|\Omega| - 4C_2 \int_0^t s\left[\left\|\mathbf{v}^\delta_{\mathbf{N}(\delta),\mathbf{m}}(s)\right\|_{L^1} + \frac{1}{p}\left\|\mathbf{v}^\delta_{\mathbf{N}(\delta),\mathbf{m}}(s)\right\|^p_{L^p}\right]ds \\
&\geq -2T^2\overline{m}|\Omega| - 4TC_2\left[T\left\|\mathbf{v}^\delta_{\mathbf{N}(\delta),\mathbf{m}}\right\|_{L^\infty(0,T;L^2)} + \frac{1}{p}\frac{1}{2C_1}\mathbf{S}^\delta_m(t)\right] \\
&\geq -B_2(\delta, T).
\end{aligned} \tag{6.28}$$

Estimate I_3. Using (6.19), we have the following estimate

$$I_3 \leq 2Tb_1 \int_0^t \|\mathbf{v}^\delta_{\mathbf{N}(\delta),\mathbf{m}}(s)\|^2_{H^1} ds \leq \frac{2Tb_1}{2b_0}\mathbf{S}^\delta_m(t). \tag{6.29}$$

Estimate I_4. Let us set

$$\widetilde{a}_T = \sup_{(x,t)\in[0,1]\times[0,T]} \frac{\partial}{\partial t}\mathbf{b}^{\text{obs}}_\delta(x,t),$$

and then I_4 is bounded by

$$I_4 \leq \widetilde{a}_T \int_0^t \left\|s\mathbf{v}^\delta_{\mathbf{N}(\delta),\mathbf{m}}(s)\right\|^2_{H^1} ds \leq T^2\widetilde{a}_T \int_0^t \left\|\mathbf{v}^\delta_{\mathbf{N}(\delta),\mathbf{m}}(s)\right\|^2_{H^1} ds \leq \frac{T^2\widetilde{a}_T}{a_0}\mathbf{S}^\delta_m(t). \tag{6.30}$$

Estimate I_5. Using Lemma 5.2, we obtain the following estimate for I_5:

$$\begin{aligned}
I_5 &\leq 2\int_0^t \|s\mathbf{P}^\delta_{\beta_{\mathbf{N}(\delta)}}(\mathbf{v}^\delta_{\mathbf{N}(\delta),\mathbf{m}}(s))\|\|s\frac{d}{ds}\mathbf{v}^\delta_{\mathbf{N}(\delta),\mathbf{m}}(s)\|ds \\
&\leq \int_0^t \|s\mathbf{P}^\delta_{\beta_{\mathbf{N}(\delta)}}(\mathbf{v}^\delta_{\mathbf{N}(\delta),\mathbf{m}}(s))\|^2 ds + \int_0^t \|s\frac{d}{ds}\mathbf{v}^\delta_{\mathbf{N}(\delta),\mathbf{m}}(s)\|^2 ds \\
&\leq \ln^2\left(\frac{1}{\beta_{\mathbf{N}(\delta)}}\right)\int_0^t \|\mathbf{v}^\delta_{\mathbf{N}(\delta),\mathbf{m}}(s)\|^2 ds + \int_0^t \|s\frac{d}{ds}\mathbf{v}^\delta_{\mathbf{N}(\delta),\mathbf{m}}(s)\|^2 ds \\
&\leq \ln^2\left(\frac{1}{\beta_{\mathbf{N}(\delta)}}\right)\frac{\mathbf{S}^\delta_m(t)}{a_0} + \int_0^t \|s\frac{d}{ds}\mathbf{v}^\delta_{\mathbf{N}(\delta),\mathbf{m}}(s)\|^2 ds
\end{aligned} \tag{6.31}$$

Combining (6.26), (6.28), (6.29), (6.30),we obtain

$$2\int_0^t \left\|s\frac{d}{ds}\mathbf{v}^\delta_{\mathbf{N}(\delta),m}(s)\right\|^2_{L^2(\Omega)} ds + b_0\left\|t\mathbf{v}^\delta_{\mathbf{N}(\delta),m}(t)\right\|^2_{H^1}$$

$$\leq B_2(\delta,T)+\frac{2Tb_1}{2b_0}\mathbf{S}_m^\delta(t)+\frac{T^2\widetilde{a}_T}{a_0}\mathbf{S}_m^\delta(t)$$
$$+\ln^2\left(\frac{1}{\beta_{\mathbf{N}(\delta)}}\right)\frac{\mathbf{S}_m^\delta(t)}{a_0}+\int_0^t \|s\frac{d}{ds}\mathbf{v}^\delta_{\mathbf{N}(\delta),\mathbf{m}}(s)\|^2 ds. \tag{6.32}$$

