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Research article

Revisiting Taibleson's theorem

Humberto Rafeiro[1] **and Joel E. Restrepo**[2,*]

[1] Department of Mathematical Sciences, United Arab Emirates University, UAE
[2] Department of Mathematics, Nazarbayev University, Kazakhstan & Regional Mathematical Center, Southern Federal University, Russia

* **Correspondence:** Email: cocojoel89@yahoo.es.

Abstract: A new characterization of the weighted Taibleson's theorem for generalized Hölder spaces is given via a Hadamard-Liouville type operator (Djrbashian's generalized fractional operator).

Keywords: generalized Hölder space; Bari-Stechkin class; modulus of continuity; generalized fractional operator; harmonic function; poisson integral

1. Introduction

Classical Hölder spaces and their applications are well known, we refer the reader to [1–4] and references therein. There has been increasing interest in the theory of the so-called non-standard spaces during last decades, see the monograph [5] and the references given there. In this paper, we study function spaces of Hölder type, defined by means of the Bari–Stechkin class (BSC for short), that are harmonic in the half-space. Almost monotonic functions satisfying conditions (2.1) and (2.2) were studied in conjunction with Lozinskii condition in the foundational paper [6]. The BSC, as well as its modifications and generalizations, proved to be essential in the study of mapping properties of some operators in spaces of continuous functions with prescribed behavior of the modulus of continuity and in the theory of Fredholm solvability of singular integral equations with piecewise continuous coefficients, see [5] and references therein. Use of functions in the BSC, instead of just power functions, allows us to generalize Hölder spaces. This generalization can be used in rough path theory and its connection with Brownian motion, in the study of boundary value problems for partial differential equations with different behavior in the boundary, among others.

Fractional calculus, although a quite old topic going back to Euler, Laplace, Abel, Liouville to name a few, is gaining popularity and has attracted attention in many academic fields due to its wide applicability. For an encyclopedic treatment of fractional calculus, up to the 1990s, we refer the reader to [7]. We are interested in the Djrbashian generalized fractional operator, which is a half-plane analog of the generalized Hadamard operator $L^{(\omega)}$ of M.M. Djrbashian, see [7] 344–346, 432, 435.

Recall that, loosely speaking, Taibleson's theorem for Hölder spaces asserts that

$$f \in \Lambda_{x^\alpha}(\mathbb{R}^n) \Longleftrightarrow \|\partial_y(P_y * f)\|_\infty \lesssim y^{-1+\alpha}.$$

In this short note, we investigate the validity of a Taibleson type theorem for a generalized Hölder space and with the partial differentiation replaced by Djrbashian's generalized fractional operator.

2. Preliminaries

A non-negative function $\omega : I \to [0, \infty)$, defined on a real interval I, is called *almost increasing* if there is a constant $C \geqslant 1$ such that $\omega(t) \leqslant C\omega(s)$ for all $t, s \in I$ with $t \leqslant s$. The notion of *almost decreasing* is similarly defined.

Definition 2.1. *The* Bari-Stechkin class*, denoted by* Ω*, is defined as the set of functions* $\gamma : [0, \infty) \to [0, \infty)$ *with the property that there exists a number* $\delta_0 > 0$ *for which the following conditions hold:*

(1) $\gamma(0) = 0$ *and* $\gamma(t)$ *is continuous at* $t = 0$*;*

(2) γ *is almost increasing on* $[0, \delta_0]$*;*

(3) $\frac{\gamma(t)}{t}$ *is almost decreasing on* $t \in [0, \delta_0]$;

(4) there exists a constant C such that for all $t \in [0, \delta_0]$

$$\int_t^{\delta_0} \frac{\gamma(x)}{x^2} dx \leqslant C \frac{\gamma(t)}{t}, \tag{2.1}$$

where C does not depend on t; and

(5) there exists a constant C such that for all $t \in [0, \delta_0]$*,*

$$\int_0^t \frac{\gamma(x)}{x} dx \leqslant C\gamma(t), \tag{2.2}$$

where C does not depend on t.

It should be pointed out that the most relevant behavior of the functions above is near zero. However, we need to do some estimations on $[\delta_0, \infty)$. Thus, everywhere in this paper, we assume that $\gamma(x)/x^2 \in L^1([\delta_0, \infty))$, without loosing the generality of the results.

