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Research article

SAV Galerkin-Legendre spectral method for the nonlinear Schrödinger-Possion equations

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Abstract: In this paper, a fully discrete scheme is proposed to solve the nonlinear Schrödinger-Possion equations. The scheme is developed by the scalar auxiliary variable (SAV) approach, the Crank-Nicolson temproal discretization and the Galerkin-Legendre spectral spatial discretization. The fully discrete scheme is proved to be mass- and energy- conserved. Moreover, unconditional energy stability and convergence of the scheme are obtained by the use of the Gagliardo-Nirenberg and some Sobolev inequalities. Numerical results are presented to confirm our theoretical findings.

Keywords: nonlinear Schrödinger-Possion equations; energy stability; error estimates; Galerkin-Legendre spectral method; scalar auxiliary variable (SAV)

1. Introduction

In this paper, we present a fully-discrete structure-preserving scheme for the following nonlinear Schrödinger-Possion equations

$$\begin{cases} \mathrm{i}u_t + u_{xx} + \beta uv = f(|u|^2)u + \psi(x)u, & \text{in } \Omega\times[0,T], \\ -v_{xx} = |u|^2, & \text{in } \Omega\times[0,T], \\ u = v = 0, & \text{on } \partial\Omega\times[0,T], \\ u(x,0) = u_0(x), & \text{in } \bar{\Omega}, \end{cases} \tag{1.1}$$

where $\mathrm{i} = \sqrt{-1}$, $\Omega = (-1,1)$. $\psi(x)$ is the real-valued potential function that represents the external field. $\beta \in R$ is a coupling constant that represents the relative strength of the Poisson potential, $\beta \geq 0$ holds in the case of attracting forces and $\beta < 0$ holds in the case of repulsive forces. The Schrödinger-Possion system is a local single particle approximation of time-dependent Hartree-Fock system. This nonlinear system (1.1) has important applications in many quantum systems [1, 2], where the complex-valued function $u(x,t)$ denotes the single particle wave function, and $v(x,t)$ represents the Poisson potential relative to the boundary condition.

In recent years, much attention has been paid to developing efficient numerical schemes for solving the Schrödinger-Possion equations (1.1). For instance, Soler and Ringhofer [3] proposed a modified Crank-Nicolson scheme, which is mass- and energy-conserving in the discrete level. Dong [4] improved the numerical methods for the general 3D case. And the computational cost was significantly reduced. Zhang and Dong [5] applied the backward Euler and time-splitting sine pseudo-spectral methods to study the ground state and dynamics of 3D system in different setups. Auzinger et al. [6] used operator splitting methods combined with finite element spatial discretizations to solve the problem. Cheng et al. [7] proposed a fast spectral element method which can reach exponential accuracy for the Schrödinger-Poisson system. Lubich [8] gave an error analysis of Strang-type splitting integrators for the Schrödinger-Poisson equations. Furthermore, there exists many other numerical schemes for Schrödinger-type equations. Li et al. [9] developed efficient numerical schemes for the coupled fractional Klein-Gordon-Schrödinger equation by combining the Crank-Nicolson/leap-frog difference methods for the temporal discretization and the Galerkin finite element methods for the spatial discretization. Li et al. [10] constructed the conservative linearized Galerkin finite element methods (FEMs) for the nonlinear Klein-Gordon-Schrödinger equations. Li et al. [11] proposed a fully discrete scheme for the nonlinear fractional Schrödinger equation with wave operator by combining the Crank-Nicolson method in temporal direction with the Galerkin finite element method in the spatial direction. Antoine et al. [12] applied the finite difference time domain methods and time-splitting spectral method to solve the nonlinear Schrödinger/Gross-Pitaevskii equations. More details about the related papers can be found in [13–19]. Up to now, most schemes are proved to be mass- and energy-conserving. The convergence results are missing in most references.

In this paper, we aim to develop structure-preserving schemes as well as their error analysis for the Schrödinger-Poisson system. We firstly introduce a scalar auxiliary variable (SAV) and rewrite the equations as a new family of partial differential equations. Then, we apply the Legendre-Galerkin spectral method in the spatial discretization and the Crank-Nicolson method in the temporal discretization. We show the fully-discrete scheme is mass- and energy-conserved. Moreover, we obtain the boundedness of the numerical solutions based on the Gagliardo-Nirenberg and some Sobolev inequalities.

In terms of the bounded numerical solutions, we present a rigourous error analysis of the fully discrete numerical scheme. The convergence results indicate that the fully-discrete scheme is of order 2 in the temporal direction and decreases exponentially in the spatial direction.

Compared with other numerical schemes for the nonlinear Schrödinger-Possion equations, the proposed scheme is proved to be mass- and energy-conserved without any time steps restrictions. Furthermore, by applying spectral method in space, the proposed scheme can reach exponential convergence in space. We also give the unconditional convergence in the L^∞-norm in the final section.

It is remarkable that the key to developing structure-preserving method is the so called scalar auxiliary variable (SAV) approach. The approach was firstly proposed by Shen et al. in [20, 21] and has a successful application in developing energy stable or energy-conserving schemes for time-dependent partial differential equations, such as gradient flows [22–24], wave equations [25–27], Schrödinger equations [28, 29] and so on [30]. Up to now, most numerical results are obtained in the real spaces and much attention is paid on energy-conserving properties of the scheme. In this paper, the study is extended to a coupled system in the complex space and the unconditional converge results are investigated.

The rest of the paper is organized as follows. In Section 2, a fully discrete scheme for the generalized nonlinear Schrödinger-Possion equation is introduced. In Section 3, the convergence of the proposed scheme is discussed. In Section 4, some numerical experiments are provided to demonstrate the theoretical results. Finally, some concluding remarks are presented in Section 5.

Throughout the paper we use C and $C_i (i \in N)$ to denote positive constants, which could have different values in different places.

2. Fully discrete scheme

In this section, we present the fully discrete method based on SAV approach.

2.1. Preparation

Through the paper, we use the following notations:

$$(u, v) = \int_\Omega u(x)\bar{v}(x)dx,\ \|u\|^2 = (u, u),\ \|u\|_l^2 = \int_\Omega |\partial_x^l u|^2 dx,\ \|u\|_{H^k}^2 = \sum_{l=0}^{k} \|u\|_l^2,$$

where $0 \le l \le k$, $k \in N$ and $\bar{v}$ denotes the conjugate of v. Firstly, we assume that the exact solutions of Schrödinger-Possion equations (1.1) satisfy

$$\left\|\frac{\partial u^{(l_1)}}{\partial t}\right\|_{H^\sigma} + \left\|\frac{\partial v^{(l_2)}}{\partial t}\right\|_{H^\sigma} \le K^*,\ 0 \le t \le T, \tag{2.1}$$

where $l_1 = 0,\ 1,\ 2,\ 1_2 = 0, 1$, $\sigma \ge 2$ and $K^* > 0$ is a positive constant.

We also let $\phi_k(x) = L_k(x) - L_{k+2}(x)$, where $\{L_k(x)\}_{k=0}^N$ represents the Legendre polynomials. Define

$$X_N = \{\phi_k(x) :\ k = 0,\ 1,\ \cdots,\ N-2\}.$$

Denote the polynomial space by

$$P_N = \{p(x)|p(x) = \sum_{i=0}^{N} c_i x^i,\ c_i \in R\}.$$

For all $u \in L^2(\Omega)$, we define $P_L : L^2(\Omega) \mapsto X_N$ as:

$$(P_L u - u, v) = 0,\ \forall v \in X_N.$$

Let $v = P_L u$ and $v = \Delta P_L u$ respectively, we can obtain the following stability properties:

$$\|P_L u\| \le \|u\|,\ \forall u \in L^2(\Omega),$$
$$\|\nabla P_L u\| \le \|\nabla u\|,\ \forall u \in H_0^1(\Omega).$$

Thus $\|P_L u\|_{H^1} \le \|u\|_{H^1}$, $\forall u \in H_0^1(\Omega)$. Meanwhile, we define $\Pi_N : H_0^1(\Omega) \mapsto X_N$ as:

$$(\nabla \Pi_N u - \nabla u, \nabla v) = 0,\ \forall v \in X_N.$$

Then we have the following lemma.

