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# From short exact sequences of abelian categories to short exact sequences of homotopy categories and derived categories

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**Abstract:** We show that a short exact sequence of abelian categories gives rise to short exact sequences of abelian categories of complexes, homotopy categories and unbounded derived categories, refining a result of J. Miyachi.

**Keywords:** abelian categories; homotopy categories; unbounded derived categories; short exact sequences; recollements

## 1. Introduction

As two important kinds of additive categories, abelian categories and triangulated categories are ubiquitous in mathematics. A natural problem is to consider the relationship between them. It is well known that homotopy categories and derived categories of abelian categories are triangulated categories.

Let us recall some background. For details, we refer the reader to the online notes [1] and we will recall the relevant notions in Section 2. Recall that for an abelian category $\mathscr{A}$, a full additive subcategory $\mathscr{C}$ of $\mathscr{A}$ is called a *Serre subcategory* if $\mathscr{C}$ is closed under taking subobjects, quotients and extensions. In this case, we can form the the *Serre quotient* of $\mathscr{A}$ by $\mathscr{C}$ by inverting each morphism $f$ in $\mathscr{A}$ such that its kernel and cokernel belong to $\mathscr{C}$. The Serre quotient $\mathscr{A}/\mathscr{C}$ is also an abelian category and the quotient functor $Q : \mathscr{A} \to \mathscr{A}/\mathscr{C}$ is an exact functor. In this case, we will say that

$$0 \to \mathscr{C} \xrightarrow{i} \mathscr{A} \xrightarrow{Q} \mathscr{A}/\mathscr{C} \to 0 \tag{1.1}$$

is a *short exact sequence of abelian categories*, where $i : \mathscr{C} \to \mathscr{A}$ is the inclusion functor.

Similarly one can define short exact sequences of triangulated categories. Let $\mathcal{T}$ be a triangulated category. A triangulated subcategory $\mathcal{T}'$ of $\mathcal{T}$ is called a *thick subcategory* if it is closed under taking direct summands ( [2] and [3, Proposition 1.3]). Then we can define the *Verdier quotient* $\mathcal{T}/\mathcal{T}'$, which is the localization of $\mathcal{T}$ by inverting all morphisms $f$ in $\mathcal{T}$ whose cones lie in $\mathcal{T}'$. The Verdier quotient $\mathcal{T}/\mathcal{T}'$ is still a triangulated category, and the quotient functor $Q : \mathcal{T} \to \mathcal{T}/\mathcal{T}'$ is a triangle functor. In this case, we will say that

$$\mathcal{T}' \xrightarrow{i} \mathcal{T} \xrightarrow{Q} \mathcal{T}/\mathcal{T}'$$

is a *short exact sequence of triangulated categories*, where $i : \mathcal{T}' \to \mathcal{T}$ is the inclusion functor.

It is a natural question to see whether a short exact sequence of abelian categories (1.1) gives rise to short exact sequences of triangulated categories by taking their homotopy categories or derived categories.

Let us introduce some notations. For $* = +, -, b, \emptyset$, we denote by $C^*(\mathscr{A})$ the category of left bounded, right bounded, bounded and unbounded complexes of $\mathscr{A}$, respectively. Let $K^*(\mathscr{A})$ and $D^*(\mathscr{A})$ be the corresponding versions for homotopy categories and derived categories. Note that for unbounded versions, we usually delete $\emptyset$ in the notations, although $K^{\emptyset}(\mathscr{A})$ has a different meaning in [2].

A naive question is whether (1.1) induces short exact sequences of derived categories

$$D^*(\mathscr{C}) \to D^*(\mathscr{A}) \to D^*(\mathscr{A}/\mathscr{C}),$$

for $* = +, -, b$ or $\emptyset$. It is easy to see by examples that this naive version is wrong and one reason is that $D^*(\mathscr{C})$ is NOT necessarily a triangulated subcategory of $D^*(\mathscr{A})$. We should replace $D^*(\mathscr{C})$ by $D^*_{\mathscr{C}}(\mathscr{A})$, i.e. the full subcategory of $D^*(\mathscr{A})$ consisting of complexes whose cohomology groups belong to $\mathscr{C}$. In [4, Theorem 3.2], J. Miyachi provided such a short exact sequence for left bounded, right bounded and bounded derived categories, respectively.

**Theorem 1.1.** [4, Theorem 3.2] Let $0 \longrightarrow \mathscr{C} \xrightarrow{i} \mathscr{A} \xrightarrow{Q} \mathscr{A}/\mathscr{C} \longrightarrow 0$ be a short exact sequence of abelian categories. For each $* \in \{+, -, b\}$, we have an induced short exact sequence of triangulated categories

$$D^*_{\mathscr{C}}(\mathscr{A}) \xrightarrow{i^*} D^*(\mathscr{A}) \xrightarrow{Q^*} D^*(\mathscr{A}/\mathscr{C}),$$

where $D^*_{\mathscr{C}}(\mathscr{A}) = \{X^\bullet \in D^*(\mathscr{A}) \mid \forall i \in \mathbb{Z}, \mathrm{H}^i(X^\bullet) \in \mathscr{C}\}$.

The objective of this paper is to refine the above result of J. Miyachi, more precisely, we want to consider the corresponding versions for complex categories, homotopy categories and unbounded derived categories.

It reveals that the key point is to produce short exact sequences of complexes categories.

**Theorem 1.2** (Theorems 3.1 and 3.5)**.** *A short exact sequence of abelian categories*

$$0 \longrightarrow \mathscr{C} \xrightarrow{i} \mathscr{A} \xrightarrow{Q} \mathscr{A}/\mathscr{C} \longrightarrow 0$$

*induces short exact sequences of abelian categories*

$$0 \longrightarrow C^*(\mathscr{C}) \xrightarrow{i^*} C^*(\mathscr{A}) \xrightarrow{Q^*} C^*(\mathscr{A}/\mathscr{C}) \longrightarrow 0$$

*for* $* \in \{+, -, b, \emptyset\}$.

The left bounded /right bounded/bounded versions of the above result and their proofs have been implicit in the work of J. Miyachi [4]. It is surprising to notice that for the unbounded version, we do NOT need any extra conditions.

Based on Theorem 1.2, we obtain short exact sequences of homotopy categories.

**Theorem 1.3** (Theorem 4.3)**.** *Let* $0 \longrightarrow \mathscr{C} \xrightarrow{i} \mathscr{A} \xrightarrow{Q} \mathscr{A}/\mathscr{C} \longrightarrow 0$ *be a short exact sequence of abelian categories. We have induced short exact sequences of triangulated categories*

$$\mathrm{Ker}(Q^*) \xrightarrow{i^*} K^*(\mathscr{A}) \xrightarrow{Q^*} K^*(\mathscr{A}/\mathscr{C})$$

*for* $* \in \{+, -, b, \emptyset\}$.

It seems that there is no natural description of the kernel of the functor $Q^* : K^*(\mathscr{A}) \to K^*(\mathscr{A}/\mathscr{C})$, although we provide such one in Proposition 4.2.

Now we can easily deduce Miyachi's result and its unbounded version.

**Theorem 1.4** (Theorem 5.2)**.** *Let* $0 \longrightarrow \mathscr{C} \xrightarrow{i} \mathscr{A} \xrightarrow{Q} \mathscr{A}/\mathscr{C} \longrightarrow 0$ *be a short exact sequence of abelian categories. We have induced short exact sequences of triangulated categories*

$$D^*_{\mathscr{C}}(\mathscr{A}) \xrightarrow{i^*} D^*(\mathscr{A}) \xrightarrow{Q^*} D^*(\mathscr{A}/\mathscr{C})$$

*for* $* \in \{+, -, b, \emptyset\}$.

Notice that based on the induced short exact sequences of complex categories, the proofs for homotopy categories and derived categories are rather direct.

Next we consider the question when the natural functor $D^*(\mathscr{C}) \to D^*_{\mathscr{C}}(\mathscr{A})$ is an equivalence. D. Yao showed that the fullness implies that dense property for bounded derived categories [5, Theorem 2.1]. We consider the left/right bounded cases and the unbounded case under the following conditions ($\bigstar_1$) and ($\bigstar_2$) on the inclusion functor $i : \mathscr{C} \to \mathscr{A}$:

($\bigstar_1$) $\mathscr{A}$ has countable coproducts, $\mathscr{C}$ is closed under countable coproducts in $\mathscr{A}$ and countable coproducts of exact sequences are exact;

($\bigstar_2$) $\mathscr{A}$ has countable products, $\mathscr{C}$ is closed under countable products in $\mathscr{A}$ and countable products of exact sequences are exact.

We also show that the dense property also implies the fully faithfulness, as far as we assume a slighter stronger dense property; see Proposition 6.7.

This paper is organised as follows. Section 2 contains some preliminaries, including basic notions about calculi of fractions, and we also recall basic facts about localisations of abelian and triangulated categories as well as exact sequences, (co)localisation sequences and recollements. Section 3 is the core part of this paper, in which we study the induced short exact sequences of complex categories. It contains much technical details. In Sections 4 and 5, we consider induced short exact sequences for homotopy categories and derived categories, respectively. In particular, we show Theorem 5.2 which is an unbounded version of the result of J. Miyachi. In Section 6, we continue the line of research begun by D. Yao. We consider several criterions for the natural functor $D^*(\mathscr{C}) \to D^*_{\mathscr{C}}(\mathscr{A})$ to be an equivalence. In the last Section 7, we present some applications and examples, some known, while others unknown.

In this paper, we don't care about set theoretical difficulties, that is, we always assume that the categories involved exist.

## 2. Preliminaries

### *2.1. Calculi of fractions*

In this subsection, we recall basic facts about calculi of fractions. For an introduction to localisations of categories, we refer the reader to [6].

**Definition 2.1.** [6, Section 2.2] Let $\mathscr{A}$ be a category. A class of morphisms $\mathscr{S}$ in $\mathscr{A}$ admits a *calculus of left fractions* (or is a *left localizing class*) if it satisfies the following conditions (morphisms in $\mathscr{S}$ will be denoted by $\Longrightarrow$):

(i) For every object $C \in \mathscr{A}$, the identity morphism $\mathrm{id}_C$ belongs to $\mathscr{S}$;

(ii) the composition of two morphisms in $\mathscr{S}$ is again an element of $\mathscr{S}$, whenever they are composable;

(iii) every diagram $C \xleftarrow{f} C' \xRightarrow{\omega} D$ with $\omega \in \mathscr{S}$ can be completed to a commutative square

$$\begin{tikzcd} C' \arrow[r, "f"] \arrow[d, Rightarrow, "\omega"'] & C \arrow[d, Rightarrow, "\omega'"] \\ D \arrow[r, "f'"'] & D' \end{tikzcd}$$

with $\omega' \in \mathscr{S}$;

(iv) if for two morphisms $f, g$ in $\mathscr{A}$ and $\omega$ in $\mathscr{S}$ such that $f \circ \omega = g \circ \omega$, then there exists $\vartheta \in \mathscr{S}$ such that $\vartheta \circ f = \vartheta \circ g$.

Dually, we define *calculi of right fractions* or *right localising classes*. A class of morphisms which admits both a left and right calculus of fractions is called a *multiplicative system* (or a *localising class*).

Let $\mathscr{S}$ be a left localising class of a category $\mathscr{A}$ and $X, Y \in \mathscr{A}$. A left fraction $(s, b)$ in $\mathscr{A}$ from $X$ to $Y$ is a diagram

$$X \xrightarrow{b} Z \xLeftarrow{s} Y$$

with $s \in \mathscr{S}$. Two left fractions $(s, b)$ and $(r, a)$ from $X$ to $Y$ are equivalent, denoted by $(s, b) \sim (r, a)$, if there exists a commutative diagram in $\mathscr{A}$

$$\begin{tikzcd} & Z \arrow[d] & \\ X \arrow[ur, "b"] \arrow[r] \arrow[dr, "a"'] & \cdot & Y \arrow[ul, Rightarrow, "s"'] \arrow[l, Rightarrow] \arrow[dl, Rightarrow, "r"] \\ & W \arrow[u] & \end{tikzcd} .$$

This is an equivalence relation. We denote the equivalence class of $(s, b)$ by $s^{-1}b$. Let $r^{-1}a$ be an equivalence class of left fractions from $X$ to $Y$, $s^{-1}b$ be an equivalence class of left fractions from $Y$ to $Z$. Then their composition is defined as

$$s^{-1}b \circ r^{-1}a = (ts)^{-1}ca,$$

which can be illustrated by the following diagram:

$$\begin{tikzcd}
& & \cdot & & \\
& \cdot \arrow[ur, "c"] & & \cdot \arrow[ul, Rightarrow, "t"'] & \\
X \arrow[ur, "a"] & & Y \arrow[ul, Rightarrow, "r"'] \arrow[ur, "b"] & & Z, \arrow[ul, Rightarrow, "s"']
\end{tikzcd}$$

where the new morphisms $t$ and $c$ are constructed using Definition 2.1 (iii). It is not difficult to see that the definition of compositions does not depend on the choices of $t$ and $c$.

We define a new category, denoted by $[\mathscr{S}^{-1}]\mathscr{A}$, as follows. Its objects are the same as $\mathscr{A}$, the morphisms from one object $X$ to another one $Y$ are the equivalence classes of left fractions from $X$ to $Y$, and compositions of morphisms are defined above. Moreover, there exists a natural quotient functor

$$Q : \mathscr{A} \to [\mathscr{S}^{-1}]\mathscr{A}$$

sending $X$ to $X$ and $f : X \to Y$ to $1_Y^{-1}f$. It can be shown that $[\mathscr{S}^{-1}]\mathscr{A}$ is the localisation of $\mathscr{A}$ with respect to $\mathscr{S}$. Similarly, given a right localisation class $\mathscr{S}$ in $\mathscr{A}$, one can define the category $\mathscr{A}[\mathscr{S}^{-1}]$, which is the localisation of $\mathscr{A}$ with respect to $\mathscr{S}$.

### 2.2. *Exact sequences, (co)localisation sequences and recollements of abelian categories*

Basic references of this subsection are [6, Section 3.6] and [1].

Let $\mathscr{A}$ be an abelian category. A full additive subcategory $\mathscr{C}$ of $\mathscr{A}$ is called a *Serre subcategory* if for every short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathscr{A}$, the following condition holds: $X, Z \in \mathscr{C}$ if and only if $Y \in \mathscr{C}$. It is easy to see that the class of morphisms

$$\mathscr{M}_{\mathscr{C}}(\mathscr{A}) = \{f \in \mathrm{Mor}(\mathscr{A}) \mid \mathrm{Ker}(f), \mathrm{Coker}(f) \in \mathscr{C}\}$$

is a multiplicative system, where $\mathrm{Ker}(f)$ and $\mathrm{Coker}(f)$ denote the kernel and the cokernel of $f$, respectively.