Definition 2.2. *We introduce the* generalized Hölder space $\Lambda_{\gamma(\cdot)}(\mathbb{R}^n)$*,* $\gamma \in \Omega$*, as the set of functions* $f \in L^\infty(\mathbb{R}^n)$ *such that, for any* $x \in \mathbb{R}^n$ *and all sufficient small* $|t|$ $(t \in \mathbb{R}^n)$*, we have*

$$\|f(x+t) - f(x)\|_\infty \leqslant C\gamma(|t|), \tag{2.3}$$

with C independent of x and t. The semi-norm and norm are introduced as

$$\|f\|_{\#,\Lambda_{\gamma(\cdot)}} = \sup_{|t|>0} \frac{\|f(x+t) - f(x)\|_\infty}{\gamma(|t|)}, \quad \|f\|_{\Lambda_\gamma(\mathbb{R}^n)} = \|f\|_{\#,\Lambda_\gamma(\mathbb{R}^n)} + \|f\|_\infty.$$

The method of proof of the classical Taibleson theorem (see Proposition 7 and Lemma 4 in [11]) carries over, with corresponding modifications, to the generalized Hölder spaces as stated in Theorem 2.1. We leave the reader to check the details.

Theorem 2.1. *Let $f \in L^\infty(\mathbb{R}^n)$ and $\gamma \in \Omega$. Then the following results are equivalent:*

(1) $f \in \Lambda_{\gamma(\cdot)}(\mathbb{R}^n)$;

(2) there exists a constant $C > 0$ such that, for all sufficiently small $y > 0$, we have $\|\partial_y u(x,y)\|_\infty \leqslant C\frac{\gamma(y)}{y}$; and

(3) there exists a constant $C > 0$ such that, for all sufficiently small $y > 0$ and $i = \overline{1,n}$, we have $\|\partial_{x_i} u(x,y)\|_\infty \leqslant C\frac{\gamma(y)}{y}$.

3. Characterization of the generalized Hölder spaces

In this section, we characterize generalized Hölder spaces $\Lambda_{\gamma(\cdot)}(\mathbb{R}^n)$ by means of Djrbashian's generalized fractional operator, which is a twofold generalization of Taibleson's theorem, both in the function space as well as in the differential operator.

Throughout the rest of the paper, we assume that ω is a function in the class Θ; i.e., $\omega(x) > 0$ and it is decreasing on $(0,\infty)$.

Moreover, in this section, we just consider those functions from both classes Ω and Θ, which satisfy the following weighted inequality:

$$\int_t^\infty \frac{\gamma(s+t)}{s^2}\left(\int_0^s \omega(r)dr\right)ds \leqslant C\frac{\gamma(t)}{t}\int_0^t \omega(s)ds, \quad t \in (0,\infty), \tag{3.1}$$

for some constant $C > 0$. Note that condition (3.1) resembles the one considered in [5, §2.2].

We also assume that

$$\omega_1(x) = \int_0^x \omega(t)dt < \infty, \quad 0 < x < \infty.$$

Definition 3.1. *The* Djrbashian generalized fractional operator L_ω *is introduced, with $\omega \in \Theta$, as*

$$L_\omega u(x,y) := \int_0^\infty \partial_y u(x,y+t)\omega(t)dt = \int_y^\infty \partial_s u(x,s)\omega(s-y)ds, \tag{3.2}$$

for any function $u(x,y)$ $(x \in \mathbb{R}^n, y > 0)$ in $\mathbb{R}^{n+1}_+$.

For simplicity, we write $L_\omega P_y(x) := L_\omega P(x,y)$, where

$$P_y(x) = c_n y(|x|^2+y^2)^{-\frac{n+1}{2}},\ y > 0,\ x \in \mathbb{R}^n,\ c_n = \pi^{-\frac{n+1}{2}}\Gamma\left(\frac{n+1}{2}\right),$$

is the Poisson kernel.

Lemma 3.1. *Let ω be in the class Θ. Then the following assertions hold:*

(1) there is a constant $C > 0$, such that for any $x \in \mathbb{R}^n \setminus \{0\}$ and for all sufficient small $y > 0$ we have

$$\left|L_\omega P_y(x)\right| \leqslant C\frac{1}{y^{n+1}}\int_0^y \omega(s)ds,$$

(2) there is a constant $C > 0$, such that for any $x \in \mathbb{R}^n \setminus \{0\}$ and for all sufficient small $y > 0$ we have

$$\left|L_\omega P_y(x)\right| \leqslant C\frac{1}{|x|^{n+1}}\int_0^{|x|}\omega(s)ds,$$

and

(3) there is a constant $C > 0$, such that for any $x \in \mathbb{R}^n \setminus \{0\}$ and for all sufficient small $y > 0$ we have

$$\left\|L_\omega P_y(x)\right\|_1 \leqslant C\frac{1}{y}\int_0^{y}\omega(s)ds.$$