Lemma 1. *([31]) For all $u \in H_0^1(\Omega) \cap H^m(\Omega)(m \geq 1)$, there holds*

$$\|u - \Pi_N u\|_l \leq CN^{l-m}\|u\|_m, l = 0,\ 1,$$

where $C > 0$ is a constant independent of N.

We collect some lemmas here. They play an important role in the proof of the main results.

Lemma 2. *([32]) Suppose that $v(x) \in C^1[a, b]$, and $v(a) = v(b) = 0$, then*

$$\|v\|_\infty \leq \frac{\sqrt{b-a}}{2}|v|_1,$$
$$\|v\| \leq \frac{b-a}{\sqrt{6}}|v|_1.$$

Lemma 3. *([33]) Let m be a nonnegative integer, and assume $g \in H^m(\Omega)$ and $\partial\Omega \in C^{m+2}$. Suppose that $v \in H_0^1(\Omega)$ is the unique solution of the boundary problem*

$$\alpha v - \beta^2 \Delta v = g \ \ in\ \ \Omega,$$
$$v = 0 \ \ on\ \ \partial\Omega.$$

Then $v \in H^{m+2}(\Omega)$ and the following estimate holds

$$\|v\|_{H^{m+2}} \leq C\|g\|_{H^m},$$

where C depending on m, Ω and α, β.

Lemma 4. *([32]) Suppose that $u(t) \in C^3[0, T]$, then we have*

$$\Big|u_t(t_{n+\frac{1}{2}}) - \frac{u(t_{n+1}) - u(t_n)}{t_{n+1} - t_n}\Big| \leq C(t_{n+1} - t_n)^2,$$

where C is a constant independent of $t_{n+1} - t_n$.

Lemma 5. *([34]) Suppose that the discrete mesh function $\{\varpi^n | n = 1,\ 2,\ \cdots,\ M; M\tau = T\}$ is nonnegative and satisfies recurrence formula*

$$\varpi^{n+1} - \varpi^n \leq A\tau\varpi^{n+1} + B\tau\varpi^n + C_n\tau,$$

where A, B and $C_n(n = 1,\ 2,\ \cdots,\ M)$ are nonnegative constants. Then

$$\varpi^n \leq (\varpi^0 + \tau\sum_{l=1}^{M} C_l)e^{2(A+B)T},\ n = 1,\ 2,\ \cdots,\ M,$$

where τ is small such that $(A + B)\tau \leq \frac{M-1}{2M}(M > 1)$.

Lemma 6. *([35]) The Legendre polynomials $\{L_n(x)\}$ satisfy the following three-term recurrence relation*

$$(n + 1)L_{n+1}(x) = (2n + 1)xL_n(x) - nL_{n-1}(x), \quad n \geq 1,$$
$$L_0(x) = 1, \quad L_1(x) = x,$$

and the orthogonality relation

$$\int_{-1}^{1} L_k(x)L_j(x)dx = \begin{cases} 1/(k + \frac{1}{2}), & j = k \\ 0, & j \neq k, \end{cases}$$

where $k,\ j \geq 1$.

2.2. The SAV-CN scheme

In system (1.1), a scalar auxiliary variable can be introduced:

$$w(t) = \sqrt{\mathcal{H}(t) + C_0},$$

where

$$\mathcal{H}(t) = \int_\Omega (F(|u|^2) + \psi(x)|u|^2)dx \, with \, F(s) = \int_0^s f(z)dz.$$

Thus (1.1) can be rewritten as

$$\mathrm{i}u_t + u_{xx} + \beta uv = b(u)w, \tag{2.2}$$

$$v_{xx} = -|u|^2, \tag{2.3}$$

$$w_t = \Re\,(b(u), u_t), \tag{2.4}$$

where $b(u) = (f(|u|^2)u + \psi(x)u)/\sqrt{\mathcal{H}(t) + C_0}$ and $\Re(\cdot)$ means to take the real part of a complex number.

Then, we present the fully-discrete scheme. Let $t_n = n\tau, n = 0, 1, \cdots, N_t$, where $\tau = \frac{T}{N_t}$ and N_t is a positive integer. For a sequence of functions $\{\varphi^i\}_{i=0}^{N_t}$, we denote

$$D_\tau \varphi^n = \frac{\varphi^{n+1} - \varphi^n}{\tau}, \; n = 0,\ 1,\ 2,\ \cdots,\ N_t - 1,$$

$$\varphi^{n+\frac{1}{2}} = \frac{\varphi^{n+1} + \varphi^n}{2}, \; n = 0,\ 1,\ 2,\ \cdots,\ N_t - 1.$$

The SAV-CN scheme is to find $u_N^n, v_N^n \in X_N$ and $w_N^n \in R^1$ such that, for $n = 0,\ 1,\ 2,\ \cdots,\ N_t - 1$:

$$\mathrm{i}(D_\tau u_N^n, \phi) + ((u_N^{n+\frac{1}{2}})_{xx}, \phi) + (\beta u_N^{n+\frac{1}{2}} v_N^{n+\frac{1}{2}}, \phi) = (b(u_N^{n+\frac{1}{2}}) w_N^{n+\frac{1}{2}}, \phi), \forall \phi \in X_N, \tag{2.5}$$

$$((v_N^{n+1})_x, \phi_x) = (|u_N^{n+1}|^2, \phi), \forall \phi \in X_N, \tag{2.6}$$

$$w_N^{n+1} - w_N^n = \Re\,(b(u_N^{n+\frac{1}{2}}), u_N^{n+1} - u_N^n), \tag{2.7}$$

where $b(u_N^{n+\frac{1}{2}}) = (f(|u_N^{n+\frac{1}{2}}|^2)u_N^{n+\frac{1}{2}} + \psi(x)u_N^{n+\frac{1}{2}})/\sqrt{H(t_{n+\frac{1}{2}}) + C_0}$.

Firstly, we have the following continuous and discrete mass and energy conservation laws, respectively.

Theorem 1. *Suppose that u, v, and w are the solutions of system* (2.2)–(2.4)*, then the continuous mass and energy conservation laws can be obtained as follows:*

$$(I)\ \frac{d}{dt}\hat{\mathcal{M}} = 0, \; with \quad \hat{\mathcal{M}} = \|u\|^2,$$

$$(II)\ \frac{d}{dt}\hat{\mathcal{Q}} = 0, \quad with \quad \hat{\mathcal{Q}} = \|u_x\|^2 - \frac{\beta}{2}\|v_x\|^2 + (w(t))^2.$$

Besides, suppose that u_N^n, v_N^n and w_N^n are the solutions of system (2.5)–(2.7)*, we have*

$$(III)\ \tilde{\mathcal{M}}^{n+1} = \tilde{\mathcal{M}}^n, \; with \quad \tilde{\mathcal{M}}^n = \|u_N^n\|^2, \; 0 \le n \le N_t - 1,$$

$$(IV)\ \tilde{\mathcal{Q}}^{n+1} = \tilde{\mathcal{Q}}^n, \quad with \quad \tilde{\mathcal{Q}}^n = \|(u_N^n)_x\|^2 - \frac{\beta}{2}\|(v_N^n)_x\|^2 + (w_N^n)^2, \; 0 \le n \le N_t - 1.$$

Proof. Case (a): Proof of (I) and (II).

