Electronic Research Archive

http://www.aimspress.com/journal/era

ERA, 30(4): 1354–1373.
DOI: 10.3934/era.2022071
Received: 21 December 2021
Revised: 18 February 2022
Accepted: 21 February 2022
Published: 18 March 2022

Research article

On a time-space fractional diffusion equation with a semilinear source of exponential type

Anh Tuan Nguyen[1,2] **and Chao Yang**[3,*]

[1] Division of Applied Mathematics, Science and Technology Advanced Institute, Van Lang University, Ho Chi Minh City, Vietnam
[2] Faculty of Technology, Van Lang University, Ho Chi Minh City, Vietnam
[3] College of Mathematical Sciences, Harbin Engineering University, Harbin, HLJ 150001, China

* **Correspondence:** yangchao_@hrbeu.edu.cn; Tel: +86-15146449220.

Abstract: In the current paper, we are concerned with the existence and uniqueness of mild solutions to a Cauchy problem involving a time-space fractional diffusion equation with an exponential semilinear source. By using the iteration method and some $L^p - L^q$-type estimates of fundamental solutions associated with the Mittag-Leffler function, we study the well-posedness of the problem in two different cases corresponding to two assumptions on the Cauchy data. On the one hand, when considering initial data in $L^p(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, the problem possesses a local-in-time solution. On the other hand, we obtain a global existence result for a mild solution with small data in an Orlicz space.

Keywords: Exponential nonlinearity; time-space fractional diffusion; Caputo derivative; global well-posedness

1. Introduction

In this paper, we consider the following initial value problem for a time-space fractional diffusion equation

$$\begin{cases} \partial_t^\alpha u(t,x) + (-\Delta)^{\frac{\sigma}{2}} u(t,x) = F(u(t,x)), & (t,x) \in (0,\infty)\times\mathbb{R}^N, \\ u(t,x) = h(x), & (t,x) \in \{0\}\times\mathbb{R}^N, \end{cases} \tag{1.1}$$

where $N \geq 1$, $0 < \sigma \leq 2$, h is the initial data function, and the symbol ∂_t^α stands for the Caputo derivative of fractional order $\alpha \in (0,1)$ (Section 2). In Problem (1.1), we are mainly focus on the semilinear case in which the function F satisfies the following assumptions

$$\Big|F(u) - F(v)\Big| \leq L\left(\big|u\big|^{\nu-1}\exp(|u|^p) + \big|v\big|^{\nu-1}\exp(|v|^p)\right)\big|u - v\big|, \quad u, v \in \mathbb{R}, \tag{1.2}$$

and

$$\Big|F(u)\Big| \leq L\Big|u\Big|^{\nu}\exp(|u|^{p}), \quad u \in \mathbb{R}, \tag{1.3}$$

where $\nu, p > 1$ and L is a positive constant. The reason why we study this source function comes from the great interest of the PDEs community with a polynomial source of the form $G_p(u) = |u|^{p-1}u$ or $G_p(u) = u^p$ and some similar forms. Many good papers about this topic have attracted our attention. Wang-Xu [1] and Xu-Su [2] used the potential well method to investigate the well-posedness of a pseudo-parabolic equation with nonlinear function G_p. Lian et al. [3] studied a Schrödinger equation with polynomial nonlinearity. They used infinite Nehari manifolds with geometric features to provide infinite sharp conditions for global existence and blowup results of solutions. A modified form of G_p was considered by Chen et al. [4] in the Gierer–Meinhardt system. The authors applied a functional method to obtain a bound of some ratios of the solution, and then, the existence of global and blowup solutions were proved.

From strong interest of PDEs with polynomial non-linearity through the above mentioned papers and related works, we consider the following heat equation

$$\begin{cases} \partial_t u(t,x) - \Delta u(t,x) = |u(t,x)|^{p-1}u(t,x), & (t,x) \in (0,\infty) \times \mathbb{R}^N, \\ u(t,x) = u_0(x), & (t,x) \in \{0\} \times \mathbb{R}^N, \end{cases} \tag{1.4}$$

where $u_0 \in L^q(\mathbb{R}^N)$, $1 < q < \infty$, $p > 1$. Recall that (1.4) admits a scale solution

$$u_\lambda(t,x) := \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x), \quad \lambda > 0.$$

and the value $q_c = \frac{N(p-1)}{2}$ called the critical exponent plays an important role in investigating the existence and uniqueness results. Considering the case when $q = p = q_c = \frac{N}{N-2}$ in $\mathbb{R}^2$, the power exponent is approximated ∞. Therefore, it seems reasonable to replace the source function in (1.4) with the nonlinearity of the exponential type. As far as we know, the attention on nonlinear functions satisfying the assumptions (1.2) and (1.3) is derived and developed by several works in the literature [5–9]. Two groups in [5, 8] and [7, 9] studied, respectively, parabolic equations and the Schrödinger equation (NLS) with exponential nonlinearities. More precisely, in [8], Ioku proved the global-in-time existence of a mild solution to a semilinear heat equation with exponential nonlinearity under some smallness assumptions on the initial data. Meanwhile, Furioli [5] showed that the notions of weak and mild solutions are equivalent and investigated decay estimates and the asymptotic behavior of small-data global solutions. Nakamura and Ozawa in [9] provided global-in-time results for solutions in homogeneous Sobolev spaces and homogeneous Besov space to a NLS with a source function of exponential type. A source function $f(u) = (e^{4\pi|u|^2} - 1 - 4\pi|u|^2)$ was considered for a two-dimensional NLS problem by Ibrahim et al. [7]. The authors showed that the solution to their problem tends to be a free Schrödinger under certain conditions. Also, we refer to some other works by Nakamura and Ozawa [10] and Ibrahim et al. [6] for wave equations with the nonlinearity of exponential growth.

During the past decades, fractional calculus has received increased attention due to its wide applications in diverse fields of science and engineering such as stochastic processes [11], fluid mechanics [12, 13], chemotaxis in biology [14], viscoelasticity [15], etc. Apart from that, many interesting mathematical models and results on this topic have been done [16–25]. A strong inspiration for studying Problem (1.1) with the presence of the operator ∂_t^α comes from the fact that many physical phenomena carry the substance of diffusion processes, many studies about diffusion equations have been done [26–30], and fractional calculus is very effective in modeling anomalous diffusion processes. In fact, while a diffusive particle in the usual diffusion process possesses the mean square displacement behaving like $C_1 t$ for $t \to \infty$, such behavior of a particle in an anomalous diffusion process is $C_2 t^\alpha$ [31], C_1, C_2 are positive constants. Starting from the above characteristics of an anomalous diffusion process, many good works about fractional diffusion equations have been done. Because it is very difficult and lengthy to present all the related works, we would like to present only the works, which motivated us. In [32], a general time-fractional diffusion equation subject to the Dirichlet boundary condition was studied by Vergara and Zacher. By using energy estimates and a powerful inequality for integrodifferential operators, the authors proved sharp estimates for the decay in time of solutions. Andrade et al. [33] considered the following non-local initial boundary value problem associated with a time-fractional heat equation

$$\begin{cases} \partial_t^\alpha u(t,x) + (-\Delta)^{\frac{\sigma}{2}} u(t,x) = f(t, u(t,x)), & (t,x) \in (0,\infty) \times \Omega, \\ u(t,x) = 0 & (t,x) \in \{0\} \times \partial\Omega, \\ u(t,x) = u_0(x) + \sum_{i=1}^{k} \beta_i(x) u(T_i, x), & (t,x) \in \{0\} \times \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $\alpha \in (0,1)$, $\sigma \in (0,2]$, $T_i \in \mathbb{R}$, $\beta_i : \Omega \to \mathbb{R}$, and the continuous function $f : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ satisfies, $|f(t,s)| \le c(1 + |s|^p)$ and

$$\left| f(t,s) - f(t,r) \right| \le c(1 + |s|^{p-1} + |r|^{p-1})|s - r|,$$

where $p > 1$ and c is a positive constant. The existence and regularity of mild solutions were established with some sufficient conditions. In [34], Tuan et al. were concerned in a terminal value problem for a time-space fractional diffusion equation. For the problem with linear source function, regularity properties of solutions were studied. The existence, uniqueness and regularity to solution were proved in the case of nonlinear source.

The main results of this paper are providing the existence and uniqueness of mild solutions to Problem (1.1) with function F satisfying (1.2) and (1.3). Corresponding to two different cases of initial data, we obtain a local-in-time solution and a global small-data solution. With usual initial data, by the Picard iteration method and some $L^p - L^q$ or $L^p - L^\Xi$ estimates of fundamental solutions involving the Mittag-Leffler function, we provide the existence and uniqueness of mild solutions to (1.1) on a reasonable time interval $(0, T]$. Apart from that, we also show that solutions are continuous from $(0, T]$ to $L^p(\mathbb{R}^N)$. Global-in-time results are obtained by making use of the norm (3.14). From the technical point of view, we split the second term of the right-hand side of (2.1) into two parts, while the part with small-time is easy to handle, the controlling of the large time part requires small assumptions on the initial data in an Orlicz space to be achieved.

The structure of the paper is as follows. Firstly, we provided some preliminaries in section 2 including some function spaces, fractional settings and formula, linear estimates for a mild solution. The main results about the local and global well-posedness are stated in section 3.