We denote by $\mathscr{A}/\mathscr{C}$ the localization of $\mathscr{A}$ with respect to $\mathscr{M}_{\mathscr{C}}(\mathscr{A})$ and call it the quotient category of $\mathscr{A}$ by $\mathscr{C}$. The quotient category $\mathscr{A}/\mathscr{C}$ is an abelian category and the quotient functor $Q : \mathscr{A} \to \mathscr{A}/\mathscr{C}$ is an exact functor. We denote by $i : \mathscr{C} \to \mathscr{A}$ the inclusion functor.

In this case, we say that

$$0 \to \mathscr{C} \xrightarrow{i} \mathscr{A} \xrightarrow{Q} \mathscr{A}/\mathscr{C} \to 0$$

is a *short exact sequence of abelian categories*.

If we assume that the inclusion functor $i : \mathscr{C} \to \mathscr{A}$ satisfies the condition ($\bigstar_1$), then $\mathscr{A}/\mathscr{C}$ has countable coproducts and the exact functor $Q : \mathscr{A} \to \mathscr{A}/\mathscr{C}$ commutes with countable coproducts [7, Lemma A.2.21]. Dually, if the condition ($\bigstar_2$) holds for the inclusion functor $i : \mathscr{C} \to \mathscr{A}$, then $\mathscr{A}/\mathscr{C}$ has countable products and the exact functor $Q : \mathscr{A} \to \mathscr{A}/\mathscr{C}$ commutes with countable products.

Let $F : \mathscr{A} \to \mathscr{B}$ be an exact functor between abelian categories. Then

$$\mathrm{Ker}(F) = \{X \in \mathscr{A} \mid F(X) \simeq 0\}$$

is a Serre subcategory of $\mathscr{A}$ and so $F$ induces an exact functor $\overline{F} : \mathscr{A}/\mathrm{Ker}(F) \to \mathscr{B}$. This functor $\overline{F}$ is necessarily faithful, whose simple proof is left to the reader.

Given a short exact sequence of abelian categories

$$0 \to \mathscr{C} \xrightarrow{i} \mathscr{A} \xrightarrow{Q} \mathscr{A}/\mathscr{C} \to 0,$$

if $Q$ has a right adjoint $S$ (or equivalently $i$ has right adjoint; cf [6]), the diagram

$$\begin{tikzcd} \mathscr{C} \arrow[r, "i"] & \mathscr{A} \arrow[l, bend left] \arrow[r, "Q"] & \mathscr{A}/\mathscr{C} \arrow[l, bend left, "S"] \end{tikzcd}$$

is called a *localisation sequence* (or a *right recollement*) of abelian categories. Obviously, $S$ is left exact. However, it is generally not an exact functor.

Dually, if $Q$ has a left adjoint $R$ (or equivalently $i$ has left adjoint), the diagram

$$\begin{tikzcd} \mathscr{C} \arrow[r, "i"'] & \mathscr{A} \arrow[l, bend right] \arrow[r, "Q"'] & \mathscr{A}/\mathscr{C} \arrow[l, bend right, "R"'] \end{tikzcd}$$

is called a *colocalisation sequence* (or a *left recollement*) of abelian categories. Obviously, $R$ is right exact. However, it is generally not an exact functor.

When $Q$ has both a left adjoint $R$ and a right adjoint $S$ (or equivalently so does $i$), the diagram

$$\begin{tikzcd} \mathscr{C} \arrow[r, "i" description] & \mathscr{A} \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "Q" description] & \mathscr{A}/\mathscr{C} \arrow[l, bend right, "R"'] \arrow[l, bend left, "S"] \end{tikzcd}$$

will be called a *recollement* of abelian categories.

There also exists a generalisation of the above notions, say, ladders of abelian categories; see, for instance, [8].

*2.3. Exact sequences, (co)localisation sequences and recollements of triangulated categories*

Let $\mathcal{T}$ be a triangulated category with shift functor $\Sigma$. A triangulated subcategory $\mathcal{T}'$ of $\mathcal{T}$ is called a *thick subcategory* if it is closed under taking direct summands; see [2] and [3, Proposition 1.3]. Then the *Verdier quotient* $\mathcal{T}/\mathcal{T}'$ is defined as the localization of $\mathcal{T}$ by the multiplicative system

$$\mathcal{M}_{\mathcal{T}'}(\mathcal{T}) = \{f : X \to Y \in \mathrm{Mor}(\mathcal{T}) \mid \mathrm{Cone}(f) \in \mathcal{T}'\},$$

where $\mathrm{Cone}(f)$ is the third object appearing in the distinguished triangle

$$X \xrightarrow{f} Y \to \mathrm{Cone}(f) \to \Sigma X.$$

The Verdier quotient $\mathcal{T}/\mathcal{T}'$ is still a triangulated category and the quotient functor $Q : \mathcal{T} \to \mathcal{T}/\mathcal{T}'$ is a triangle functor. In this case, let $i : \mathcal{T}' \to \mathcal{T}$ be the inclusion functor. We say that $\mathcal{T}' \xrightarrow{i} \mathcal{T} \xrightarrow{Q} \mathcal{T}/\mathcal{T}'$ is a *short exact sequence of triangulated categories*.

Given a short exact sequence of triangulated categories $\mathcal{T}' \xrightarrow{i} \mathcal{T} \xrightarrow{Q} \mathcal{T}/\mathcal{T}'$, if $i$ (or $Q$) has a left adjoint, then the diagram

$$\begin{tikzcd} \mathcal{T}' \arrow[r, "i"'] & \mathcal{T} \arrow[l, bend right] \arrow[r, "Q"'] & \mathcal{T}/\mathcal{T}' \arrow[l, bend right] \end{tikzcd}$$

is called a *colocalisation sequence* (or a *left recollement*) of triangulated categories. Dually, given a short exact sequence of triangulated categories $\mathcal{T}' \xrightarrow{i} \mathcal{T} \xrightarrow{Q} \mathcal{T}/\mathcal{T}'$, if $i$ (or $Q$) has a right adjoint, then the diagram

$$\begin{tikzcd} \mathcal{T}' \arrow[r, "i"] & \mathcal{T} \arrow[l, bend left] \arrow[r, "Q"] & \mathcal{T}'' \arrow[l, bend left] \end{tikzcd}$$

is called a *localisation sequence* or a *right recollement* of triangulated categories.

When $i$ (or $Q$) has a left adjoint and a right adjoint, the diagram

$$\begin{tikzcd} \mathcal{T}' \arrow[r, "i" description] & \mathcal{T} \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "Q" description] & \mathcal{T}/\mathcal{T}' \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

is called a recollement of triangulated categories.

There exists a generalisation of the above notions, say, ladders of triangulated categories; see [9].

The main goal of this paper is to produce short exact sequences, left recollements, right recollements, recollements of homotopy categories and derived categories from short exact sequences, left recollements, right recollements, recollements of abelian categories, respectively.

## 3. Constructing short exact sequences of abelian categories of complexes

In this section, we will show that a short exact sequence of abelian categories gives rise to short exact sequences of various complex categories.

Let $\mathscr{A}$ be an abelian category. Recall that for $* = +, -, b, \emptyset$, we denote by $C^*(\mathscr{A})$ the category of left bounded, right bounded, bounded or unbounded complexes, respectively.

**Theorem 3.1.** *A short exact sequence of abelian categories*

$$0 \longrightarrow \mathscr{C} \xrightarrow{\ i\ } \mathscr{A} \xrightarrow{\ Q\ } \mathscr{A}/\mathscr{C} \longrightarrow 0$$

*induces short exact sequences of abelian categories*

$$0 \longrightarrow C^*(\mathscr{C}) \xrightarrow{\ i^*\ } C^*(\mathscr{A}) \xrightarrow{\ Q^*\ } C^*(\mathscr{A}/\mathscr{C}) \longrightarrow 0$$

*for* $* \in \{+, -, b\}$.

We split the proof of the above result into several lemmas.

The first lemma shows that it suffices to show that $Q^* : C^*(\mathscr{A}) \to C^*(\mathscr{A}/\mathscr{C})$ is dense and full. This lemma is the result for abelian categories analogous to [4, Lemma 3.1], which itself deals with triangulated categories.

**Lemma 3.2.** *Let $F : \mathscr{C} \to \mathscr{D}$ be an exact functor between abelian categories. Suppose that $F$ is dense and full. Then the induced functor*

$$\overline{F} : \mathscr{C}/\mathrm{Ker}(F) \to \mathscr{D}$$

*is an equivalence, where* $\mathrm{Ker}(F) := \{X \in \mathscr{C} \mid F(X) \simeq 0\}$.

**Proof** It is obvious that $\overline{F}$ is full and dense, so it suffices to show that $\overline{F}$ is faithful. Given a morphism $s^{-1}f$ in $\mathscr{C}/\mathrm{Ker}(F)$ presented by a left fraction $X \xrightarrow{f} Y' \overset{s}{\Leftarrow} Y$. Suppose that it is sent to zero by $\overline{F}$.

Then $\overline{F}(s^{-1}f) = F(s)^{-1}F(f) = 0$ and thus $F(f) = 0$. Since $F$ is exact, $F(\mathrm{Im}(f)) = \mathrm{Im}(F(f)) = 0$. Hence, $\mathrm{Im}(f)$ belongs to $\mathrm{Ker}(F)$. As $f$ can be written as the composition $X \to \mathrm{Im}(f) \to Y'$, it becomes the zero morphism in $\mathscr{C}/\mathrm{Ker}(F)$.

**Lemma 3.3.** *[10, Tag 06XL] Given a short exact sequence of abelian categories*

$$0 \longrightarrow \mathscr{C} \xrightarrow{\ i\ } \mathscr{A} \xrightarrow{\ Q\ } \mathscr{A}/\mathscr{C} \longrightarrow 0$$

*then for $* \in \{+, -, b\}$, the induced functor $Q^* : C^*(\mathscr{A}) \to C^*(\mathscr{A}/\mathscr{C})$ is dense.*

Let us remark that the above result and also the following one have been implicit in the proof of [4, Theorem 3.2].

**Lemma 3.4.** *Let $\mathscr{C}$ be a Serre subcategory of an abelian category $\mathscr{A}$ and let $* \in \{+, -, b\}$. Then the induced functor $Q^* : C^*(\mathscr{A})/C^*(\mathscr{C}) \to C^*(\mathscr{A}/\mathscr{C})$ is full.*

**Proof** The result follows obviously from the following statement.

*For a chain map $f : X^\bullet \to Y^\bullet$ in $C^*(\mathscr{A}/\mathscr{C})$, there exist a complex $Z^\bullet$ in $C^*(\mathscr{A})$ and chain maps*

$$g : X^\bullet \to Z^\bullet \quad \text{and} \quad u : Y^\bullet \to Z^\bullet$$

*in $C^*(\mathscr{A})$ such that $Q^*(u) \circ f = Q^*(g)$ and that $Q^*(u)$ is an isomorphism in $C^*(\mathscr{A}/\mathscr{C})$.*

Now we prove this statement. By Lemma 3.3, the functor $Q^* : C^*(\mathscr{A}) \to C^*(\mathscr{A}/\mathscr{C})$ is dense. We can assume that $X^\bullet$ and $Y^\bullet$ are complexes in $C^*(\mathscr{A})$.

*(1) The left bounded case.*

Suppose that $f = (f^i)_{i\in\mathbb{Z}}$ has the following representation diagram

$$\begin{tikzcd}
X^\bullet : & \cdots 0 \arrow[r] & X^0 \arrow[r, "\partial^0"] \arrow[d, "f'^0"] & X^1 \arrow[r, "\partial^1"] \arrow[d, "f'^1"] & \cdots \arrow[r, "\partial^{n-1}"] & X^n \arrow[r] \arrow[d, "f'^n"] & \cdots \\
 & & W^0 & W^1 & \cdots & W^n & \\
Y^\bullet : & \cdots 0 \arrow[r] & Y^0 \arrow[r, "d^0"] \arrow[u, Rightarrow, "s^0"] & Y^1 \arrow[r, "d^1"] \arrow[u, Rightarrow, "s^1"] & \cdots \arrow[r, "d^{n-1}"] & Y^n \arrow[r] \arrow[u, Rightarrow, "s^n"] & \cdots
\end{tikzcd}$$

with $s^i \in \mathscr{M}_{\mathscr{C}}(\mathscr{A})$, $\forall i \in \mathbb{Z}$.

When $i < 0$, let $Z^i = 0$. For $i = 0$, let $Z^0 = W^0$, $g^0 = f'^0$ and $u^0 = s^0$. Since $s^0 \in \mathscr{M}_{\mathscr{C}}(\mathscr{A})$, by Definition 2.1 (iii), we have the following commutative diagram in $\mathscr{A}$

$$\begin{tikzcd}
X^0 \arrow[r, "\partial^0"] \arrow[d, "g^0"] & X^1 \arrow[dr, "f'^1"] & \\
Z^0 \arrow[r, "\omega^0"] & A^1 & W^1 \\
Y^0 \arrow[r, "d^0"] \arrow[u, Rightarrow, "u^0"'] & Y^1 \arrow[u, Rightarrow, "v^1"'] \arrow[ur, Rightarrow, "s^1"] &
\end{tikzcd},$$

where $v^1 : Y^1 \to A^1$ belongs to $\mathscr{M}_{\mathscr{C}}(\mathscr{A})$.