Let

$$\mathbf{R}_m^\delta(t)=\int_0^t \left\|s\frac{d}{ds}\mathbf{v}^\delta_{\mathbf{N}(\delta),m}(s)\right\|^2_{L^2(\Omega)} ds+\left\|t\mathbf{v}^\delta_{\mathbf{N}(\delta),m}(t)\right\|^2_{H^1(\Omega)},$$

and then since

$$\int_0^t \mathbf{R}_m^\delta(s)ds\geq \int_0^t \|s\frac{d}{ds}\mathbf{v}^\delta_{\mathbf{N}(\delta),\mathbf{m}}(s)\|^2 ds$$

together with (6.32), we deduce that

$$\mathbf{R}_m^\delta(t)\leq \frac{B(3,\delta)}{\min(2,b_0)}+\frac{1}{\min(2,b_0)}\int_0^t \mathbf{R}_m^\delta(s)ds \tag{6.33}$$

where

$$B(3,\delta)=B_2(\delta,T)+\frac{2Tb_1}{2b_0}B(2,\delta)+\frac{T^2\widetilde{a}_T}{a_0}B(2,\delta)+\ln^2\left(\frac{1}{\beta_{\mathbf{N}(\delta)}}\right)\frac{B(2,\delta)}{a_0}.$$

Applying Gronwall's inequlality, we obtain that

$$\int_0^t \left\|s\frac{d}{ds}\mathbf{v}^\delta_{\mathbf{N}(\delta),m}(s)\right\|^2_{L^2(\Omega)} ds+\left\|t\mathbf{v}^\delta_{\mathbf{N}(\delta),m}(t)\right\|^2_{H^1(\Omega)}\leq B_4(\delta,T), \tag{6.34}$$

where $B(4,\delta)$ depends only on δ,T and does not depend on m.

Step 3. *The limiting process*.

Combining (6.19), (6.21) and (6.34), we deduce that, there exists a subsequence of $\{\mathbf{v}^\delta_{\mathbf{N}(\delta),m}\}$ still denoted by $\{\mathbf{v}^\delta_{\mathbf{N}(\delta),m}\}$ such that (see [22]), say,

$$\begin{cases} \mathbf{v}^\delta_{\mathbf{N}(\delta),m}\to \mathbf{v}^\delta_{\mathbf{N}(\delta)} & \text{in } \; L^\infty(0,T;L^2) \; \text{weak*}, \\ \mathbf{v}^\delta_{\mathbf{N}(\delta),m}\to \mathbf{v}^\delta_{\mathbf{N}(\delta)} & \text{in } \; L^2(0,T;H^1) \; \text{weak}, \\ t\mathbf{v}^\delta_{\mathbf{N}(\delta),m}\to t\mathbf{v}^\delta_{\mathbf{N}(\delta)} & \text{in } \; L^\infty(0,T;H^1) \; \text{weak*}, \\ \left(t\mathbf{v}^\delta_{\mathbf{N}(\delta),m}\right)'\to \left(t\mathbf{v}^\delta_{\mathbf{N}(\delta)}\right)' & \text{in } \; L^2(Q_T) \; \text{weak}, \\ \mathbf{v}^\delta_{\mathbf{N}(\delta),m}\to \mathbf{v}^\delta_{\mathbf{N}(\delta)} & \text{in } \; L^p(Q_T) \; \text{weak}; \end{cases} \tag{6.35}$$

here $Q_T=\Omega\times(0,T)$. Using a compactness lemma ([22], Lions, p. 57) applied to (6.35), we can extract from the sequence $\{\mathbf{v}^\delta_{\mathbf{N}(\delta),m}\}$ a subsequence still denoted by $\{\mathbf{v}^\delta_{\mathbf{N}(\delta),m}\}$ such that

$$\left(t\mathbf{v}^\delta_{\mathbf{N}(\delta),m}\right)'\to \left(t\mathbf{v}^\delta_{\mathbf{N}(\delta)}\right)' \;\text{ strongly in } L^2(Q_T). \tag{6.36}$$