2. Preliminary

2.1. Some basic setups

We first introduce some function spaces. Let $(B, \|\cdot\|_B)$ be a Banach space. For $T > 0$, we denote by $C([0,T]; B)$ the space of all continuous functions u from $[0,T]$ to B and define the following space

$$L^\infty(0,T;B) := \left\{ u : [0,T] \to B \,\middle|\, u \text{ is bounded almost everywhere on } [0,T] \right\}.$$

Recall that $L^\infty(0,T;B)$ is a Banach space with respect to the norm

$$\|u\|_{L^\infty(0,T;B)} := \operatorname*{ess\,sup}_{t\in(0,T)} \left\|u(t)\right\|_B < \infty, \quad u \in L^\infty(0,T;B).$$

Let $\Xi(z) := e^{z^p} - 1$. The Orlicz space $L^\Xi(\mathbb{R}^N)$ is defined as the space of all functions satisfying the following converging result for some $\kappa > 0$

$$\int_{\mathbb{R}^N} \Xi\Big(\kappa^{-1}\big|u(x)\big|\Big)\mathrm{d}x < \infty.$$

The space $L^\Xi(\mathbb{R}^N)$ is a Banach space with the Luxemburg norm given as follows

$$\left\|u\right\|_{L^\Xi(\mathbb{R}^N)} := \inf\left\{ \kappa > 0 \,\middle|\, \int_{\mathbb{R}^N} \Xi\left(\kappa^{-1}\big|u(x)\big|\right)\mathrm{d}x \le 1 \right\}, \quad u \in L^\Xi(\mathbb{R}^N).$$

Note that the space $C_0^\infty(\mathbb{R}^N)$ is not dense in $L^\Xi(\mathbb{R}^N)$. In fact, we have the following embeddings

$$L^p(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \hookrightarrow \mathrm{cl}_{L^\Xi(\mathbb{R}^N)}(C_0^\infty(\mathbb{R}^N)) \hookrightarrow L^\Xi(\mathbb{R}^N).$$

For more details about Orlicz spaces, we refer the reader to [37, Chapter 8] and references given there.

Next, let us provide some fractional settings. For $a, b > 0$, the Beta function $\mathbf{B}$ and the Gamma function Γ are defined respectively as follows

$$\mathbf{B}(a,b) := \int_0^1 (1-m)^{a-1} m^{b-1}\mathrm{d}m,$$
$$\Gamma(a) := \int_0^\infty m^{a-1}e^{-m}\mathrm{d}m.$$

Let $\alpha \in (0,1)$. By considering the following memory kernel

$$k_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad t > 0,$$

for a smooth enough function u, we can define the Caputo derivative of order α by

$$\partial_t^\alpha u(t) := k_{1-\alpha}(\cdot) * \frac{\mathrm{d}}{\mathrm{d}t}u(t).$$

Also, for two real constants α_1 and α_2, we define the Mittag-Leffler function $E_{\alpha_1,\alpha_2} : \mathbb{C} \to \mathbb{C}$ in the following way

$$E_{\alpha_1,\alpha_2}(z) := \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha_1 k + \alpha_2)}.$$

2.2. Fundamental solutions

The fractional Laplace operator can be defined via the Fourier multiplier [5, Section 2]

$$(-\Delta)^{\frac{\beta}{2}}u(x) := \mathcal{F}^{-1}\left(|\xi|^{\beta}\mathcal{F}(u)(\xi)\right)(x),$$

where the Fourier transform is recalled as follows

$$\mathcal{F}(u)(\xi) := \int_{\mathbb{R}^N} u(x)e^{-i<x,\xi>}\mathrm{d}x,$$

and $\mathcal{F}^{-1}$ is the inverse Fourier transform. From the above definitions, for any $\alpha \in (0,1)$ and $\sigma \in (1,2]$, we define two functions $\mathscr{A}_1(\alpha,\sigma)(t)$ and $\mathscr{A}_2(\alpha,\sigma)(t)$ by

$$\mathscr{A}_1(\alpha,\sigma)(t)u(x) := \mathcal{F}^{-1}\Big(E_{\alpha,1}\left(-|\xi|^{\sigma}t^{\alpha}\right)\Big) * u(x),$$
$$\mathscr{A}_2(\alpha,\sigma)(t)u(x) := \mathcal{F}^{-1}\Big(E_{\alpha,\alpha}\left(-|\xi|^{\sigma}t^{\alpha}\right)\Big) * u(x).$$

From [35, Section 1.3] or [14, Section 2], we see that the mild solution to Problem (1.1) satisfies the following Duhamel integral equality

$$u(t,x) = \mathscr{A}_1(\alpha,\sigma)(t)h(x) + \int_0^t (t-m)^{\alpha-1}\mathscr{A}_2(\alpha,\sigma)(t-m)F(u(m,x))\mathrm{d}m. \tag{2.1}$$

It turns out that handling the operators $\mathscr{A}_1(\alpha,\sigma)(t)$ and $\mathscr{A}_2(\alpha,\sigma)(t)$ plays an important role in controlling norms of the mild solution u. Therefore, we provide some linear estimates of $\mathscr{A}_1(\alpha,\sigma)(t)$ and $\mathscr{A}_2(\alpha,\sigma)(t)$ in the following lemma.

Lemma 2.1. *[14, Proposition 3.3] Let $r \in [1,\infty)$. Then, there exists a positive constant C_0 such that the following statements hold*

(i) if $N > r\sigma$ and $s \in [r, \frac{Nr}{N-r\sigma})$, for any $u \in L^r(\mathbb{R}^N)$, we have

$$\left\|\mathscr{A}_1(\alpha,\sigma)(t)u\right\|_{L^s(\mathbb{R}^N)} \le C_0 t^{\frac{\alpha N}{\sigma}\left(\frac{1}{s}-\frac{1}{r}\right)}\left\|u\right\|_{L^r(\mathbb{R}^N)}, \tag{2.2}$$

(ii) If $N > 2r\sigma$ and $s \in [r, \frac{Nr}{N-2r\sigma})$, for any $u \in L^r(\mathbb{R}^N)$, we have

$$\left\|\mathscr{A}_2(\alpha,\sigma)(t)u\right\|_{L^s(\mathbb{R}^N)} \le C_0 t^{\frac{\alpha N}{\sigma}\left(\frac{1}{s}-\frac{1}{r}\right)}\left\|u\right\|_{L^r(\mathbb{R}^N)}. \tag{2.3}$$

Furthermore, (2.2) *(resp.* (2.3)*) holds for any $s \in [r,\infty)$ if $N = r\sigma$ (resp. $N = 2r\sigma$) and $s \in [r,\infty]$ if $N < r\sigma$ (resp. $N < 2r\sigma$).*

Corollary 2.2. For any $u \in L^{\Xi}(\mathbb{R}^N)$, we have

$$\left\|\mathscr{A}_1(\alpha,\sigma)(t)u\right\|_{L^{\Xi}(\mathbb{R}^N)} \le C_0\left\|u\right\|_{L^{\Xi}(\mathbb{R}^N)},$$
$$\left\|\mathscr{A}_2(\alpha,\sigma)(t)u\right\|_{L^{\Xi}(\mathbb{R}^N)} \le C_0\left\|u\right\|_{L^{\Xi}(\mathbb{R}^N)}.$$

Proof of Corollary 2.2. First, for a positive real number a, we have the following observation

$$\inf\left\{\kappa>0 \,\middle|\, \int_{\mathbb{R}^N} \Xi\left(\frac{a|u(x)|}{\kappa}\right)\mathrm{d}x \le 1\right\} = \inf\left\{a\kappa>0 \,\middle|\, \int_{\mathbb{R}^N} \Xi\left(\frac{|u(x)|}{\kappa}\right)\mathrm{d}x \le 1\right\}.$$

Therefore, the following equality holds

$$\inf\left\{\kappa>0 \,\middle|\, \int_{\mathbb{R}^N} \Xi\left(\frac{a|u(x)|}{\kappa}\right)\mathrm{d}x \le 1\right\} = a\inf\left\{\kappa>0 \,\middle|\, \int_{\mathbb{R}^N} \Xi\left(\frac{|u(x)|}{\kappa}\right)\mathrm{d}x \le 1\right\}. \tag{2.4}$$

Next, from the expansion of the exponential function, for any $\kappa>0$, we have

$$\begin{aligned}\int_{\mathbb{R}^N} \Xi\left(\frac{|\mathscr{A}_1(\alpha,\sigma)(t)u(x)|}{\kappa}\right)\mathrm{d}x &= \sum_{j\in\mathbb{N}} \frac{\left\|\mathscr{A}_1(\alpha,\sigma)(t)u\right\|^{2j}_{L^{2j}(\mathbb{R}^N)}}{j!\kappa^{2k}} \\ &\le \sum_{j\in\mathbb{N}} \frac{C_0^{2j}\left\|u\right\|^{2j}_{L^{2j}(\mathbb{R}^N)}}{j!\kappa^{2k}} \\ &= \int_{\mathbb{R}^N} \Xi\left(\frac{C_0|u(x)|}{\kappa}\right)\mathrm{d}x.\end{aligned}$$

Combining this result and (2.4) yields the desired estimate. Similarly, we can find the estimate of the norm for $\mathscr{A}_2(\alpha,\sigma)(t)u$. The proof is completed. □

3. Existence and uniqueness

This section is used to present the main results of this paper including the existence and uniqueness of mild solutions satisfying Eq (2.1). We provide two different results about local well-posedness and global well-posedness according to two cases of initial data.

- For $u_0 \in L^p(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$, Problem (1.1) possesses a unique mild solution u on $[0,T]$ where T is small enough. In addition, this solution is also continuous on $(0,T]$.
- By making some small assumptions on the initial data in $L^\Xi(\mathbb{R}^N)$, we can prove that the solution u to Problem (1.1) exists globally in time.