Since $f$ is a chain map, there exist $r^1 : A^1 \to Z^1$ and $k^1 : W^1 \to Z^1$ in $\mathscr{M}_{\mathscr{C}}(\mathscr{A})$ such that the following diagram commute in $\mathscr{A}$

$$\begin{tikzcd}
 & W^1 \arrow[d, Rightarrow, "k^1"'] & \\
X^0 \arrow[ur, "f'^1\circ\partial^0"] \arrow[r] \arrow[dr, "\omega^0\circ g^0"'] & Z^1 & Y^1 \arrow[l, Rightarrow] \arrow[ul, Rightarrow, "s^1"'] \arrow[dl, Rightarrow, "v^1"] \\
 & A^1 \arrow[u, Rightarrow, "r^1"] &
\end{tikzcd}.$$

Let $\partial'^0 = r^1 \circ \omega^0$, $g^1 = k^1 \circ f'^1$ and $u^1 = k^1 \circ s^1$. Then we get the following commutative diagram in $\mathscr{A}$:

$$\begin{tikzcd}
0 \arrow[r, "0"] \arrow[d, "0"] & X^0 \arrow[r, "\partial^0"] \arrow[d, "g^0"] & X^1 \arrow[d, "g^1"] \\
0 \arrow[r, "0"] & Z^0 \arrow[r, "\partial'^0"] & Z^1 \\
0 \arrow[r, "0"] \arrow[u, Rightarrow, "id"'] & Y^0 \arrow[r, "d^0"] \arrow[u, Rightarrow, "u^0"'] & Y^1 \arrow[u, Rightarrow, "u^1"']
\end{tikzcd}.$$

Thus, we have constructed the object $Z^1$, the maps $u^1$, $g^1$ and the differential $\partial'^0$. Moreover, we have

$$f^1 = (s^1)^{-1} f'^1 = (k^1 \circ s^1)^{-1}(k^1 \circ f'^1) = (u^1)^{-1} g^1.$$

Since $s^1 \in \mathscr{M}_{\mathscr{C}}(\mathscr{A})$, by Definition 2.1 (iii), we have a commutative diagram in $\mathscr{A}$

$$\begin{tikzcd}
X^1 \arrow[r, "\partial^1"] \arrow[d, "g^1"] & X^2 \arrow[dr, "f'^2"] & \\
Z^1 \arrow[r, "\omega^1"] & A^2 & W^2 \\
Y^1 \arrow[r, "d^1"] \arrow[u, Rightarrow, "u^1"'] & Y^2 \arrow[u, Rightarrow, "v^2"'] \arrow[ur, Rightarrow, "s^2"] &
\end{tikzcd},$$

where $v^2 : Y^2 \to A^2$ belongs to $\mathscr{M}_{\mathscr{C}}(\mathscr{A})$. Since $f$ is a chain map, there exist $r^2 : A^2 \to W'^2$ and $k^2 : W^2 \to W'^2$ belonging to $\mathscr{M}_{\mathscr{C}}(\mathscr{A})$ such that the following diagram commutes in $\mathscr{A}$

$$\begin{tikzcd}
 & W^2 \arrow[d, Rightarrow, "k^2"] & \\
X^1 \arrow[ur, "f'^2 \circ \partial^1"] \arrow[r] \arrow[dr, "\omega^1 \circ g^1"'] & W'^2 & Y^2 \arrow[l, Rightarrow] \arrow[ul, Rightarrow, "s^2"'] \arrow[dl, Rightarrow, "v^2"] \\
 & A^2 \arrow[u, Rightarrow, "r^2"] &
\end{tikzcd}.$$

We set $\partial''^1 = r^2 \circ \omega^1$, $g'^2 = k^2 \circ f'^2$ and $u'^2 = k^2 \circ s^2$. Then the following diagram commutes in $\mathscr{A}$

$$\begin{tikzcd}
X^0 \arrow[r, "\partial^0"] \arrow[d, "g^0"] & X^1 \arrow[r, "\partial^1"] \arrow[d, "g^1"] & X^2 \arrow[d, "g'^2"] \\
Z^0 \arrow[r, "\partial'^0"] & Z^1 \arrow[r, "\partial''^1"] & W'^2 \\
Y^0 \arrow[r, "d^0"] \arrow[u, Rightarrow, "u^0"] & Y^1 \arrow[r, "d^1"] \arrow[u, Rightarrow, "u^1"] & Y^2. \arrow[u, Rightarrow, "u'^2"]
\end{tikzcd}$$

It is easy to see that $\partial''^1 \circ \partial'^0 \circ u^0 = u'^2 \circ d^1 \circ d^0 = 0 = 0 \circ u^0$.

Since $u^0 \in \mathscr{M}_{\mathscr{C}}(\mathscr{A})$, by Definition 2.1 (iv), there exists $a : W'^2 \Rightarrow Z^2$ in $\mathscr{M}_{\mathscr{C}}(\mathscr{A})$ such that $a \circ \partial''^1 \circ \partial'^0 = 0$.

Let $u^2 = a \circ u'^2, g^2 = a \circ g'^2$ and $\partial'^1 = a \circ \partial''^1$. So $\partial'^1 \circ \partial'^0 = 0$. Thus, we have constructed $Z^2$, $u^2$, $g^2$ and the differential $\partial'^1$. Moreover, we have $f^2 = (u^2)^{-1} g^2$.

Repeating this process, we construct the complex $Z^\bullet \in C^+(\mathscr{A})$ and the chain map $g : X^\bullet \to Z^\bullet$, $u : Y^\bullet \to Z^\bullet$ in $C^+(\mathscr{A})$ such that $Q^+(u) \circ f = Q^+(g)$ and that $Q^+(u)$ is isomorphism in $C^+(\mathscr{A})$.

*(2) The right bounded case.*

In this situation, we use right fractions to construct the complex $Z^\bullet$ and the chain maps $g$ and $u$.

*(3) The bounded case.*

We use left fractions (or right fractions) to construct the complex $Z^\bullet$ and the chain maps $g$ and $u$. It is easy to see that we can choose $Z^n = 0$ for $|n| \gg 0$.

Hence, our statement is proved.

**Proof of Theorem 3.1** Let Ker($Q^*$) be the kernel of $Q^* : C^*(\mathscr{A}) \to C^*(\mathscr{A}/\mathscr{C})$. It is obvious that Ker($Q^*$) $= C^*(\mathscr{C})$. The above three lemmas show that the induced functor $C^*(\mathscr{A})/C^*(\mathscr{C}) \to C^*(\mathscr{A}/\mathscr{C})$ is an equivalence.

Now we consider the unbounded case.

**Theorem 3.5.** *Given a short exact sequence of abelian categories*

$$0 \longrightarrow \mathscr{C} \xrightarrow{\ i\ } \mathscr{A} \xrightarrow{\ Q\ } \mathscr{A}/\mathscr{C} \longrightarrow 0\,,$$

*it induces another short exact sequence of abelian categories of unbounded complexes*

$$0 \longrightarrow C(\mathscr{C}) \xrightarrow{\ i\ } C(\mathscr{A}) \xrightarrow{\ Q\ } C(\mathscr{A}/\mathscr{C}) \longrightarrow 0\,.$$

By Lemma 3.2, this theorem follows from the two lemmas below.

**Lemma 3.6.** *[10, Tag 06XL] The induced functor $Q : C(\mathscr{A}) \to C(\mathscr{A}/\mathscr{C})$ in Theorem 3.5 is dense.*

**Lemma 3.7.** *The induced functor $Q : C(\mathscr{A}) \to C(\mathscr{A}/\mathscr{C})$ in Theorem 3.5 is full.*

**Proof** Let $f : X^\bullet \to Y^\bullet$ be a chain map in $C(\mathscr{A}/\mathscr{C})$. By Lemma 3.6, we can assume that $X^\bullet$ and $Y^\bullet$ are in $C(\mathscr{A})$. By Lemma 3.4, we construct a positive complex $(Z^\bullet, \partial'^\bullet) \in C^+(\mathscr{A})$ such that the following diagram commute

$$\begin{tikzcd}
\cdots \arrow[r] & X^{-1} \arrow[r, "\partial^{-1}"] & X^0 \arrow[r, "\partial^0"] \arrow[d, "g^0"] & X^1 \arrow[r, "\partial^1"] \arrow[d, "g^1"] & X^2 \arrow[r] \arrow[d, "g^2"] & \cdots \\
 & & Z^0 \arrow[r, "\partial'^0"] & Z^1 \arrow[r, "\partial'^1"] & Z^2 \arrow[r] & \cdots \\
\cdots \arrow[r] & Y^{-1} \arrow[r, "d^{-1}"] & Y^0 \arrow[r, "d^0"] \arrow[u, Rightarrow, "u^0"'] & Y^1 \arrow[r, "d^1"] \arrow[u, Rightarrow, "u^1"'] & Y^2 \arrow[r] \arrow[u, Rightarrow, "u^2"'] & \cdots ,
\end{tikzcd}$$

where $Q(u^i)^{-1}Q(g^i) = f^i$ for all $i \geqslant 0$.

Dually, by Lemma 3.4, we construct a negative complex $(Z^\bullet, d'^\bullet)$ such that the following diagram commutes

$$\begin{tikzcd}
\cdots \arrow[r] & X^{-2} \arrow[r, "\partial^{-2}"] & X^{-1} \arrow[r, "\partial^{-1}"] & X^0 \arrow[r, "\partial^0"] \arrow[d, "g^0"] & X^1 \arrow[r, "\partial^1"] \arrow[d, "g^1"] & X^2 \arrow[r] \arrow[d, "g^2"] & \cdots \\
\cdots Z^{-2} \arrow[r, "d'^{-2}"] \arrow[ur, Rightarrow, "v^{-2}"] \arrow[dr, "h^{-2}"'] & Z^{-1} \arrow[r, "d'^{-1}"] \arrow[ur, Rightarrow, "v^{-1}"] \arrow[dr, "h^{-1}"'] & Z'^0 \arrow[ur, Rightarrow, "v^0"] \arrow[dr, "h^0"'] & Z^0 \arrow[r, "\partial'^0"] & Z^1 \arrow[r, "\partial'^1"] & Z^2 \arrow[r] & \cdots \\
\cdots \arrow[r] & Y^{-2} \arrow[r, "d^{-2}"] & Y^{-1} \arrow[r, "d^{-1}"] & Y^0 \arrow[r, "d^0"] \arrow[u, Rightarrow, "u^0"'] & Y^1 \arrow[r, "d^1"] \arrow[u, Rightarrow, "u^1"'] & Y^2 \arrow[r] \arrow[u, Rightarrow, "u^2"'] & \cdots ,
\end{tikzcd}$$

where $Q(h^i)Q(v^i)^{-1} = f^i$ for all $i \leqslant 0$. An interesting point is that the morphisms $v^0$ and $h^0$ are constructed by using the commuting square

$$\begin{tikzcd}
Z^0 & Y^0 \arrow[l, Leftarrow, "u^0"'] \\
X^0 \arrow[u, "g^0"] & Z'^0 \arrow[u, "h^0"'] \arrow[l, Leftarrow, "v^0"]
\end{tikzcd}$$

whose existence is guaranteed by (the dual) of Definition 2.1 (iii). This will not influence the proof of Lemma 3.4.

Observe that, by construction, the following two complexes

$$(M^\bullet, d'^\bullet):\ \cdots \longrightarrow Z^{-2} \xrightarrow{d'^{-2}} Z^{-1} \xrightarrow{d'^{-1}} Z'^0 \xrightarrow{\partial^0 \circ v^0} X^1 \xrightarrow{\partial^1} X^2 \longrightarrow \cdots$$

and

$$(N^\bullet, \partial'^\bullet):\ \cdots \longrightarrow Y^{-2} \xrightarrow{d^{-2}} Y^{-1} \xrightarrow{u^0 \circ d^{-1}} Z^0 \xrightarrow{\partial'^0} Z^1 \xrightarrow{\partial'^1} Z^2 \longrightarrow \cdots$$

are in $C(\mathscr{A})$.

Thus, we have the following commutative diagram in $\mathscr{A}$

$$\begin{tikzcd}
X^\bullet & \cdots \arrow[r] & X^{-2} \arrow[r, "\partial^{-2}"] & X^{-1} \arrow[r, "\partial^{-1}"] & X^0 \arrow[r, "\partial^0"] & X^1 \arrow[r, "\partial^1"] & X^2 \arrow[r] & \cdots \\
M^\bullet \arrow[u, "v"] \arrow[d, "f'"'] & \cdots \arrow[r, "d'^{-3}"] & Z^{-2} \arrow[u, Rightarrow, "v^{-2}"'] \arrow[r, "d'^{-2}"] \arrow[d, "h^{-2}"] & Z^{-1} \arrow[u, Rightarrow, "v^{-1}"'] \arrow[r, "d'^{-1}"] \arrow[d, "h^{-1}"] & Z'^0 \arrow[u, Rightarrow, "v^0"'] \arrow[r, "\partial^0 \circ v^0"] \arrow[d, "u^0 \circ h^0"] & X^1 \arrow[u, equal] \arrow[r, "\partial^1"] \arrow[d, "g^1"] & X^2 \arrow[u, equal] \arrow[r] \arrow[d, "g^2"] & \cdots \\
N^\bullet & \cdots \arrow[r, "d^{-3}"] & Y^{-2} \arrow[r, "d^{-2}"] & Y^{-1} \arrow[r, "u^0 \circ d^{-1}"] & Z^0 \arrow[r, "\partial'^0"] & Z^1 \arrow[r, "\partial'^1"] & Z^2 \arrow[r, "\partial'^2"] & \cdots \\
Y^\bullet \arrow[u, "u"] & \cdots \arrow[r] & Y^{-2} \arrow[u, equal] \arrow[r, "d^{-2}"] & Y^{-1} \arrow[u, equal] \arrow[r, "d^{-1}"] & Y^0 \arrow[u, Rightarrow, "u^0"'] \arrow[r, "d^0"] & Y^1 \arrow[u, Rightarrow, "u^1"'] \arrow[r, "d^1"] & Y^2 \arrow[u, Rightarrow, "u^2"'] \arrow[r] & \cdots .
\end{tikzcd}$$

Therefore, we have chain maps $f' : M^\bullet \to N^\bullet$, $v : (M^\bullet, \partial'^\bullet) \to (X^\bullet, \partial^\bullet)$ and $u : (Y^\bullet, d^\bullet) \to (N^\bullet, d'^\bullet)$ in $C(\mathscr{A})$. It is easy to see that $v$ and $u$ are isomorphisms in $C(\mathscr{A})/C(\mathscr{C})$. Then $Q(u)^{-1}Q(f')Q(v)^{-1} = f$. So the induced functor $Q : C(\mathscr{A})/C(\mathscr{C}) \to C(\mathscr{A}/\mathscr{C})$ is full.

## 4. Constructing short exact sequences of homotopy categories

In this section, we show that a short exact sequence of abelian categories gives rise to short exact sequences of homotopy categories.