By the Riesz-Fischer theorem, we can extract from $\{\mathbf{v}^\delta_{\mathbf{N}(\delta),m}\}$ a subsequence still denoted by $\{\mathbf{v}^\delta_{\mathbf{N}(\delta),m}\}$ such that

$$\mathbf{v}^\delta_{\mathbf{N}(\delta),m}(x,t)\to \mathbf{v}^\delta_{\mathbf{N}(\delta)}(x,t) \;\text{ a.e. } \;(x,t)\; \text{ in } \; Q_T=\Omega\times(0,T). \tag{6.37}$$

Because F is continuous, then

$$F\left(x,t,\mathbf{v}^\delta_{\mathbf{N}(\delta),m}(x,t)\right)\to F\left(x,t,\mathbf{v}^\delta_{\mathbf{N}(\delta)}(x,t)\right) \;\text{ a.e. } \;(x,t)\; \text{ in } \; Q_T=\Omega\times(0,T). \tag{6.38}$$

On the other hand, using (6.3), (6.19), (6.21) , we obtain

$$\left\| F\left(\mathbf{v}^{\delta}_{\mathbf{N}(\delta),m}(x,t)\right)\right\|_{L^{p'}(Q_T)} \leq B_5(\delta,T), \tag{6.39}$$

where $B_5(\delta,T)$ is a constant independent of m. We shall now require the following lemma, the proof of which can be found in [22] (see Lemma 1.3).

Lemma 6.2. *Let Q be a bounded open subset of $\mathbb{R}^N$ and $G_m, G \in L^q(Q)$, $1 < q < \infty$, such that*

$$\|G_m\|_{L^q(Q)} \leq C, \text{ where } C \text{ is a constant independent of } m \tag{6.40}$$

and

$$G_m \to G \text{ a.e. } (x,t) \text{ in } Q.$$

Then

$$G_m \to G \text{ in } L^q(Q) \text{weakly.}$$

Applying Lemma 6.2 with $q = p' = \frac{p}{p-1}$, $G_m = F\left(\mathbf{v}^{\delta}_{\mathbf{N}(\delta),m}(x,t)\right)$, $G = F\left(\mathbf{v}^{\delta}_{\mathbf{N}(\delta)}(x,t)\right)$, we deduce from (6.38) and (6.39) that

$$F\left(\mathbf{v}^{\delta}_{\mathbf{N}(\delta),m}\right) \to F\left(\mathbf{v}^{\delta}_{\mathbf{N}(\delta)}\right) \text{ in } L^{p'}(Q) \text{ weakly.} \tag{6.41}$$

Passing to the limit in (6.12) and (6.10) by (6.35) and (6.41), we have established a solution of Problem (6.1) .

6.1.2. Proof of the uniqueness of solution of Problem (6.1)

Assume that the Problem (6.1) has two solution $\mathbf{v}^{\delta}_{\mathbf{N}(\delta)}$ and $\mathbf{w}^{\delta}_{\mathbf{N}(\delta)}$. We have to show that $\mathbf{v}^{\delta}_{\mathbf{N}(\delta)} = \mathbf{w}^{\delta}_{\mathbf{N}(\delta)}$. We recall that

$$\begin{cases} \dfrac{\partial \mathbf{v}^{\delta}_{\mathbf{N}(\delta)}}{\partial t} + \nabla\Big(\mathbf{b}^{\text{obs}}_{\delta}(x,t)\nabla \mathbf{v}^{\delta}_{\mathbf{N}(\delta)}\Big) = F\left(x,t,\mathbf{v}^{\delta}_{\mathbf{N}(\delta)}(x,t)\right) + \mathbf{P}^{\delta}_{\beta_{\mathbf{N}(\delta)}}\mathbf{v}^{\delta}_{\mathbf{N}(\delta)}, \\ \dfrac{\partial \mathbf{w}^{\delta}_{\mathbf{N}(\delta)}}{\partial t} + \nabla\Big(\mathbf{b}^{\text{obs}}_{\delta}(x,t)\nabla \mathbf{w}^{\delta}_{\mathbf{N}(\delta)}\Big) = F\left(x,t,\mathbf{w}^{\delta}_{\mathbf{N}(\delta)}(x,t)\right) + \mathbf{P}^{\delta}_{\beta_{\mathbf{N}(\delta)}}\mathbf{w}^{\delta}_{\mathbf{N}(\delta)} \\ \mathbf{u}^{\delta}_{\mathbf{N}(\delta)}(x,T) = \mathbf{w}^{\delta}_{\mathbf{N}(\delta)} = \overline{G}_{\delta,\mathbf{N}(\delta)}(x), \end{cases} \tag{6.42}$$