Taking the inner product of Eq (2.2) with $\bar{u}$, we can obtain that

$$\mathrm{i}(u_t, u) + (u_{xx}, u) + \beta(uv, u) = (b(u)w, u). \tag{2.8}$$

Meanwhile, multiplying Eq (2.3) with $-\beta\bar{v}$, and integrating it over Ω, we can get

$$-\beta(v_{xx}, v) = \beta(|u|^2, v). \tag{2.9}$$

Summing up Eqs (2.8) and (2.9), we can get

$$\mathrm{i}(u_t, u) - \|u_x\|^2 + \beta\|v_x\|^2 = (b(u)w, u).$$

Taking the imaginary part, we can obtain

$$\Re(u_t, u) = \frac{1}{2}\frac{d}{dt}\|u\|^2 = 0.$$

Besides, multiplying Eq (2.2) with $\bar{u}_t$, and integrating it over Ω, we can get:

$$\mathrm{i}(u_t, u_t) + (u_{xx}, u_t) + \beta(uv, u_t) = (b(u)w, u_t),$$

Taking the real part, we can obtain

$$\Re\,(u_{xx}, u_t) + \Re\,\beta(uv, u_t) = \Re\,(b(u)w, u_t),$$

which is equivalent to

$$-\frac{d}{dt}\|u_x\|^2 + \frac{\beta}{2}\frac{d}{dt}(v, |u|^2) = \frac{d}{dt}(w)^2,$$

Since $(v, |u|^2) = \|v_x\|^2$, we can obtain the mass and energy conservation laws

$$\frac{d}{dt}\hat{\mathcal{M}} = 0, \; with \quad \hat{\mathcal{M}} = \|u\|^2,$$
$$\frac{d}{dt}\hat{Q} = 0, \quad with \quad \hat{Q} = \|u_x\|^2 - \frac{\beta}{2}\|v_x\|^2 + (w)^2.$$

Case (b): Proof of (III) and (IV).

Let $\phi = u_N^{n+\frac{1}{2}}$ in Eq (2.5). Taking the imaginary part, we have

$$\frac{1}{2\tau}(\|u_N^{n+1}\|^2 - \|u_N^n\|^2) = 0,$$

which implies $\tilde{\mathcal{M}}^{n+1} = \tilde{\mathcal{M}}^n$.

Let $\phi = D_\tau u_N^n$ in Eq (2.5). Consider the real part, one has

$$I_1 + I_2 + I_3 + I_4 = 0,$$

where

$$I_1 = \Re\,(\mathrm{i}D_\tau u_N^n, D_\tau u_N^n) = \Re\,\frac{\mathrm{i}}{\tau^2}\|u_N^{n+1} - u_N^n\|^2 = 0,$$

$$
\begin{aligned}
I_2 &= \Re\,((u_N^{n+\frac{1}{2}})_{xx}, D_\tau u_N^n) = \frac{1}{2\tau}(-\|(u_N^{n+1})_x\|^2 + \|(u_N^n)_x\|^2),\\
I_3 &= \Re\,(\beta u_N^{n+\frac{1}{2}} v_N^{n+\frac{1}{2}}, D_\tau u_N^n) = \Re\,\frac{\beta}{2\tau}(v_N^{n+\frac{1}{2}}, |u_N^{n+1}|^2 - |u_N^n|^2)\\
&= \Re\,\frac{\beta}{4\tau}(v_N^{n+1} + v_N^n, (v_N^{n+1})_{xx} - (v_N^n)_{xx})\\
&= \frac{\beta}{4\tau}(\|(v_N^{n+1})_x\|^2 - \|(v_N^n)_x\|^2),\\
I_4 &= -\Re\,(b(u_N^{n+\frac{1}{2}})w_N^{n+\frac{1}{2}}, D_\tau u_N^n) = -\frac{w_N^{n+\frac{1}{2}}}{\tau}\,\Re\,(b(u_N^{n+\frac{1}{2}}), u_N^{n+1} - u_N^n)\\
&= -\frac{1}{2\tau}(w_N^{n+1} + w_N^n)(w_N^{n+1} - w_N^n) = -\frac{1}{2\tau}((w_N^{n+1})^2 - (w_N^n)^2).
\end{aligned}
$$

Therefore, we have

$$
\|(u_N^{n+1})_x\|^2 - \frac{\beta}{2}\|(v_N^{n+1})_x\|^2 + (w_N^{n+1})^2 = \|(u_N^n)_x\|^2 - \frac{\beta}{2}\|(v_N^n)_x\|^2 + (w_N^n)^2.
$$

This completes the proof.

Then we have the following unconditional convergence results.

Theorem 2. *Suppose that the exact solutions of Schrödinger-Possion equations* (1.1) *satisfy* (2.1). *Then, there exists a positive constant τ_0, such that when $\tau \le \tau_0$, the fully-discrete system defined in Eqs* (2.5)–(2.7) *has the numerical approximations $\{u_N^m, v_N^m, w_N^m\}$, $m = 1, 2, 3, \ldots, N_t$, satisfying*

$$
(I)\ \|u^m - u_N^m\|^2 + \|v^m - v_N^m\|^2 + (w^m - w_N^m)^2 \le C(N^{2-2\sigma} + \tau^4), \tag{2.10}
$$
$$
(II)\ \|u^m - u_N^m\|_{L^\infty}^2 + \|v^m - v_N^m\|_{L^\infty}^2 \le C(N^{2-2\sigma} + \tau^4), \tag{2.11}
$$

where C is a positive constant independent of τ and N.

3. Numerical analysis

This section is concerned with the proof of convergence analysis of the fully conservative discrete scheme.

3.1. Boundedness

Firstly, we present the boundedness of the numerical solutions u_N^n, v_N^n and w_N^n in L^∞-norm and H^1-norm respectively.

Lemma 7. *Suppose that the exact solutions satisfy* (2.1)*, then the following estimates hold:*

$$
\|u_N^n\|_{H^1} \le K_2,\ \|v_N^n\|_{H^2} \le K_3,\ |w_N^n| \le K_1,
$$
$$
\|u_N^n\|_{L^\infty} \le K_4,\ \|v_N^n\|_{L^\infty} \le K_5,
$$

where $K_i (i = 1, 2, \cdots, 5)$ are positive constants independent of τ and N.

Proof. By the discrete mass conservation $\tilde{M}^n = \|u_N^n\|^2 = \tilde{M}^0$, we have $\|u_N^n\| \le C_1$.
Let $\phi = v_N^n$ in Eq (2.6), we can obtain that

$$-((v_N^n)_{xx}, v_N^n) = \|(v_N^n)_x\|^2 = (|u_N^n|^2, v_N^n).$$

According to Lemma 3, we can get $\|v_N^n\|_{H^2} \le C_2 \||u_N^n|^2\| = C_2\|u_N^n\|_{L^4}^2$.
By using Gagliardo-Nirenberg inequalities, we can obtain

$$\|v_N^n\|_{H^2}^2 \le C_2\|u_N^n\|_{L^4}^4 \le C_2\|(u_N^n)_x\|\|u_N^n\|^3 \le \varepsilon\|(u_N^n)_x\|^2 + C_3\|u_N^n\|^6,$$

where C_3 is dependent on ε. Next, we will show the boundedness of $\|(u_N^n)_x\|$, $\|(v_N^n)_x\|$ and w_N^n respectively. Firstly, if $\beta \le 0$, we can conclude from Theorem 1 that $\|(u_N^n)_x\|^2 \le C_4$, $\|(v_N^n)_x\|^2 \le C_5$ and $|w_N^n| \le C_6$.
Secondly, if $\beta > 0$, by Theorem 1, we have

$$\tilde{Q}^n = \|(u_N^n)_x\|^2 - \frac{\beta}{2}\|(v_N^n)_x\|^2 + (w_N^n)^2 = \tilde{Q}^0 = C_7.$$

Thus $\|(u_N^n)_x\|^2 + (w_N^n)^2 = C_7 + \frac{\beta}{2}\|(v_N^n)_x\|^2 \le \frac{\beta}{2}(\varepsilon\|(u_N^n)_x\|^2 + C_3\|u_N^n\|^6) + C_7$, which can be rewritten as

$$(1 - \frac{\beta}{2}\varepsilon)\|(u_N^n)_x\|^2 + (w_N^n)^2 \le C_7 + C_3\frac{\beta}{2}\|u_N^n\|^6 \le C_8.$$

Let $\varepsilon < 1/\beta$, we can get $\|(u_N^n)_x\|^2 \le C_9$, $|w_N^n| \le K_1$. Then we can infer that $\|u_N^n\|_{H^1} \le K_2$, $\|v_N^n\|_{H^2} \le K_3$.
By using the Soblev embedding theorem, we can obtain

$$\|u_N^n\|_{L^\infty} \le K_4, \quad \|v_N^n\|_{L^\infty} \le K_5.$$

The proof is completed.