Theorem 3.1 (Local-in-time solution)**.** *Let $\nu, p>1$. Suppose that h belongs to $L^p(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$. Then, we can find a reasonable number T such that Problem* (1.1) *possesses a unique mild solution*

$$u \in L^\infty\Big(0,T;L^p(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)\Big)\cap C\Big((0,T];L^p(\mathbb{R}^N)\Big),$$

where $\|\cdot\|_{L^p(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)} := \|\cdot\|_{L^p(\mathbb{R}^N)} + \|\cdot\|_{L^\infty(\mathbb{R}^N)}$.

Proof. To begin, we consider the sequence $\{u_l\}_{l\in\mathbb{N}}$ as follows

$$\begin{aligned} u_1(t,x) &:= \mathscr{A}_1(\alpha,\sigma)(t)h(x), \\ u_{l+1}(t,x) &:= \mathscr{A}_1(\alpha,\sigma)(t)h(x) + \int_0^t (t-m)^{\alpha-1}\mathscr{A}_2(\alpha,\sigma)(t-m)F(u_l(m,x))\mathrm{d}m. \end{aligned}$$

We aim to prove that $\{u_l\}_{l\in\mathbb{N}}$ is a Cauchy sequence in $L^\infty(0,T;L^p(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N))$. Then, the completeness of this space ensures the existence of a limit function u that can be shown to be the unique mild solution to Problem (1.1). To this end, the first task is to check whether $\{u_l\}_{l\in\mathbb{N}}$ is in $L^\infty(0,T;L^p(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N))$ or not. Indeed, we apply Lemma 2.1 to get

$$\Big\|\mathscr{A}_1(\alpha,\sigma)(t)h(\cdot)\Big\|_{L^\infty(\mathbb{R}^N)} \le C_0\big\|h\big\|_{L^\infty(\mathbb{R}^N)},$$

and

$$\Big\|\mathscr{A}_1(\alpha,\sigma)(t)h(\cdot)\Big\|_{L^p(\mathbb{R}^N)} \le C_0\big\|h\big\|_{L^p(\mathbb{R}^N)}.$$

Since $h\in L^p(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$, we easily find that

$$\big\|u_1\big\|_{L^\infty(0,T;L^p(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N))} \le C_0\big\|h\big\|_{L^p(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)}. \tag{3.1}$$

This result implies that $u_1\in L^\infty(0,T;L^p(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N))$. Before moving to the second step, we provide some nonlinear estimates of the source function. For functions $w,v\in L^p(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$, we find that

$$\begin{aligned}\Big\|F(w)-F(v)\Big\|_{L^\infty(\mathbb{R}^N)} \le & L\big\|w\big\|^{\nu-1}_{L^\infty(\mathbb{R}^N)}\exp\left(\big\|w\big\|^p_{L^\infty(\mathbb{R}^N)}\right)\big\|w-v\big\|_{L^\infty(\mathbb{R}^N)} \\ &+ L\big\|v\big\|^{\nu-1}_{L^\infty(\mathbb{R}^N)}\exp\left(\big\|v\big\|^p_{L^\infty(\mathbb{R}^N)}\right)\big\|w-v\big\|_{L^\infty(\mathbb{R}^N)}\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}\Big\|F(w)-F(v)\Big\|_{L^p(\mathbb{R}^N)} \le & L\big\|w\big\|^{\nu-1}_{L^\infty(\mathbb{R}^N)}\exp\left(\big\|w\big\|^p_{L^\infty(\mathbb{R}^N)}\right)\big\|w-v\big\|_{L^p(\mathbb{R}^N)} \\ &+ L\big\|v\big\|^{\nu-1}_{L^\infty(\mathbb{R}^N)}\exp\left(\big\|v\big\|^p_{L^\infty(\mathbb{R}^N)}\right)\big\|w-v\big\|_{L^p(\mathbb{R}^N)}.\end{aligned} \tag{3.3}$$

We are now ready to consider the remaining elements of $\{u_l\}_{l\in\mathbb{N}}$. Let $R_1 = 2C_0\big\|h\big\|_{L^p(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)}$. Suppose that u_l is in the open ball $B(0,R_1)\subset L^\infty(0,T;L^p(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N))$ for any $l\in\mathbb{N}$. From Lemma 2.1, the following estimate is satisfied for any $t>0$

$$\begin{aligned}\Big\|u_{l+1}(t)-u_1(t)\Big\|_{L^\infty(\mathbb{R}^N)} &\le \int_0^t (t-m)^{\alpha-1}\Big\|\mathscr{A}_2(\alpha,\sigma)(t-m)F(u_l(m))\Big\|_{L^\infty(\mathbb{R}^N)}\mathrm{d}m \\ &\le C_0\int_0^t (t-m)^{\alpha-1}\Big\|F(u_l(m))\Big\|_{L^\infty(\mathbb{R}^N)}\mathrm{d}m.\end{aligned}$$

Apply (3.2) and the assumption (1.3), we find that

$$\begin{aligned}\Big\|u_{l+1}(t)-u_1(t)\Big\|_{L^\infty(\mathbb{R}^N)} &\le C_0L\int_0^t (t-m)^{\alpha-1}\big\|u_l(m)\big\|^\nu_{L^\infty(\mathbb{R}^N)}\exp\left(\big\|u_l(m)\big\|^p_{L^\infty(\mathbb{R}^N)}\right)\mathrm{d}m \\ &\le \frac{C_0LT^\alpha}{\alpha}\big\|u_l\big\|^\nu_{L^\infty(0,T;L^\infty(\mathbb{R}^N))}\exp\left(\big\|u_l\big\|^p_{L^\infty(0,T;L^\infty(\mathbb{R}^N))}\right).\end{aligned} \tag{3.4}$$

By similar arguments, we also get a same result for the L^p-norm as follows

$$\Big\|u_{l+1}(t)-u_1(t)\Big\|_{L^p(\mathbb{R}^N)} \le C_0L\int_0^t (t-m)^{\alpha-1}\big\|u_l(m)\big\|^\nu_{L^\infty(\mathbb{R}^N)}\exp\left(\big\|u_l(m)\big\|^p_{L^\infty(\mathbb{R}^N)}\right)\mathrm{d}m$$

$$\leq \frac{C_0 L T^\alpha}{\alpha} \left\| u_l \right\|^{\nu}_{L^\infty\left(0,T;L^\infty(\mathbb{R}^N)\right)} \exp\left(\left\| u_l \right\|^{p}_{L^\infty\left(0,T;L^\infty(\mathbb{R}^N)\right)} \right), \tag{3.5}$$

where we have used (3.3) with $w = u_l$ and (1.3).

Combining the above two estimates and choosing

$$T < \left(\frac{2L \left\| h \right\|^{\nu-\alpha}_{L^p(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)}}{\alpha} \exp\left(2C_0 \left\| h \right\|^{p}_{L^p(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)} \right) \right)^{\frac{-1}{\alpha}},$$

we obtain the following result

$$\left\| u_{l+1}(t) - u_1(t) \right\|_{L^p(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)} < \frac{R_1}{2}. \tag{3.6}$$

In view of (3.1) and (3.6), for any $l > 1$, if $u_l \in B(0, R_1)$, we obtain the estimate below

$$\begin{aligned} \operatorname*{ess\,sup}_{t\in(0,T)} \left\| u_{l+1}(t) \right\|_{L^p(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)} \leq & \operatorname*{ess\,sup}_{t\in(0,T)} \left\| u_{l+1}(t) - u_1(t) \right\|_{L^p(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)} \\ & + \operatorname*{ess\,sup}_{t\in(0,T)} \left\| u_1(t) \right\|_{L^p(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)} \\ \leq & R_1. \end{aligned} \tag{3.7}$$

From (3.1) and (3.7), the induction method can be applied to conclude that $\{u_l\}_{l\in\mathbb{N}} \subset B(0, R_1)$.

In addition, we can check that $\{u_l\}_{l\in\mathbb{N}}$ is a Cauchy sequence in $B(0, R_1)$. In fact, presume for $l \geq 2$ that u_l and u_{l-1} are elements of $B(0, R_1)$, the techniques as in (3.4) and (3.5) enable us to find for any $t \in (0, T)$ that

$$\begin{aligned} & \left\| u_{l+1}(t) - u_l(t) \right\|_{L^\infty(\mathbb{R}^N)} \\ \leq & C_0 \int_0^t (t-m)^{\alpha-1} \left\| F(u_l(m)) - F(u_{l-1}(m)) \right\|_{L^\infty(\mathbb{R}^N)} \mathrm{d}m \\ \leq & C_0 L \sum_{k\in\{l-1,l\}} \int_0^t \frac{\left\| u_k(m) \right\|^{\nu-1}_{L^\infty(\mathbb{R}^N)}}{(t-m)^{1-\alpha}} \exp\left(\left\| u_k(m) \right\|^{p}_{L^\infty(\mathbb{R}^N)} \right) \left\| u_l(m) - u_{l-1}(m) \right\|_{L^\infty(\mathbb{R}^N)} \mathrm{d}m \\ \leq & \frac{C_0 L T^\alpha}{\alpha} \sum_{k\in\{l-1,l\}} \left\| u_k \right\|^{\nu-1}_{L^\infty\left(0,T;L^\infty(\mathbb{R}^N)\right)} \exp\left(\left\| u_l \right\|^{p}_{L^\infty\left(0,T;L^\infty(\mathbb{R}^N)\right)} \right) \left\| u_l - u_{l-1} \right\|_{L^\infty\left(0,T;L^\infty(\mathbb{R}^N)\right)} \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} & \left\| u_{l+1}(t) - u_l(t) \right\|_{L^p(\mathbb{R}^N)} \\ \leq & C_0 \int_0^t (t-m)^{\alpha-1} \left\| F(u_l(m)) - F(u_{l-1}(m)) \right\|_{L^p(\mathbb{R}^N)} \mathrm{d}m \\ \leq & C_0 L \sum_{k\in\{l-1,l\}} \int_0^t \frac{\left\| u_k(m) \right\|^{\nu-1}_{L^\infty(\mathbb{R}^N)}}{(t-m)^{1-\alpha}} \exp\left(\left\| u_k(m) \right\|^{p}_{L^\infty(\mathbb{R}^N)} \right) \left\| u_l(m) - u_{l-1}(m) \right\|_{L^p(\mathbb{R}^N)} \mathrm{d}m \end{aligned}$$

$$\leq \frac{C_0 L T^\alpha}{\alpha} \sum_{k\in\{l-1,l\}} \left\|u_k\right\|^{\nu-1}_{L^\infty\left(0,T;L^\infty(\mathbb{R}^N)\right)} \exp\left(\left\|u_l\right\|^{p}_{L^\infty\left(0,T;L^\infty(\mathbb{R}^N)\right)}\right) \left\|u_l - u_{l-1}\right\|_{L^\infty\left(0,T;L^p(\mathbb{R}^N)\right)}. \tag{3.9}$$