For $\overline{R}_\delta > 0$, we put

$$\mathbf{W}^{\delta}_{\mathbf{N}(\delta)}(x,t) = e^{\overline{R}_\delta(t-T)}\left[\mathbf{v}^{\delta}_{\mathbf{N}(\delta)}(x,t) - \mathbf{w}^{\delta}_{\mathbf{N}(\delta)}(x,t)\right].$$

Then for $(x,t) \in \Omega \times (0,T)$, we get

$$\begin{aligned} &\frac{\partial \mathbf{W}^{\delta}_{\mathbf{N}(\delta)}}{\partial t} + \nabla\Big(\mathbf{b}^{\text{obs}}_{\delta}(x,t)\nabla \mathbf{W}^{\delta}_{\mathbf{N}(\delta)}\Big) - \overline{R}_\delta \mathbf{W}^{\delta}_{\mathbf{N}(\delta)} \\ &\qquad = \mathbf{P}^{\delta}_{\beta_{\mathbf{N}(\delta)}}\mathbf{W}^{\delta}_{\mathbf{N}(\delta)} + e^{\overline{R}_\delta(t-T)}\left[F\left(x,t,\mathbf{v}^{\delta}_{\mathbf{N}(\delta)}(x,t)\right) - F\left(x,t,\mathbf{w}^{\delta}_{\mathbf{N}(\delta)}(x,t)\right)\right], \end{aligned} \tag{6.43}$$

and

$$\mathbf{W}^{\delta}_{\mathbf{N}(\delta)}|_{\partial\Omega} = 0, \ \mathbf{W}^{\delta}_{\mathbf{N}(\delta)}(x,T) = 0.$$

By taking the inner product of both sides of (6.43) with $\mathbf{W}^{\delta}_{\mathbf{N}(\delta)}$ then taking the integral from t to T and noting the equality

$$\int_\Omega \nabla\Big(\mathbf{b}^{\text{obs}}_{\delta}(x,t)\nabla \mathbf{W}^{\delta}_{\mathbf{N}(\delta)}\Big)\mathbf{V}^{\delta}_{\mathbf{N}(\delta)}dx = -\int_\Omega \mathbf{b}^{\text{obs}}_{\delta}(x,t)|\nabla \mathbf{W}^{\delta}_{\mathbf{N}(\delta)}|^2 dx,$$

we deduce

$$
\begin{aligned}
&\|\mathbf{W}^{\delta}_{\mathbf{N}(\delta)}(.,T)\|^2_{L^2(\Omega)} - \|\mathbf{W}^{\delta}_{\mathbf{N}(\delta)}(.,t)\|^2_{L^2(\Omega)} \\
&= 2\int_t^T \int_\Omega \mathbf{P}^{\delta}_{\beta_{\mathbf{N}(\delta)}} \mathbf{W}^{\delta}_{\mathbf{N}(\delta)}(x,s)dxds + 2\int_t^T \int_\Omega \mathbf{b}^{\text{obs}}_{\delta}(x,s)|\nabla \mathbf{W}^{\delta}_{\mathbf{N}(\delta)}|^2 dxds + 2\overline{R}_\delta \int_t^T \|\mathbf{W}^{\delta}_{\mathbf{N}(\delta)}(.,s)\|^2_{L^2(\Omega)}ds \\
&+ 2\int_t^T \int_\Omega e^{\overline{R}_\delta(t-T)} \left[F\left(x,s,\mathbf{v}^{\delta}_{\mathbf{N}(\delta)}(x,s)\right) - F\left(x,s,\mathbf{w}^{\delta}_{\mathbf{N}(\delta)}(x,s)\right)\right] \mathbf{W}^{\delta}_{\mathbf{N}(\delta)}(x,s)dxds \\
&\geq 2\int_t^T \int_\Omega \mathbf{P}^{\delta}_{\beta_{\mathbf{N}(\delta)}} \mathbf{W}^{\delta}_{\mathbf{N}(\delta)}(x,s)dxds + 2\overline{R}_\delta \int_t^T \|\mathbf{W}^{\delta}_{\mathbf{N}(\delta)}(.,s)\|^2_{L^2(\Omega)}ds \\
&+ 2\int_t^T \int_\Omega e^{\overline{R}_\delta(t-T)} \left[F\left(x,s,\mathbf{v}^{\delta}_{\mathbf{N}(\delta)}(x,s)\right) - F\left(x,s,\mathbf{w}^{\delta}_{\mathbf{N}(\delta)}(x,s)\right)\right] \mathbf{W}^{\delta}_{\mathbf{N}(\delta)}(x,s)dxds. \qquad (6.44)
\end{aligned}
$$