We begin with an interesting observation which should be well known, but we could not find it in the literature. Let $\mathscr{A}$ be an abelian category. Let $A \in \mathscr{A}$ and $n \in \mathbb{Z}$. Denote by $D^n(A)$ the *elementary contractible complex*

$$\cdots \to 0 \to A \xrightarrow{1_A} A \to 0 \to \cdots$$

which concentrates in degrees $n$ and $n+1$. Elementary contractible complexes are contractible as shown by the following commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & A \arrow[r, "1_A"] \arrow[d, "1_A"] \arrow[dl] & A \arrow[r] \arrow[d, "1_A"] \arrow[dl, "1_A"'] & 0 \arrow[r] \arrow[dl] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & A \arrow[r, "1_A"'] & A \arrow[r] & 0 \arrow[r] & \cdots .
\end{tikzcd}$$

**Proposition 4.1.** *A complex $X^\bullet$ is contractible if and only if $X^\bullet$ is a direct summand of direct sums of elementary contractible complexes. More precisely, $X^\bullet$ is a direct summand of $\oplus_{n\in\mathbb{Z}} D^n(X^n)$.*

**Proof** The if-part is obvious. We now prove the only-if-part. Let $(X^\bullet, d^\bullet)$ be a chain complex. Suppose that it is contractible. Then there exist a family of morphisms $\{s^n : X^n \to X^{n-1} | n \in \mathbb{Z}\}$ such that for each $n \in \mathbb{Z}$,

$$d^{n-1} \circ s^n + s^{n+1} \circ d^n = 1_{X^n}.$$

Let $Y^\bullet$ be be the direct sum of elementary contractible complexes $\oplus_{n\in\mathbb{Z}}D^n(X^n)$. For each $n$, we have $Y^n = X^n \oplus X^{n-1}$ and the differential of $Y^\bullet$ has the form $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

For each $n$, let

$$f^n = (1_{X^n} \;\; d^{n-1}) : Y^n \to X^n \quad \text{and} \quad g^n = (s^{n+1} \circ d^n \;\; s^n)^T : X^n \to Y^n.$$

Then $f^\bullet \circ g^\bullet = 1_{X^\bullet}$. Thus, $X^\bullet$ is a direct summand of $\oplus_{n\in\mathbb{Z}}D_n(X^n)$.

Notice that when $X^\bullet \in C^*(\mathscr{A})$ for $* \in \{+, -, b, \emptyset\}$, we still have $\oplus_{n\in\mathbb{Z}}D^n(X^n) \in C^*(\mathscr{A})$.

Let $\mathscr{C}$ be a Serre subcategory of $\mathscr{A}$. For $* \in \{+, -, b, \emptyset\}$, we will describe the kernel of $Q^* : K^*(\mathscr{A}) \to K^*(\mathscr{A}/\mathscr{C})$. We say that a complex $(X^\bullet, d_X^\bullet)$ in $C^*(\mathscr{A})$ is *null homotopic modulo* $\mathscr{C}$ if there exist chain maps

$$s^\bullet : (Z^\bullet, \partial^\bullet) \to (X^\bullet, d^\bullet) \quad \text{and} \quad v^\bullet : (X^\bullet, d^\bullet) \to (W^\bullet, \delta^\bullet)$$

in $C^*(\mathscr{A})$ and a collection of morphisms $h = \{h^i : Z^i \to W^{i-1}, i \in \mathbb{Z}\}$ in $\mathscr{A}$ such that $s^i, v^i \in \mathscr{M}_{\mathscr{C}}(\mathscr{A})$ and $h^i\partial^i + \delta^{i-1}h^{i-1} = v^i s^i$ for all $i \in \mathbb{Z}$.

**Proposition 4.2.** *Let $\mathscr{A}$ be an abelian category and $\mathscr{C}$ a Serre subcategory of $\mathscr{A}$. Then a complex $X^\bullet$ belongs to* $\mathrm{Ker}(Q^*)$ *if and only if $X^\bullet$ is null homotopic modulo $\mathscr{C}$.*

**Proof** The condition is obviously sufficient, so we only need to show the necessity. We deal with the left, right bounded cases and the unbounded case, respectively.

*(1) The left bounded case.*

Suppose that a complex $(X^\bullet, d^\bullet)$ lies in Ker($Q^+$). Without loss of generality, we assume that

$$(X^\bullet, d^\bullet) = 0 \to X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} X^2 \xrightarrow{d^2} \cdots .$$

Since $X^\bullet$ is null homotopic in $C^+(\mathscr{A}/\mathscr{C})$, we have the following representation diagram in $\mathscr{A}$:

$$\begin{tikzcd}
X^\bullet & 0 \arrow[r] & X^0 \arrow[r, "d^0"] \arrow[dl, "u^0=0"'] & X^1 \arrow[r, "d^1"] \arrow[dl, "u^1"'] & X^2 \arrow[r, "d^2"] \arrow[dl, "u^2"'] & X^3 \arrow[r] \arrow[dl, "u^3"'] & \cdots \\
 & 0 & W^0 & W^1 & W^2 & & \\
X^\bullet & 0 \arrow[r] \arrow[u, Rightarrow, "v^{-1}=\mathrm{id}"] & X^0 \arrow[r, "d^0"] \arrow[u, Rightarrow, "v^0"] & X^1 \arrow[r, "d^1"] \arrow[u, Rightarrow, "v^1"] & X^2 \arrow[r, "d^2"] \arrow[u, Rightarrow, "v^2"] & \cdots ,
\end{tikzcd}$$

where $v^i, \forall i \in \mathbb{Z}$, is in $\mathscr{M}_{\mathscr{A}}(\mathscr{C})$ and satisfy $Q(d^{i-1})Q(v^{i-1})^{-1}Q(u^i) + Q(v^i)^{-1}Q(u^{i+1})Q(d^i) = \mathrm{id}_{X^i}$.

Next we will construct a complex $(\mathrm{Y}^\bullet, \partial^\bullet) : \cdots \to 0 \to \mathrm{Y}^0 \to \mathrm{Y}^1 \to \mathrm{Y}^2 \to \cdots$ together with morphisms $s^i : X^i \to Y^{i-1}$ and $t^i : X^i \to Y^i \in \mathscr{M}_{\mathscr{A}}(\mathscr{C})$, $\forall i \in \mathbb{Z}$ , such that the following diagram commutes in $\mathscr{A}$

$$\begin{tikzcd}
X^\bullet \arrow[d, "s"'] & 0 \arrow[r] & X^0 \arrow[r, "d^0"] \arrow[dl, "s^0"'] & X^1 \arrow[r, "d^1"] \arrow[dl, "s^1"'] & X^2 \arrow[r, "d^2"] \arrow[dl, "s^2"'] & \cdots \\
Y^\bullet & 0 \arrow[r] & Y^0 \arrow[r, "\partial^0"] & Y^1 \arrow[r, "\partial^1"] & Y^2 & \\
X^\bullet \arrow[u, Rightarrow, "t"] & 0 \arrow[r] \arrow[u, Rightarrow, "t^{-1}"] & X^0 \arrow[r, "d^0"] \arrow[u, Rightarrow, "t^0"] & X^1 \arrow[r, "d^1"] \arrow[u, Rightarrow, "t^1"] & X^2 \arrow[r, "d^2"] \arrow[u, Rightarrow, "t^2"] & \cdots .
\end{tikzcd}$$

For all $i \in \mathbb{Z}$, we have formulas

$$Q(t^i)^{-1}Q(s^{i+1}) = Q(v^i)^{-1}Q(u^{i+1}), \quad s^{i+1}d^i + \partial^{i-1}s^i = t^i \in \mathscr{M}_{\mathscr{A}}(\mathscr{C}) \quad \text{and} \quad t^i d^{i-1} = \partial^{i-1}t^{i-1}.$$

When $i \leqslant 0$, let $s^i = 0$, $t^{i-1} = \mathrm{id}_0$ and $\partial^{i-1} = 0$. For $i = 0$, since $Q(v^0)^{-1}Q(u^1)Q(d^0) = \mathrm{id}_{X^0}$, we take $Y^0 = W^0$ and let $s^1 = u^1$, $t^0 = v^0$. Then the desired conditions hold obviously.

For $i = 1$, by Definition 2.1 (iii), we can obtain two commutative squares

$$\begin{tikzcd}
Y^0 \arrow[r, "e^0"] & V^0 \\
X^0 \arrow[u, Rightarrow, "t^0"] \arrow[r, "d^0"] & X^1 \arrow[u, Rightarrow, "f^0"']
\end{tikzcd}$$

with $f^0 \in \mathscr{M}_{\mathscr{C}}(\mathscr{A})$ and

$$\begin{tikzcd}
 & K^1 & \\
V^0 \arrow[ur, "\beta^1"] & & W^1 \arrow[ul, Rightarrow, "\alpha^0"'] \\
 & X^1 \arrow[ul, Rightarrow, "f^0"] \arrow[ur, "v^1"'] &
\end{tikzcd}$$

with $\alpha^0 \in \mathscr{M}_{\mathscr{C}}(\mathscr{A})$. Therefore, the sum $Q(d^0)Q(v^1)^{-1}Q(u^1) + Q(v^2)^{-1}Q(u^2)Q(d^1)$ can be represented as the following diagram in $\mathscr{A}$

$$\begin{tikzcd}
 & K^1 & \\
X^1 \arrow[ur, "\beta^1 e^0 s^1 + \alpha^0 u^2 d^1"] & & X^1. \arrow[ul, Rightarrow, "\alpha^0 v^1 = \beta^1 f^0"']
\end{tikzcd}$$

Since we have formula

$$Q(d^0)Q(v^0)^{-1}Q(u^1) + Q(v^1)^{-1}Q(u^2)Q(d^1) = \mathrm{id}_{X^1},$$

there exist morphisms $b^1 : K^1 \to Y^1$ and $a^1 : X^1 \to Y^1$ in $\mathscr{M}_{\mathscr{A}}(\mathscr{C})$ such that the following diagram commutes in $\mathscr{A}$

$$\begin{tikzcd}
 & K^1 \arrow[d, Rightarrow, "b^1"] & \\
X^1 \arrow[ur, "\beta^1 e^0 s^1 + \alpha^0 u^2 d^1"] \arrow[r, "a^1"] \arrow[dr, "id"'] & Y^1 & X^1 \arrow[ul, Rightarrow, "\alpha^0 v^1 = \beta^1 f^0"'] \arrow[l, Rightarrow, "a^1"'] \arrow[dl, Rightarrow, "id"] \\
 & X^1 \arrow[u, Rightarrow, "a^1"'] &
\end{tikzcd}$$

.

We set $s^2 = b^1\alpha^0u^2, t^1 = b^1\alpha^0v^1$ and $\partial^0 = b^1\beta^1e^0$. Then we have

$$s^2d^1 + \partial^0 s^1 = a^1 \in \mathscr{M}_{\mathscr{A}}(\mathscr{C}), \quad Q(t^1)^{-1}Q(s^2) = Q(v^1)^{-1}Q(u^2) \quad \text{and} \quad t^1d^0 = \partial^0 t^0.$$

For $i = 2$, by the same construction in the case $i = 0$, we construct the following commutative diagram in $\mathscr{A}$

$$\begin{tikzcd}
X^0 \arrow[r, "d^0"] & X^1 \arrow[r, "d^1"] \arrow[dl, "s^1"'] & X^2 \arrow[r, "d^2"] \arrow[dl, "s^2"'] & X^3 \arrow[dl, "j^3"'] \\
Y^0 \arrow[r, "\partial^0"] & Y^1 \arrow[r, "\sigma^1"] & K^2 & \\
X^0 \arrow[r, "d^0"] \arrow[u, Rightarrow, "t^0"] & X^1 \arrow[r, "d^1"] \arrow[u, Rightarrow, "t^1"] & X^2 \arrow[r, "d^2"] \arrow[u, Rightarrow, "\rho^2"] & X^3.
\end{tikzcd}$$

However, the map $\sigma^1\partial^0$ maybe not 0. As $\sigma^1\partial^0 t^0 = \rho^2 d^1 d^0 = 0$ and $t^0 \in \mathscr{M}_{\mathscr{A}}(\mathscr{C})$, by Definition 2.1 (iv), there exists a map $w^1 : K^2 \to Y^2$ in $\mathscr{M}_{\mathscr{A}}(\mathscr{C})$ such that $w^1\sigma^1\partial^0 = 0$.

We set $\partial^1 = w^1\sigma^1$, $s^3 = w^1 j^3$ and $t^2 = w^1\rho^2$. Then $Y^2$, $s^3$ and $t^2$ satisfy the conditions that we need. Repeating this same process, we construct the desired complex $(Y^\bullet, \partial^\bullet)$. Therefore, $X^\bullet$ is null homotopic modulo $\mathscr{C}$.

*(2) The right bounded case.*

In this situation, we should use right fractions to construct the right bounded complex $(Z^\bullet, \delta^\bullet)$ and maps $\{t^i : Z^{i+1} \to X^i \,|\, i \in \mathbb{Z}_{\leqslant 0}\}$

$$\begin{tikzcd}
X^\bullet & \cdots \arrow[r] & X^{-2} \arrow[r, "d^{-2}"] & X^{-1} \arrow[r, "d^{-1}"] & X^0 \arrow[r, "d^0"] & 0 \\
Z^\bullet \arrow[u, Rightarrow, "u"] \arrow[d, "t"] & \cdots \arrow[r] & Z^{-2} \arrow[r, "\delta^{-2}"] \arrow[u, Rightarrow, "u^{-2}"] \arrow[dl, "t^{-3}"'] & Z^{-1} \arrow[r, "\delta^{-1}"] \arrow[u, Rightarrow, "u^{-1}"] \arrow[dl, "t^{-2}"'] & Z^0 \arrow[r, "\delta^0"] \arrow[u, Rightarrow, "u^0"] \arrow[dl, "t^{-1}"'] & 0 \arrow[u, Rightarrow, "id_0"] \arrow[dl, "t^0"'] \\
X^\bullet & \cdots X^{-3} \arrow[r, "d^{-3}"] & X^{-2} \arrow[r, "d^{-2}"] & X^{-1} \arrow[r, "d^{-1}"] & X^0 \arrow[r, "d^0"] & 0
\end{tikzcd}$$

such that for each $i$ in $\mathbb{Z}$, the map $t^i\delta^i + d^{i-1}t^{i-1} = u^i$ belongs to $\mathscr{M}_{\mathscr{A}}(\mathscr{C})$ and $u^i\delta^{i-1} = d^{i-1}u^{i-1}$. Therefore, the complex $X^\bullet$ is null homotopic modulo $\mathscr{C}$.