Therefore, if we choose

$$T \leq \left(\frac{8L\left\|h\right\|^{\nu-\alpha}_{L^p(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)}}{\alpha} \exp\left(2C_0\left\|h\right\|^{p}_{L^p(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)}\right) \right)^{\frac{-1}{\alpha}},$$

the following estimate can be drawn from (3.8) and (3.9)

$$\begin{aligned}
\left\|u_{l+1} - u_l\right\|_{L^\infty(0,T;L^p(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N))} &\leq \frac{4C_0LT^\alpha}{\alpha} R_1^{\nu-1} \exp\left(R_1^p\right) \left\|u_l - u_{l-1}\right\|_{L^\infty(0,T;L^p(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N))} \\
&\leq \frac{1}{2}\left\|u_l - u_{l-1}\right\|_{L^\infty(0,T;L^p(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N))},
\end{aligned}$$

for any $l \geq 2$. Based on this result, for any $l_2 > l_1 \geq 2$, we have

$$\begin{aligned}
\left\|u_{l_2} - u_{l_1}\right\|_{L^\infty(0,T;L^p(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N))} &\leq \sum_{l=l_1}^{l_2-1} \left\|u_{l+1} - u_l\right\|_{L^\infty(0,T;L^p(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N))} \\
&\leq \sum_{l=l_1}^{l_2-1} 2^{1-l}\left\|u_2 - u_l\right\|_{L^\infty(0,T;L^p(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N))} \\
&\leq \sum_{l=l_1}^{l_2-1} 2^{2-l}\left\|u_2 - u_l\right\|_{L^\infty(0,T;L^p(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N))}.
\end{aligned}$$

It means $\{u_l\}_{l\in\mathbb{N}}$ is a Cauchy sequence in $B(0, R_1)$, provided that we have already shown that $\{u_l\}_{l\in\mathbb{N}} \subset B(0, R_1)$. By the completeness of the space $L^\infty\left(0, T; L^p(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)\right)$ and the dominated convergence theorem, there exists a unique limit function u satisfying

$$\begin{aligned}
u &= \lim_{l\to\infty} \mathscr{A}_1(\alpha, \sigma)(t)h(x) + \int_0^t (t - m)^{\alpha-1} \mathscr{A}_2(\alpha, \sigma)(t - m)F(u_l(m, x))\mathrm{d}m \\
&= \mathscr{A}_1(\alpha, \sigma)(t)h(x) + \int_0^t (t - m)^{\alpha-1} \mathscr{A}_2(\alpha, \sigma)(t - m)F(u(m, x))\mathrm{d}m.
\end{aligned}$$

In addition, we can also show that $u \in C\left((0, T]; L_0^\Xi(\mathbb{R}^N)\right)$. For $t, \varepsilon > 0$, it is easy to check that

$$\begin{aligned}
\left\|u(t + \varepsilon, \cdot) - u(t, \cdot)\right\|_{L^p(\mathbb{R}^N)} \leq &\left\|(\mathscr{A}_1(\alpha, \sigma)(t + \varepsilon) - \mathscr{A}_1(\alpha, \sigma)(t))h(\cdot)\right\|_{L^p(\mathbb{R}^N)} \\
&+ \int_0^t \left\|\mathscr{Q}(t + \varepsilon - m, t - m)F(u(m, \cdot))\right\|_{L^p(\mathbb{R}^N)} \mathrm{d}m \\
&+ \int_t^{t+\varepsilon} \left\|(t + \varepsilon - m)^{\alpha-1} \mathscr{A}_2(\alpha, \sigma)(t + \varepsilon - m)F(u(m, \cdot))\right\|_{L^p(\mathbb{R}^N)} \mathrm{d}m,
\end{aligned} \tag{3.10}$$

where we define

$$\mathscr{Q}(t + \varepsilon, t)u := (t + \varepsilon)^{\alpha-1} \mathscr{A}_2(\alpha, \sigma)(t + \varepsilon)u - t^{\alpha-1} \mathscr{A}_2(\alpha, \sigma)(t - m)u.$$

By Theorem 3.2 and Remark 1.6 in [36], we deduce

$$\lim_{\varepsilon\to 0}\left\|(\mathscr{A}_1(\alpha,\sigma)(t+\varepsilon)-\mathscr{A}_1(\alpha,\sigma)(t))h(\cdot)\right\|_{L^p(\mathbb{R}^N)}=0 \tag{3.11}$$

and

$$\lim_{\varepsilon\to 0}\left\|\mathscr{Q}(t+\varepsilon-m,t-m)F(u(m,\cdot))\right\|_{L^p(\mathbb{R}^N)}=0.$$

Furthermore, by using Lemma 2.1, we obtain

$$\left\|\mathscr{Q}(t+\varepsilon-m,t-m)F(u(m,\cdot))\right\|_{L^p(\mathbb{R}^N)}\le 2C_0(t-m)^{\alpha-1}\left\|F(u(m,\cdot))\right\|_{L^p(\mathbb{R}^N)}.$$

From the fact that $u\in B(0,R_1)\subset L^\infty\left(0,T;L^p(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)\right)$, it follows immediately

$$\left\|\mathscr{Q}(t+\varepsilon-m,t-m)F(u(m,\cdot))\right\|_{L^p(\mathbb{R}^N)}\le 2C_0L(t-m)^{\alpha-1}R_1^\nu\exp\left(R_1^p\right).$$

In sum, we have

$$\lim_{\varepsilon\to 0}\int_0^t\left\|\mathscr{Q}(t+\varepsilon-m,t-m)F(u(m,\cdot))\right\|_{L^p(\mathbb{R}^N)}\mathrm{d}m=0. \tag{3.12}$$

We next consider the third term on the right hand side of (3.10) as follows

$$\begin{aligned}&\int_t^{t+\varepsilon}\left\|(t+\varepsilon-m)^{\alpha-1}\mathscr{A}_2(\alpha,\sigma)(t+\varepsilon-m)F(u(m,\cdot))\right\|_{L^p(\mathbb{R}^N)}\mathrm{d}m\\ \le&C_0\int_t^{t+\varepsilon}(t+\varepsilon-m)^{\alpha-1}\left\|F(u(m,\cdot))\right\|_{L^p(\mathbb{R}^N)}\mathrm{d}m\\ \le&\frac{C_0\varepsilon^\alpha}{\alpha}R_1^\nu\exp\left(R_1^p\right),\end{aligned}$$

where we apply Lemma 2.1. Therefore, there holds

$$\lim_{\varepsilon\to 0}\int_t^{t+\varepsilon}\left\|(t+\varepsilon-m)^{\alpha-1}\mathscr{A}_2(\alpha,\sigma)(t+\varepsilon-m)F(u(m,\cdot))\right\|_{L^p(\mathbb{R}^N)}\mathrm{d}m=0. \tag{3.13}$$

Combining (3.10), (3.11), (3.12) and (3.13) yields the desired result. The theorem is thus proved. □

Theorem 3.2. *[Global small-data solution] Let $\nu>\frac{4}{3}$ and $p\ge 2$. Suppose that one of the following assumptions is satisfied,*

- $\nu<2,\ \sigma<N<\frac{\sigma}{\nu-1}$, *and there exists a constant $q\ge 2$ satisfying*

$$\max\left\{\frac{N}{\sigma},\frac{p}{3\nu-4}\right\}<q<\min\left\{\frac{N}{\sigma\left(1-\frac{\nu-1}{\alpha\nu}\right)},\frac{N}{\sigma-N(\nu-1)}\right\};$$

- $\nu\ge 2,\ \sigma<N$, *and there exists a constant $q\ge 2$ satisfying*

$$q>\max\left\{\frac{N}{\sigma},\frac{p}{2}\right\}.$$

Let $\beta = \frac{\alpha\left(1-\frac{N}{\sigma q}\right)}{\nu-1}$ *and* $\eta = \frac{\alpha N}{\sigma\beta}$. *Then, if the data of* $h \in L^{\Xi}(\mathbb{R}^N) \cap L^{\eta}(\mathbb{R}^N) \cap L^{\frac{p\eta}{p+\eta}}(\mathbb{R}^N)$ *is small enough, then Problem* (1.1) *possesses a unique mild solution in* $L^{\infty}(0, \infty; L^{\Xi}(\mathbb{R}^N))$.

