Electronic Research Archive

http://www.aimspress.com/journal/era

ERA, 30(11): 4038–4065.
DOI: 10.3934/era.2022205
Received: 22 June 2022
Revised: 29 August 2022
Accepted: 31 August 2022
Published: 13 September 2022

Research article

Existence and stability results of a plate equation with nonlinear damping and source term

Mohammad M. Al-Gharabli[1,2] **and Adel M. Al-Mahdi**[1,2,*]

[1] The Preparatory Year Program, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia

[2] The Interdisciplinary Research Center in Construction and Building Materials, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia

* **Correspondence:** Email: almahdi@kfupm.edu.sa.

Abstract: The main goal of this work is to investigate the following nonlinear plate equation

$$u_{tt} + \Delta^2 u + \alpha(t)g(u_t) = u|u|^\beta,$$

which models suspension bridges. Firstly, we prove the local existence using Faedo-Galerkin method and Banach fixed point theorem. Secondly, we prove the global existence by using the well-depth method. Finally, we establish explicit and general decay results for the energy of solutions of the problem. Our decay results depend on the functions α and g and obtained without any restriction growth assumption on g at the origin. The multiplier method, properties of the convex functions, Jensen's inequality and the generalized Young inequality are used to establish the stability results.

Keywords: plate equation; Galerkin method; Banach fixed point theorem; general decay; nonlinear frictional damping

1. Introduction

The importance of bridges is undeniable and their presence in human daily life goes back a long time. With the presence of the bridges, road and railway traffic runs without any interruption over rivers and hazardous areas, time and fuel are saved, congestion on roads is minimized, distances between places are reduced, and many accidents have been avoided, as the bridges have reduced the number of bends and zig-zags in roads. As a result, many economies have grown and many societies have become connected. However, bridges have brought some challenges, such as collapse and instability due to natural hazards such as wind, earthquakes, etc. To overcome these difficulties, engineers and scientists have made efforts to find the best designs and possible models. Our aim in this work is to investigate the following plate problem

$$\begin{cases} u_{tt} + \Delta^2 u + \alpha(t) g(u_t) = u|u|^{\beta}, \text{ in } \Omega \times (0,T), \\ u(0,y,t) = u_{xx}(0,y,t) = u(\pi,y,t) = u_{xx}(\pi,y,t) = 0, \ (y,t) \in (-d,d) \times (0,T), \\ u_{yy}(x,\pm d,t) + \sigma u_{xx}(x,\pm d,t) = 0, \ (x,t) \in (0,\pi) \times (0,T), \\ u_{yyy}(x,\pm d,t) + (2-\sigma) u_{xxy}(x,\pm d,t) = 0, \ (x,t) \in (0,\pi) \times (0,T), \\ u(x,y,0) = u_0(x,y), \ u_t(x,y,0) = u_1(x,y), \ \text{ in } \Omega \times (0,T), \end{cases} \tag{1.1}$$

where $\Omega = (0,\pi) \times (-d,d)$, $d,\beta > 0$, $g : \mathbb{R} \to \mathbb{R}$ and $\alpha : [0,+\infty) \to (0,+\infty)$ is a nonincreasing differentiable function, u is the vertical displacement of the bridge and σ is the Poisson ratio. This is a weakly damped nonlinear suspension-bridge problem, in which the damping is modulated by a time dependent-coefficient $\alpha(t)$. Firstly, we prove the local existence using the Faedo-Gherkin method and Banach fixed point theorem. Secondly, we prove the global existence by using the well-depth method. Finally, we establish an explicit and general decay result, depending on g and α, for which the exponential and polynomial decay rate estimates are only special cases. The proof is based on the multiplier method and makes use of some properties of convex functions, including the use of the general Young inequality and Jensen's inequality.

2. Literature review

The famous report by Claude-Louis Navier [1] was the only mathematical treatise of suspension bridges for several decades. Another milestone theoretical contribution was the monograph by Melan [2]. After the Tacoma collapse, engineers felt the necessary to introduce the time variable in mathematical models and equations in order to attempt explanations of what had occurred. As a matter of fact, in Appendix VI of the Federal Report [3], a model of inextensible cables is derived and the linearized Melan equation was obtained. Other important contributions were the works by Smith-Vincent and the analysis of vibrations in suspension bridges presented by Bleich-McCullough-Rosecrans-Vincent [4]. In all these historical references, the bridge was modelled linearly as a beam suspended to a cable. Hence, all the equations were linear. Mathematicians have not shown any interest in suspension bridges until recently. McKenna, in 1987, introduced the first nonlinear models to study them from a theoretical point of view, and he was followed by several other mathematicians (see [5,6]). McKenna's main idea was to consider the slackening of the hangers as a nonlinear phenomenon, a statement which is by now well-known also among engineers [7,8]. The slackening phenomenon was analyzed in various complex beam models by several authors (see [9–11]). Motivated by the wonderful book of Rocard [12], where it was pointed out that the correct way to model a suspension bridge is through a thin plate, Ferrero-Gazzola [13] introduced the following hyperbolic problem:

$$\begin{cases} u_{tt}(x,y,t) + \eta u_t + \Delta^2 u(x,y,t) + h(x,y,u) = f, \text{ in } \Omega \times \mathbb{R}^+, \\ u(0,y,t) = u_{xx}(0,y,t) = u(\pi,y,t) = u_{xx}(\pi,y,t) = 0, \ (y,t) \in (-\ell,\ell) \times \mathbb{R}^+, \\ u_{yy}(x,\pm\ell,t) + \sigma u_{xx}(x,\pm\ell,t) = 0, \ (x,t) \in (0,\pi) \times \mathbb{R}^+, \\ u_{yyy}(x,\pm\ell,t) + (2-\sigma) u_{xxy}(x,\pm\ell,t) = 0, \ (x,t) \in (0,\pi) \times \mathbb{R}^+, \\ u(x,y,0) = u_0(x,y), \ u_t(x,y,0) = u_1(x,y), \ \text{ in } \Omega \times \mathbb{R}^+, \end{cases} \tag{2.1}$$

where $\Omega = (0,\pi) \times (-\ell,\ell)$ is a planar rectangular plate, σ is the well-known Poisson ratio, η is the damping coefficient, h is the nonlinear restoring force of the hangers and f is an external force. After the appearance of the above model, many mathematicians showed interest in investigating variants of it, using different kinds of damping with the aim to obtain stability of the bridge modeled through the above problem. Messaoudi [14] considered the following nonlinear Petrovsky equation