By the assumption we have

$$
\begin{aligned}
&\int_t^T \int_\Omega e^{\overline{R}_\delta(s-T)} \left[F\left(x,s,\mathbf{v}^{\delta}_{\mathbf{N}(\delta)}(x,s)\right) - F\left(x,s,\mathbf{w}^{\delta}_{\mathbf{N}(\delta)}(x,s)\right)\right] \mathbf{W}^{\delta}_{\mathbf{N}(\delta)}(x,s)dxds \\
&= \int_t^T \int_\Omega e^{\overline{R}_\delta(s-T)} \left[F\left(x,s,\mathbf{v}^{\delta}_{\mathbf{N}(\delta)}(x,s)\right) - F\left(x,s,\mathbf{w}^{\delta}_{\mathbf{N}(\delta)}(x,s)\right)\right] e^{\overline{R}_\delta(s-T)}\left[\mathbf{v}^{\delta}_{\mathbf{N}(\delta)}(x,s) - \mathbf{w}^{\delta}_{\mathbf{N}(\delta)}(x,s)\right] dxds \\
&\geq -\overline{\gamma} \int_t^T \int_\Omega e^{2\overline{R}_\delta(s-T)} \left[\mathbf{v}^{\delta}_{\mathbf{N}(\delta)}(x,s) - \mathbf{w}^{\delta}_{\mathbf{N}(\delta)}(x,s)\right]^2 dxds \\
&= -\overline{\gamma} \int_t^T \|\mathbf{W}^{\delta}_{\mathbf{N}(\delta)}(.,s)\|^2_{L^2(\Omega)}ds. \qquad (6.45)
\end{aligned}
$$

Using the inequality (5.8), we get the following estimate

$$
\int_t^T \int_\Omega \mathbf{P}^{\delta}_{\beta_{\mathbf{N}(\delta)}} \mathbf{W}^{\delta}_{\mathbf{N}(\delta)}(x,s)dxds \geq -\frac{2}{T}\ln\left(\frac{1}{\beta_{\mathbf{N}_\delta}}\right) \int_t^T \|\mathbf{W}^{\delta}_{\mathbf{N}(\delta)}(.,s)\|^2_{L^2(\Omega)}ds. \qquad (6.46)
$$

Combine equations (6.44), (6.48), (6.46) and choose

$$
\overline{R}_\delta = \frac{1}{T}\ln\left(\frac{1}{\beta_{\mathbf{N}_\delta}}\right) + \gamma
$$

to obtain

$$
\|\mathbf{W}^{\delta}_{\mathbf{N}(\delta)}(.,T)\|^2_{L^2(\Omega)} - \|\mathbf{W}^{\delta}_{\mathbf{N}(\delta)}(.,t)\|^2_{L^2(\Omega)} \geq 0
$$

This implies that for all $t \in [0,T]$ then $\|\mathbf{W}^{\delta}_{\mathbf{N}(\delta)}(.,t)\|^2_{L^2(\Omega)} = 0$ since $\mathbf{W}^{\delta}_{\mathbf{N}(\delta)}(x,T) = 0$. The proof is completed.

6.1.3. Convergence estimate

Our analysis and proof is short and similar to the proof of Theorem 5.1. Indeed, let us also set

$$
\mathbf{V}^{\delta}_{\mathbf{N}(\delta)}(x,t) = e^{\rho_\delta(t-T)}\left[\mathbf{u}^{\delta}_{\mathbf{N}(\delta)}(x,t) - \mathbf{u}(x,t)\right].
$$

By using some of the above steps we obtain

$$
\|\mathbf{V}^{\delta}_{\mathbf{N}(\delta)}(\cdot,T)\|^2_{L^2(\Omega)} - \|\mathbf{V}^{\delta}_{\mathbf{N}(\delta)}(\cdot,t)\|^2_{L^2(\Omega)}
$$

$$= \widetilde{A}_4 + \widetilde{A}_5 + \widetilde{A}_6 + \underbrace{2\int_t^T \left\langle e^{\rho_\delta(s-T)}\left[F\left(x,s,\mathbf{u}^\delta_{\mathbf{N}(\delta)}(x,s)\right) - F(x,s;\mathbf{u}(x,s))\right], \mathbf{V}^\delta_{\mathbf{N}(\delta)}\right\rangle_{L^2(\Omega)} ds}_{=:\widetilde{A}_8} \tag{6.47}$$