Remark 3.1. *It's not too difficult to find a non-empty set of parameters meeting the assumptions of Theorem 3.2. Indeed, it can be pointed out some examples as follows*

(i) if $\alpha = 0.4,\ \nu = 1.66,\ \sigma = 1.5$ *and* $p = 2$ *and* $N = 2$, *we can choose* $q = 2.1$;

(ii) if $\alpha = 0.7,\ \nu = 2.5,\ \sigma = 2$ *and* $p = 3$ *and* $N = 3$, *we can choose* $q = 4$.

Remark 3.2. *By the embedding* $L^{\Xi}(\mathbb{R}^N) \hookrightarrow L^{\eta}(\mathbb{R}^N)$ *for any* $\eta \geq p$, *the assumption of h becomes* $h \in L^{\Xi}(\mathbb{R}^N) \cap L^{\frac{p\eta}{p+\eta}}(\mathbb{R}^N)$ *whenever* $\eta \geq p$. *For example, if* $\alpha = 0.7,\ \nu = 2.5,\ p = N = 3$ *and* $q = 4$, *we have* $\eta = 3.6 > p$. *Therefore, we only need* $h \in L^{\Xi}(\mathbb{R}^N) \cap L^{\frac{p\eta}{p+\eta}}(\mathbb{R}^N)$.

Proof. We first introduce a function space for the existence of solutions as follows

$$L^{\infty}_{\beta}\left(0, T; L^{\Xi}(\mathbb{R}^N)\right) := \left\{u \in L^{\infty}(0, T; L^{\Xi}(\mathbb{R}^N)) \,\middle|\, \left\|u\right\|_{L^{\infty}_{\beta}(0,T;L^{\Xi}(\mathbb{R}^N))} < \infty\right\},$$

where $\|\cdot\|_{L^{\infty}_{\beta}(0,T;L^{\Xi}(\mathbb{R}^N))}$ is defined by

$$\left\|u\right\|_{L^{\infty}_{\beta}(0,T;L^{\Xi}(\mathbb{R}^N))} := \max\left\{\operatorname*{ess\,sup}_{t\in(0,T)} \left\|u(t)\right\|_{L^{\Xi}(\mathbb{R}^N)}, \operatorname*{ess\,sup}_{t\in(0,T)} t^{\beta}\left\|u(t)\right\|_{L^{\Xi}(\mathbb{R}^N)}\right\}. \tag{3.14}$$

Next, we consider the estimate of the source term. Suppose that $w, v \in L^{\Xi}(\mathbb{R}^N)$, we can deduce from Taylor's expansion of the exponential function and Hölder's inequality with $\frac{1}{q} = \frac{1}{3q} + \frac{1}{3q} + \frac{1}{3q}$ that

$$\left\|F(w) - F(v)\right\|_{L^q(\mathbb{R}^N)} \leq L \sum_{g\in\{w,v\}} \left(\sum_{j\in\mathbb{N}} \frac{1}{j!} \left\|g\right\|^{\nu-1}_{L^{3q(\nu-1)}(\mathbb{R}^N)} \left\|g\right\|^{jp}_{L^{3pqj}(\mathbb{R}^N)}\right) \left\|w - v\right\|_{L^{3q}(\mathbb{R}^N)}. \tag{3.15}$$

Thanks to the definition of the Luxemburg norm and the monotone convergence theorem, for any $u \in L^{\Xi}(\mathbb{R}^N)$ there holds

$$\int_{\mathbb{R}^N} \frac{\left|u(t,x)\right|^q}{\left\|u(t)\right\|^q_{L^{\Xi}(\mathbb{R}^N)} \Gamma(\frac{q}{p} + 1)} \mathrm{d}x \leq \int_{\mathbb{R}^N} \left(e^{\left|\frac{u(t,x)}{\kappa}\right|^p} - 1\right) \mathrm{d}x \leq 1,$$

provided that

$$\frac{z^q}{\Gamma(q+1)} < e^z - 1$$

for any $q > 1$ and $z > 0$. Therefore, we obtain the following estimate

$$\left\|u\right\|_{L^q(\mathbb{R}^N)} \leq \sqrt[q]{\Gamma\left(\frac{q}{p} + 1\right)} \left\|u\right\|_{L^{\Xi}(\mathbb{R}^N)}. \tag{3.16}$$

Applying (3.16) to (3.15), we get immediately that

$$\begin{aligned}\left\|F(w)-F(v)\right\|_{L^q(\mathbb{R}^N)} \le & L\left(\Gamma\left(\frac{3q(\nu-1)}{p}+1\right)\right)^{\frac{1}{3q(\nu-1)}}\left(\Gamma\left(\frac{3q}{p}+1\right)\right)^{\frac{1}{3q}} \\ & \times \sum_{g\in\{w,v\}}\|g\|_{L^\Xi(\mathbb{R}^N)}^{\nu-1}\sum_{j\in\mathbb{N}}\frac{\left(\Gamma(3qj+1)\right)^{\frac{1}{3q}}}{j!}\|g\|_{L^\Xi(\mathbb{R}^N)}^{jp}\|w-v\|_{L^\Xi(\mathbb{R}^N)}.\end{aligned}$$

Using [5, Lemma 3.3], we derive

$$\left\|F(w)-F(v)\right\|_{L^q(\mathbb{R}^N)} \le C(q)\sum_{g\in\{w,v\}}\|g\|_{L^\Xi(\mathbb{R}^N)}^{\nu-1}\sum_{j\in\mathbb{N}}\left(3q\|g\|_{L^\Xi(\mathbb{R}^N)}^{p}\right)^j\|w-v\|_{L^\Xi(\mathbb{R}^N)}, \tag{3.17}$$

where we use the fact that for $j\in\mathbb{N}$, there holds $\Gamma(j+1)=j!$ and denote

$$C(q) := C_1L\left(\Gamma\left(\frac{3q(\nu-1)}{p}+1\right)\right)^{\frac{1}{3q(\nu-1)}}\left(\Gamma\left(\frac{3q}{p}+1\right)\right)^{\frac{1}{3q}},$$

where C_1 is a positive constant that is independent of w, v. Let R_2 be a sufficiently small constant. Suppose that u_l is in an open ball $B(0,R_2)\subset L_\beta^\infty(0,T;L^\Xi(\mathbb{R}^N))$ for any $l\in\mathbb{N}$, we can show that $u_{l+1}\in B(0,R_2)$. In fact, on the one hand, by applying Lemma 2.1, we derive

$$\begin{aligned}\left\|u_{l+1}(t)-u_1(t)\right\|_{L^\infty(\mathbb{R}^N)} &\le \int_0^t (t-m)^{\alpha-1}\left\|\mathscr{A}_2(\alpha,\sigma)(t-m)F(u_l(m))\right\|_{L^\infty(\mathbb{R}^N)}\mathrm{d}m \\ &\le C_0\int_0^t (t-m)^{\alpha\left(1-\frac{N}{\sigma q}\right)-1}\left\|F(u_l(m))\right\|_{L^q(\mathbb{R}^N)}\mathrm{d}m.\end{aligned}$$

Then, we use (3.17) with $w=u_l$ and (1.3) to find

$$\left\|u_{l+1}(t)-u_1(t)\right\|_{L^\infty(\mathbb{R}^N)} \le C_0C(q)\left(\int_0^t (t-m)^{\alpha\left(1-\frac{N}{\sigma q}\right)-1}\|u(m)\|_{L^\Xi(\mathbb{R}^N)}^{\nu}\mathrm{d}m\right)\frac{1}{1-3qR_2^p}, \tag{3.18}$$

where we presume that $R_2<(3q)^{-\frac{1}{p}}$. On the other hand, repeat application of Lemma (2.1) with $s=p$ and $r=\frac{pq}{p+q}$ yields

$$\begin{aligned}\left\|u_{l+1}(t)-u_1(t)\right\|_{L^p(\mathbb{R}^N)} &\le \int_0^t (t-m)^{\alpha-1}\left\|\mathscr{A}_2(\alpha,\sigma)(t-m)F(u_l(m))\right\|_{L^p(\mathbb{R}^N)}\mathrm{d}m \\ &\le C_0\int_0^t (t-m)^{\alpha\left(1-\frac{N}{\sigma q}\right)-1}\left\|F(u_l(m))\right\|_{L^r(\mathbb{R}^N)}\mathrm{d}m,\end{aligned}$$

where we note that if $N<\sigma q<2\sigma q$, we deduce $p<\frac{Nr}{N-2r\sigma}$. As a consequence, if $R_2<(3r)^{-\frac{1}{p}}$, it follows from the above estimate and (3.17) that

$$\left\|u_{l+1}(t)-u_1(t)\right\|_{L^p(\mathbb{R}^N)} \le C_0C(r)\left(\int_0^t (t-m)^{\alpha\left(1-\frac{N}{\sigma q}\right)-1}\|u(m)\|_{L^\Xi(\mathbb{R}^N)}^{\nu}\mathrm{d}m\right)\frac{1}{1-3rR^p}, \tag{3.19}$$

provided that $\max\{3r(\nu-1), 3r\} \geq p$. Combining (3.18), (3.19) and the embedding $L^{\infty}(\mathbb{R}^N)\cap L^p(\mathbb{R}^N) \hookrightarrow L^{\Xi}(\mathbb{R}^N)$ gives

$$\begin{aligned}&\left\|u_{l+1}(t) - u_1(t)\right\|_{L^{\Xi}(\mathbb{R}^N)}\\ \leq&\frac{C_0C_2\left(2-3R^p(C(r)r+qC(q))\right)}{(1-3qR^p)(1-3rR^p)}\left(\int_0^t (t-m)^{\alpha\left(1-\frac{N}{\sigma q}\right)-1}\left\|u_l(m)\right\|^{\nu}_{L^{\Xi}(\mathbb{R}^N)}\mathrm{d}m\right),\end{aligned} \tag{3.20}$$

where C_2 is a positive constant coming from the embedding. According to the definition of the Beta function, we have

$$t^{\beta}\int_0^t (t-m)^{\alpha\left(1-\frac{N}{\sigma q}\right)-1}m^{-\beta\nu}\mathrm{d}m = \mathbf{B}\left(\alpha\left(1-\frac{N}{\sigma q}\right), 1-\beta\nu\right). \tag{3.21}$$