We now present the following unconditionally optimal error estimates of numerical scheme (2.5)–(2.7).

3.2. *Proof of Theorem 2*

Denote $\tilde{u}^n = \Pi_N u^n$, $e^n = \tilde{u}^n - u_N^n$, $e_0^{n+\frac{1}{2}} = \tilde{u}(t_{n+\frac{1}{2}}) - u_N^{n+\frac{1}{2}}$, $\eta^n = v^n - v_N^n$ and $\zeta^n = w_n - w_N^n$. Let $u^n = u(x, t_n)$, $v^n = (x, t_n)$, $w^n = w(t_n)$. Subtracting Eqs (2.5)–(2.7) from Eqs (2.2)–(2.4), we have

$$\mathrm{i}(D_\tau e^n, \phi) = ((e_0^{n+\frac{1}{2}})_x, \phi_x) + (G_1^{n+\frac{1}{2}}, \phi) + (R_1^{n+\frac{1}{2}}, \phi),\ \forall \phi \in X_N, \tag{3.1}$$

$$-(\eta_{xx}^{n+1}, \phi) = (G_2^{n+1}, \phi),\ \forall \phi \in X_N, \tag{3.2}$$

$$\zeta^{n+1} - \zeta^n = \Re(b(u^{n+\frac{1}{2}}), u^{n+1} - u^n) - \Re(b(u_N^{n+\frac{1}{2}}), u_N^{n+1} - u_N^n) + \tau R_2^{n+\frac{1}{2}}, \tag{3.3}$$

where

$$\begin{aligned} G_1^{n+\frac{1}{2}} &= -\beta(\tilde{u}^{n+\frac{1}{2}} v^{n+\frac{1}{2}} - u_N^{n+\frac{1}{2}} v_N^{n+\frac{1}{2}}) \\ &\quad + (b(\tilde{u}^{n+\frac{1}{2}}) w^{n+\frac{1}{2}} - b(u_N^{n+\frac{1}{2}}) w_N^{n+\frac{1}{2}}), \\ G_2^{n+1} &= |u^{n+1}|^2 - |u_N^{n+1}|^2, \end{aligned}$$

$$R_1^{n+\frac{1}{2}} = \mathrm{i}(D_\tau u^n - u_t(t_{n+\frac{1}{2}}) + \mathrm{i}(D_\tau \tilde{u}^n - D_\tau u^n) + \beta(\tilde{u}^{n+\frac{1}{2}} - u^{n+\frac{1}{2}})v^{n+\frac{1}{2}}$$
$$+ (b(u^{n+\frac{1}{2}}) - b(\tilde{u}^{n+\frac{1}{2}}))w^{n+\frac{1}{2}},$$
$$R_2^{n+\frac{1}{2}} = D_\tau w^n - w_t(t_{n+\frac{1}{2}}) + \Re(b(u^{n+\frac{1}{2}}), u_t(t_{n+\frac{1}{2}}) - D_\tau u^n).$$

Let $\phi = e^{n+\frac{1}{2}}$ in Eq (3.1), we can obtain

$$\begin{aligned} \mathrm{i}(D_\tau e^n, e^{n+\frac{1}{2}}) &= ((e_0^{n+\frac{1}{2}})_x, e_x^{n+\frac{1}{2}}) + (G_1^{n+\frac{1}{2}}, e^{n+\frac{1}{2}}) + (R_1^n, e^{n+\frac{1}{2}}) \\ &= ((\tilde{u}(t_{n+\frac{1}{2}}) - \tilde{u}^{n+\frac{1}{2}})_x + e_x^{n+\frac{1}{2}}, e_x^{n+\frac{1}{2}}) + (G_1^{n+\frac{1}{2}}, e^{n+\frac{1}{2}}) + (R_1^n, e^{n+\frac{1}{2}}), \end{aligned} \tag{3.4}$$

where we have used

$$e_0^{n+\frac{1}{2}} = \tilde{u}(t_{n+\frac{1}{2}}) - \tilde{u}^{n+\frac{1}{2}} + \tilde{u}^{n+\frac{1}{2}} - u_N^{n+\frac{1}{2}}. \tag{3.5}$$

Thus Eq (3.4) can be rewritten as

$$\begin{aligned} \frac{\mathrm{i}}{2\tau}(\|e^{n+1}\|^2 - \|e^n\|^2) =& \frac{1}{\tau}\Im(e^{n+1}, e^n) + ((\tilde{u}(t_{n+\frac{1}{2}}) - \tilde{u}^{n+\frac{1}{2}})_x, e_x^{n+\frac{1}{2}}) \\ &+ \|e_x^{n+\frac{1}{2}}\|^2 + (G_1^{n+\frac{1}{2}}, e^{n+\frac{1}{2}}) + (R_1, e^{n+\frac{1}{2}}). \end{aligned} \tag{3.6}$$

where $\Im(\cdot)$ means to take the imaginary part of a complex number. Taking the imaginary part of Eq (3.6), we have

$$\frac{1}{2\tau}(\|e^{n+1}\|^2 - \|e^n\|^2) = \Im((\tilde{u}(t_{n+\frac{1}{2}}) - \tilde{u}^{n+\frac{1}{2}}))_{xx}, e^{n+\frac{1}{2}}) + \Im(G_1^{n+\frac{1}{2}}, e^{n+\frac{1}{2}}) + \Im(R_1, e^{n+\frac{1}{2}}). \tag{3.7}$$

We first estimate $\Im(G_1^{n+\frac{1}{2}}, \overline{e}^{n+\frac{1}{2}})$. Utilizing Cauchy-Schwarz inequality, we have

$$\Im(G_1^{n+\frac{1}{2}}, e^{n+\frac{1}{2}}) \leq \frac{1}{2}(\|G_1^{n+\frac{1}{2}}\|^2 + \|e^{n+\frac{1}{2}}\|^2). \tag{3.8}$$

We know that

$$\begin{aligned} |G_1^{n+\frac{1}{2}}| \leq& |\beta||\tilde{u}^{n+\frac{1}{2}}v^{n+\frac{1}{2}} - u_N^{n+\frac{1}{2}}v_N^{n+\frac{1}{2}}| + |b(\tilde{u}^{n+\frac{1}{2}})w^{n+\frac{1}{2}} - b(u_N^{n+\frac{1}{2}})w_N^{n+\frac{1}{2}}| \\ \leq& |\beta|(|(\tilde{u}^{n+\frac{1}{2}} - u_N^{n+\frac{1}{2}})v^{n+\frac{1}{2}}| + |(v^{n+\frac{1}{2}} - v_N^{n+\frac{1}{2}})u_N^{n+\frac{1}{2}}|) \\ &+ |(b(\tilde{u}^{n+\frac{1}{2}}) - b(u_N^{n+\frac{1}{2}}))w^{n+\frac{1}{2}}| + |(w^{n+\frac{1}{2}} - w_N^{n+\frac{1}{2}})b(u_N^{n+\frac{1}{2}})|. \end{aligned}$$