The terms $\widetilde{A}_4, \widetilde{A}_5, \widetilde{A}_6$ are similar to (5.22). Now, we consider $\widetilde{A}_8$. By assumption (6.4), we have

$$\begin{aligned}
&\int_t^T \int_\Omega e^{\overline{R}_\delta(s-T)}\left[F\left(x,s,\mathbf{u}^\delta_{\mathbf{N}(\delta)}(x,s)\right) - F(x,s,\mathbf{u}(x,s))\right]\mathbf{V}^\delta_{\mathbf{N}(\delta)}(x,s)dxds\\
&= \int_t^T \int_\Omega e^{\overline{R}_\delta(s-T)}\left[F\left(x,s,\mathbf{u}^\delta_{\mathbf{N}(\delta)}(x,s)\right) - F(x,s,\mathbf{u}(x,s))\right] e^{\overline{R}_\delta(s-T)}\left[\mathbf{u}^\delta_{\mathbf{N}(\delta)}(x,s) - \mathbf{u}(x,s)\right]dxds\\
&\geq -\overline{\gamma}\int_t^T \int_\Omega e^{2\overline{R}_\delta(s-T)}\left[\mathbf{u}^\delta_{\mathbf{N}(\delta)}(x,s) - \mathbf{u}(x,s)\right]^2 dxds\\
&= -\overline{\gamma}\int_t^T \|\mathbf{V}^\delta_{\mathbf{N}(\delta)}(.,s)\|^2_{L^2(\Omega)}ds.
\end{aligned} \tag{6.48}$$

After using the results of the proof of Theorem 5.1, we get

$$\begin{aligned}
\mathbf{E}\|\mathbf{V}^\delta_{\mathbf{N}(\delta)}(\cdot,t)\|^2_{L^2(\Omega)} \leq{}& \mathbf{E}\|\overline{G}_{\delta,\mathbf{N}(\delta)}(x) - g(x)\|^2_{L^2(\Omega)}\\
&+ \beta_{\mathbf{N}_\delta}\|\mathbf{u}\|^2_{C([0,T];\mathcal{W}_{MT}(\Omega))} + \frac{\overline{M}^2\delta^2T^3}{b_0}\|\mathbf{u}\|^2_{L^\infty(0,T;\mathcal{H}^1_0(\Omega))}\\
&+ \mathbf{E}\int_t^T\left(-2\rho_\delta + \frac{2}{T}\ln\left(\frac{1}{\beta_{\mathbf{N}_\delta}}\right) + 2\overline{\gamma} + 1\right)\|\mathbf{V}^\delta_{\mathbf{N}(\delta)}(\cdot,s)\|^2_{L^2(\Omega)}ds.
\end{aligned} \tag{6.49}$$

Since

$$\mathbf{V}^\delta_{\mathbf{N}(\delta)}(x,t) = e^{\rho_\delta(t-T)}\left(\mathbf{u}^\delta_{\mathbf{N}(\delta)}(x,t) - \mathbf{u}(x,t)\right)$$

and applying Lemma 4.1, we observe that

$$\begin{aligned}
e^{2\rho_\delta(t-T)}\mathbf{E}\left\|\mathbf{u}^\delta_{\mathbf{N}(\delta)}(x,t) - \mathbf{u}(x,t)\right\|^2_{L^2(\Omega)} \leq{}& \delta^2\mathbf{N}(\delta) + \frac{1}{\lambda^{2\gamma}_{\mathbf{N}(\delta)}}\|g\|_{H^{2\gamma}(\Omega)}\\
&+ \beta_{\mathbf{N}_\delta}\|\mathbf{u}\|^2_{C([0,T];\mathcal{W}_{MT}(\Omega))} + \frac{\overline{M}^2\delta^2T^3}{b_0}\|\mathbf{u}\|^2_{L^\infty(0,T;\mathcal{H}^1_0(\Omega))}\\
&+ (2\overline{\gamma}+1)\int_t^T e^{2\rho_\delta(s-T)}\mathbf{E}\left\|\mathbf{u}^\delta_{\mathbf{N}(\delta)}(x,s) - \mathbf{u}(x,s)\right\|^2_{L^2(\Omega)}ds.
\end{aligned} \tag{6.50}$$