*(3) The unbounded case.*

Suppose $(X^\bullet, d^\bullet)$ is in Ker(Q). We assume that

$$(X^\bullet, d^\bullet) = \cdots \to X^{-2} \to X^{-1} \to X^0 \to X^1 \to X^2 \to \cdots .$$

By the proof of the right and left bounded cases, we can construct the following diagram in $\mathscr{A}$:

$$\begin{tikzcd}
X^\bullet : & \cdots \arrow[r] & X^{-2} \arrow[r, "d^{-2}"] & X^{-1} \arrow[r, "d^{-1}"] & X^0 \arrow[r, "d^0"] \arrow[d, "s^{-1}"'] & X^1 \arrow[r, "d^1"] \arrow[d, "s^0"'] & X^2 \arrow[r, "d^2"] \arrow[d, "s^1"'] & \cdots \\
& \cdots Z^{-2} \arrow[r, "\delta^{-2}"] \arrow[ur, Rightarrow, "u^{-2}"] \arrow[d, "t^{-3}"'] & Z^{-1} \arrow[r, "\delta^{-1}"] \arrow[ur, Rightarrow, "u^{-1}"] \arrow[d, "t^{-2}"'] & Z^0 \arrow[ur, Rightarrow, "u^0"] \arrow[d, "t^{-1}"'] & W^{-1} \arrow[r, "\partial^{-1}"] & W^0 \arrow[r, "\partial^0"] & W^1 \cdots & \\
X^\bullet : & \cdots X^{-3} \arrow[r, "d^{-3}"] & X^{-2} \arrow[r, "d^{-2}"] & X^{-1} \arrow[r, "d^{-1}"] \arrow[ur, Rightarrow, "\alpha^{-1}"] & X^0 \arrow[r, "d^0"] \arrow[ur, Rightarrow, "\alpha^0"] & X^1 \arrow[r, "d^1"] \arrow[ur, Rightarrow, "\alpha^1"] & X^2 \arrow[r, "d^2"] & \cdots
\end{tikzcd}$$

such that $t^i\delta^i + d^{i-1}t^{i-1} = u^i \in \mathscr{M}_{\mathscr{A}}(\mathscr{C})$ for $i \leqslant 0$ and $s^j d^j + \partial^{-(j+1)} s^{-(j+1)} = \alpha^j \in \mathscr{M}_{\mathscr{A}}(\mathscr{C})$ for $j \geqslant 0$.

The morphisms $u^0$ and $t^{-1}$ are constructed by using the commuting square

$$\begin{tikzcd} W^{-1} & X^{-1} \arrow[l, Leftarrow, "\alpha^{-1}"'] \\ X^0 \arrow[u, "s^{-1}"] & Z^0. \arrow[l, Leftarrow, "u^0"] \arrow[u, "t^{-1}"'] \end{tikzcd}$$

Its existence is guaranteed by (the dual of) Definition 2.1 (iii).

Observe that the following complexes

$$(Z^\bullet, \partial^\bullet): \ \cdots \longrightarrow Z^{-2} \xrightarrow{\delta^{-2}} Z^{-1} \xrightarrow{u^0\delta^{-1}} X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} X^2 \longrightarrow \cdots$$

and

$$(W^\bullet, \delta^\bullet): \ \cdots \longrightarrow X^{-3} \xrightarrow{d^{-3}} X^{-2} \xrightarrow{\alpha^{-1}d^{-2}} W^{-1} \xrightarrow{\partial^{-1}} W^0 \xrightarrow{\partial^0} W^1 \longrightarrow \cdots$$

are in $C(\mathscr{A})$.

Thus, we have the following diagram in $C(\mathscr{A})$

$$\begin{tikzcd}
(X^\bullet, d^\bullet): & \cdots \arrow[r] & X^{-2} \arrow[r, "d^{-2}"] & X^{-1} \arrow[r, "d^{-1}"] & X^0 \arrow[r, "d^0"] & X^1 \arrow[r, "d^1"] & X^2 \arrow[r] & \cdots \\
(Z^\bullet, \partial^\bullet): \arrow[u, Rightarrow, "u"] \arrow[d, "h"'] & \cdots \arrow[r] & Z^{-2} \arrow[r, "\delta^{-2}"] \arrow[u, Rightarrow, "u^{-2}"] \arrow[d, "t^{-3}"'] & Z^{-1} \arrow[r, "u^0\delta^{-1}"] \arrow[u, Rightarrow, "u^{-1}"] \arrow[d, "t^{-2}"'] & X^0 \arrow[r, "d^0"] \arrow[u, Rightarrow, "\mathrm{id}_{X^0}"] \arrow[d, "s^{-1}"'] & X^1 \arrow[r, "d^1"] \arrow[u, Rightarrow, "\mathrm{id}_{X^1}"] \arrow[d, "s^0"'] & X^2 \arrow[r] \arrow[u, Rightarrow, "\mathrm{id}_{X^2}"] \arrow[d, "s^1"'] & \cdots \\
(W^\bullet, \delta^\bullet): & \cdots \arrow[r] & X^{-3} \arrow[r, "d^{-3}"] & X^{-2} \arrow[r, "\alpha^{-1}d^{-2}"] & W^{-1} \arrow[r, "\partial^{-1}"] & W^0 \arrow[r, "\partial^0"] & W^1 \arrow[r] & \cdots \\
(X^\bullet, d^\bullet): \arrow[u, Rightarrow, "\alpha"] & \cdots \arrow[r] & X^{-3} \arrow[r, "d^{-3}"] \arrow[u, Rightarrow, "\mathrm{id}_{X^{-3}}"] & X^{-2} \arrow[r, "d^{-2}"] \arrow[u, Rightarrow, "\mathrm{id}_{X^{-2}}"] & X^{-1} \arrow[r, "d^{-1}"] \arrow[u, Rightarrow, "\alpha^{-1}"] & X^0 \arrow[r, "\alpha^0"] \arrow[u, Rightarrow, "v^0"] & X^1 \arrow[r] \arrow[u, Rightarrow, "\alpha^1"] & \cdots ,
\end{tikzcd}$$

where $u$ and $\alpha$ are morphisms of chain complexes in $\mathscr{M}_{\mathscr{A}}(\mathscr{C})$ and $\{h^i : Z^i \to W^{i-1} \mid i \in \mathbb{Z}\}$ are a collection of maps. Moreover, we have

$$h^i\partial^i + \delta^{i-1}h^{i-1} = \alpha^i u^i \in \mathscr{M}_{\mathscr{A}}(\mathscr{C})$$

for each $i \in \mathbb{Z}$. Therefore, $X^\bullet$ is null homotopic modulo $\mathscr{C}$.

**Theorem 4.3.** *Let* $0 \longrightarrow \mathscr{C} \xrightarrow{i} \mathscr{A} \xrightarrow{Q} \mathscr{A}/\mathscr{C} \longrightarrow 0$ *be an short exact sequence of abelian categories. Then for each* $* \in \{+, -, b, \emptyset\}$, *the induced triangle functor* $Q^* : K^*(\mathscr{A}) \to K^*(\mathscr{A}/\mathscr{C})$ *is fully faithful and we have an induced short exact sequence of triangulated categories*

$$\mathrm{Ker}(Q^*) \xrightarrow{i^*} K^*(\mathscr{A}) \xrightarrow{Q^*} K^*(\mathscr{A}/\mathscr{C}).$$

**Proof** By [4, Lemma 3.1], it suffices to show that the induced functor $Q^* : K^*(\mathscr{A})/\mathrm{Ker}(Q^*) \to K^*(\mathscr{A}/\mathscr{C})$ is full and dense.

The dense property is clear by Lemma 3.6. It is enough to show that

$$Q^* : K(\mathscr{A})/\mathrm{Ker}(Q^*) \to K(\mathscr{A}/\mathscr{C})$$

is full. We give the proof for the case $* = \emptyset$, the other cases $* = +, -, b$ being similar.

For any morphism $f : M^\bullet \to N^\bullet \in K(\mathscr{A}/\mathscr{C})$, by the statement in the proof of Lemma 3.7, there exist chain maps

$$v : M^\bullet \to X^\bullet \quad \text{and} \quad u : Y^\bullet \to N^\bullet$$

in $\mathscr{M}_{C(\mathscr{C})}(C(\mathscr{A}))$ and chain map $f' : M^\bullet \to N^\bullet$in $C(\mathscr{A})$ such that the chain map $u^{-1}f'v^{-1}$ in $C(\mathscr{A})/C(\mathscr{C})$ is sent to $f$ in $C(\mathscr{A}/\mathscr{C})$. Obviously $Q : K(\mathscr{A})/\mathrm{Ker}(Q^*) \to K(\mathscr{A}/\mathscr{C})$ sends its corresponding morphism in $K(\mathscr{A})/\mathrm{Ker}(Q^*)$ to $f$.

## 5. An unbounded version of Miyachi's theorem

Let $\mathscr{A}$ be an abelian category and $\mathscr{C}$ a Serre subcategory of $\mathscr{A}$. For $* \in \{+, -, b, \emptyset\}$, recall that $D^*_{\mathscr{C}}(\mathscr{A})$ denotes the full subcategory of $D^*(\mathscr{A})$ whose objects are complexes $X^\bullet$ such that the $n$-th homology $H^n(X)$ belongs to $\mathscr{C}$ for all $n \in \mathbb{Z}$.

J. Miyachi showed the following result, which is the starting point of this paper.

**Theorem 5.1.** [4, Theorem 3.2] Let $0 \longrightarrow \mathscr{C} \xrightarrow{i} \mathscr{A} \xrightarrow{Q} \mathscr{A}/\mathscr{C} \longrightarrow 0$ be an short exact sequence of abelian categories. Then we have induced short exact sequences of triangulated categories

$$D^*_{\mathscr{C}}(\mathscr{A}) \xrightarrow{i^*} D^*(\mathscr{A}) \xrightarrow{Q^*} D^*(\mathscr{A}/\mathscr{C})$$

for $* \in \{+, -, b\}$.

We would like to generalize Miyachi's result to unbounded derived categories without imposing any other condition.

**Theorem 5.2.** *Let $0 \longrightarrow \mathscr{C} \xrightarrow{i} \mathscr{A} \xrightarrow{Q} \mathscr{A}/\mathscr{C} \longrightarrow 0$ be a short exact sequence of abelian categories. Then we have an induced short exact sequence of unbounded derived categories*

$$D_{\mathscr{C}}(\mathscr{A}) \xrightarrow{i} D(\mathscr{A}) \xrightarrow{Q} D(\mathscr{A}/\mathscr{C}).$$

**Proof** We still denote by $Q : D(\mathscr{A}) \to D(\mathscr{A}/\mathscr{C})$ the derived functor induced by the exact quotient functor $Q : \mathscr{A} \to \mathscr{A}/\mathscr{C}$. Note that $\mathrm{Ker}(Q) = D_{\mathscr{C}}(\mathscr{A})$.

By [4, Lemma 3.1], it suffices to show that the induced functor $\overline{Q} : D(\mathscr{A})/D_{\mathscr{C}}(\mathscr{A}) \to D(\mathscr{A}/\mathscr{C})$ is full and dense.

As in the case of homotopy categories, the dense property is clear by Lemma 3.6. We need to show that $\overline{Q} : D(\mathscr{A})/D_{\mathscr{C}}(\mathscr{A}) \to D(\mathscr{A}/\mathscr{C})$ is full.

Let $f : X^\bullet \to Y^\bullet$ be a morphism in $D(\mathscr{A}/\mathscr{C})$. It has a presentation by left fraction

$$\begin{tikzcd} & X_1^\bullet & \\ X^\bullet \arrow[ur, "f'"] & & Y^\bullet, \arrow[ul, Rightarrow, "t"'] \end{tikzcd}$$

where $f'$ and $t$ are morphisms in $K(\mathscr{A}/\mathscr{C})$ and $t$ is a quasi-isomorphism.

By Theorem 4.3, there exist morphisms $s_1, s_2, t_1$ in $K(\mathscr{A})$ such that the quotient functor

$$\overline{Q} : K(\mathscr{A})/\mathrm{Ker}Q \to K(\mathscr{A}/\mathscr{C})$$

sends the left fractions $s_1^{-1}f_1$ and $s_2^{-1}t_1$ in $K(\mathscr{A})/\mathrm{Ker}Q$ to $f'$ and $t$ respectively. Since $D(\mathscr{A})/D_{\mathscr{C}}(\mathscr{A})$ is a quotient of $K(\mathscr{A})$, the image of $(s_2^{-1}t_1)^{-1}(s_1^{-1}f_1)$ under the functor $\overline{Q} : D(\mathscr{A})/D_{\mathscr{C}}(\mathscr{A}) \to D(\mathscr{A}/\mathscr{C})$ is $t^{-1}f'$.

We are done.

Next, we consider the cases when the given short exact sequence of abelian categories is in fact part of a (co)localization sequence, or even a recollement.

**Theorem 5.3.** *Let*

$$\begin{tikzcd} \mathscr{C} \arrow[r, "i"] & \mathscr{A} \arrow[r, "Q"] \arrow[l, bend left] & \mathscr{A}/\mathscr{C} \arrow[l, bend left, "S"] \end{tikzcd}$$

*be a localization sequence of abelian categories. Assume that the right total derived functor* $\mathbf{R}S : D(\mathscr{A}/\mathscr{C}) \to D(\mathscr{A})$ *of* $S$ *exists (for instance, when* $\mathscr{A}/\mathscr{C}$ *has enough injectives and satisfies the axiom* $AB4^*$*, i.e. it has arbitrary small products which preserve exactness). Then there exists a localization sequence of unbounded derived categories:*

$$\begin{tikzcd} D_{\mathscr{C}}(\mathscr{A}) \arrow[r, "i"] & D(\mathscr{A}) \arrow[r, "Q"] \arrow[l, bend left] & D(\mathscr{A}/\mathscr{C}). \arrow[l, bend left, "\mathbf{R}S"] \end{tikzcd}$$

Similarly, one can show the following two results.