By Lemma 1, Lemma 4 and Theorem 7, we have $|G_1^{n+\frac{1}{2}}| \leq C(\tau^2 + |\eta^{n+\frac{1}{2}}| + |\zeta^{n+\frac{1}{2}}| + |e^{n+\frac{1}{2}}|)$. Using Lemma 4, we obtain $|R_1^{n+\frac{1}{2}}| \leq C\tau^2 + CN^{-\sigma}$ and $\|(\tilde{u}(t_{n+\frac{1}{2}}) - \tilde{u}^{n+\frac{1}{2}})_{xx}\|^2 \leq C\tau^2$. Thus

$$\|e^{n+1}\|^2 - \|e^n\|^2 \leq C\tau((\zeta^{n+\frac{1}{2}})^2 + \|e^{n+\frac{1}{2}}\|^2 + \|\eta^{n+\frac{1}{2}}\|^2) + C\tau(N^{-2\sigma} + \tau^4). \tag{3.9}$$

Similarly, we can obtain

$$\|G_1^{n+\frac{1}{2}}\|_{H^1} \leq |\beta|\|\tilde{v}^{n+\frac{1}{2}}u^{n+\frac{1}{2}} - u_N^{n+\frac{1}{2}}v_N^{n+\frac{1}{2}}\|_{H^1} + \|b(\tilde{u}^{n+\frac{1}{2}})w^{n+\frac{1}{2}} - b(u_N^{n+\frac{1}{2}})w_N^{n+\frac{1}{2}}\|_{H^1}$$

$$\leq C(\tau^2 + \|\eta^{n+\frac{1}{2}}\|_{H^1} + \|e^{n+\frac{1}{2}}\|_{H^1} + |\zeta^{n+\frac{1}{2}}|),$$

and $\|R_1^{n+\frac{1}{2}}\|_{H^1} \leq C(N^{1-\sigma} + \tau^2)$.
Then, setting $\phi = e^{n+1} - e^n$ in Eq (3.1), we get

$$\mathrm{i}(D_\tau e^n, e^{n+1} - e^n) = ((e_0^{n+\frac{1}{2}})_x, e_x^{n+1} - e_x^n) + (G_1^{n+\frac{1}{2}}, e^{n+1} - e^n) + (R_1^{n+\frac{1}{2}}, e^{n+1} - e^n). \tag{3.10}$$

Taking the real part in Eq (3.10), we have

$$\begin{aligned}&\Re(e_x^{n+\frac{1}{2}}, e_x^{n+1} - e_x^n) + \Re[(\tilde{u}(t_{n+\frac{1}{2}}) - \tilde{u}^{n+\frac{1}{2}}, e^{n+1} - e^n)\\ &+ (G_1^{n+\frac{1}{2}}, e^{n+1} - e^n) + (R_1^{n+\frac{1}{2}}, e^{n+1} - e^n)] = 0,\end{aligned} \tag{3.11}$$

which can be written as

$$\|e_x^{n+1}\|^2 - \|e_x^n\|^2 = -2\tau[\Re(\tilde{u}(t_{n+\frac{1}{2}}) - \tilde{u}^{n+\frac{1}{2}}, D_\tau e^n) + (G_1^{n+\frac{1}{2}}, D_\tau e^n) + (R_1^{n+\frac{1}{2}}, D_\tau e^n)], \tag{3.12}$$

According to the Sobolev inequalities and the estimations of the truncation errors, we have

$$\|e_x^{n+1}\|^2 - \|e_x^n\|^2 \leq C_\epsilon \tau((\zeta^{n+\frac{1}{2}})^2 + \|e^{n+\frac{1}{2}}\|_{H^1}^2 + \|\eta^{n+\frac{1}{2}}\|_{H^1}^2 + N^{2-2\sigma} + \tau^4) + \epsilon\tau\|D_\tau e^n\|_{H^{-1}}^2, \tag{3.13}$$

where $\epsilon > 0$ is a positive constant.

In order to estimate $\|D_\tau e^n\|_{H^{-1}}$ above, we consider Eq (3.1), from which we can get the following estimate for any test functions $v \in X_N \subseteq H_0^1(\Omega)$:

$$\begin{aligned}|(D_\tau e^n, v)| &= | - \mathrm{i}((e_0^{n+\frac{1}{2}})_x, (P_L v)_x) - \mathrm{i}(G_1^{n+\frac{1}{2}}, P_L v) - \mathrm{i}(R_1^{n+\frac{1}{2}}, P_L v)|\\ &\leq C(\|e^{n+\frac{1}{2}}\|_{H^1} + \|\eta^{n+\frac{1}{2}}\| + |\zeta^{n+\frac{1}{2}}| + \|G_1^{n+\frac{1}{2}}\|_{H^{-1}} + \|R_1^{n+\frac{1}{2}}\|_{H^{-1}})\|P_L v\|_{H^1}\\ &\leq C(\|e^{n+\frac{1}{2}}\|_{H^1} + \|\eta^{n+\frac{1}{2}}\| + |\zeta^{n+\frac{1}{2}}| + (\tau^2 + N^{1-\sigma}))\|v\|_{H^1},\end{aligned} \tag{3.14}$$

where $P_L v$ represents the L^2 projection operator of v defined in Section 2.1.
By the duality between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$, we can derive that

$$\|D_\tau e^n\|_{H^{-1}}^2 \leq C(\|e^{n+\frac{1}{2}}\|_{H^1}^2 + \|\eta^{n+\frac{1}{2}}\|^2 + |\zeta^{n+\frac{1}{2}}|^2 + (\tau^4 + N^{2-2\sigma})). \tag{3.15}$$

Then, multiplying (3.15) with $\epsilon\tau$ and summing up (3.13), (3.15), we get

$$\|e_x^{n+1}\|^2 - \|e_x^n\|^2 \leq C\tau((\zeta^{n+\frac{1}{2}})^2 + \|e^{n+\frac{1}{2}}\|_{H^1}^2 + \|\eta^{n+\frac{1}{2}}\|_{H^1}^2) + C\tau(N^{2-2\sigma} + \tau^4). \tag{3.16}$$

Let $\phi = \eta^{n+1}$ in Eq (3.2), one has

$$\begin{aligned}\|\eta_x^{n+1}\|^2 &= (G_2^{n+1}, \eta^{n+1})\\ &\leq \frac{1}{2}(\|G_2^{n+1}\|^2 + \|\eta^{n+1}\|^2)\\ &\leq \frac{1}{2}\||u^{n+1}|^2 - |u_N^{n+1}|^2\|^2 + \frac{1}{2}\|\eta_x^{n+1}\|^2,\end{aligned}$$

where we have used Lemma 2. Then

$$\begin{aligned}\|\eta_x^{n+1}\|^2 &\le \||u^{n+1}|^2 - |u_N^{n+1}|^2\|^2\\ &\le \|(|u^{n+1}| + |u_N^{n+1}|)(|u^{n+1}| - |u_N^{n+1}|)\|^2\\ &\le C\||u^{n+1}| - |u_N^{n+1}|\|^2\\ &\le C\||u^{n+1} - \tilde{u}^{n+1} + \tilde{u}^{n+1} - u_N^{n+1}|\|^2\\ &\le CN^{-2\sigma} + C\|e^{n+1}\|^2,\end{aligned}\tag{3.17}$$

which also implies

$$\|\eta^{n+1}\|^2 \le CN^{-2\sigma} + C\|e^{n+1}\|^2.\tag{3.18}$$

Further, taking the inner product of the Eq (3.3) with $2\zeta^{n+\frac{1}{2}}$, we can obtain

$$\begin{aligned}(\zeta^{n+1})^2 - (\zeta^n)^2 =&2\Re(b(u^{n+\frac{1}{2}}), u^{n+1} - u^n)\zeta^{n+\frac{1}{2}} - 2\Re(b(u_N^{n+\frac{1}{2}}), u_N^{n+1} - u_N^n)\zeta^{n+\frac{1}{2}}\\ &+ 2\tau R_2^n\zeta^{n+\frac{1}{2}},\end{aligned}\tag{3.19}$$

where

$$\begin{aligned}u^{n+1} - u^n &= \mathrm{i}\tau(u_{xx}^{n+\frac{1}{2}} + \beta u^{n+\frac{1}{2}}v^{n+\frac{1}{2}} - b(u^{n+\frac{1}{2}})w^{n+\frac{1}{2}}),\\ u_N^{n+1} - u_N^n &= \mathrm{i}\tau((u_N^{n+\frac{1}{2}})_{xx} + \beta u_N^{n+\frac{1}{2}}v_N^{n+\frac{1}{2}} - b(u_N^{n+\frac{1}{2}})w_N^{n+\frac{1}{2}}).\end{aligned}\tag{3.20}$$