Gronwall's lemma allows us to obtain

$$\begin{aligned}
&e^{2\rho_\delta(t-T)}\mathbf{E}\left\|\mathbf{u}^\delta_{\mathbf{N}(\delta)}(x,t) - \mathbf{u}(x,t)\right\|^2_{L^2(\Omega)}\\
&\leq \underbrace{\left[\delta^2\mathbf{N}(\delta) + \frac{1}{\lambda^{2\gamma}_{\mathbf{N}(\delta)}}\|g\|_{H^{2\gamma}(\Omega)} + \beta_{\mathbf{N}_\delta}\|\mathbf{u}\|^2_{C([0,T];\mathcal{W}_{MT}(\Omega))} + \frac{\overline{M}^2\delta^2T^3}{b_0}\|\mathbf{u}\|^2_{L^\infty(0,T;\mathcal{H}^1_0(\Omega))}\right]}_{\widetilde{C}(\delta)} e^{(2\overline{\gamma}+1)(T-t)}.
\end{aligned} \tag{6.51}$$

By choosing $\rho_\delta = \frac{1}{T}\ln\left(\frac{1}{\beta_{\mathbf{N}_\delta}}\right) > 0$ we have

$$\mathbf{E}\left\|\mathbf{u}^\delta_{\mathbf{N}(\delta)}(x,t) - \mathbf{u}(x,t)\right\|^2_{L^2(\Omega)} \leq \beta^{\frac{2t}{T}}_{\mathbf{N}(\delta)}e^{(2\overline{\gamma}+1)T}\widetilde{C}(\delta). \tag{6.52}$$

7. Application to some specific equations

7.1. Ginzburg-Landau equation

Here we consider a special source function $F(u) = u - u^3$ for Problem (1.1). This is called the Ginzburg-Landau equation. This function satisfies the condition of section 5 and does not satisfy the condition in section 4. For all $\mathcal{R} > 0$, we approximate F by $\mathcal{F}_{\mathcal{R}}$ defined by

$$\mathcal{F}_{\mathcal{R}}(x,t;w) := \begin{cases} \mathcal{R}^3 - \mathcal{R}, & w \in (-\infty, -\mathcal{R}) \\ u - u^3, & w \in [-\mathcal{R}, \mathcal{R}], \\ \mathcal{R} - \mathcal{R}^3, & w \in (\mathcal{R}, +\infty). \end{cases} \tag{7.1}$$

We consider the problem

$$\begin{cases} \dfrac{\partial \mathbf{u}^{\delta}_{\mathbf{N}(\delta)}}{\partial t} - \nabla\Big(\mathbf{a}^{\text{obs}}_{\delta}(x,t)\nabla \mathbf{u}^{\delta}_{\mathbf{N}(\delta)}\Big) - \mathbf{Q}^{\delta}_{\beta_{\mathbf{N}(\delta)}}(\mathbf{u}^{\delta}_{\mathbf{N}(\delta)})(x,t) \\ \qquad\qquad = \mathcal{F}_{R_\delta}\left(\mathbf{u}^{\delta}_{\mathbf{N}(\delta)}(x,t)\right), \quad (x,t) \in \Omega \times (0,T), \\ \mathbf{u}^{\delta}_{\mathbf{N}(\delta)}|_{\partial\Omega} = 0, \quad t \in (0,T), \\ \mathbf{u}^{\delta}_{\mathbf{N}}(x,T) = \overline{G}_{\delta,\mathbf{N}(\delta)}(x), \quad (x,t) \in \Omega \times (0,T). \end{cases} \tag{7.2}$$

It is easy to see that $K(\mathcal{R}_\delta) = 1 + 3\mathcal{R}^2_\delta$. Choose $\beta_{\mathbf{N}(\delta)} = \mathbf{N}(\delta)^{-c}$ for any $0 < c < \min(\frac{1}{2}, \frac{2\gamma}{d})$, and $\mathbf{N}(\delta)$ is chosen as

$$\mathbf{N}(\delta) = \left(\frac{1}{\delta}\right)^{m(\frac{1}{2}-c)}, \beta_{\mathbf{N}(\delta)} = \left(\frac{1}{\delta}\right)^{-mc(\frac{1}{2}-c)} \quad 0 < m < 1. \tag{7.3}$$

Choose $\mathcal{R}_\delta$ such that

$$\mathcal{R}_\delta = \sqrt{\frac{K(\mathcal{R}_\delta) - 1}{3}} = \sqrt{\frac{\frac{1}{kT}\ln\left(m(\frac{1}{2}-c)\ln\left(\frac{1}{\delta}\right)\right) - 1}{3}}.$$

Then applying Theorem 5.1, the error $\mathbf{E}\left\|\mathbf{u}^{\delta}_{\mathbf{N}(\delta)}(x,t) - \mathbf{u}(x,t)\right\|^2_{L^2(\Omega)}$ is of the order $\ln^2\left(\frac{1}{\delta}\right)\left(\delta\right)^{2mc(\frac{1}{2}-c)\frac{t}{T}}$.