**Theorem 5.4.** *Let*

$$\begin{tikzcd} \mathscr{C} \arrow[r, "i"'] & \mathscr{A} \arrow[r, "Q"'] \arrow[l, bend right] & \mathscr{A}/\mathscr{C} \arrow[l, bend right, "T"'] \end{tikzcd}$$

*be a colocalization sequence of abelian categories. Assume that the left total derived functor* $\mathbf{L}T : D(\mathscr{A}/\mathscr{C}) \to D(\mathscr{A})$ *of* $T$ *exists (for instance, when* $\mathscr{A}/\mathscr{C}$ *has enough projectives and satisfies the axiom* $AB4$*, i.e. it has arbitrary small coproducts which preserve exactness). Then there exists a colocalisation sequence of unbounded derived categories*

$$\begin{tikzcd} D_{\mathscr{C}}(\mathscr{A}) \arrow[r, "i"'] & D(\mathscr{A}) \arrow[r, "Q"'] \arrow[l, bend right] & D(\mathscr{A}/\mathscr{C}). \arrow[l, bend right, "\mathbf{L}T"'] \end{tikzcd}$$

**Theorem 5.5.** *Let*

$$\begin{tikzcd} \mathscr{C} \arrow[r, "i" description] & \mathscr{A} \arrow[r, "Q" description] \arrow[l, bend right] \arrow[l, bend left] & \mathscr{A}/\mathscr{C} \arrow[l, bend right, "T"'] \arrow[l, bend left, "S"] \end{tikzcd}$$

*be a recollement of abelian categories. Assume that the right total derived functor* $\mathbf{R}S : D(\mathscr{A}/\mathscr{C}) \to D(\mathscr{A})$ *of* $S$ *and the left total derived functor* $\mathbf{L}T : D(\mathscr{A}/\mathscr{C}) \to D(\mathscr{A})$ *of* $T$ *exist. Then there exists a recollement of unbounded derived categories*

$$\begin{tikzcd} D_{\mathscr{C}}(\mathscr{A}) \arrow[r, "i" description] & D(\mathscr{A}) \arrow[r, "Q" description] \arrow[l, bend right] \arrow[l, bend left] & D(\mathscr{A}/\mathscr{C}). \arrow[l, bend right, "\mathbf{L}T"'] \arrow[l, bend left, "\mathbf{R}S"] \end{tikzcd}$$

## 6. Criterions for the canonical functor $D^*(\mathscr{C}) \to D^*_{\mathscr{C}}(\mathscr{A})$ to be an equivalence

Let $\mathscr{C}$ be a Serre subcategory of an abelian category $\mathscr{A}$. It is obvious that the triangle functor $D^*(\mathscr{C}) \to D^*(\mathscr{A})$ factors through $D^*(\mathscr{C}) \to D^*_{\mathscr{C}}(\mathscr{A})$, which is still denoted by $i^*$ by abuse of notations. In this section, we consider the question when the natural functor

$$i^* : D^*(\mathscr{C}) \to D^*_{\mathscr{C}}(\mathscr{A})$$

is an equivalence with $* \in \{+, -, b, \emptyset\}$.

An interesting observation is that the fullness of the functor $i^*$ implies its dense property. This has been obtained by D. Yao [5] for bounded derived categories.

**Proposition 6.1.** *[5, Theorem 2.1] Let $\mathscr{A}$ be an abelian category and $\mathscr{C}$ a Serre subcategory of $\mathscr{A}$. If the canonical functor $i^b : D^b(\mathscr{C}) \to D^b_{\mathscr{C}}(\mathscr{A})$ is full, then it is dense.*

Remark that the canonical functor $i^b : D^b(\mathscr{C}) \to D^b_{\mathscr{C}}(\mathscr{A})$ is also faithful in Proposition 6.1. In fact, since $i : \mathscr{C} \to \mathscr{A}$ is fully faithful and exact, $i^b : D^b(\mathscr{C}) \to D^b_{\mathscr{C}}(\mathscr{A})$ takes any non-zero object to non-zero object. By the following lemma, it is faithful as well.

**Lemma 6.2.** *[11, pp.446] Any full triangle functor between triangulated categories is faithful as long as it does not take any non-zero object to zero.*

We generalise Yao's result to left bounded/right unbounded/unbounded cases.

**Proposition 6.3** (The left bounded case)**.** *Let $\mathscr{A}$ be an abelian category and $\mathscr{C}$ a Serre subcategory of $\mathscr{A}$. Suppose that the inclusion functor $i : \mathscr{C} \to \mathscr{A}$ satisfies the condition ($\bigstar_1$). If the canonical functor $i^+ : D^+(\mathscr{C}) \to D^+_{\mathscr{C}}(\mathscr{A})$ is full, then it is faithful and dense, hence an equivalence.*

**Proof** Obviously, the functor $i^b : D^b(\mathscr{C}) \to D^b_{\mathscr{C}}(\mathscr{A})$ is full. By Proposition 6.1, $i^b : D^b(\mathscr{C}) \to D^b_{\mathscr{C}}(\mathscr{A})$ is dense. Let $E^\bullet \in D^+_{\mathscr{C}}(\mathscr{A})$ and suppose that $E^\bullet$ can be represented as follows:

$$E^\bullet : \cdots \to 0 \to E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} E^n \xrightarrow{d^n} \cdots .$$

Suppose that the condition ($\bigstar_1$) holds on the inclusion functor $i : \mathscr{C} \to \mathscr{A}$. We know that

$$E^\bullet \cong \varinjlim_{n \geqslant 0} E^\bullet_{\leqslant n},$$

where

$$E^\bullet_{\leqslant n} : \cdots \to 0 \to E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} \mathrm{Ker}(d^n) \to 0 \to \cdots$$

is the right mild truncation of $E^\bullet$ at the $n$-th place together with canonical morphisms $u_{n,n+1} : E^\bullet_{\leqslant n} \to E^\bullet_{\leqslant n+1}$.

Since $E^\bullet_{\leqslant n}$ belongs to $D^b_{\mathscr{C}}(\mathscr{A})$, then there exists a complex $F^\bullet_n \in D^b(\mathscr{C})$ together with an isomorphism $f_n : i^b(F^\bullet_n) \to E^\bullet_{\leqslant n}$ in $D^b_{\mathscr{C}}(\mathscr{A})$. Define $g_{n,n+1} = f^{-1}_{n+1} \circ u_{n,n+1} \circ f_n$ in $D^b_{\mathscr{C}}(\mathscr{A})$ for each $n \in \mathbb{Z}_{\geqslant 0}$ and we have the following commutative diagram in $D^+_{\mathscr{C}}(\mathscr{A})$:

$$\begin{tikzcd}
i^b(F^\bullet_0) \arrow[r, "g_{01}"] \arrow[d, "f_0"'] & i^b(F^\bullet_1) \arrow[r, "g_{12}"] \arrow[d, "f_1"'] & \cdots \arrow[r] & i^b(F^\bullet_n) \arrow[r, "g_{n,n+1}"] \arrow[d, "f_n"'] & i^b(F^\bullet_{n+1}) \arrow[r] \arrow[d, "f_{n+1}"'] & \cdots \\
E^\bullet_{\leqslant 0} \arrow[r, "u_{01}"] & E^\bullet_{\leqslant 1} \arrow[r, "u_{12}"] & \cdots \arrow[r] & E^\bullet_{\leqslant n} \arrow[r, "u_{n,n+1}"] & E^\bullet_{\leqslant n+1} \arrow[r] & \cdots .
\end{tikzcd}$$

Since $i^+ : D^+(\mathscr{C}) \to D^+_{\mathscr{C}}(\mathscr{A})$ is full, there exists $h_{n,n+1} : F^\bullet_n \to F^\bullet_{n+1}$ in $D^+(\mathscr{C})$ such that $i^+(h_{n,n+1}) = g_{n,n+1}$ for each $n \geq 0$.

By our assumption, the subcategory $\mathscr{C}$ has countable direct sums. One can form the homotopy colimit

$$\oplus_{i\geqslant 0} F^\bullet_i \xrightarrow{h} \oplus_{i\geqslant 0} F^\bullet_i \to \mathrm{homcolim}(h) \to \Sigma \oplus_{i\geqslant 0} F^\bullet_i,$$

where the restriction of $h$ to $F^\bullet_i$ is $\mathrm{id} - h_{i,i+1}$.

The condition ($\bigstar_1$) implies that $i^+$ commutes with countable direct sums. So we have the following morphism of distinguished triangles in $D^+_{\mathscr{C}}(\mathscr{A})$

$$\begin{tikzcd}
\bigoplus_{i\geqslant 0} i^+(F^\bullet_i) \arrow[r, "i^+(h)=g"] \arrow[d, "\oplus_{i\geqslant 0} f_i"'] & \bigoplus_{i\geqslant 0} i^+(F^\bullet_i) \arrow[r] \arrow[d, "\oplus_{i\geqslant 0} f_i"'] & i^+(\mathrm{homcolim}(h)) \arrow[r] \arrow[d, dashed, "\theta"'] & {} \\
\bigoplus_{i\geqslant 0} E^\bullet_{\leqslant i} \arrow[r, "u"] & \bigoplus_{i\geqslant 0} E^\bullet_{\leqslant i} \arrow[r] & E^\bullet \arrow[r] & {} .
\end{tikzcd}$$

Since countable direct sums of quasi-isomorphisms are still quasi-isomorphisms, $\theta$ is an isomorphism in $D^+_{\mathscr{C}}(\mathscr{A})$. We are done.

By using a similar method, we have the following results.

**Proposition 6.4** (The right bounded case). *Let $\mathscr{A}$ be an abelian category and $\mathscr{C}$ a Serre subcategory of $\mathscr{A}$. Suppose that the inclusion functor $i : \mathscr{C} \to \mathscr{A}$ satisfies the condition ($\bigstar_2$). If the canonical functor $i^- : D^-(\mathscr{C}) \to D^-_{\mathscr{C}}(\mathscr{A})$ is full, then it is faithful and dense, hence an equivalence.*

**Proposition 6.5** (The unbounded case). *Let $\mathscr{A}$ be an abelian category and $\mathscr{C}$ a Serre subcategory of $\mathscr{A}$. Suppose that the inclusion functor $i : \mathscr{C} \to \mathscr{A}$ satisfies the condition the conditions ($\bigstar_1$) and ($\bigstar_2$). If the canonical functor $i : D(\mathscr{C}) \to D_{\mathscr{C}}(\mathscr{A})$ is full, then it is faithful and dense, hence an equivalence.*

Now we consider the inverse problem whether the dense property implies that $i^*$ is fully faithful. We could only show that this holds under slightly stronger property. We need a well known criterion.

**Proposition 6.6.** *[12, Proposition 1.6.5] Let $\mathscr{C}$ be a category and $\mathscr{C}'$ a full subcategory of $\mathscr{C}$. Let $S$ be a multiplicative system in $\mathscr{C}$, and let $S'$ be the family of morphisms of $\mathscr{C}'$ which belong to $S$. Assume that one of the following conditions holds:*

*(1) whenever $f : X \to Y$ is a morphism in $S$, with $Y \in Ob(\mathscr{C}')$, there exists $g : W \to X$, with $W \in Ob(\mathscr{C}')$ and $f \circ g \in S$,*

*(2) the same as (1) with the arrows reversed.*

*Then the localisation $\mathscr{C}'[S'^{-1}]$ is a full subcategory of $\mathscr{C}[S^{-1}]$.*

**Proposition 6.7.** *Let $\mathscr{A}$ be an abelian category and $\mathscr{C}$ a Serre subcategory of $\mathscr{A}$. Suppose that the canonical functor $i^* : D^*(\mathscr{C}) \to D^*_{\mathscr{C}}(\mathscr{A})$ is dense "with fixed direction", i.e. one of the following conditions holds:*

*(1) for any $X^\bullet \in D^*_{\mathscr{C}}(\mathscr{A})$, there exists a quasi-isomorphism $X^\bullet \to Y^\bullet$ with $Y^\bullet \in D^*(\mathscr{C})$;*

*(2) for any $X^\bullet \in D^*_{\mathscr{C}}(\mathscr{A})$, there exists a quasi-isomorphism $Y^\bullet \to X^\bullet$ with $Y^\bullet \in D^*(\mathscr{C})$.*

*Then $i^*$ is fully faithful, hence an equivalence.*

**Proof** This follows immediately from Proposition 6.6.

Recall that a full abelian subcategory $\mathscr{C}$ of an abelian category $\mathscr{A}$ is thick if it is closed under extensions. The next result is another criterion, which is in fact a special case of [12, Proposition 1.7.11].

**Proposition 6.8.** *Let $\mathscr{A}$ be an abelian category with enough injective objects and $\mathscr{C}$ a thick subcategory of $\mathscr{A}$. Suppose that each object $C \in \mathscr{C}$ can be embedded into an object $I$ of $\mathscr{C}$ which is an injective object of $\mathscr{A}$. Then the natural functor $i^* : D^*(\mathscr{C}) \to D^*_{\mathscr{C}}(\mathscr{A})$ is an equivalence for $* = +, b$. This is also true for $* = \emptyset$, if we suppose that the inclusion functor $i : \mathscr{C} \to \mathscr{A}$ satisfies the condition $(\bigstar_1)$ or $(\bigstar_2)$.*

**Proof** The cases for $* = +, b$ are exactly [13, Proposition 2.42]. In fact, by [12, Proposition 1.7.11], for each complex $X^\bullet \in D^+_{\mathscr{C}}(\mathscr{A})$, one can find a quasi-isomorphism $X^\bullet \to Y^\bullet$ with $Y^\bullet \in D^+(\mathscr{C})$.

For the unbounded case, let $X^\bullet$ be a complex in $D_{\mathscr{C}}(\mathscr{A})$. Under the condition $(\bigstar_1)$ we realise $X^\bullet$ as a direct limit of brutal truncations and then use the homotopy colimits, or under the condition $(\bigstar_2)$ realise $X^\bullet$ as an inverse limit of mild truncations and use homotopy limits. The details are left to the reader.

Similarly, the dual statement of the above proposition is also true.

**Proposition 6.9.** *Let $\mathscr{A}$ be an abelian category with enough projective objects and $\mathscr{C}$ a thick subcategory of $\mathscr{A}$. Suppose that any $C$ in $\mathscr{C}$ is a quotient of an object $A$ in $\mathscr{C}$ which is projective as an object of $\mathscr{A}$. Then the natural functor $i^* : D^*(\mathscr{C}) \to D^*_{\mathscr{C}}(\mathscr{A})$ is an equivalence for $* = -, b$. This is also true for $* = \emptyset$, if we suppose that the inclusion functor $i : \mathscr{C} \to \mathscr{A}$ satisfies the condition $(\bigstar_1)$ or $(\bigstar_2)$.*

## 7. Examples

In this section, we will present some examples, most of which come from [17] and [8].

**Example 7.1.** [14, pp.205] Let $\mathscr{A}$ be an abelian category. Recall that an additive contravariant functor $F$ from $\mathscr{A}$ to the category of abelian groups $Ab$ is finitely presented if it fits into an exact sequence

$$\mathrm{Hom}_{\mathscr{A}}(-, X) \xrightarrow{f_*} \mathrm{Hom}_{\mathscr{A}}(-, Y) \to F \to 0 \tag{7.1}$$

for some morphism $X \xrightarrow{f} Y$ in $\mathscr{A}$. Denote by $\mathrm{mod}(\mathscr{A})$ the category of all finitely presented contravariant functors from $\mathscr{A}$ to $Ab$.