$$u_{tt} + \Delta^2 u + a u_t |u_t|^{m-2} = b u |u|^{p-2}, \tag{2.2}$$

and proved the existence of a local weak solution, showed that this solution is global if $m \geq p$ and blows up in finite time if $p > m$ and the energy is negative. Wang [15] considered the equation

$$u_{tt} + \delta u_t + \Delta^2 u + au = u|u|^{p-2}, \tag{2.3}$$

where $a = a(x, y, t)$ together with the above initial and boundary conditions. After showing the uniqueness and existence of local solutions, he gave sufficient conditions for global existence and finite-time blow-up of solutions. Mukiawa [16] considered a plate equation modeling a suspension bridge with weak damping and hanger restoring force. He proved the well-posedness and established an explicit and general decay result without putting restrictive growth conditions on the frictional damping term. Messaoudi and Mukiawa [17] studied problem (2.3), where the linear frictional damping was replaced by nonlinear frictional damping and established the existence of a global weak solution and proved exponential and polynomial stability results. Audu et al. [18] considered a plate equation as a model for a suspension bridge with a general nonlinear internal feedback and time-varying weight. Under some conditions on the feedback and the coefficient functions, the authors established a general decay estimate. For more results related to the existence of work on similar problems, we mention the work of Xu et al. [19], in which they proved the local existence of a weak solution by the Galerkin method and the global existence by the potential well method. He et al. [20] considered the following Kirchhoff type equation

$$-\left(a + b\int_{\Omega} |\nabla u|^2 dx\right)\Delta u = f(u) + h, \quad \text{in } \Omega, \tag{2.4}$$

where $\Omega \subseteq \mathbb{R}^3$ is a bounded domain or $\Omega = \mathbb{R}^3$, $0 \leq h \in L^2(\Omega)$ and $f \in C(\mathbb{R}, \mathbb{R})$. The authors proved the existence of at least one or two positive solutions by using the monotonicity trick, and nonexistence criterion is also established by virtue of the corresponding Pohoaev identity. Recently, Wang et al. [21] considered the fractional Rayleigh-Stokes problem where the nonlinearity term satisfied certain critical conditions and proved the local existence, uniqueness and continuous dependence upon the initial data of ε-regular mild solutions. More results in this direction can be found in [22–27]. The paper is organized as follows. In Section 3, we present some preliminaries and essential lemmas. We prove the local existence in Section 4 and the global existence in Section 5. The statement and the proof of our stability result will be given in Section 6.

3. Preliminaries and essential lemmas

In this section, we present some material needed in the proofs of our results. First, we introduce the following space

$$H_*^2(\Omega) = \{w \in H^2(\Omega) : w = 0 \ on \ \{0, \pi\} \times (-d, d)\}, \tag{3.1}$$

together with the inner product

$$(u,v)_{H_*^2} = \int\limits_\Omega \Big(\Delta u \Delta v + (1-\sigma)(2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx})\Big)\,dx. \tag{3.2}$$

It is well known that $(H_*^2(\Omega), (\cdot,\cdot)_{H_*^2})$ is a Hilbert space, and the norm $\|.\|_{H_*^2}^2$ is equivalent to the usual H^2, see [13]. Throughout this paper, c is used to denote a generic positive constant.

Lemma 3.1. *[15] Let $u \in H_*^2(\Omega)$ and assume that $1 \le p < \infty$, then, there exists a positive constant $C_e = C_e(\Omega, p) > 0$ such that*

$$\|u\|_p \le C_e \|u\|_{H_*^2(\Omega)}.$$

Lemma 3.2. *(Jensen's inequality) Let $\psi : [a,b] \longrightarrow \mathbb{R}$ be a convex function. Assume that the functions $f : (0,L) \longrightarrow [a,b]$ and $r : (0,L) \longrightarrow \mathbb{R}$ are integrable such that $r(x) \ge 0$, for any $x \in (0,L)$ and $\int_0^L r(x)dx = k > 0$. Then,*

$$\psi\left(\frac{1}{k}\int_0^L f(x)r(x)dx\right) \le \frac{1}{k}\int_0^L \psi(f(x))r(x)dx. \tag{3.3}$$

We consider the following hypotheses:

(H1). The function $g : \mathbb{R} \to \mathbb{R}$ is nondecreasing C^0 function satisfying for $\varepsilon, c_1, c_2 > 0$,

$$\begin{aligned} &c_1|s| \le |g(s)| \le c_2|s|, \ \text{if } |s| \ge \varepsilon, \\ &|s|^2 + g^2(s) \le G^{-1}(sg(s)), \ \text{if } |s| \le \varepsilon, \end{aligned} \tag{3.4}$$

where $G : \mathbb{R}^+ \to \mathbb{R}^+$ is a C^1 function which is linear or strictly increasing and strictly convex C^2 function on $[0,\varepsilon]$ with $G(0) = 0$ and $G'(0) = 0$. In addition, the function g satisfies, for $\vartheta > 0$,

$$(g(s_1) - g(s_2))(s_1 - s_2) \ge \vartheta|s_1 - s_2|^2. \tag{3.5}$$

(H2). The function $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ is a nonincreasing differentiable function such that $\int_0^\infty \alpha(t)dt = \infty$.

Remark 3.3. *Hypothesis (H1) implies that $sg(s) > 0$, for all $s \ne 0$ and it was introduced and employed by Lasiecka and Tataru [28]. It was shown there that the monotonicity and continuity of g guarantee the existence of the function G with the properties stated in (H1).*

Remark 3.4. *As in [28], we use Condition* (3.5) *to prove the uniqueness of the solution.*

The following lemmas will be of essential use in establishing our main results.