7.2. The nonlinear Fisher–KPP equation

In this subsection, we are concerned with the backward problem for a nonlinear parabolic equation of Fisher–Kolmogorov–Petrovsky–Piskunov type

$$\mathbf{u}_t - \nabla\Big(a(x,t)\nabla\mathbf{u}\Big) = \gamma(x)\mathbf{u}^2 - \mu(x)\mathbf{u}, \quad (x,t) \in \Omega \times (0,T), \tag{7.4}$$

with the following condition

$$\begin{cases} \mathbf{u}(x,T) = g(x), \quad (x,t) \in \Omega \times (0,T), \\ \mathbf{u}|_{\partial\Omega} = 0, \quad t \in (0,T). \end{cases} \tag{7.5}$$

By Skellam [33], Eq (7.4) has many applications in population dynamics and periodic environments. In these references, the quantity $\mathbf{u}(x,t)$ generally stands for a population density, and the coefficients $a(x,t)$, $\gamma(x)$, $\mu(x)$ respectively, correspond to the diffusion coefficient, the intrinsic growth rate coefficient and a coefficient measuring the effects of competition on the birth and death rates. Our method that can be applied to this model is similar to example 7.1. However, since the ideas of Example 7.1 and 7.2 are the same, we only state the model without giving the errors.

7.3. The second equation

Taking the function $F(u) = u^{\frac{1}{3}}$ it is easy to see that F satisfy (6.2), (6.3) and (6.4). Moreover, we can show that F is not locally Lipschitz function. So, we cannot regularize the problem in this case with Problem (5.6). We consider the problem

$$\begin{cases} \dfrac{\partial \mathbf{u}^\delta_{\mathbf{N}(\delta)}}{\partial t} - \nabla\Big(\mathbf{a}^{\text{obs}}_\delta(x,t)\nabla \mathbf{u}^\delta_{\mathbf{N}(\delta)}\Big) - \mathbf{Q}^\delta_{\beta_{\mathbf{N}(\delta)}}(\mathbf{u}^\delta_{\mathbf{N}(\delta)})(x,t) \\ \qquad\qquad\qquad = \Big(\mathbf{u}^\delta_{\mathbf{N}(\delta)}(x,t)\Big)^{\frac{1}{3}}, \quad (x,t) \in \Omega \times (0,T), \\ \mathbf{u}^\delta_{\mathbf{N}(\delta)}|_{\partial\Omega} = 0, \quad t \in (0,T), \\ \mathbf{u}^\delta_{\mathbf{N}}(x,T) = \overline{G}_{\delta,\mathbf{N}(\delta)}(x), \quad (x,t) \in \Omega \times (0,T). \end{cases} \tag{7.6}$$

Choose $\beta_{\mathbf{N}_\delta}$ and $\mathbf{N}_\delta$ as in subsection 6.1. Applying Theorem 5.1, the error between the solution of Problem (7.6) and $\mathbf{u}$, $\mathbf{E}\left\|\mathbf{u}^\delta_{\mathbf{N}(\delta)}(x,t) - \mathbf{u}(x,t)\right\|^2_{L^2(\Omega)}$, is of the order $\delta^{2mc(\frac{1}{2}-c)\frac{t}{T}}$.

Remark 7.1. *In the following, we give a comparison of the method and results in this paper with the results in [30, 31]. All methods are truncation methods, but our problem is complicated due to the data being noised by random data. We need Lemma 4.1 to determine the correct set up according to the measured data. The coefficients $N(\delta)$ should be chosen appropriately so that the error between the sought solution and the correct solution converges. There are two advantages to this article that were not explored in [30, 31]*

- *In Theorem 4.3, we give a regularization result in the case of a weaker assumption for u, i.e., $u \in C([0,T]; L^2(\Omega))$. This is one of the first results obtained in this case and was not considered in [30, 31]. In those papers, to investigate the error, the exact solution is assumed in a Gevrey space, which limits the number of functions than if one considered the function space $C([0,T]; L^2(\Omega))$.*
- *In [30, 31], the source functions must satisfy a global lipschitz condition. However, in our article, we deal with a fairly broad function class, consisting of the local Lipschitz function class and some local non-Lipschitz function class (see Section 6).*

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Conflict of interest

The authors declare there is no conflicts of interest.

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