Proof. By the definition of L_ω and the well-known property of the Poisson kernel, we get

$$\left|L_\omega P_y(x)\right| \leqslant \int_y^\infty |\partial_s P_s(x)|\,\omega(s-y)ds \lesssim \int_y^\infty \frac{\omega(s-y)}{s^{n+1}}ds.$$

We split the last integral as $\left(\int_0^y + \int_y^\infty\right)\frac{\omega(r)}{(r+y)^{n+1}}dr =: R_1 + R_2$.

The integral R_1 is estimated by

$$R_1 \leqslant \frac{1}{y^{n+1}}\int_0^y \omega(r)dr.$$

For R_2, since ω is decreasing on $(0,\infty)$ and $y\,\omega(y) \leqslant \int_0^y \omega(r)dr$, we obtain

$$R_2 \lesssim \omega(y)\int_y^\infty (r+y)^{-n-1}dr \lesssim \frac{\omega(y)}{y^n} \lesssim \frac{1}{y^{n+1}}\int_0^y \omega(r)dr.$$

Hence, assertion (1) follows.

To obtain assertion (2), we need to consider two cases.

First case: $|x| \leqslant y$. From the previous pointwise estimate for $L_\omega P_y$ and the fact that $\frac{1}{s}\int_0^s \omega(r)dr$ is a non-increasing function on $(0,\infty)$, which can be checked by a standard calculus argument, yields

$$\left|L_\omega P_y(x)\right| \lesssim \frac{1}{y^{n+1}}\int_0^y \omega(s)ds \lesssim \frac{1}{|x|^{n+1}}\int_0^{|x|}\omega(r)dr.$$

Second case: $|x| > y$. We have

$$\begin{aligned}\left|L_\omega P_y(x)\right| &\leqslant \int_y^\infty |\partial_s P_s(x)|\,\omega(s-y)ds \\ &= \left(\int_y^{|x|} + \int_{|x|}^\infty\right)|\partial_s P_s(x)|\,\omega(s-y)ds =: J_1 + J_2.\end{aligned}$$

Notice that

$$J_1 \lesssim \frac{1}{|x|^{n+1}}\int_y^{|x|}\omega(s-y)ds \lesssim \frac{1}{|x|^{n+1}}\int_0^{|x|}\omega(r)dr.$$

Furthermore, we have

$$J_2 \lesssim \int_{|x|}^\infty \frac{\omega(s-y)}{s^{n+1}}ds = \int_{|x|-y}^\infty \frac{\omega(r)dr}{(r+y)^{n+1}}$$

$$\lesssim \omega(|x|-y)\int_{|x|-y}^{\infty}(r+y)^{-n-1}dr \lesssim \frac{\omega(|x|)}{|x|^n} \lesssim \frac{1}{|x|^{n+1}}\int_0^{|x|}\omega(t)dt$$

since ω is decreasing on $(0,\infty)$ and $|x|\omega(|x|) \leqslant \int_0^{|x|}\omega(t)dt$, which ends assertion (2).

To get estimate (3), we have

$$\begin{aligned}
\|L_\omega P_y(x)\|_1 &= \int_{|x|\leqslant y}\left|L_\omega P_y(x)\right|dx + \int_{|x|>y}\left|L_\omega P_y(x)\right|dx \\
&\lesssim \frac{1}{y^{n+1}}\int_0^y \omega(s)ds\int_{|x|\leqslant y}dx + \int_{|x|>y}\left(\frac{1}{|x|^{n+1}}\int_0^{|x|}\omega(s)ds\right)dx \\
&\lesssim \frac{1}{y}\int_0^y\omega(s)ds + \int_y^\infty\left(\frac{1}{r^2}\int_0^r\omega(s)ds\right)dr.
\end{aligned}$$

From Eq (3.1), with $\gamma = 1$, the assertion (3) follows. □

It is known that for $f \in L^\infty(\mathbb{R}^n)$, we have $u(x,y) = (P_y * f)(x)$ is a harmonic function for any $x \in \mathbb{R}^n$ and for all sufficiently small $y > 0$. Since $\|L_\omega P_y(x)\|_1 < \infty$ for any $y > 0$, by Lemma 3.1, then Fubini's theorem and Young's convolution inequality it follows that $L_\omega u(x,y) = (L_\omega P_y * f)(x)$.