Introduce a functor $\omega : \mathrm{mod}(\mathscr{A}) \to \mathscr{A}$ by imposing

$$\omega(F) = \mathrm{Coker}(X \xrightarrow{f} Y),$$

for a finitely presented contravariant functor $F : \mathscr{A}^{op} \to Ab$ with a presentation (7.1). The functor $\omega$ is exact and is left adjoint to the Yoneda functor $\mathcal{Y} : \mathscr{A} \to \mathrm{mod}(\mathscr{A})$. Moreover there exists a localization sequence of abelian categories

$$\begin{tikzcd} \mathrm{eff}(\mathscr{A}) \arrow[r, "i"] & \mathrm{mod}(\mathscr{A}) \arrow[l, bend left] \arrow[r, "\omega"] & \mathscr{A} \arrow[l, bend left, "\mathcal{Y}"] \end{tikzcd}$$

with $\mathrm{eff}(\mathscr{A}) := \mathrm{Ker}(\omega)$.

By Theorems 5.1 and 5.2, we obtain the following exact sequences of triangulated categories

$$D^{*}_{\mathrm{eff}(\mathscr{C})}(\mathrm{mod}(\mathscr{A})) \xrightarrow{i^{*}} D^{*}(\mathrm{mod}(\mathscr{A})) \xrightarrow{\omega^{*}} D^{*}(\mathscr{A})$$

for $* = b, -, +, \emptyset$.

It would be very interesting to determine when $i^{*} : D^{*}(\mathrm{eff}(\mathscr{A})) \to \mathrm{D}^{*}_{\mathrm{eff}(\mathscr{A})}(\mathrm{mod}(\mathscr{A}))$ is an equivalence.

**Example 7.2.** Let $\mathscr{A}$ be a small abelian category. Denote by $\mathrm{Mod}(\mathscr{A}) = \mathrm{Add}(\mathscr{A}^{op}, Ab)$ the category of all additive contravariant functor from $\mathscr{A}$ to $Ab$ and by $\mathrm{Lex}(\mathscr{A})$ the full subcategory of $\mathrm{Mod}(\mathscr{A})$ consisting of all left exact functors. By [15, Theorem 2.3], there exists a localisation sequence of abelian categories

$$\begin{tikzcd} \mathrm{Eff}(\mathscr{A}) \arrow[r, "i"] & \mathrm{Mod}(\mathscr{A}) \arrow[l, bend left] \arrow[r, "Q"] & \mathrm{Lex}(\mathscr{A}), \arrow[l, bend left, "T"] \end{tikzcd}$$

where $i$ is the inclusion functor and $\mathrm{Eff}(\mathscr{A}) := \mathrm{Ker(Q)}$ which can be seen as the ind-completion of $\mathrm{eff}(\mathscr{A})$ [16].

It is easy to see that the adjunction pair $(Q, T)$ satisfies the conditions in [17, Theorem 10]. Then the above localisation sequence can be extended to be a recollement of abelian categories

$$\begin{tikzcd} \mathrm{Eff}(\mathscr{A}) \arrow[r, "i"] & \mathrm{Mod}(\mathscr{A}) \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "Q"] & \mathrm{Lex}(\mathscr{A}), \arrow[l, bend right, "\mathbf{L}_0 T"'] \arrow[l, bend left, "T"] \end{tikzcd}$$

where $\mathbf{L}_0 T$ is the 0-th left derived functor of $T$.

By Theorem 5.1, we obtain the following exact sequence of derived categories

$$D^{b}_{\mathrm{Eff}(\mathscr{A})}(\mathrm{Mod}(\mathscr{A})) \xrightarrow{i^{b}} D^{b}(\mathrm{Mod}(\mathscr{A})) \xrightarrow{Q^{b}} D^{b}(\mathrm{Lex}(\mathscr{A})).$$

Since $\mathscr{A}$ is small, $\mathrm{Lex}(\mathscr{A})$ is a Grothendieck category [16, Theorem 8.6.5]. Hence, by Theorem 5.3, 5.4 and 5.5, we have the following right recollement, left recollement and recollement of triangulated categories:

$$\begin{tikzcd} D^{+}_{\mathrm{Eff}(\mathscr{A})}(\mathrm{Mod}(\mathscr{A})) \arrow[r, "i^{+}"] & D^{+}(\mathrm{Mod}(\mathscr{A})) \arrow[l, bend left] \arrow[r, "Q^{+}"] & D^{+}(\mathrm{Lex}(\mathscr{A})), \arrow[l, bend left, "\mathbf{R}^{+} T"] \end{tikzcd}$$

$$\begin{tikzcd} D^{-}_{\mathrm{Eff}(\mathscr{A})}(\mathrm{Mod}(\mathscr{A})) \arrow[r, "i^{-}"'] & D^{+}(\mathrm{Mod}(\mathscr{A})) \arrow[l, bend right] \arrow[r, "Q^{+}"'] & D^{+}(\mathrm{Lex}(\mathscr{A})) \arrow[l, bend right, "\mathbf{L}^{-} T"'] \end{tikzcd}$$

and

$$\begin{tikzcd} D_{\mathrm{Eff}(\mathscr{A})}(\mathrm{Mod}(\mathscr{A})) \arrow[r, "i"] & D(\mathrm{Mod}(\mathscr{A})) \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "Q"] & D(\mathrm{Lex}(\mathscr{A})). \arrow[l, bend right, "\mathbf{L}T"'] \arrow[l, bend left, "\mathbf{R}T"] \end{tikzcd}$$

It would be very interesting to determine when $i^* : D^*(\mathrm{Eff}(\mathscr{C})) \to \mathrm{D}^*_{\mathrm{Eff}(\mathscr{C})}(\mathrm{Mod}(\mathscr{C}))$ is an equivalence.

**Example 7.3.** [17, Section 4.7] Let $\mathscr{C}$ be a finitely accessible additive category. We denote by $\mathrm{fp}\mathscr{C}$ the full subcategory of $\mathscr{C}$ consisting of all finitely presented objects and recall that $\mathrm{mod}(\mathscr{C})$ denotes the category of finitely presented functors as in Example 7.1. The category $\mathrm{mod}(\mathscr{C})$ is an abelian category with all small colimits. Any additive functor $\mathrm{G} \in \mathrm{Mod}(\mathrm{fp}\mathscr{C})$ can be extended to a unique functor $\overleftarrow{G} : \mathscr{C}^{op} \to Ab$ which preserves inverse limits. Therefore, there is a functor

$$\overleftarrow{-} : \mathrm{Mod}(\mathrm{fp}\mathscr{C}) \to \mathrm{mod}(\mathscr{C})$$

which is right adjoint to the restriction functor

$$\mathrm{R} : \mathrm{mod}(\mathscr{C}) \to \mathrm{Mod}(\mathrm{fp}\mathscr{C}),\ \mathrm{F} \mapsto \mathrm{F}|_{\mathrm{fp}\mathscr{C}}.$$

By [17, Corollary 20], we have in fact a recollement of abelian categories

$$\begin{tikzcd} \mathrm{Ker}(R) \arrow[r, "i"] & \mathrm{mod}(\mathscr{C}) \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "R"] & \mathrm{Mod}(\mathrm{fp}\mathscr{C}), \arrow[l, bend right, "\mathbf{L}_0 S"'] \arrow[l, bend left, "S"] \end{tikzcd}$$

where $S := \overleftarrow{-}$ and $\mathbf{L}_0 S$ is the 0-th left derived functor of $S$.

By Theorems 5.1, we have the following exact sequence of derived categories

$$D^b_{\mathrm{Ker(R)}}(\mathrm{mod}(\mathscr{C})) \xrightarrow{i^b} D^b(\mathrm{mod}(\mathscr{C})) \xrightarrow{R^b} D^b(\mathrm{Mod}(\mathrm{fp}\mathscr{C})).$$

Since $\mathrm{Mod}(\mathrm{fp}\mathscr{C})$ has enough projectives and enough injectives, by Theorems 5.3, 5.4 and 5.5, we obtain the following right recollement, left recollement and recollement of triangulated categories:

$$\begin{tikzcd} D^+_{\mathrm{Ker(R)}}(\mathrm{mod}(\mathscr{C})) \arrow[r] & D^+(\mathrm{mod}(\mathscr{C})) \arrow[l, bend left] \arrow[r, "R^+"] & D^+(\mathrm{Mod}(\mathrm{fp}\mathscr{C})), \arrow[l, bend left, "\mathbf{R}^+ S"] \end{tikzcd}$$

$$\begin{tikzcd} D^-_{\mathrm{Ker(R)}}(\mathrm{mod}(\mathscr{C})) \arrow[r] & D^-(\mathrm{mod}(\mathscr{C})) \arrow[l, bend right] \arrow[r, "R^-"'] & D^-(\mathrm{Mod}(\mathrm{fp}\mathscr{C})), \arrow[l, bend right, "\mathbf{L}^- S"'] \end{tikzcd}$$

$$\begin{tikzcd} D_{\mathrm{Ker(R)}}(\mathrm{mod}(\mathscr{C})) \arrow[r] & D(\mathrm{mod}(\mathscr{C})) \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "R"] & D(\mathrm{Mod}(\mathrm{fp}\mathscr{C})). \arrow[l, bend right, "\mathbf{L}S"'] \arrow[l, bend left, "\mathbf{R}S"] \end{tikzcd}$$

It would be very interesting to determine when $i^* : D^*(\mathrm{Ker}(R)) \to D^*_{\mathrm{Ker(R)}}(\mathrm{mod}(\mathscr{C}))$ is an equivalence.

**Example 7.4.** Let $A$ be an associative ring with unit and $e \in A$ an idempotent. Then there exists a recollement of abelian categories

$$\begin{tikzcd} \mathrm{Mod}(A/AeA) \arrow[r, "i"] & \mathrm{Mod}(A) \arrow[l, bend right, "u"'] \arrow[l, bend left, "v"] \arrow[r, "R"] & \mathrm{Mod}(eAe), \arrow[l, bend right, "T"'] \arrow[l, bend left, "S"] \end{tikzcd}$$

where $i$ is the inclusion, $u = - \otimes_A A/AeA$, $v = \mathrm{Hom}_A(A/AeA, -)$, $R = \mathrm{Hom}_A(eA, -)$, $T = - \otimes_{eAe} eA$, $S = \mathrm{Hom}_{eAe}(Ae, -)$.

By Theorem 5.1, we have an exact sequence of triangulated categories

$$D^b_{\mathrm{Ker(R)}}(\mathrm{Mod(A)}) \xrightarrow{i^b} D^b(\mathrm{Mod(A)}) \xrightarrow{R^b} D^b(\mathrm{Mod(eAe)}).$$

By Theorems 5.3, 5.4 and 5.5, we obtain the following right recollement, left recollement and recollement of triangulated categories:

$$\begin{tikzcd} D^+_{\mathrm{Ker(R)}}(\mathrm{Mod(A)}) \arrow[r] & D^+(\mathrm{Mod(A)}) \arrow[l, bend left] \arrow[r, "R^+"] & D^+((\mathrm{Mod(eAe)}), \arrow[l, bend left, "\mathbf{R}^+S"] \end{tikzcd}$$

$$\begin{tikzcd} D^-_{\mathrm{Ker(R)}}(\mathrm{Mod(A)}) \arrow[r] & D^-(\mathrm{Mod(A)}) \arrow[l, bend right] \arrow[r, "R^-"'] & D^-(\mathrm{Mod(eAe)}) \arrow[l, bend right, "\mathbf{L}^-T"'] \end{tikzcd}$$

and

$$\begin{tikzcd} D_{\mathrm{Ker(R)}}(\mathrm{Mod(A)}) \arrow[r] & D(\mathrm{Mod(A)}) \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "R"] & D(\mathrm{Mod(eAe)}). \arrow[l, bend right, "\mathbf{L}T"'] \arrow[l, bend left, "\mathbf{R}S"] \end{tikzcd}$$

The natural functor $i^* : D^*(\mathrm{Ker}(R)) \to D^*_{\mathrm{Ker(R)}}(\mathrm{Mod}(A))$ is an equivalence if and only if the surjection $A \to A/AeA$ is a homological epimorphism; see [18].

**Example 7.5.** Let $A$ be a right Noetherian ring. Denote by $\mathrm{Mod}(A)$ the category of all $A$-modules and by $\mathrm{mod}(A)$ the full subcategory of finitely generated modules. Then $\mathrm{mod}(A)$ is a Serre subcategory of $\mathrm{Mod}(A)$. We have an exact sequence of abelian categories

$$0 \longrightarrow \mathrm{mod}(A) \xrightarrow{i} \mathrm{Mod}(A) \xrightarrow{Q} \mathrm{Mod}(A)/\mathrm{mod}(A) \longrightarrow 0,$$

where $i$ and $Q$ are the canonical inclusion functor and the quotient functor respectively. Notice that by [8, Section 4.2], this short exact sequence can NOT be extended to a left or right recollement.

By [16, Theorem 15.3.1], we know that

$$i^b : D^b(\mathrm{mod}(A)) \to D^b_{\mathrm{mod}(A)}(\mathrm{Mod}(A)) \quad \text{and} \quad i^- : D^-(\mathrm{mod}(A)) \to D^-_{\mathrm{mod}(A)}(\mathrm{Mod}(A))$$

are equivalences. Thus, by Theorem 5.1 and 5.2, we have the following exact sequences of triangulated categories:

$$D^*(\mathrm{mod}(A)) \xrightarrow{i^*} D^*(\mathrm{Mod}(A)) \xrightarrow{Q^*} D^*(\mathrm{Mod}(A)/\mathrm{mod}(A))$$

for $* = b, -$,

$$D^+_{\mathrm{mod}A}(\mathrm{Mod}(A)) \xrightarrow{i^+} D^+(\mathrm{Mod}(A)) \xrightarrow{Q^+} D^+(\mathrm{Mod}(A)/\mathrm{mod}(A)),$$

and

$$D_{\mathrm{mod}A}(\mathrm{Mod}(A)) \xrightarrow{i} D(\mathrm{Mod}(A)) \xrightarrow{Q} D(\mathrm{Mod}(A)/\mathrm{mod}(A)).$$

It would be very interesting to see when $i^+ : D^+(\mathrm{mod}(A)) \to D^+_{\mathrm{mod}(A)}(\mathrm{Mod}(A))$ and $i : D(\mathrm{mod}(A)) \to D_{\mathrm{mod}(A)}(\mathrm{Mod}(A))$ are equivalences; for recent progress, see [19].