Lemma 3.5. *[29] Let $E : \mathbb{R}^+ \to \mathbb{R}^+$ be a nonincreasing function and $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ be a strictly increasing C^1-function, with $\gamma(t) \to +\infty$ as $t \to +\infty$. Assume that there exists $c > 0$ such that*

$$\int\limits_S^\infty \gamma'(t)E(t)dt \le cE(S) \qquad 1 \le S < +\infty.$$

Then there exist positive constants k and ω such that

$$E(t) \le ke^{-\omega\gamma(t)}.$$

Lemma 3.6. *[30] Let $E : \mathbb{R}^+ \to \mathbb{R}^+$ be a differentiable and nonincreasing function and $\chi : \mathbb{R}^+ \to \mathbb{R}^+$ be a convex and increasing function such that $\chi(0) = 0$. Assume that*

$$\int_s^{+\infty} \chi(E(t))\, dt \leq E(s), \quad \forall s \geq 0. \tag{3.6}$$

Then, E satisfies the following estimate

$$E(t) \leq \psi^{-1}\left(h(t) + \psi(E(0))\right), \quad \forall t \geq 0, \tag{3.7}$$

where $\psi(t) = \int_t^1 \frac{1}{\chi(s)} ds$, and

$$\begin{cases} h(t) = 0, \quad 0 \leq t \leq \frac{E(0)}{\chi(E(0))}, \\ h^{-1}(t) = t + \frac{\psi^{-1}(t+\psi(E(0)))}{\chi\left(\psi^{-1}(t+\psi(E(0)))\right)}, \quad t > 0. \end{cases}$$

4. Local existence

In this section, we state and prove the local existence of weak solutions of problem (1.1). Similar results can be found in [31, 32]. To this end, we consider the following problem

$$\begin{cases} u_{tt}(x,y,t) + \Delta^2 u(x,y,t) + \alpha(t) g(u_t) = f(x,t), \text{ in } \Omega \times (0,T), \\ u(0,y,t) = u_{xx}(0,y,t) = u(\pi,y,t) = u_{xx}(\pi,y,t) = 0, \ (y,t) \in (-d,d) \times (0,T), \\ u_{yy}(x,\pm d,t) + \sigma u_{xx}(x,\pm d,t) = 0, \ (x,t) \in (0,\pi) \times (0,T), \\ u_{yyy}(x,\pm d,t) + (2-\sigma) u_{xxy}(x,\pm d,t) = 0, \ (x,t) \in (0,\pi) \times (0,T), \\ u(x,y,0) = u_0(x,y), \ u_t(x,y,0) = u_1(x,y), \text{ in } \Omega \times (0,T), \end{cases} \tag{4.1}$$

where $f \in L^2(\Omega \times (0,T))$ and $(u_0, u_1) \in H^2_*(\Omega) \times L^2(\Omega)$. Then, we prove the following theorem:

Theorem 4.1. *Let $(u_0, u_1) \in H^2_*(\Omega) \times L^2(\Omega)$. Assume that (H1) and (H2) hold. Then, problem* (4.1) *has a unique local weak solution*

$$u \in L^\infty([0,T), H^2_*(\Omega)), \quad u_t \in L^\infty([0,T), L^2(\Omega)), \quad u_{tt} \in L^\infty([0,T), \mathcal{H}(\Omega)),$$

where $\mathcal{H}(\Omega)$ is the dual space of $H^2_*(\Omega)$.

Proof. Uniqueness: Suppose that (4.1) has two weak solutions (u, v). Then, $w = u - v$ satisfies

$$\begin{cases} w_{tt}(x,y,t) + \Delta^2 w(x,y,t) + \alpha(t) g(u_t) - \alpha(t) g(v_t) = 0, \text{ in } \Omega \times (0,T), \\ w(0,y,t) = w_{xx}(0,y,t) = w(\pi,y,t) = w_{xx}(\pi,y,t) = 0, \ (y,t) \in (-d,d) \times (0,T), \\ w_{yy}(x,\pm d,t) + \sigma w_{xx}(x,\pm d,t) = 0, \ (x,t) \in (0,\pi) \times (0,T), \\ w_{yyy}(x,\pm d,t) + (2-\sigma) w_{xxy}(x,\pm d,t) = 0, \ (x,t) \in (0,\pi) \times (0,T), \\ w(x,y,0) = w_t(x,y,0) = 0, \text{ in } \Omega \times (0,T). \end{cases} \tag{4.2}$$

Multiplying (4.2) by w_t and integrating over $(0,t)$, we get

$$\frac{1}{2}\frac{d}{dt}\left[\int_\Omega \left(w_t^2 + |\Delta w|^2\right) dx\right] + \alpha(t) \int_\Omega \left(g(u_t) - g(v_t)\right)(u_t - v_t) dx = 0. \tag{4.3}$$

Integrating (4.3) over $(0, t)$, we obtain

$$\int_{\Omega}\left(w_t^2+|\Delta w|^2\right)dx+2\alpha(t)\int_0^t\int_{\Omega}\left(g(u_t)-g(v_t)\right)(u_t-v_t)dxds=0. \tag{4.4}$$

Using Condition (3.5) and (H2), for $a.e.\ x\in\Omega$, we have

$$\int_{\Omega}\left(w_t^2+|\Delta w|^2dx\right)=0, \tag{4.5}$$

We conclude $u=v=0$ on $\Omega\times(0,T)$, which proves the uniqueness of the solution of problem (4.1). Existence: To prove the existence of the solution for problem (4.1), we use the Faedo-Galerkin method as follows: First, we consider $\left\{v_j\right\}_{j=1}^{\infty}$ an orthonormal basis of $H_*^2(\Omega)$ and define, for all $k\geq 1$, a sequence v^k in $\mathcal{V}_k=span\{v_1,v_2,...,v_k\}\subset H_*^2(\Omega)$, given by

$$u^k(x,t)=\Sigma_{j=1}^k a_j(t)v_j(x),$$

for all $x\in\Omega$ and $t\in(0,T)$ and satisfies the following approximate problem

$$\begin{cases}\int_{\Omega}u_{tt}^k(x,t)v_jdx+\int_{\Omega}\Delta u^k(x,t)\Delta v_jdx+\alpha(t)\int_{\Omega}g(u_t^k)v_j=\int_{\Omega}f(x,t)v_jdx, \text{in } \Omega\times(0,T),\\ u^k(x,y,0)=u_0^k(x,y),\ u_t^k(x,y,0)=u_1^k(x,y),\ \text{ in } \ \Omega\times(0,T),\end{cases} \tag{4.6}$$

for all $j=1,2,...,k$,

$$u^k(0)=u_0^k=\Sigma_{i=1}^k\langle u_0,v_i\rangle v_i,\ u_t^k(0)=u_1^k=\Sigma_{i=1}^k\langle u_1,v_i\rangle v_i, \tag{4.7}$$

such that

$$\begin{aligned}u_0^k&\longrightarrow u_0\in H_*^2(\Omega),\\ u_1^k&\longrightarrow u_1\in L^2(\Omega).\end{aligned} \tag{4.8}$$