Next, we give an estimation for any function in the space $\Lambda_{\gamma(\cdot)}(\mathbb{R}^n)$ by means of the operator L_ω near the boundary of the half-space $\mathbb{R}^{n+1}_+$.

Theorem 3.1. *Let $\gamma \in \Omega$, $\omega \in \Theta$. If $f \in \Lambda_{\gamma(\cdot)}(\mathbb{R}^n)$ then there exists a constant $C > 0$ such that for all sufficiently small $y > 0$ and $x \in \mathbb{R}^n$ it follows*

$$\|L_\omega u(x,y)\|_\infty \leqslant C\frac{\gamma(y)}{y}\int_0^y\omega(s)ds, \quad u(x,y) = (P_y * f)(x). \tag{3.3}$$

Proof. By the definition of the operator L_ω and Lemma 2.1, we obtain

$$\begin{aligned}
\|L_\omega u(x,y)\|_\infty &\leqslant \int_y^\infty \|\partial_s u(x,s)\|_\infty\,\omega(s-y)ds \lesssim \int_y^\infty\frac{\gamma(s)}{s}\omega(s-y)ds \\
&= A\left(\int_0^y + \int_y^\infty\right)\frac{\gamma(t+y)}{t+y}\omega(t)dt =: K_1 + K_2.
\end{aligned}$$

Observe that $K_1 \lesssim \frac{\gamma(y)}{y}\int_0^y\omega(t)dt$, since $\frac{\gamma(t)}{t}$ is almost decreasing on $(0,\infty)$. The integral K_2 can be estimated by

$$K_2 \lesssim \int_y^\infty\frac{\gamma(t+y)}{t}\left(\frac{1}{t}\int_0^t\omega(s)ds\right)dt \lesssim \frac{\gamma(y)}{y}\int_0^y\omega(s)ds,$$

due to $\omega(t) \leqslant \frac{1}{t}\int_0^t\omega(s)ds$ and condition (3.1). Hence, the desired result follows using the above estimates. □

One way to prove the converse statement of Theorem 3.1 could be by establishing the inverse operator of L_ω, as was done in [8]. At this moment, we can not say anything about the existence and form of such inverse operator of L_ω on the half-space $\mathbb{R}^{n+1}_+$. Thus, the converse of Theorem 3.1 is, at present, far from being solved. Nevertheless, in Theorem 3.2, we prove it for a special class of functions, viz., for $\omega(x) = x^{-\alpha}$ with $0 < \alpha < 1$.

Recall the *Liouville fractional integro-differential type operators*:

$$I^\alpha g(y) = \frac{1}{\Gamma(\alpha)} \int_y^\infty (t-y)^{\alpha-1} g(t)dt, \quad D^\alpha g(y) := I^{1-\alpha} g'(y),$$

where $0 < \alpha < 1$ and $-\infty < y < \infty$, with the convention $I^0 g(t) = g(t)$.

Remark 3.1. *One can prove, by changing the region of integration, that if* $\lim_{s\to\infty} g(s) = 0$*, then* $I^\alpha D^\alpha g(y) = -g(y)$*. Under this notation, we have that* $L_{x^{-\alpha}} u(x,y) = \Gamma(1-\alpha) D^\alpha u(x,y)$.

Now we give some results on properties of these operators. For simplicity, we use the following notations:

$$I^\alpha u(x,y) := I^\alpha u_x(y), \qquad I^\alpha P_y(x) := I^\alpha P_x(y),$$
$$D^\alpha u(x,y) := D^\alpha u_x(y), \qquad D^\alpha P_y(x) := D^\alpha P_x(y).$$

Lemma 3.2. *Let* $0 < \alpha < 1$*. Then there exists a constant* $C > 0$ *such that* $\left\| I^\alpha \left(\partial_{x_i} P_y\right)(x) \right\|_1 \leqslant C y^{\alpha-1}$*, for all sufficiently small* $y > 0$ *and* $x \in \mathbb{R}^n$.