In view of (3.20) and (3.21), for any $t > 0$, the following estimate is satisfied

$$t^{\beta}\left\|u_{l+1}(t) - u_1(t)\right\|_{L^{\Xi}(\mathbb{R}^N)} \leq \frac{C_0C_2\left(2-3R_2^p(C(r)r+C(q)q)\right)}{(1-3qR_2^p)(1-3rR_2^p)}R_2^{\nu}\mathbf{B}\left(\alpha\left(1-\frac{N}{\sigma q}\right), 1-\beta\nu\right), \tag{3.22}$$

provided that

$$\left\|u(t)\right\|_{L^{\Xi}(\mathbb{R}^N)} \leq t^{-\beta}\operatorname*{ess\,sup}_{t\in(0,T)} t^{\beta}\left\|u(t)\right\|_{L^{\Xi}(\mathbb{R}^N)}$$

for any $t > 0$ and $u \in L^{\infty}_{\beta}\left(0, T; L^{\Xi}(\mathbb{R}^N)\right)$. Then, if R_2 is small enough such that

$$\frac{\left(2-3R_2^p(C(r)r+C(q)q)\right)}{(1-3qR_2^p)(1-3rR_2^p)}R_2^{\nu-1} < \left(2C_0C_2\mathbf{B}\left(\alpha\left(1-\frac{N}{\sigma q}\right), 1-\beta\nu\right)\right)^{-1},$$

we get immediately

$$t^{\beta}\left\|u_{l+1}(t) - u_1(t)\right\|_{L^{\Xi}(\mathbb{R}^N)} < \frac{R_2}{2} \quad \text{for all } t > 0.$$

Next, we set

$$\mathscr{T} = \frac{R_2}{4}\left(\frac{C_0C_2\left(2-3R_2^p(C(r)r+C(q)q)\right)}{(1-3qR_2^p)(1-3rR_2^p)\alpha\left(1-\frac{N}{\sigma q}\right)}R_2^{\nu}\right)^{\frac{-1}{\alpha\left(1-\frac{N}{\sigma q}\right)}}.$$

Then, for any $t \leq \mathscr{T}$, (3.20) implies

$$\left\|u_{l+1}(t) - u_1(t)\right\|_{L^{\Xi}(\mathbb{R}^N)} \leq \frac{C_0C_2\left(2-3R_2^p(C(r)r+C(q)q)\right)R_2^{\nu}}{(1-3qR_2^p)(1-3rR_2^p)}\left(\int_0^t (t-m)^{\alpha\left(1-\frac{N}{\sigma q}\right)-1}\mathrm{d}m\right),$$

provided that $u_l \in B(0, R_2)$. Since $N < \sigma q$, we obtain

$$\left\|u_{l+1}(t) - u_1(t)\right\|_{L^{\Xi}(\mathbb{R}^N)} \leq \frac{C_0C_2\left(2-3R_2^p(C(r)r+C(q)q)\right)}{(1-3qR_2^p)(1-3rR_2^p)\alpha\left(1-\frac{N}{\sigma q}\right)}R_2^{\nu}\mathscr{T}^{\alpha\left(1-\frac{N}{\sigma q}\right)}$$

$$\leq \frac{R_2}{4}.$$

At the same time, if $t > \mathscr{T}$, we deduce

$$\begin{aligned}
&\left\|u_{l+1}(t) - u_1(t)\right\|_{L^{\Xi}(\mathbb{R}^N)} \\
\leq &\mathscr{T}^{-\beta} t^{\beta} \left\|u_{l+1}(t) - u_1(t)\right\|_{L^{\Xi}(\mathbb{R}^N)} \\
\leq &\mathscr{T}^{-\beta} \frac{C_0 C_2 \left(2 - 3R_2^p (C(r)r + C(q)q)\right)}{(1 - 3qR_2^p)(1 - 3rR_2^p)} R_2^{\nu} \mathbf{B}\left(\alpha\left(1 - \frac{N}{\sigma q}\right), 1 - \beta\nu\right) \\
= &4^{\beta} \left(\frac{C_0 C_2 \left(2 - 3R_2^p (C(r)r + C(q)q)\right)}{(1 - 3qR_2^p)(1 - 3rR_2^p)}\right)^{1 + \frac{\beta}{\alpha\left(1 - \frac{N}{\sigma q}\right)}} R_2^{\nu\left(1 + \frac{\beta}{\alpha\left(1 - \frac{N}{\sigma q}\right)}\right) - \beta} \mathbf{B}\left(\alpha\left(1 - \frac{N}{\sigma q}\right), 1 - \beta\nu\right),
\end{aligned}$$

where we apply (3.22). If R_2 satisfies

$$\frac{\left(2 - 3R_2^p (C(r)r + C(q)q)\right)^{1 + \frac{\beta}{\alpha\left(1 - \frac{N}{\sigma q}\right)}} R_2^{\nu\left(1 + \frac{\beta}{\alpha\left(1 - \frac{N}{\sigma q}\right)}\right) - \beta - 1}}{\left((1 - 3qR_2^p)(1 - 3rR_2^p)\right)^{1 + \frac{\beta}{\alpha\left(1 - \frac{N}{\sigma q}\right)}}} < \frac{\left(\mathbf{B}\left(\alpha\left(1 - \frac{N}{\sigma q}\right), 1 - \beta\nu\right)\right)^{-1}}{4^{\beta+1} C_0 C_2},$$

there holds immediately

$$\left\|u_{l+1}(t) - u_1(t)\right\|_{L^{\Xi}(\mathbb{R}^N)} < \frac{R_2}{4}, \quad \text{for all } t > \mathscr{T}.$$

From the above results, whether t is greater than $\mathscr{T}$ or t is less than $\mathscr{T}$, we always get the following result

$$\left\|u_{l+1}(t) - u_1(t)\right\|_{L^{\Xi}(\mathbb{R}^N)} < \frac{R_2}{2}$$

as long as $u_l \in B(0, R_2)$ and R_2 is small enough. For the purpose of proving $\{u_l\}_{l \in \mathbb{N}}$ is a subset of $B(0, R_2)$, we also need to consider the initial data h. On the one hand, by using Corollary 2.2, we get easily that

$$\left\|u_1\right\|_{L^{\Xi}(\mathbb{R}^N)} = \left\|\mathscr{A}_1(\alpha, \sigma)(t)h\right\|_{L^{\Xi}(\mathbb{R}^N)} \leq C_0 \left\|h\right\|_{L^{\Xi}(\mathbb{R}^N)}.$$

On the other hand, Lemma 2.1 shows that

$$\begin{aligned}
\left\|\mathscr{A}_1(\alpha, \sigma)(t)h\right\|_{L^{\infty}(\mathbb{R}^N)} &\leq C_0 t^{-\frac{\alpha N}{\sigma \eta}} \left\|h\right\|_{L^{\eta}(\mathbb{R}^N)}, \\
\left\|\mathscr{A}_1(\alpha, \sigma)(t)h\right\|_{L^{p}(\mathbb{R}^N)} &\leq C_0 t^{-\frac{\alpha N}{\sigma \eta}} \left\|h\right\|_{L^{\frac{p\eta}{p+\eta}}(\mathbb{R}^N)},
\end{aligned}$$

where $\eta = \frac{\alpha N}{\sigma \beta} > 1$. Then, we get

$$t^{\beta} \left\|\mathscr{A}_1(\alpha, \sigma)(t)h\right\|_{L^{\Xi}(\mathbb{R}^N)} \leq C_0 C_2 \left\|h\right\|_{L^{\eta}(\mathbb{R}^N)}.$$

Presume that the initial data is small enough, precisely,

$$\begin{cases} \left\|h\right\|_{L^{\Xi}(\mathbb{R}^N)} \le \dfrac{R_2}{2C_0}, \\ \left\|h\right\|_{L^{\eta}(\mathbb{R}^N)\cap L^{\frac{p\eta}{p+\eta}}(\mathbb{R}^N)} \le \dfrac{R_2}{4C_0C_2}. \end{cases}$$

Then, we can conclude that $u_1 \in B(0, R_2)$. Hence, if $u_l \in B(0, R_2)$ for any $l \ge 2$, we have $u_{l+1} \in B(0, R_2)$, provided that R_2 and the initial data are sufficiently small. Summarily, we have $\{u_l\}_{l\in\mathbb{N}} \in B(0, R_2)$.