For any $k\geq 1$, problem (4.6) generates a system of k nonlinear ordinary differential equations. The ODE's standard existence theory assures the existence of a unique local solution u^k for problem (4.6) on $[0,T_k)$, with $0<T_k\leq T$. Next, we have to show, by a *priori* estimates, that $T_k=T,\forall k\geq 1$. Now, multiplying (4.6) by $a_j'(t)$, using Green's formula and the boundary conditions, and then summing each result over j we obtain, for all $0<t\leq T_k$,

$$\frac{1}{2}\frac{d}{dt}\left[\int_{\Omega}\left(|u_t^k|^2+(\Delta u^k)^2\right)dx\right]+\alpha(t)\int_{\Omega}u_t^kg\left(u_t^k\right)dx=\int_{\Omega}f(x,t)u_t^k(x,t)dx. \tag{4.9}$$

Then, integrating (4.9) over $(0,t)$ leads to

$$\begin{aligned}&\frac{1}{2}\int_{\Omega}\left(|u_t^k|^2+|\Delta u^k|^2\right)dx+\int_0^t\int_{\Omega}\alpha(s)u_t^kg\left(u_t^k\right)dxds\\ &=\frac{1}{2}\int_{\Omega}\left(|u_1^k|^2+|\Delta u_0^k|^2\right)dx+\int_0^t\int_{\Omega}f(x,t)u_t^k(x,t)dxds.\end{aligned} \tag{4.10}$$

From the convergence (4.8), using the fact that $f \in L^2(\Omega \times (0,T))$, and exploiting Young's inequality, then (4.10) becomes, for some $C > 0$, and for any $t \in [0, t_k)$

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \left[|u_t^k|^2 + |\Delta u^k|^2 dx\right] + \int_0^t \int_\Omega \alpha(s) u_t^k g\left(u_t^k\right) dxds \\
&\le \frac{1}{2}\int_\Omega \left[|u_1^k|^2 + |\Delta u_0^k|^2\right] dx + \varepsilon \int_0^t \int_\Omega |u_t^k|^2 dxds + C_\varepsilon \int_0^t \int_\Omega |f(x,s)|^2 dxds \\
&\le C_\varepsilon + \varepsilon \sup_{(0,T_k)} \int_\Omega |u_t^k|^2 dx.
\end{aligned} \tag{4.11}$$

Therefore, we obtain

$$\begin{aligned}
&\frac{1}{2}\sup_{(0,T_k)} \int_\Omega |u_t^k|^2 dx + \frac{1}{2}\sup_{(0,T_k)} \int_\Omega |\Delta u^k|^2 dx + \frac{1}{2}\sup_{(0,T_k)} \int_0^{t_k} \int_\Omega \alpha(s) u_t^k(x,s) g\left(u_t^k(x,s)\right) dxds \\
&\le C_\varepsilon + \varepsilon \sup_{(0,T_k)} \int_\Omega |u_t^k|^2 dx.
\end{aligned} \tag{4.12}$$

Choosing $\varepsilon = \frac{1}{4}$, estimate (4.12) yields, for all $T_k \le T$ and $C > 0$,

$$\sup_{(0,T_k)} \int_\Omega |u_t^k|^2 dx + \sup_{(0,T_k)} \int_\Omega |\Delta u^k|^2 dx + \sup_{(0,T_k)} \int_0^{t_k} \int_\Omega \alpha(s) u_t^k(x,s) g\left(u_t^k(x,s)\right) dxds \le C. \tag{4.13}$$

Consequently, the solution u^k can be extended to $(0,T)$, for any $k \ge 1$. In addition, we have

$$(u^k) \text{ is bounded in } L^\infty((0,T), H_*^2(\Omega)) \text{ and } (u_t^k) \text{ is bounded in } L^\infty((0,T), L^2(\Omega)).$$

Therefore, we can extract a subsequence, denoted by (u^ℓ) such that, when $\ell \to \infty$, we have

$$u^\ell \to u \ \text{ weakly * in } L^\infty((0,T), H_*^2(\Omega)) \text{ and } u_t^\ell \to u_t \text{ weakly * in } L^\infty((0,T), L^2(\Omega)).$$

Next, we prove that $g(u_t^\ell)$ is bounded in $L^2\left((0,T); L^2(\Omega)\right)$. For this purpose, we consider two cases:
Case 1. G is linear on $[0,\varepsilon]$. Then using (H1) and Young's inequality, we get

$$\begin{aligned}
\int_\Omega g^2(u_t^\ell)dx &\le c\int_\Omega u_t^\ell g(u_t^\ell)dx - \int_\Omega |u_t^\ell|^2 dx \\
&\le \frac{c}{4\delta_0}\int_\Omega |u_t^\ell|^2 dx + \delta_0 \int_\Omega g^2(u_t^\ell)dx,
\end{aligned} \tag{4.14}$$

for a suitable choice of δ_0 and using the fact that u_t^ℓ is bounded in $L^2((0,T), L^2(\Omega))$, we obtain

$$\int_0^T \int_\Omega g^2(u_t^\ell)dxdt \le c. \tag{4.15}$$

Case 2. G is nonlinear. Let $0 < \varepsilon_1 \le \varepsilon$ such that

$$sg(s) \le \min\{\varepsilon, G(\varepsilon)\} \text{ for all } |s| \le \varepsilon_1. \tag{4.16}$$

Then, one can show that

$$\begin{cases} s^2 + g^2(s) \le G^{-1}(sg(s)) & \text{for all } |s| \le \varepsilon_1 \\ c_1'|s| \le |g(s)| \le c_2'|s| & \text{for all } |s| \ge \varepsilon_1. \end{cases} \tag{4.17}$$

Define the following sets

$$\Omega_1 = \{x \in \Omega : |u_t^\ell| \le \varepsilon_1\}, \text{ and } \Omega_2 = \{x \in \Omega : |u_t^\ell| > \varepsilon_1\}. \tag{4.18}$$