Proof. We split

$$\left\| I^\alpha \left(\partial_{x_i} P_y\right) \right\|_1 = \left(\int_{y\leqslant|x|} + \int_{y>|x|} \right) \left| I^\alpha \left(\partial_{x_i} P_y\right)(x) \right| dx := L_1 + L_2.$$

Since $\left|\partial_{x_i} P_t(x)\right| \lesssim t^{-n-1}$, we have

$$\begin{aligned}\Gamma(\alpha) \left| I^\alpha \left(\partial_{x_i} P_y\right)(x) \right| &\leqslant \int_y^\infty (t-y)^{\alpha-1} \left|\partial_{x_i} P_t(x)\right| dt \lesssim \int_y^\infty (t-y)^{\alpha-1} \frac{dt}{t^{n+1}} \\ &\lesssim \left(\frac{1}{y^{n+1}} \int_0^y u^{\alpha-1} du + \int_y^\infty u^{\alpha-1} \frac{du}{u^{n+1}} \right) \lesssim y^{\alpha-n-1},\end{aligned}$$

thus, $L_1 \lesssim y^{\alpha-n-1} \int_0^y r^{n-1} dr \lesssim \frac{y^{\alpha-1}}{n}$.

Also, if $y > |x|$, then $L_2 \lesssim \frac{y^{\alpha-1}}{1-\alpha}$, and the lemma follows. □

Lemma 3.3. *If* $f \in L^\infty(\mathbb{R}^n)$*,* $0 < \alpha < 1$ *and* $\|L_{x^{-\alpha}} u(x, y/2)\|_\infty < \infty$*, then there exists a constant* $C > 0$ *such that* $\|\partial_{x_i} u(x,y)\|_\infty \leqslant C y^{\alpha-1} \left\| L_{x^{-\alpha}} u(x, y/2) \right\|_\infty$ *for all sufficiently small* $y > 0$ *and* $x \in \mathbb{R}^n$.

Proof. Since $\lim_{y\to\infty} \partial_{x_i} u(x,y) = 0$ and Remark 3.1 we get

$$-\partial_{x_i} u(x,y) = I_y^\alpha D_y^\alpha (\partial_{x_i} P_y * f)(x),$$

where I_y^α, D_y^α are the same operators of I^α, D^α with respect to the variable y. This notation is necessary and useful to prove the affirmation. For $y = y_1 + y_2$ with $y_{1,2} > 0$ and Fubini's theorem it follows that

$$D_y^\alpha (\partial_{x_i} P_y * f)(x)$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(1-\alpha)} \int_y^\infty (s-y)^{-\alpha} \partial_s(\partial_{x_i} P_s * f)(x) ds \\
&= \frac{1}{\Gamma(1-\alpha)} \int_{y_1}^\infty (u-y_1)^{-\alpha} \partial_u(\partial_{x_i} P_{u+y_2} * f)(x) du \\
&= \partial_{x_i} P_{y_2} * \left(\frac{1}{\Gamma(1-\alpha)} \int_{y_1}^\infty (u-y_1)^{-\alpha} (\partial_u P_u) * f)(x) du \right) \\
&= (\partial_{x_i} P_{y_2}) * D_{y_1}^\alpha u(y_1, x).
\end{aligned}
$$

We also have that

$$
\begin{aligned}
I_y^\alpha D_y^\alpha (\partial_{x_i} P_y * f)(x) &= I_y^\alpha ((\partial_{x_i} P_{y_2}) * D_{y_1}^\alpha u(y_1, x)) \\
&= I_y^\alpha ((\partial_{x_i} P_{y-y_1}) * D_{y_1}^\alpha u(y_1, x)) \\
&= \frac{1}{\Gamma(\alpha)} \int_{y_1+y_2}^\infty (s-y_1-y_2)^{\alpha-1} [(\partial_{x_i} P_{s-y_1}) * D_{y_1}^\alpha u(y_1, x)] ds \\
&= \frac{1}{\Gamma(\alpha)} \int_{y_2}^\infty (u-y_2)^{\alpha-1} [(\partial_{x_i} P_u) * D_{y_1}^\alpha u(y_1, x)] du \\
&= I_{y_2}^\alpha \left(\partial_{x_i} P_{y_2} \right) * D_{y_1}^\alpha u(x, y_1).
\end{aligned}
$$

Therefore

$$
-\partial_{x_i} u(x,y) = I_{y_2}^\alpha \left(\partial_{x_i} P_{y_2} \right) * D_{y_1}^\alpha u(x, y_1).
$$

Notice that, for $y_1 = y_2 = y/2$, by Young's convolution inequality and Lemma 3.2, we obtain

$$
\begin{aligned}
\left\| \partial_{x_i} u(x,y) \right\|_\infty &\leqslant \| I_{y/2}^\alpha \left(\partial_{x_i} P_{y/2}(x) \right) \|_1 \| D_{y/2}^\alpha u(x, y/2) \|_\infty \\
&\lesssim y^{\alpha-1} \| D_{y/2}^\alpha u(\cdot, y/2)(x) \|_\infty \lesssim y^{\alpha-1} \| L_{x^{-\alpha}} u(x, y/2) \|_\infty,
\end{aligned}
$$

which ends the proof. □

Now we establish a new characterization of the generalized Hölder space by means of the Djrbashian generalized fractional operator.