According to Eq (3.20), we can transform Eq (3.19) into

$$\begin{aligned}(\zeta^{n+1})^2 - (\zeta^n)^2 =& 2\zeta^{n+\frac{1}{2}}(\Re(b(u^{n+\frac{1}{2}}) - b(u_N^{n+\frac{1}{2}}), u^{n+1} - u^n)\\ &- \Re\,\tau((b(u_N^{n+\frac{1}{2}}))_x, \mathrm{i}(u_x^{n+\frac{1}{2}} - \tilde{u}_x^{n+\frac{1}{2}}))\\ &- \Re\,\tau((b(u_N^{n+\frac{1}{2}}))_x, \mathrm{i}e_x^{n+\frac{1}{2}}) + \tau R_2^n\\ &+ \Re\,\tau(b(u_N^{n+\frac{1}{2}}), \mathrm{i}(G_1^{n+\frac{1}{2}} + R_1^n))).\end{aligned}\tag{3.21}$$

Similar to the analysis above, we can obtain

$$\begin{aligned}(\zeta^{n+1})^2 - (\zeta^n)^2 \le& C\tau(\|b(u_N^{n+\frac{1}{2}})\| + \|(b(u_N^{n+\frac{1}{2}}))_x\| + \|u_t\|_{L^\infty(0,T;H^\sigma)})((\zeta^{n+\frac{1}{2}})^2 + \|e^{n+\frac{1}{2}}\|_{H^1}^2\\ &+ \|\eta^{n+\frac{1}{2}}\|^2 + \|G_1^{n+\frac{1}{2}}\|^2) + C\tau(N^{2-2\sigma} + \tau^4).\end{aligned}$$

Based on inequality (3.18) and Theorem 7, we can conclude that

$$(\zeta^{n+1})^2 - (\zeta^n)^2 \le C\tau(\|e^{n+\frac{1}{2}}\|_{H^1}^2 + (\zeta^{n+\frac{1}{2}})^2) + C\tau(N^{2-2\sigma} + \tau^4).\tag{3.22}$$

Combining the inequalities (3.9), (3.16), (3.18) and (3.22), we have

$$\begin{aligned}\|e^{n+1}\|_{H^1}^2 - \|e^n\|_{H^1}^2 + (\zeta^{n+1})^2 - (\zeta^n)^2 \le& C\tau(N^{2-2\sigma} + \tau^4) + C\tau(\|e^n\|_{H^1}^2 + \|e^{n+1}\|_{H^1}^2\\ &+ (\zeta^n)^2 + (\zeta^{n+1})^2).\end{aligned}\tag{3.23}$$

By using discrete Gronwall's inequality (Lemma 5), we can obtain

$$\|e^{n+1}\|_{H^1}^2 + (\zeta^{n+1})^2 \le C(N^{2-2\sigma} + \tau^4).\tag{3.24}$$

Then it follows that

$$\|\eta^{n+1}\|^2 \le C(N^{2-2\sigma} + \tau^4). \tag{3.25}$$

Using the triangle inequality, we have

$$\|u^m - u_N^m\|^2 \le \|u^m - \tilde{u}^m\|^2 + \|\tilde{u}^m - u_N^m\|^2 \le C(N^{2-2\sigma} + \tau^4), \tag{3.26}$$

$$\|v^m - v_N^m\|^2 \le \|v^m - \tilde{v}^m\|^2 + \|\tilde{v}^m - v_N^m\|^2 \le C(N^{2-2\sigma} + \tau^4), \tag{3.27}$$

which completes the proof.

We further have the error estimates in L^∞-norm.

According to Lemma 2 and (3.24), we can obtain that

$$\|e^m\|_{L^\infty}^2 \le C\|e_x^m\|^2 \le C(N^{2-2\sigma} + \tau^4). \tag{3.28}$$

Based on (3.17), we can get

$$\|\eta^m\|_{L^\infty}^2 \le C\|\eta_x^m\|^2 \le C(N^{2-2\sigma} + \tau^4). \tag{3.29}$$

By the triangle inequalities (3.28) and (3.29), we obtain the desired results (2.11).

4. Numerical results

In this section, two numerical examples are presented to verify the theoretical results. In the numerical examples, the errors are defined as follows

$$\eta_1 = \|u(x,t_n) - u_N^n\|_{L^\infty},\ \eta_2 = \|v(x,t_n) - v_N^n\|_{L^\infty},$$

where $u(x,t_n)$, $v(x,t_n)$ represent the analytical solutions and u_N^n, v_N^n represent the numerical solutions respectively. In the numerical simulations, we apply the iterative algorithms [36] to solve the nonlinear algebraic equations and the iterative tolerance is 10^{-15}.

Example 1. *Consider the following nonlinear Schrödinger equations*

$$\begin{aligned}
&iu_t + u_{xx} + uv = 3|u|^2u - |u|^6u + \psi_1(x)u,\ x \in \Omega,\ t \in [0,T],\\
&v_{xx} = -|u|^2,\ x \in \Omega,\ t \in [0,T],\\
&u(x,t) = v(x,t) = 0,\ x \in \partial\Omega, t \in [0,T],\\
&u(x,0) = \cos(\pi x/2),\ x \in \Omega,
\end{aligned}$$

where $\Omega = (-1,1)$ *and* $\psi_1(x)$ *is a real potential function obtained by the following exact solutions*

$$u(x,t) = e^{-it}\cos(\pi x/2),\ v(x,t) = 1/2\pi^2\cos(\pi x) - 1/4x^2 + 1/2\pi^2 + 1/4.$$

We set $N = 15$, $T = 1$ and test the convergence orders in the temporal directions. We show the maximum numerical errors at $T = 1$ in Table 1. The numerical results indicate that the numerical method is second-order accurate in temporal direction. Then, we set $\tau = 0.001$ and solve the problem with different N. We show the maximum numerical errors in Figure 1, respectively. One can see that when N increases, the error decreases exponentially.

Table 1. Errors and convergence orders with $N = 15$ for example 1.

τ	η_1	Order	η_2	Order
1/8	$3.4522e-03$	-	$1.6242e-05$	-
1/16	$8.8208e-04$	1.9686	$4.2440e-06$	1.9363
1/24	$3.9049e-04$	2.0097	$1.9204e-06$	1.9565
1/32	$2.1809e-04$	2.0247	$1.0625e-06$	2.0562
1/40	$1.3862e-04$	2.0310	$6.9399e-07$	1.9091

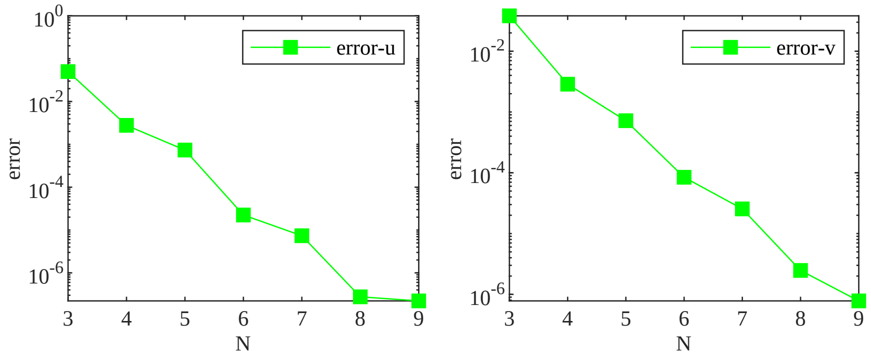

Figure 1. The L^∞-error of u(left) and v(right) with $\tau = 0.001$ when N takes different values for example 1.