Then, using (4.17) and (4.18) leads for some $c_2' > 0$,

$$\begin{aligned} \int_\Omega g^2(u_t^\ell)dx &= \int_{\Omega_2} g^2(u_t^\ell)dx + \int_{\Omega_1} g^2(u_t^\ell)dx \\ &\le c_2' \int_{\Omega_2} |u_t^\ell|^2 dx + \int_{\Omega_1} \left(|u^\ell|_t^2 + g^2(u_t^\ell)\right) dx - \int_{\Omega_1} |u_t^\ell|^2 dx \\ &\le c_2' \int_{\Omega_2} |u_t^\ell|^2 dx + \int_{\Omega_1} G^{-1}(u_t^\ell g(u_t^\ell))dx. \end{aligned} \tag{4.19}$$

Let

$$J^\ell(t) := \int_{\Omega_1} u_t^\ell g(u_t^\ell)dx,$$

$$E^\ell(t) = \frac{1}{2}\left(\|u_t^\ell\|_2^2 + \|u^\ell\|_{H_*^2(\Omega)}^2\right) - \frac{1}{\beta+2}\|u^\ell\|_{\beta+2}^{\beta+2}, \tag{4.20}$$

and

$$\left(E^\ell\right)'(t) = -\alpha(t) \int_\Omega u_t^\ell g(u_t^\ell)dx \le 0. \tag{4.21}$$

Using (4.19) and Jensen's inequality, we obtain

$$\begin{aligned} \int_\Omega g^2(u_t^\ell)dx &\le c \int_\Omega |u_t^\ell|^2 dx + G^{-1}(J^\ell(t)) \\ &= c \int_\Omega |u_t^\ell|^2 dx + \frac{G'\left(\varepsilon_0 \frac{E^\ell(t)}{E^\ell(0)}\right)}{G'\left(\varepsilon_0 \frac{E^\ell(t)}{E^\ell(0)}\right)} G^{-1}\left(J^\ell(t)\right). \end{aligned} \tag{4.22}$$

Using the convexity of G (G' is increasing), we obtain for $t \in (0, T)$,

$$G'\left(\varepsilon_0 \frac{E^\ell(t)}{E^\ell(0)}\right) \ge G'\left(\varepsilon_0 \frac{E^\ell(T)}{E^\ell(0)}\right) = c.$$

Let G^* be the convex conjugate of G in the sense of Young (see [33], pp. 61–64), then, for $s \in (0, G'(\varepsilon)]$,

$$G^*(s) = s(G')^{-1}(s) - G[(G')^{-1}(s)] \le s(G')^{-1}(s). \tag{4.23}$$

Using the general Young inequality

$$AB \le G^*(A) + G(B), \quad \text{if } A \in (0, G'(\varepsilon)],\ B \in (0, \varepsilon],$$

for

$$A = G'\left(\varepsilon_0 \frac{E^\ell(t)}{E^\ell(0)}\right) \quad \text{and} \quad B = G^{-1}\left(J^\ell(t)\right),$$

and using the fact that $E^\ell(t) \leq E^\ell(0)$, we get

$$\begin{aligned}
\int_\Omega g^2(u_t^\ell)dx &\leq c\int_\Omega |u_t^\ell|^2 dx + c\varepsilon_0 \frac{E^\ell(t)}{E^\ell(0)} G'\left(\varepsilon_0 \frac{E^\ell(t)}{E^l(0)}\right) - C(E^\ell)'(t) \\
&\leq c\int_\Omega |u_t^\ell|^2 dx + c\varepsilon_0 \frac{E^\ell(t)}{E^\ell(0)} G'\left(\varepsilon_0 \frac{E^\ell(t)}{E^\ell(0)}\right) - C(E^\ell)'(t) \\
&\leq c\int_\Omega |u_t^\ell|^2 dx + c - C(E^\ell)'(t).
\end{aligned} \tag{4.24}$$

Integrating (4.24) over $(0, T)$, we obtain

$$\int_0^T \int_\Omega g^2(u_t^\ell)dxdt \leq c\int_0^T \int_\Omega |u_t^\ell|^2 dxdt + cT - C\left(E^\ell(T) - E^\ell(0)\right). \tag{4.25}$$

Using (4.21) and the fact that u_t^ℓ is bounded in $L^2\left((0,T); L^2(\Omega)\right)$, we conclude that $g(u_t^\ell)$ is bounded in $L^2\left((0,T); L^2(\Omega)\right)$. So, we find, up to a subsequence, that

$$g(u_t^\ell) \rightharpoonup \chi \text{ in } L^2\left((0,T); L^2(\Omega)\right). \tag{4.26}$$

Now, we have to show that $\chi = g(u_t)$. In (4.6), we use u^ℓ instead of u^k and then integrate over $(0, t)$ to get

$$\int_\Omega u_t^\ell v_j dx - \int_\Omega u_1^\ell v_j dx + \int_0^t \int_\Omega \Delta u^\ell \Delta v_j dxds + \int_0^t \int_\Omega \alpha(s) g(u_t^\ell) v_j dxds = \int_0^t \int_\Omega f v_j dxds, \ j < \ell. \tag{4.27}$$

As $\ell \to +\infty$, we easily check that

$$\int_\Omega u_t v_j dx - \int_\Omega u_1 v_j dx + \int_0^t \int_\Omega \Delta u \Delta v_j dxds + \int_0^t \int_\Omega \alpha(s) \chi v_j dxds = \int_0^t \int_\Omega f v_j dxds, \ j \geq 1. \tag{4.28}$$

Hence, for $v \in H_*^2(\Omega)$, we have

$$\int_\Omega u_t v dx - \int_\Omega u_1 v dx + \int_0^t \int_\Omega \Delta u \Delta v dxds + \int_0^t \int_\Omega \alpha(s) \chi v dxds = \int_0^t \int_\Omega f v dxds. \tag{4.29}$$

Since all terms define absolute continuous functions, we get, for *a.e.* $t \in [0, T]$ and for $v \in H_*^2(\Omega)$, the following

$$\frac{d}{dt} \int_\Omega u_t v dx + \int_\Omega \Delta u \Delta v dx + \int_\Omega \alpha(t) \chi v dx = \int_\Omega f v dxds. \tag{4.30}$$

This implies that

$$u_{tt} + \Delta^2 u + \alpha(t)\chi = f, \text{ in } D'\left(\Omega \times (0,T)\right). \tag{4.31}$$

Using (H1), we see that

$$X^\ell := \int_0^T \int_\Omega \alpha(s)\left(u_t^\ell - v\right)\left(g(u_t^\ell) - g(v)\right) dxdt \geq 0, \ v \in L^2\left((0,T); L^2(\Omega)\right). \tag{4.32}$$