To complete the Banach principle argument, we need also to show that $\{u_l\}_{l\in\mathbb{N}}$ is a Cauchy sequence in $B(0, R_2)$. Since the techniques are not too different from those in the results above, we only briefly present the main estimates. For u_l and u_{l-1} in $B(0, R_2)$, $l \ge 2$, Lemma 2.1 yields

$$\begin{aligned} &\left\|u_{l+1}(t) - u_l(t)\right\|_{L^{\infty}(\mathbb{R}^N)} \\ \le& C_0 \int_0^t (t-m)^{\alpha\left(1-\frac{N}{\sigma q}\right)-1} \left\|F(u_l(m)) - F(u_{l-1}(m))\right\|_{L^q(\mathbb{R}^N)} \mathrm{d}m. \\ \le& \frac{C_0C(q)}{1-3qR^p} \sum_{k\in\{l-1,l\}} \int_0^t (t-m)^{\alpha\left(1-\frac{N}{\sigma q}\right)-1} \|u_k(m)\|^{\nu-1}_{L^{\Xi}(\mathbb{R}^N)} \left\|u_l(m) - u_{l-1}(m)\right\|_{L^{\Xi}(\mathbb{R}^N)} \mathrm{d}m \end{aligned}$$

and

$$\begin{aligned} &\left\|u_{l+1}(t) - u_l(t)\right\|_{L^{p}(\mathbb{R}^N)} \\ \le& C_0 \int_0^t (t-m)^{\alpha\left(1-\frac{N}{\sigma q}\right)-1} \left\|F(u_l(m)) - F(u_{l-1}(m))\right\|_{L^r(\mathbb{R}^N)} \mathrm{d}m. \\ \le& \frac{C_0C(r)}{1-3rR^p} \sum_{k\in\{l-1,l\}} \int_0^t (t-m)^{\alpha\left(1-\frac{N}{\sigma q}\right)-1} \|u_k(m)\|^{\nu-1}_{L^{\Xi}(\mathbb{R}^N)} \left\|u_l(m) - u_{l-1}(m)\right\|_{L^{\Xi}(\mathbb{R}^N)} \mathrm{d}m, \end{aligned}$$

provided that $R_2 < \min\left\{(3q)^{-\frac{1}{p}}, (3r)^{-\frac{1}{p}}\right\}$. Therefore, since $L^p(\mathbb{R}^N) \cap L^{\infty}(\mathbb{R}^N)$ embeds into $L^{\Xi}(\mathbb{R}^N)$, we have

$$\begin{aligned} \left\|u_{l+1}(t) - u_l(t)\right\|_{L^{\Xi}(\mathbb{R}^N)} \le& \frac{C_0C_2\left(2 - 3R_2^p(C(r)r + qC(q))\right)}{(1-3qR_2^p)(1-3rR_2^p)} \\ &\times \sum_{k\in\{l-1,l\}} \int_0^t \frac{\|u_k(m)\|^{\nu-1}_{L^{\Xi}(\mathbb{R}^N)}}{(t-m)^{1-\alpha\left(1-\frac{N}{\sigma q}\right)}} \left\|u_l(m) - u_{l-1}(m)\right\|_{L^{\Xi}(\mathbb{R}^N)} \mathrm{d}m. \end{aligned} \tag{3.23}$$

It follows immediately that

$$\begin{aligned} t^{\beta}\left\|u_{l+1}(t) - u_l(t)\right\|_{L^{\Xi}(\mathbb{R}^N)} \le& \frac{2C_0C_2\left(2 - 3R_2^p(C(r)r + qC(q))\right)R_2^{\nu-1}}{(1-3qR_2^p)(1-3rR_2^p)} \\ &\times \mathbf{B}\left(\alpha\left(1-\frac{N}{\sigma q}\right), 1-\beta\nu\right) \operatorname*{ess\,sup}_{t\in(0,T)} t^{\beta}\left\|u_l(t) - u_{l-1}(t)\right\|_{L^{\Xi}(\mathbb{R}^N)}. \end{aligned}$$

Presume that R_2 is small enough such that

$$\frac{\left(2-3R_2^p(C(r)r+C(q)q)\right)}{(1-3qR_2^p)(1-3rR_2^p)}R_2^{\nu-1}<\left(4C_0C_2\mathbf{B}\left(\alpha\left(1-\frac{N}{\sigma q}\right),1-\beta\nu\right)\right)^{-1}.$$

We then find that

$$\operatorname*{ess\,sup}_{t\in(0,T)} t^{\beta}\left\|u_{l+1}(t)-u_l(t)\right\|_{L^{\Xi}(\mathbb{R}^N)}\leq\frac{1}{2}\operatorname*{ess\,sup}_{t\in(0,T)} t^{\beta}\left\|u_l(t)-u_{l-1}(t)\right\|_{L^{\Xi}(\mathbb{R}^N)}. \tag{3.24}$$

Set

$$\overline{\mathscr{T}}=\frac{1}{8}\left(\frac{C_0C_2\left(2-3R_2^p(C(r)r+C(q)q)\right)}{(1-3qR_2^p)(1-3rR_2^p)\alpha\left(1-\frac{N}{\sigma q}\right)}R_2^{\nu-1}\right)^{\frac{-1}{\alpha\left(1-\frac{N}{\sigma q}\right)}}.$$

On the one hand, for any $t\leq\overline{\mathscr{T}}$ and $l\geq 2$, we derive (3.23) that

$$\begin{aligned}
&\left\|u_{l+1}(t)-u_l(t)\right\|_{L^{\Xi}(\mathbb{R}^N)}\\
\leq&\frac{2C_0C_2\left(2-3R_2^p(C(r)r+C(q)q)\right)}{(1-3qR_2^p)(1-3rR_2^p)\alpha\left(1-\frac{N}{\sigma q}\right)}R_2^{\nu-1}\,\mathscr{T}^{\alpha\left(1-\frac{N}{\sigma q}\right)}\operatorname*{ess\,sup}_{t\in(0,T)}\left\|u_l(t)-u_{l-1}(t)\right\|_{L^{\Xi}(\mathbb{R}^N)}\\
\leq&\frac{1}{4}\operatorname*{ess\,sup}_{t\in(0,T)}\left\|u_l(t)-u_{l-1}(t)\right\|_{L^{\Xi}(\mathbb{R}^N)}.
\end{aligned} \tag{3.25}$$

On the other hand, if $t>\overline{\mathscr{T}}$, the following estimate holds

$$\begin{aligned}
&\left\|u_{l+1}(t)-u_l(t)\right\|_{L^{\Xi}(\mathbb{R}^N)}\\
\leq&\frac{2C_0C_2\left(2-3R_2^p(C(r)r+C(q)q)\right)}{\overline{\mathscr{T}}^{\beta}(1-3qR_2^p)(1-3rR_2^p)}R_2^{\nu-1}\mathbf{B}\left(\alpha\left(1-\frac{N}{\sigma q}\right),1-\beta\nu\right)\\
=&4^{\beta}\left(\frac{2C_0C_2\left(2-3R_2^p(C(r)r+C(q)q)\right)}{(1-3qR_2^p)(1-3rR_2^p)}\right)^{1+\frac{\beta}{\alpha\left(1-\frac{N}{\sigma q}\right)}}R_2^{(\nu-1)\left(1+\frac{\beta}{\alpha\left(1-\frac{N}{\sigma q}\right)}\right)-\beta}\\
&\times\mathbf{B}\left(\alpha\left(1-\frac{N}{\sigma q}\right),1-\beta\nu\right)\operatorname*{ess\,sup}_{t\in(0,T)}\left\|u_l(t)-u_{l-1}(t)\right\|_{L^{\Xi}(\mathbb{R}^N)}\\
\leq&\frac{1}{4}\operatorname*{ess\,sup}_{t\in(0,T)}\left\|u_l(t)-u_{l-1}(t)\right\|_{L^{\Xi}(\mathbb{R}^N)},
\end{aligned} \tag{3.26}$$

as long as R_2 satisfies

$$\frac{\left(2-3R_2^p(C(r)r+C(q)q)\right)^{1+\frac{\beta}{\alpha\left(1-\frac{N}{\sigma q}\right)}}R_2^{(\nu-1)\left(1+\frac{\beta}{\alpha\left(1-\frac{N}{\sigma q}\right)}\right)-\beta}}{\left((1-3qR_2^p)(1-3rR_2^p)\right)^{1+\frac{\beta}{\alpha\left(1-\frac{N}{\sigma q}\right)}}}<\frac{\left(\mathbf{B}\left(\alpha\left(1-\frac{N}{\sigma q}\right),1-\beta\nu\right)\right)^{-1}}{4^{\beta+\frac{3}{2}}C_0C_2}.$$

Combining (3.25) and (3.26) yields

$$\operatorname*{ess\,sup}_{t\in(0,T)} \left\| u_{l+1}(t) - u_l(t) \right\|_{L^{\Xi}(\mathbb{R}^N)} \le \frac{1}{2} \operatorname*{ess\,sup}_{t\in(0,T)} \left\| u_l(t) - u_{l-1}(t) \right\|_{L^{\Xi}(\mathbb{R}^N)} \tag{3.27}$$

for any $T \in (0, \infty)$ and $l \ge 2$. As the result of (3.24) and (3.27), for $u_l, u_{l-1} \in B(0, R_2)$ and $l \ge 2$, we obtain

$$\left\| u_{l+1} - u_l \right\|_{L^\infty_\beta\left(0,T;L^{\Xi}(\mathbb{R}^N)\right)} \le \frac{1}{2} \left\| u_l - u_{l-1} \right\|_{L^\infty_\beta\left(0,T;L^{\Xi}(\mathbb{R}^N)\right)}.$$

Then, by similar arguments of Theorem 3.1, we can show that $\{u_l\}_{l\in\mathbb{N}}$ is a Cauchy sequence. In addition, since $L^\infty_\beta\left(0, T; L^{\Xi}(\mathbb{R}^N)\right)$ is a complete space with the metric

$$d(u, v) := \left\| u - v \right\|_{L^\infty_\beta\left(0,T;L^{\Xi}(\mathbb{R}^N)\right)},$$

there exists a unique limit function of $\{u_l\}_{l\in\mathbb{N}}$ in $B(0, R_2)$, which is the unique mild solution to Problem (1.1). The proof is completed. □

4. Conclusions

This paper considers a Cauchy Problem for a time-space fractional diffusion equation with exponential source term. By iteration method, a unique local mild solution is derived for initial data in $L^p(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. The existence and uniqueness are extended to be global in time when we suppose additionally that initial data in an Orlicz space are small enough. However, since the space $C_0^\infty(\mathbb{R}^N)$ is not dense in $L^{\Xi}(\mathbb{R}^N)$, the continuity of solutions in the term of time-variable is not considered for the global case. This will be a potential approach to improve the results of this work in the future.