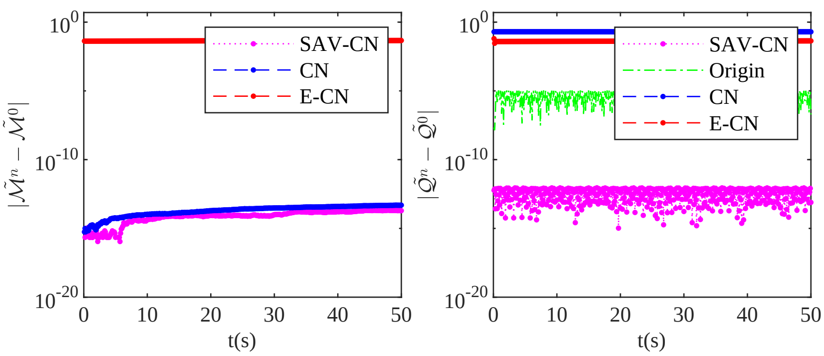

Figure 2. The discrepancies of the discrete mass(left) and energy(right) with $N = 20$ and $\tau = 0.1$ for example 1.

We test the structure-preserving properties of the numerical scheme(i.e., SAV-CN). We also discretized original equations (1.1) in space by Legendre-Galerkin spectral method and in temporal direction by the extrapolated Crank-Nicolson (ECN) method and the standard Crank-Nicolson (CN) method. We let $N = 20$ and $\tau = 0.1$ and show the discrepancies of the discrete mass and energy in Figure 2. One can see that the discrepancies of the discrete mass are sufficiently small for both the SAV-CN and CN methods. Furthermore, we also compare the discrepancies of the discrete energy between SAV-CN and the original energy (Origin) in Figure 2. We can find that the the discrepancies of the original energy is between 10^{-6} and 10^{-4}. In contrast, the discrepancies of the discrete energy are of the order of the machine precision only for the SAV-CN methods.

Example 2. *Consider the following nonlinear Schrödinger equations*

$$\begin{aligned} &iu_t + u_{xx} - uv = |u|^2 u - |u|^4 u + \psi_2(x)u,\ x \in \Omega,\ t \in [0,T],\\ &v_{xx} = -|u|^2,\ x \in \Omega,\ t \in [0,T],\\ &u(x,t) = v(x,t) = 0,\ x \in \partial\Omega, t \in [0,T],\\ &u|_{t=0} = \sin(\pi x/2),\ x \in \Omega, \end{aligned}$$

where $\Omega = (0,2)$*, and* $\psi_2(x)$ *is a real potential function obtained by the following exact solutions*

$$u(x,t) = e^{it}\sin(\pi x/2),\ v(x,t) = 1/2\pi^2\cos(\pi x) - 1/2(x-1)^2 - 1/2\pi^2 + 1/2.$$

In this example, we firstly use the linear transformation $x = s+1 (s \in (-1,1))$ and solve the resulting problem by using the proposed method. To test the convergence orders in the temporal direction, we set $N = 15$ and refine the temporal stepsizes. The maximum numerical errors at $t = 1$ and convergence orders are shown in Table 2. The numerical results show that the convergence orders in the temporal directions are of 2. Then, we set $\tau = 0.001$ and solve the problem with different N. We present the maximum numerical errors in Figure 3, respectively. Again, we find that when N increases, the error decreases exponentially.

Table 2. Errors and convergence orders with $N = 15$ for example 2.

τ	η_1	Order	η_2	Order
1/8	$4.2277e-03$	-	$2.8882e-05$	-
1/16	$1.0385e-03$	2.0254	$7.3369e-06$	1.9770
1/24	$4.6725e-04$	1.9696	$3.2959e-06$	1.9736
1/32	$2.6437e-04$	1.9797	$1.8469e-06$	2.0133
1/40	$1.7003e-04$	1.9780	$1.1815e-06$	2.0021

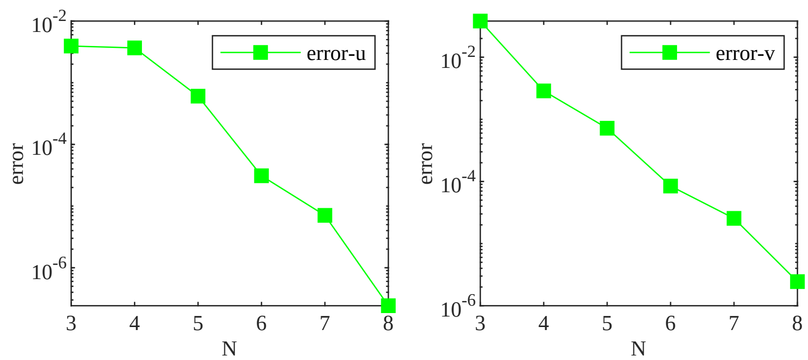

Figure 3. The L^∞-error of u(left) and v(right) with $\tau = 0.001$ when N takes different values for example 2.

Next, we let $N = 20$, $\tau = 0.1$ and solve the problem by using previous mentioned methods. We show the discrepancies of the discrete mass and energy in Figure 4, respectively. Again, the discrepancies of the discrete mass are less than 10^{-15} for both the SAV-CN and CN methods. Besides, we also compare the discrepancies of the discrete energy between SAV-CN and the original energy (Origin) in Figure 4. We can conclude that the the discrepancies of the original energy is between 10^{-7} and 10^{-5}. In contrast, the discrepancies of the discrete energy are sufficiently small only for the SAV-CN methods. These results further confirm the structure-preserving properties of the proposed method.

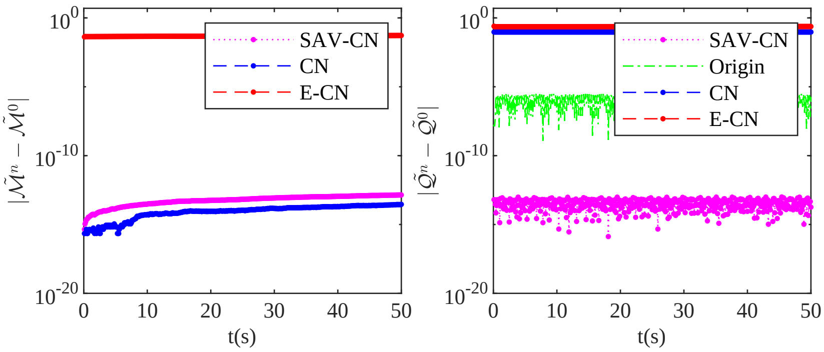

Figure 4. The discrepancies of the discrete mass(left) and energy(right) with $N = 20$ and $\tau = 0.1$ for example 2.

5. Conclusions

In this paper, we consider numerical solutions of the nonlinear Schrödinger-Possion equations. The fully-discrete scheme is developed by combing the SAV approach and the Crank-Niclson Galerkin-Legendre spectral methods. The fully discrete scheme is proved structure-preserved. The unconditional stability and convergence results are obtained. It is shown the fully-discrete scheme is of order 2 in the temporal direction and decreases exponentially in the spatial direction. Numerical results are given to confirm our theoretical findings.

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Conflict of interest

The authors declare no conflicts of interest.