So, by using (4.6) and replacing u^k by u^ℓ, we get

$$\begin{aligned} X^\ell = & \int_0^T \int_\Omega f u_t^\ell dxdt + \frac{1}{2}\int_\Omega \left(|u_t^\ell|^2 + |\Delta u^\ell|^2\right) dx - \frac{1}{2}\int_\Omega |u_t^\ell(x,T)|^2 dx \\ & - \frac{1}{2}\int_\Omega |\Delta u_t^\ell(x,T)|^2 dx - \int_0^T \int_\Omega \alpha(t) g(u_t^\ell) v dxdt - \int_0^T \int_\Omega \alpha(t) g(v)\left(u_t^\ell - v\right) dxdt. \end{aligned} \tag{4.33}$$

Taking $\ell \to +\infty$, we obtain

$$\begin{aligned} 0 \le \limsup_\ell X^\ell \le & \int_0^T \int_\Omega f u_t dxdt + \frac{1}{2}\int_\Omega \left(|u_1|^2 + |\Delta u_0|^2\right) dx \\ & - \frac{1}{2}\int_\Omega |u_t(x,T)|^2 dx - \frac{1}{2}\int_\Omega |\Delta u_t(x,T)|^2 dx - \int_0^T \int_\Omega \alpha(t)\chi v dxdt \\ & - \int_0^T \int_\Omega \alpha(t) g(v)\left(u_t - v\right) dxdt. \end{aligned} \tag{4.34}$$

Replacing v by u_t in (4.30) and integrating over $(0,T)$, we obtain

$$\begin{aligned} \int_0^T \int_\Omega f u_t dxdt = & \frac{1}{2}\int_\Omega \left(|u_t(x,T)|^2 dx + |\Delta u(x,T)|^2\right) dx - \frac{1}{2}\int_\Omega |u_1|^2 dx \\ & - \frac{1}{2}\int_\Omega |\Delta u_0|^2 dx + \int_0^T \int_\Omega \alpha(t)\chi u_t dxdt. \end{aligned} \tag{4.35}$$

Adding of (4.34) and (4.35), we get

$$0 \le \limsup_\ell X^\ell \le \int_0^T \int_\Omega \alpha(t)\chi u_t dxdt - \int_0^T \int_\Omega \alpha(t)\chi v dxdt - \int_0^T \int_\Omega \alpha(t) g(v)(u_t - v) dxdt. \tag{4.36}$$

This gives

$$\int_0^T \int_\Omega \alpha(t)\left(\chi - g(v)\right)(u_t - v) dxdt \ge 0,\ v \in L^2\left((0,T), L^2(\Omega)\right). \tag{4.37}$$

Hence,

$$\int_0^T \int_\Omega \alpha(t)\left(\chi - g(v)\right)(u_t - v) dxdt \ge 0,\ v \in L^2\left(\Omega \times (0,T)\right). \tag{4.38}$$

Let $v = \lambda w + u_t$, where $\lambda > 0$ and $w \in L^2\left(\Omega \times (0,T)\right)$. Then, we get

$$-\lambda \int_0^T \int_\Omega \alpha(t)\left(\chi - g(\lambda w + u_t)\right) w dxdt \ge 0,\ w \in L^2\left(\Omega \times (0,T)\right). \tag{4.39}$$

For $\lambda > 0$, we have

$$\lambda \int_0^T \int_\Omega \alpha(t)\left(\chi - g(\lambda w + u_t)\right) w dxdt \le 0,\ w \in L^2\left(\Omega \times (0,T)\right). \tag{4.40}$$

As $\lambda \to 0$ and using the continuity of g with respect of λ, we get

$$\lambda \int_0^T \int_\Omega \alpha(t)\left(\chi - g(u_t)\right) w dxdt \le 0,\ w \in L^2\left(\Omega \times (0,T)\right). \tag{4.41}$$

Similarly, for $\lambda < 0$, we get

$$\lambda \int_0^T \int_\Omega \alpha(t)\left(\chi - g(u_t)\right) w dt \geq 0,\ w \in L^2\left(\Omega \times (0,T)\right). \tag{4.42}$$

This implies that $\chi = g(u_t)$. Hence, (4.30) becomes

$$\int_\Omega \left(u_{tt}v + \Delta u \Delta v + \alpha(t) g(u_t) v\right) dx = \int_\Omega f v dx,\ v \in L^2\left((0,T); H_*^2(\Omega)\right). \tag{4.43}$$

which gives

$$u_{tt} + \Delta^2 u + \alpha(t) g(u_t) = f, \text{ in } D'\left(\Omega \times (0,T)\right). \tag{4.44}$$

To handle the initial conditions of problem (4.1), we first note that

$$\begin{aligned} u^\ell &\rightharpoonup u \text{ weakly } *\text{in } L^\infty(0,T; H_*^2(\Omega)) \\ u_t^\ell &\rightharpoonup u_t \text{ weakly } * \text{ in } L^\infty(0,T; L^2(\Omega)). \end{aligned} \tag{4.45}$$

Thus, using Lion's Lemma and (4.6), we easily obtain $u^\ell \to u \in C\left([0,T]; L^2(\Omega)\right)$. Therefore, $u^\ell(x,0)$ makes sense and $u^\ell(x,0) \to u(x,0) \in L^2(\Omega)$. Also, we see that

$$u^\ell(x,0) = u_0^\ell \to u_0(x) \in H_*^2(\Omega).$$

Hence, $u(x,0) = u_0(x)$. As in [34], let $\phi \in C_0^\infty(0,T)$, and replacing u^k by u^ℓ, we obtain from (4.6) and for any $j \leq \ell$

$$\begin{cases} -\int_0^T \int_\Omega u_t^\ell(x,t) v_j(x) \phi'(t) dx dt = -\int_0^T \int_\Omega \Delta u^\ell(x,t) \Delta v_j(x) \phi(t) dx dt \\ -\int_0^T \int_\Omega \alpha(t) g(u_t^\ell) v_j(x) \phi(t) dx dt + \int_0^T \int_\Omega f(x,t) v_j(x) \phi(t) dx dt. \end{cases} \tag{4.46}$$