Theorem 3.2. *Let $\gamma \in \Omega$. Then the following statements are equivalent:*

(1) $f \in \Lambda_{\gamma(\cdot)}(\mathbb{R}^n)$.

(2) There exists a constant $C > 0$ such that

$$
\| L_{x^{-\alpha}} u(x,y) \|_\infty \leqslant C \frac{\gamma(y)}{y^\alpha}, \tag{3.4}
$$

for all sufficiently small $y > 0$ and $x \in \mathbb{R}^n$.

Proof. By Theorem 3.1, with $\omega(x) = x^{-\alpha}$, the first implication follows immediately.

For the converse, assume that condition (3.4) holds. Write

$$
f(x+t) - f(x) = \big(u(x+t,y) - u(x,y)\big) + \big(f(x+t) - u(x+t,y)\big) - \big(f(x) - u(x,y)\big),
$$

for any $x \in \mathbb{R}^n$ and for all sufficiently small $|t|, y > 0$ ($t \in \mathbb{R}^n$), where again y does not depend on x or t, but it is best to choose for this proof $y = |t|$. It is easy to see that $|u(x+t,y) - u(x,y)| \leqslant \int_0^1 |\nabla u(h(r), y)||h'(r)|dr$, where $h(r) = rx + (1-r)(x+t)$, $0 \leqslant r \leqslant 1$. Hence, by Lemma 3.3 and inequality (3.4) it follows that

$$|u(x+t,y) - u(x,y)| \lesssim y^{\alpha-1}\frac{\gamma(y/2)}{y^\alpha}\int_0^1 |h'(r)|dr \leqslant C_2\gamma(y). \tag{3.5}$$

Also, $f(x+t) - u(x+t,y) = -\int_0^y \partial_s u(x+t,s)ds$. Now by Lemma 3.3 and inequality (3.4), we have

$$\begin{aligned}\left\|\partial_y u(x,y)\right\|_\infty &\leqslant \int_y^\infty \sum_{k=1}^n \|\partial_{x^k}\partial_{x^k} u(x,w)\|_\infty\, dw \lesssim \int_y^\infty \|\partial_{x^k} u(x,w)\|_\infty \frac{dw}{w} \\ &\lesssim \int_y^\infty \frac{\gamma(w)dw}{w^2} \lesssim \left(\int_y^{\delta_0} + \int_{\delta_0}^\infty\right)\frac{\gamma(w)dw}{w^2} \lesssim \frac{\gamma(y)}{y},\end{aligned}$$

where the last inequality is obtained from condition (2.1) and the fact that $\gamma(y)/y \geqslant C\gamma(1)$. Now, using condition (2.2) we get

$$|f(x+t) - u(x+t,y)| \leqslant \int_0^y \|\partial_s u(x+t,s)\|_\infty\, ds \lesssim \int_0^y \frac{\gamma(s)}{s}ds \lesssim \gamma(y). \tag{3.6}$$

Similarly, we obtain

$$|f(x) - u(x,y)| \lesssim \gamma(|t|). \tag{3.7}$$

Hence, by conditions (3.5), (3.6) and (3.7) it follows that $f \in \Lambda_{\gamma(\cdot)}(\mathbb{R}^n)$. □

Corollary 3.1. *Let* $0 < \beta < \alpha \leqslant 1$. *Then the following statements are equivalent:*

(1) $f \in \Lambda_{x^\beta}(\mathbb{R}^n)$.

(2) There exists a constant $C > 0$ *such that* $\|L_{x^{-\alpha}} u(x,y)\|_\infty \leqslant Cy^{\beta-\alpha}$ *for all sufficiently small* $y > 0$ *and* $x \in \mathbb{R}^n$. *In particular, for* $\alpha = 1$, *we recover the Taibleson's theorem.*

4. Conclusions

In this paper we have proved a general version of Taibleson's theorem by using a generalized Hölder space and with the partial differentiation replaced by Djrbashian's generalized fractional operator. In particular, we recovered the classical result when the Djrbashian's type operator becomes to the known partial derivative. Future works can be directed to investigative some other classical results in more general function spaces by means of different integro-differential operators.

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Conflict of interest

The authors declare there is no conflicts of interest.

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