Acknowledgement

The first author Anh Tuan Nguyen is funded by Van Lang University.

Conflict of interest

The authors declare there is no conflicts of interest.

References

1. X. Wang, R. Xu, Global existence and finite time blowup for a nonlocal semilinear pseudo-parabolic equation, *Adv. Nonlinear Anal.*, **10** (2021), 261–288. https://doi.org/10.1515/anona-2020-0141
2. R. Xu, J. Su, Global existence and finite time blow-up for a class of semilinear pseudo-parabolic equations, *J. Funct. Anal.*, **264** (2013), 2732–2763. https://doi.org/10.1016/j.jfa.2013.03.010
3. W. Lian, J. Shen, R. Xu, Y. Yang, Infinite sharp conditions by Nehari manifolds for nonlinear Schrödinger equations, *J. Geom. Anal.*, **30** (2020), 1865–1886. https://doi.org/10.1007/s12220-019-00281-5

4. S. Chen, B. Melnick, R. Xu, Global existence and blowup solutions for the Gierer–Meinhardt system, *Nonlinear Anal.*, **196** (2020), 111785. https://doi.org/10.1016/j.na.2020.111785
5. G. Furioli, T. Kawakami, B. Ruf, E. Terraneo, Asymptotic behavior and decay estimates of the solutions for a nonlinear parabolic equation with exponential nonlinearity, *J. Differ. Equ.*, **262** (2017), 145–180. https://doi.org/10.1016/j.jde.2016.09.024
6. S. Ibrahim, M. Majdoub, N. Masmoudi, Global solutions for a semilinear, two-dimensional Klein-Gordon equation with exponential-type nonlinearity, *Commun. Pure Appl. Math.*, **59** (2006), 1639–1658. https://doi.org/10.1002/cpa.20127
7. S. Ibrahim, M. Majdoub, N. Masmoudi, K. Nakanishi, Scattering for the two-dimensional NLS with exponential nonlinearity, *Nonlinearity*, **25** (2012), 1843. https://doi.org/10.1088/0951-7715/25/6/1843
8. N. Ioku, The Cauchy problem for heat equations with exponential nonlinearity, *J. Differ. Equ.*, **251** (2011), 1172–1194. https://doi.org/10.1016/j.jde.2011.02.015
9. M. Nakamura, T. Ozawa, Nonlinear Schrödinger equations in the Sobolev space of critical order, *J. Funct. Anal.*, **155** (1998), 364–380. https://doi.org/10.1006/jfan.1997.3236
10. M. Nakamura, T. Ozawa, Global solutions in the critical Sobolev space for the wave equations with nonlinearity of exponential growth, *Math. Zeitschrift*, **231** (1999), 479–487. https://doi.org/10.1007/PL00004737
11. Y. Xiao, Packing measure of the sample paths of fractional Brownian motion, *Trans. Am. Math. Soc.*, **348** (1996), 3193–3213. https://doi.org/10.1090/S0002-9947-96-01712-6
12. P. M. de Carvalho-Neto, G. Planas, Mild solutions to the time fractional Navier–Stokes equations in $\mathbb{R}^N$, *J. Differ. Equ.*, **259** (2015), 2948–2980. https://doi.org/10.1016/j.jde.2015.04.008
13. M. Kirane, D. Aimene, D. Seba, Local and global existence of mild solutions of time-fractional Navier-Stokes system posed on the Heisenberg group, *Z. Angew. Math. Phys.*, **72** (2021), 1–19. https://doi.org/10.1007/s00033-021-01499-6
14. L. Li, J-G. Liu, L. Wang, Cauchy problems for Keller–Segel type time–space fractional diffusion equation, *J. Differ. Equ.*, **265** (2018), 1044–1096. https://doi.org/10.1016/j.jde.2018.03.025
15. G. Del Piero, L. Deseri, On the concepts of state and free energy in linear viscoelasticity, *Arch. Ration. Mech. Anal.*, **138** (1997), 1–35. https://doi.org/10.1007/s002050050035
16. N. T. Bao, T. Caraballo, N. H. Tuan, Y. Zhou, Existence and regularity results for terminal value problem for nonlinear fractional wave equations, *Nonlinearity*, **34** (2021), 1448. https://doi.org/10.1088/1361-6544/abc4d9
17. T. Caraballo, T. B. Ngoc, T. N. Thach, N. H. Tuan, On initial value and terminal value problems for subdiffusive stochastic Rayleigh-Stokes equation, *Discrete Contin. Dyn. Syst.-B*, **26** (2021), 4299. https://doi.org/10.3934/dcdsb.2020289
18. N. H. Can, Y. Zhou, N. H. Tuan, T. N. Thach, Regularized solution approximation of a fractional pseudo-parabolic problem with a nonlinear source term and random data, *Chaos Solitons Fractals*, **136** (2020), 109847. https://doi.org/10.1016/j.chaos.2020.109847
19. R. Grande, Space-Time Fractional Nonlinear Schrödinger Equation, *SIAM J. Math. Anal.*, **51** (2019), 4172–4212. https://doi.org/10.1137/19M1247140

20. A. T. Nguyen, T. Caraballo, N. H. Tuan, On the initial value problem for a class of nonlinear biharmonic equation with time-fractional derivative, *P. Roy. Soc. Edinb. A*, (2021), 1–43. https://doi.org/10.1017/prm.2021.44
21. H. T. Nguyen, N. A. Tuan, C. Yang, Global well-posedness for fractional Sobolev-Galpern type equations, *Discrete Contin. Dyn. Syst.*, accepted. https://doi.org/10.3934/dcds.2021206
22. N. H. Tuan, V. V. Au, R. Xu, Semilinear Caputo time-fractional pseudo-parabolic equations, *Commun. Pure Appl. Anal.*, **20** (2021), 583. https://doi.org/10.3934/cpaa.2020282
23. N. H. Tuan, T. N. Thach, Y. Zhou, On a backward problem for two-dimensional time fractional wave equation with discrete random data, *Evol. Equ. Control. Theory*, **9** (2020), 561. https://doi.org/10.3934/eect.2020024
24. N. H. Tuan, N. H. Tuan, D. Baleanu, T. N. Thach, On a backward problem for fractional diffusion equation with Riemann-Liouville derivative, *Math. Method Appl. Sci.*, **43** (2020), 1292–1312. https://doi.org/10.1002/mma.5943
25. Y. Zhang, X. Xu, Inverse source problem for a fractional diffusion equation, *Inverse Probl.*, **27** (2011), 035010. https://doi.org/10.1088/0266-5611/27/3/035010
26. J. R. Cannon, P. DuChateau, An inverse problem for a nonlinear diffusion equation, *SIAM J. Appl. Math.*, **39** (1980), 272–289. https://doi.org/10.1137/0139024
27. A. N. Cohen, R. L. Pego, Stable patterns in a viscous diffusion equation, *Trans. Am. Math. Soc.*, **324** (1991), 331–351. https://doi.org/10.1090/S0002-9947-1991-1015926-7
28. Y. Liu, R. Xu, Wave equations and reaction-diffusion equations with several nonlinear source terms of different sign, *Discrete Contin. Dyn. Syst.-B*, **7** (2007), 171. https://doi.org/10.3934/dcdsb.2007.7.171
29. Y. Liu, R. Xu, T. Yu, Global existence, nonexistence and asymptotic behavior of solutions for the Cauchy problem of semilinear heat equations, *Nonlinear Anal. Theory Methods Appl.*, **68** (2008), 3332–3348. https://doi.org/10.1016/j.na.2007.03.029
30. N. H. Tuan, T. Caraballo, On initial and terminal value problems for fractional nonclassical diffusion equations, *Proc. Am. Math. Soc.*, **149** (2021), 143–161. https://doi.org/10.1090/proc/15131
31. S. D. Eidelman, A. N. Kochubei, Cauchy problem for fractional diffusion equations, *J. Differ. Equ.*, **199** (2004), 211–255. https://doi.org/10.1016/j.jde.2003.12.002
32. V. Vergara, R. Zacher, Optimal decay estimates for time-fractional and other nonlocal subdiffusion equations via energy methods, *SIAM J. Math. Anal.*, **47** (2015), 210–239. https://doi.org/10.1137/130941900
33. B. de Andrade, C. Cuevas, H. Soto, On fractional heat equations with non-local initial conditions, *P. Edinburgh Math. Soc.*, **59** (2016), 65–76. https://doi.org/10.1017/S0013091515000590
34. N. H. Tuan, T. B. Ngoc, Y. Zhou, D. O'Regan, On existence and regularity of a terminal value problem for the time fractional diffusion equation, *Inverse Probl.*, **36** (2020), 055011. https://doi.org/10.1088/1361-6420/ab730d
35. E. Bazhlekova, *Fractional evolution equations in Banach spaces*, Eindhoven: Technische Universiteit Eindhoven, 2001.

36. R. N. Wang, D. H. Chen, T. J. Xiao, Abstract fractional Cauchy problems with almost sectorial operators, *J. Differ. Equ.*, **252** (2012), 202–235. https://doi.org/10.1016/j.jde.2011.08.048

37. R. A. Adams, J. J. F. Fournier, *Sobolev spaces*. Elsevier, 2003.