As $\ell \to +\infty$, we have for any $\phi \in C_0^\infty((0,T))$,

$$\begin{cases} -\int_0^T \int_\Omega u_t(x,t) v_j(x) \phi'(t) dx dt = -\int_0^T \int_\Omega \Delta u(x,t) \Delta v_j(x) \phi(t) dx dt \\ -\int_0^T \int_\Omega \alpha(t) g(u_t) v_j(x) \phi(t) dx dt + \int_0^T \int_\Omega f(x,t) v_j(x) \phi(t) dx dt, \end{cases} \tag{4.47}$$

for all $j \geq 1$. This implies that

$$-\int_0^T \int_\Omega u_t(x,t) v(x) \phi'(t) dx dt = \int_0^T \int_\Omega \left[-\Delta^2 u(x,t) - \alpha(t) g(u_t) + f(x,t)\right] v(x) \phi(t) dx dt, \tag{4.48}$$

for all $v \in H_*^2(\Omega)$. This means that $u_{tt} \in L^\infty((0,T); \mathcal{H}(\Omega))$ and u solves the equation

$$u_{tt} + \Delta^2 u + \alpha(t) g(u_t) = f. \tag{4.49}$$

Thus

$$u_t \in L^\infty((0,T); L^2(\Omega)),\ u_{tt} \in L^\infty((0,T); \mathcal{H}(\Omega)).$$

Consequently, $u_t \in C((0,T);\mathcal{H}(\Omega))$. So, $u_t^\ell(x,0)$ makes sense and follows that

$$u_t^\ell(x,0) \to u_t(x,0) \text{ in } \mathcal{H}(\Omega)$$

and since

$$u_t^\ell(x,0) = u_1^\ell(x) \to u_1(x) \text{ in } L^2(\Omega),$$

then

$$u_t(x,0) = u_1(x).$$

This ends the proof of Theorem 4.1.

Now, we proceed to establish the local existence result for problem (1.1).

Theorem 4.2. *Let* $(u_0,u_1) \in H_*^2(\Omega) \times L^2(\Omega)$ *be given. Then problem* (1.1) *has a unique local weak solution*

$$u \in L^\infty\left([0,T), H_*^2(\Omega)\right), \quad u_t \in L^\infty\left([0,T), L^2(\Omega)\right), \quad u_{tt} \in L^\infty\left([0,T), \mathcal{H}(\Omega)\right).$$

Remark 4.3. *In this remark, we point out four cases regarding the solution of problem* (1.1)*:*

1) *If* $\beta = 0$*, g is linear and* $(u_0,u_1) \in \left(H^4(\Omega) \cap H_*^2(\Omega)\right) \times H_*^2(\Omega)$*, then problem* (1.1) *has a unique classical solution*

$$u \in C^2\left([0,T), H_*^2(\Omega)\right), \quad u_t \in C^1\left([0,T), L^2(\Omega)\right), \quad u_{tt} \in C\left([0,T), \mathcal{H}(\Omega)\right).$$

2) *If* $\beta = 0$*, g is linear and* $(u_0,u_1) \in H_*^2(\Omega) \times L^2(\Omega)$*, then problem* (1.1) *has a unique weak solution*

$$u \in C^1\left([0,T), H_*^2(\Omega)\right), \quad u_t \in C\left([0,T), L^2(\Omega)\right), \quad u_{tt} \in L^\infty\left([0,T), \mathcal{H}(\Omega)\right).$$

3) *If* $\beta > 0$ *or g is nonlinear and* $(u_0,u_1) \in H_*^2(\Omega) \times L^2(\Omega)$*, then problem* (1.1) *has a unique weak solution*

$$u \in L^\infty\left([0,T), H_*^2(\Omega)\right), \quad u_t \in L^\infty\left([0,T), L^2(\Omega)\right), \quad u_{tt} \in L^\infty\left([0,T), \mathcal{H}(\Omega)\right).$$

4) *If* $\beta > 0$ *or g is nonlinear and* $(u_0,u_1) \in \left(H^4(\Omega) \cap H_*^2(\Omega)\right) \times H_*^2(\Omega)$*, then problem* (1.1) *has a unique strong solution*

$$u \in L^\infty\left([0,T), H^4(\Omega) \cap H_*^2(\Omega)\right), \quad u_t \in L^\infty\left(([0,T), H_*^2(\Omega)\right), \quad u_{tt} \in L^\infty([0,T), L^2(\Omega)).$$

Proof. To prove Theorem 4.2, we first let $v \in L^\infty\left([0,T), H_*^2(\Omega)\right)$ and $\widetilde{f}(v) = |v|^\beta v$. Then, by the embedding Lemma 3.1, we have

$$\|\widetilde{f}(v)\|_2^2 = \int_\Omega |v|^{2(\beta+1)} dx < +\infty. \tag{4.50}$$

Hence,

$$\widetilde{f}(v) \in L^\infty([0,T), L^2(\Omega)) \subset L^2(\Omega \times (0,T)).$$

Therefore, for each $v \in L^\infty([0,T), H_*^2(\Omega))$, there exists a unique solution

$$u \in L^\infty([0,T), H_*^2(\Omega)), \quad u_t \in L^\infty([0,T), L^2(\Omega))$$

satisfying the following nonlinear problem

$$\begin{cases} u_{tt}+\Delta^2 u+\alpha(t)g(u_t)=\widetilde{f}(v), \text{ in } \Omega\times(0,T), \\ u(0,y,t)=u_{xx}(0,y,t)=u(\pi,y,t)=u_{xx}(\pi,y,t)=0, \ (y,t)\in(-d,d)\times(0,T), \\ u_{yy}(x,\pm d,t)+\sigma u_{xx}(x,\pm d,t)=0, \ (x,t)\in(0,\pi)\times(0,T), \\ u_{yyy}(x,\pm d,t)+(2-\sigma)u_{xxy}(x,\pm d,t)=0, \ (x,t)\in(0,\pi)\times(0,T), \\ u(x,y,0)=u_0(x,y), \ u_t(x,y,0)=u_1(x,y), \ \text{ in } \Omega\times(0,T), \end{cases} \tag{4.51}$$

Now, let

$$W_T=\left\{w\in L^\infty((0,T),H_*^2(\Omega))/w_t\in L^\infty((0,T),L^2(\Omega))\right\},$$

and define the map $K: W_T\longrightarrow W_T$ by $K(v)=u$. We note that W_T is a Banach space with respect to the following norm

$$\|w\|_{W_T}=\|w\|_{L^\infty((0,T),H_*^2(\Omega))}+\|w_t\|_{L^\infty((0,T),L^2(\Omega))}.$$