**Example 7.6.** Let $\mathrm{Ab}_t$ be the category of all torsion abelian groups and $\mathrm{Ab}_f$ the category of torsion-free abelian groups. Then $(Ab_t, Ab_f)$ is a torsion pair in $Ab$ and $Ab_t$ is a Serre subcategory of $Ab$. We get a right recollement of abelian categories

$$\begin{tikzcd} \mathrm{A}b_t \arrow[r, "i"] & \mathrm{A}b \arrow[r, "Q"] \arrow[l, bend left] & \mathrm{A}b/Ab_t \arrow[l, bend left, "S"] \end{tikzcd}$$

which can not be extended further; see [8, Section 4.3].

By Theorem 5.1, we have the following short exact sequences of triangulated categories

$$D^*_{\mathrm{Ab}_t}(\mathbb{Z}) \xrightarrow{i^*} D^*(\mathbb{Z}) \xrightarrow{Q^*} D^*(\mathrm{Ab}/\mathrm{Ab}_t)$$

for $* = b, -$.

It is easy to see that the quotient category $\mathrm{A}b/\mathrm{Ab}_t$ has enough injectives. By Theorem 5.3, we obtain the following right recollements of triangulated categories:

$$\begin{tikzcd} D^+_{\mathrm{Ab}_t}(\mathbb{Z}) \arrow[r, "i^+"] & D^+(\mathbb{Z}) \arrow[r, "Q^+"] \arrow[l, bend left] & D^+(\mathrm{Ab}/\mathrm{Ab}_t), \arrow[l, bend left, "\mathbf{R}^+S"] \end{tikzcd}$$

$$\begin{tikzcd} D_{\mathrm{Ab}_t}(\mathbb{Z}) \arrow[r, "i"] & D(\mathbb{Z}) \arrow[r, "Q"] \arrow[l, bend left] & D(\mathrm{Ab}/\mathrm{Ab}_t). \arrow[l, bend left, "\mathbf{R}S"] \end{tikzcd}$$

It would be very interesting to see whether $i^* : \mathrm{D}^*(\mathrm{Ab_t}) \to \mathrm{D}^*_{\mathrm{Ab_t}}(\mathbb{Z})$ is an equivalence.

**Example 7.7.** Let us recall the Gabriel-Popescu Theorem; see for example [20]. Let $\mathscr{A}$ be a Grothendieck category, $G$ a generator of $\mathscr{A}$ and $R$ the ring of all endomorphisms of $G$. Let $S$ be the functor from $\mathscr{A}$ to Mod(R) defined by $S(X) = \mathrm{Hom}_{\mathscr{A}}(G, X)$. Then $S$ is fully faithful. Moreover, $S$ has an exact left adjoint functor $T$, i.e. we have the following right recollement of abelian categories

$$\begin{tikzcd} \mathrm{Ker(T)} \arrow[r, "i"] & \mathrm{Mod(R)} \arrow[r, "T"] \arrow[l, bend left] & \mathscr{A}. \arrow[l, bend left, "S"] \end{tikzcd}$$

By Theorem 5.1, we obtain the following exact sequences of derived categories

$$D^*_{\mathrm{Ker}T}(\mathrm{Mod}(R)) \xrightarrow{i^*} D^*(\mathrm{Mod}(R)) \xrightarrow{T^*} D^*(\mathscr{A})$$

for $* = -, b$.

Since $\mathscr{A}$ is a Grothendieck category which has enough injectives, by Theorems 5.3, we have the following right recollement of triangulated categories

$$\begin{tikzcd} D^*_{\mathrm{Ker}T}(\mathrm{Mod}(R)) \arrow[r, "i^*"] & D^*(\mathrm{Mod}(R)) \arrow[r, "T^*"] \arrow[l, bend left] & D^*(\mathscr{A}) \arrow[l, bend left, "\mathbf{R}^*S"] \end{tikzcd}$$

with $* = +, \emptyset$.

We can strengthen the Gabriel-Popescu Theorem as follows.

**Claim**: Suppose that $\mathscr{A}$ has enough projectives. Assume further that $G$ is self-small, that is, for each set $X$, we have a natural bijection

$$\mathrm{Hom}_{\mathscr{A}}(G, G^{(X)}) \simeq \oplus_X \mathrm{Hom}_{\mathscr{A}}(G, G).$$

Then there exists a recollement of abelian categories

$$\begin{tikzcd} \mathrm{Ker(T)} \arrow[r, "i"] & \mathrm{Mod(R)} \arrow[r, "T"] \arrow[l, bend right] \arrow[l, bend left] & \mathscr{A} \arrow[l, bend right, "L_0S"'] \arrow[l, bend left, "S"] \end{tikzcd}$$

and a recollement of triangulated categories

$$\begin{tikzcd} D_{\mathrm{Ker}(T)}(\mathrm{Mod}(R)) \arrow[r, "i"] & D(\mathrm{Mod}(R)) \arrow[r, "T"] \arrow[l, bend right] \arrow[l, bend left] & D(\mathscr{A}) \arrow[l, bend right, "\mathbf{L}S"'] \arrow[l, bend left, "\mathbf{R}S"] \end{tikzcd} .$$

The proof of this claim follows easily from [17, Theorem 10] and Theorem 5.5, which is left to the reader. It would be very interesting to see when $i^* : D^*(\mathrm{Ker}(T)) \to D^*_{\mathrm{Ker(T)}}(\mathrm{Mod}(R))$ is an equivalence.

**Example 7.8.** Let $K$ be a field, $R = \prod_{i=1} K_i$ and $I = \oplus_{i=1}^{\infty} K_i$ with each $K_i = K$. Then $R$ is a commutative ring and I is an idempotent ideal of $R$. We set

$$\mathcal{G} := \{M \in \mathrm{Mod}(R) \mid MI = 0\} \simeq \mathrm{Mod}(R/I)$$

and

$$\mathcal{T} := \{M \in \mathrm{Mod}(R) \mid MI = M\}.$$

By [8, Example 4.3], we get the following recollement of abelian categories

$$\begin{tikzcd} \mathcal{G} \arrow[r, "j_{-1}"] & \mathrm{Mod(R)} \arrow[r, "i_{-1}"] \arrow[l, bend right, "j_0"'] \arrow[l, bend left, "j_{-2}"] & \mathcal{T} \arrow[l, bend right, "i_0"'] \arrow[l, bend left, "i_{-2}"] \end{tikzcd},$$

where $i_0$ is the fully faithful inclusion, $j_0 = - \otimes_R R/I$, $j_{-1} = \mathrm{Hom}_{R/I}(R/I, -) \cong R/I \otimes_{R/I} -$, $j_{-2} = \mathrm{Hom}_R(R/I, -)$, respectively.

Notice that for each $* \in \{+, -, b, \emptyset\}$, the functor $j^*_{-1} : D^*(\mathcal{G}) \to D^*(\mathrm{Mod}R)$ is fully faithful, and that $j_{-1} : \mathcal{G} \to \mathrm{Mod(R)}$ satisfies the the conditions ($\bigstar_1$) and ($\bigstar_2$).

Hence, by Propositions 6.1, 6.3, 6.4 and 6.5, we have triangle equivalences

$$j^*_{-1} : D^*(\mathcal{G}) \simeq D^*_{\mathcal{G}}(\mathrm{Mod}R)$$

for $* \in \{+, -, b, \emptyset\}$.

By Theorem 5.1, we obtain an exact sequence of derived categories

$$D^b(\mathcal{G}) \xrightarrow{j^b_{-1}} D^b(\mathrm{Mod}(R)) \xrightarrow{i^b_{-1}} D^b(\mathcal{T}).$$

By Theorems 5.3, 5.4 and 5.5, we have the following right recollement, left recollement and recollement of derived categories:

$$\begin{tikzcd}
D^+(\mathcal{G}) \arrow[r, "j^+_{-1}"] & D^+(\mathrm{Mod}(R)) \arrow[l, bend left, "\mathbf{R}^+ j_{-2}"] \arrow[r, "i^+_{-1}"] & D^+(\mathcal{T}), \arrow[l, bend left, "\mathbf{R}^+ i_{-2}"]
\end{tikzcd}$$

$$\begin{tikzcd}
D^-(\mathcal{G}) \arrow[r, "j^-_{-1}"'] & D^-(\mathrm{Mod}(R)) \arrow[l, bend right, "\mathbf{L}^- j_0"'] \arrow[r, "i^-_{-1}"'] & D^-(\mathcal{T}) \arrow[l, bend right, "\mathbf{L}^- i_0"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
D(\mathcal{G}) \arrow[r, "j_{-1}"] & D(\mathrm{Mod}(R)) \arrow[l, bend right, "\mathbf{L} j_0"'] \arrow[l, bend left, "\mathbf{R} j_{-2}"] \arrow[r, "i_{-1}"] & D(\mathcal{T}). \arrow[l, bend right, "\mathbf{L} i_0"'] \arrow[l, bend left, "\mathbf{R} i_{-2}"]
\end{tikzcd}$$

**Example 7.9.** Let $R$ and $S$ be rings, $M$ an $S$-$R$-bimodule, and $\Lambda = \begin{pmatrix} R & 0 \\ M & S \end{pmatrix}$ the triangular matrix ring. A right $\Lambda$-module is identified with a triple $(X, Y)_f$, where $X$ is a right $R$-module, $Y$ a right $S$-module, and $f : Y \otimes_S M \to X$ a right $R$-map. In case of no confusion, we write $(X, Y)$ instead of $(X, Y)_f$. A left $\Lambda$-module is identified with a triple $\begin{pmatrix} U \\ V \end{pmatrix}_g$, where $U$ is a left $R$-module, $V$ a left $S$-module, and $g : M \otimes_R U \to V$ a left $S$-map. In case of no confusion, we write $\begin{pmatrix} U \\ V \end{pmatrix}$ instead of $\begin{pmatrix} U \\ V \end{pmatrix}_g$.

Let $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. By [8, Section 4.5], there is a ladder of abelian categories

$$\begin{tikzcd}
\mathrm{Mod}(R) \arrow[r, "i_0"] \arrow[r, bend right=50, "i_{-2}"'] & \mathrm{Mod}(\Lambda) \arrow[l, bend right, "i_1"'] \arrow[l, bend left, "i_{-1}"] \arrow[r, "j_0"] \arrow[r, bend right=50, "j_{-2}"'] & \mathrm{Mod}(S), \arrow[l, bend right, "j_1"'] \arrow[l, bend left, "j_{-1}"]
\end{tikzcd}$$

where

$$i_1 = - \otimes_\Lambda \begin{pmatrix} R \\ 0 \end{pmatrix}, i_0 = - \otimes_R R,\ i_{-1} = - \otimes_\Lambda \Lambda e_1,\ i_{-2} = \mathrm{Hom}_R\Big(\begin{pmatrix} R \\ M \end{pmatrix}, -\Big)$$

$$j_1 = - \otimes_S (M, S), j_0 = - \otimes_\Lambda \Lambda e_2,\ j_{-1} = - \otimes_S S,\ j_{-2} = \mathrm{Hom}_\Lambda(S, -).$$

For each $* \in \{+, -, b, \emptyset\}$, the functors $i_0^* : D^*(\mathrm{Mod}(R)) \to D^*(\mathrm{Mod}(\Lambda))$ and $j_{-1}^* : D^*(\mathrm{Mod}(S)) \to D^*(\mathrm{Mod}(\Lambda))$ are fully faithful.

Notice that $i_0 : \mathrm{Mod}(R) \to \mathrm{Mod}(\Lambda)$ and $j_{-1} : \mathrm{Mod}(S) \to \mathrm{Mod}(\Lambda)$ satisfy the conditions ($\bigstar_1$) and ($\bigstar_2$). Thus, by Propositions 6.1, 6.3, 6.4 and 6.5, we have triangle equivalences

$$i_0^* : D^*(\mathrm{Mod}(R)) \simeq D^*_{\mathrm{Mod}(R)}(\mathrm{Mod}(\Lambda)) \quad \text{and} \quad j_{-1}^* : D^*(\mathrm{Mod}(S)) \simeq D^*_{\mathrm{Mod}(S)}(\mathrm{Mod}(\Lambda))$$

for $* \in \{+, -, b, \emptyset\}$.

By Theorem 5.1, we obtain an exact sequence of derived categories

$$D^b(\mathrm{Mod}(R)) \xrightarrow{i_0^b} \mathrm{D^b}(\mathrm{Mod}(\Lambda)) \xrightarrow{j_0^b} D^b(\mathrm{Mod}(S)).$$

By Theorem 5.5, we have the following ladder of derived categories

$$\begin{tikzcd}
D(\mathrm{Mod}(R)) \arrow[r, "i_0"] \arrow[r, bend right=50, "\mathbf{R}i_{-2}"'] & D(\mathrm{Mod}(\Lambda)) \arrow[l, bend right, "\mathbf{L}i_1"'] \arrow[l, bend left, "i_{-1}"] \arrow[r, "j_0"] \arrow[r, bend right=50, "\mathbf{R}j_{-2}"'] & D(\mathrm{Mod}(S)). \arrow[l, bend right, "\mathbf{L}j_1"'] \arrow[l, bend left, "j_{-1}"]
\end{tikzcd}$$

Similarly, we obtain the following recollements of derived categories:

$$\begin{tikzcd}
D^+(\mathrm{Mod}(S)) \arrow[r, "j_{-1}^+"] & D^+(\mathrm{Mod}(\Lambda)) \arrow[l, bend right, "j_0^+"'] \arrow[l, bend left, "\mathbf{R}^+ j_{-2}"] \arrow[r, "i_{-1}^+"] & D^+(\mathrm{Mod}(R)), \arrow[l, bend right, "i_0^+"'] \arrow[l, bend left, "\mathbf{R}^+ i_{-2}"]
\end{tikzcd}$$

$$\begin{tikzcd}
D^-(\mathrm{Mod}(R)) \arrow[r, "i_0^-"] & \mathrm{D}^-(\mathrm{Mod}(\Lambda)) \arrow[l, bend right, "\mathbf{L}^- i_1"'] \arrow[l, bend left, "i_{-1}^-"] \arrow[r, "j_0^-"] & D^-(\mathrm{Mod}(S)). \arrow[l, bend right, "\mathbf{L}^- j_1"'] \arrow[l, bend left, "j_{-1}^-"]
\end{tikzcd}$$

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## Conflict of interest

The authors declare there is no conflicts of interest.

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