References

1. T. Lu, W. Cai, A Fourier spectral-discontinuous Galerkin method for time-dependent 3-D Schrödinger-Possion equations with discontinuous potentials, *J. Comput. Appl. Math.*, **220** (2008), 588–614. https://doi.org/10.1016/j.cam.2007.09.025

2. M. Ehrhardt, A. Zisowsky, Fast calculation of energy and mass preserving solutions of Schrödinger-Poisson systems on unbounded domains, *J. Comput. Appl. Math.*, **187** (2006), 1–28. https://doi.org/10.1016/j.cam.2005.03.026
3. C. Ringhofer, J. Soler, Discrete Schrödinger-Poisson systems preserving energy and mass, *Appl. Math. Lett.*, **13** (2000), 27–32. https://doi.org/10.1016/S0893-9659(00)00072-0
4. X. Dong, A short note on simplified pseudospectral methods for computing ground state and dynamics of spherically symmetric Schrödinger-Possion Slater system, *J. Comput. Phys.*, **230** (2011), 7917–7922. https://doi.org/10.1016/j.jcp.2011.07.026
5. Y. Zhang, X. Dong, On the computation of ground state and dynamics of Schrödinger-Poisson Slater system, *J. Comput. Phys.*, **230** (2011), 2660–2676. https://doi.org/10.1016/j.jcp.2010.12.045
6. W. Auzinger, T. Kassebacher, O. Koch, M. Thalhammeret, Convergence of a strang splitting finite element discretization for the Schrödinger-Poisson equation, *Math. Model. Numer. Anal.*, **51** (2017), 1245–1278. https://doi.org/10.1051/m2an/2016059
7. C. Cheng, Q. Liu, J. LeeH, H. Z. Massoud, Spectral element method for the Schrödinger-Possion system, *J. Comput. Electron.*, **3** (2004), 417–421. https://doi.org/10.1007/s10825-004-7088-z
8. C. Lubich, On splitting methods for Schrödinger-Poisson and cubic nonlinear Schrödinger equations, *Math. Comput.*, **264** (2008), 2141–2153. https://doi.org/10.1090/S0025-5718-08-02101-7
9. M. Li, C. Huang, Y. Zhao, Fast conservative numerical algorithm for the coupled fractional Klein-Gordon-Schrödinger equation, *Numer. Algorithms*, **84** (2020), 1080–1119. https://doi.org/10.1007/s11075-019-00793-9
10. M. Li, D. Shi, J. Wang, J. Wang, W. Ming, Unconditional superconvergence analysis of the conservative linearized Galerkin FEMs for nonlinear Klein-Gordon-Schrödinger equation, *Appl. Numer. Math.*, **142** (2019), 47–63. https://doi.org/10.1016/j.apnum.2019.02.004
11. M. Li, Y. Zhao, A fast energy conserving finite element method for the nonlinear fractional Schrödinger equation with wave operator, *Appl. Math. Comput.*, **338** (2018), 758–773. https://doi.org/10.1016/j.amc.2018.06.010
12. X. Antoine, W. Bao, C. Besse, Computational methods for the dynamics of the nonlinear Schrödinger/Gross-Pitaevskii equations, *Comput. Phys. Commun.*, **184** (2013), 758–773. https://doi.org/10.1016/j.cpc.2013.07.012
13. W. Bao, Q. Tang, Z. Xu, Numerical methods and comparison for computing dark and bright solitons in the NLS equation, *J. Comput. Phys.*, **235** (2013), 423–445. https://doi.org/10.1016/j.jcp.2012.10.054
14. J. Hong, Y. Liu, H. Munthe-Kaas, A. Zanna, Globally conservative properties and error estimation of a multi-symplectic scheme for Schrödinger equations with variable coefficients, *Appl. Numer. Math.*, **56** (2006), 814–843. https://doi.org/10.1016/j.apnum.2005.06.006
15. T. Wang, B. Guo, Q. Xu, Fourth-order compact and energy conservative difference schemes for the nonlinear Schrödinger equation in two dimensions, *J. Comput. Phys.*, **243** (2013), 382–399. https://doi.org/10.1016/j.jcp.2013.03.007

16. C. Besse, A relaxation scheme for the nonlinear Schrödinger equation, *SIAM J. Numer. Anal.*, **42** (2004), 934–952. https://doi.org/10.1137/S0036142901396521

17. W. Liu, B. Wang, High-order implicit Galerkin-Legendre spectral method for the two-dimensional Schrödinger equation, *Appl. Math. Comput.*, **324** (2018), 59–68. https://doi.org/10.1016/j.amc.2017.12.009

18. M. Wang, D. Li, C. Zhang, Y. Tang, Long time behavior of solutions of gKdV equations, *J. Math. Anal. Appl.*, **390** (2012), 136–150. https://doi.org/10.1016/j.jmaa.2012.01.031

19. M. Li, M. Fei, N. Wang, C. Huang, A dissipation-preserving finite element method for nonlinear fractional wave equations on irregular convex domains, *Math. Comput. Simu.*, **177** (2020), 404–419. https://doi.org/10.1016/j.matcom.2020.05.005

20. J. Shen, J. Xu, J. Yang, A new class of efficient and robust energy stable schemes for gradient flows, *Math. Comput. Simu.*, **61** (2019), 474–506. https://doi.org/10.1137/17M1150153

21. J. Shen, J. Xu, J. Yang, The scalar auxiliary variable (SAV) approach for gradient flows, *J. Comput. Phys.*, **353** (2018), 407–416. https://doi.org/10.1016/j.jcp.2017.10.021

22. X. Li, J. Shen, H. Rui, Energy stability and convergence of SAV block-centered finite difference method for gradient flows, *Math. Comput.*, **319** (2019), 2047–2968. https://doi.org/10.1090/mcom/3428

23. J. Shen, J. Xu, Convergence and error analysis for the scalar auxiliary variable (SAV) schemes to gradient flows, *SIAM J. Numer. Anal.*, **56** (2018), 2895–2912. https://doi.org/10.1137/17M1159968

24. G. Akrivis, B. Li, D. Li, Energy-decaying extrapolated RK–SAV methods for the Allen–Cahn and Cahn–Hilliard equations, *SIAM J. Sci. Comput.*, **41** (2019), A3703–A3727. https://doi.org/10.1137/19M1264412

25. D. Li, W. Sun, Linearly implicit and high-order energy-conserving schemes for nonlinear wave equations, *J. Sci. Comput.*, **83** (2020). https://doi.org/10.1007/s10915-020-01245-6

26. W. Cao, D. Li, Z. Zhang, Unconditionally optimal convergence of an energy-conserving and linearly implicit scheme for nonlinear wave equations, *Sci. China. Math.*, 2021. https://doi.org/10.1007/s11425-020-1857-5

27. W. Cai, C. Jiang, Y. Wang, Y. Song, Structure-preserving algorithms for the two-dimensional sine-Gordon equation with Neumann boundary conditions, *J. Comput. Phys.*, **395** (2019), 166–185. https://doi.org/10.1016/j.jcp.2019.05.048

28. X. Li, J. Wen, D. Li, Mass-and energy-conserving difference schemes for nonlinear fractional Schrödinger equations, *Appl. Math. Lett.*, **111** (2020), 106686. https://doi.org/10.1016/j.aml.2020.106686

29. J. Cai, J. Shen, Two classes of linearly implicit local energy-preserving approach for general multi-symplectic Hamiltonian PDEs, *J. Comput. Phys.*, **401** (2019), 108975. https://doi.org/10.1016/j.jcp.2019.108975

30. R. Tang, D. Li, On symmetrical methods for charged particle dynamics, *Symmetry*, **13** (2021), 1626. https://doi.org/10.3390/sym13091626

31. C. Canuto, M. Hussaini, A. Quarteroni, T. Zang, *Spectral Methods in Fluid Dynamics*, Springer, Berlin, 1987.
32. Z. Sun, *The numerical methods for partial differential equations*, Science Press, Beijing, 2005.
33. L. Evans, *Partial Differential Equations*, 2^{nd} edition, AMS, Providence, 2010.
34. Y. Zhou, *Application of discrete functional analysis to the finite difference methods*, International Academic Publishers, Beijing, 1990.
35. Y. Maday, A. Quarteroni, Legendre and Chebyshev spectral approximations of Burgers' equation, *Numer. Math.*, **37** (1981), 321-332. https://doi.org/10.1007/BF01400311
36. D. Li, C. Zhang, Split Newton iterative algorithm and its application, *Appl. Math. Comput.*, **217** (2010), 2260-2265. https://doi.org/10.1016/j.amc.2010.07.026

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