Multiply (4.51) by u_t and integrate over $\Omega\times(0,t)$, we get for all $t\le T$,

$$\begin{aligned} &\frac{1}{2}\int_\Omega u_t^2dx+\frac{1}{2}\int_\Omega|\Delta u|^2dx+\int_0^t\int_\Omega\alpha(s)u_tg(u_t)dxds \\ &=\frac{1}{2}\int_\Omega u_1^2dx+\frac{1}{2}\int_\Omega|\Delta u_0|^2dx+\int_0^t\int_\Omega|v|^\beta vu_tdxds. \end{aligned} \tag{4.52}$$

Using Young's inequality and the embedding Lemma 3.1, we have

$$\begin{aligned} \int_\Omega|v|^\beta vu_tdx&\le\frac{\varepsilon}{4}\int_\Omega u_t^2dx+\frac{4}{\varepsilon}\int_\Omega|v|^{2(\beta+1)}dx \\ &\le\frac{\varepsilon}{4}\int_\Omega u_t^2dx+\frac{4C_e}{\varepsilon}\|v\|_{H_*^2}^{2(\beta+1)}. \end{aligned} \tag{4.53}$$

Thus, (4.52) becomes

$$\frac{1}{2}\int_\Omega u_t^2dx+\frac{1}{2}\int_\Omega|\Delta u|^2dx\le\lambda_0+\frac{\varepsilon T}{4}\sup_{(0,T)}\int_\Omega u_t^2dx+\frac{C_e}{\varepsilon}\int_0^T\|v\|_{H_*^2}^{2(\beta+1)}dt, \tag{4.54}$$

where $\lambda_0=\frac{1}{2}\|u_1\|_2^2+\frac{1}{2}\|\Delta u_0\|_2^2$ and C_e is the embedding constant. Choosing ε such that $\frac{\varepsilon T}{2}=\frac{1}{4}$, we get

$$\|u\|_{W_T}^2\le\lambda+Tb\|v\|_{W_T}^{2(\beta+1)}.$$

Suppose that $\|v\|_{W_T}\le M$ and for $M^2>\lambda$ and $T\le T_0<\frac{M^2-\lambda}{bM^{2(\beta+1)}}$, we conclude that

$$\|u\|_{W_T}^2\le\lambda+TbM^{2(\beta+1)}\le M^2.$$

Therefore, we deduce that $K: B\longrightarrow B$, where

$$B=\left\{w\in L^\infty((0,T),H_*^2(\Omega))/w_t\in L^\infty((0,T),L^2(\Omega));\|w\|_{W_T}\le M\right\}.$$

Next, we prove, for T_0(even smaller), K is a contraction. For this purpose, let $u_1=K(v_1)$ and $u_2=K(v_2)$ and set $u=u_1-u_2$, then u satisfies the following

$$u_{tt}+\Delta^2u+\alpha(t)g(u_{1t})-\alpha(t)g(u_{2t})=|v_1|^\beta v_1-|v_2|^\beta v_2. \tag{4.55}$$

Multiplying (4.55) by u_t and integrating over $\Omega \times (0,t)$ we get, for all $t \le T$,

$$\begin{aligned}\frac{1}{2}\int_\Omega u_t^2 dx + \frac{1}{2}\int_\Omega |\Delta u|^2 dx + \int_0^t \int_\Omega (\alpha(t)g(u_{1t}) - \alpha(t)g(u_{2t}))\,(u_{1t} - u_{2t})\,dxds \\ = \int_0^t \int_\Omega \left(\widetilde{f}(v_1) - \widetilde{f}(v_2)\right) u_t dxds.\end{aligned} \tag{4.56}$$

Using (3.5) and (H2), we have

$$\frac{1}{2}\int_\Omega u_t^2 dx + \frac{1}{2}\int_\Omega |\Delta u|^2 dx \le \int_0^t \int_\Omega \left(\widetilde{f}(v_1) - \widetilde{f}(v_2)\right) u_t dxds. \tag{4.57}$$

Now, we evaluate

$$\Lambda := \int_\Omega |\widetilde{f}(v_1) - \widetilde{f}(v_2)||u_t|dx = \int_\Omega |\widetilde{f}'(\xi)||v||u_t|dx, \tag{4.58}$$

where $v = v_1 - v_2$, $\xi = \tau v_1 + (1-\tau)v_2$, $0 \le \tau \le 1$, and $\widetilde{f}'(\xi) = (\beta+1)|\xi|^\beta$.

Young's inequality implies

$$\begin{aligned}\Lambda &\le \frac{\delta}{2}\int_\Omega u_t^2 dx + \frac{2}{\delta}\int_\Omega |\widetilde{f}'(\xi)|^2 |v|^2 dx \le \frac{\delta}{2}\int_\Omega u_t^2 dx + \frac{2(\beta+1)^2}{\delta}\int_\Omega |\alpha v_1 + (1-\alpha)v_2|^{2\beta}|v|^2 dx \\ &\le \frac{\delta}{2}\int_\Omega u_t^2 dx + C_\delta \left(|v|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \left(|\alpha v_1 + (1-\alpha)v_2|^{n\beta}\right)^{\frac{2}{n}}.\end{aligned} \tag{4.59}$$

Using the embedding Lemma 3.1, we arrive at

$$\begin{aligned}\Lambda &\le \frac{\delta}{2}\int_\Omega u_t^2 dx + C_\delta C_e \|v\|_{H_*^2}^2 \left(\|v_1\|_{H_*^2}^{2\beta} + \|v_2\|_{H_*^2}^{2\beta}\right) \\ &\le \frac{\delta}{2}\int_\Omega u_t^2 dx + 4C_\delta C_e M^{2\beta}\|v\|_{H_*^2}^{2\beta}.\end{aligned} \tag{4.60}$$

Therefore, (4.57) takes the form

$$\frac{1}{2}\|u\|_{W_T}^2 \le \frac{\delta T_0}{2}\|u\|_{W_T}^2 + C_\delta M^{2\beta} T_0 \|v\|_{W_T}^{2\beta}. \tag{4.61}$$

Choosing δ sufficiently small, we see that

$$\|u\|_{W_T}^2 \le 4C_\delta M^{2\beta} T_0 \|v\|_{W_T}^{2\beta} = \gamma_0 T_0 \|v\|_{W_T}^{2\beta}. \tag{4.62}$$

Taking T_0 small enough so that,

$$\|u\|_{W_T}^2 \le \nu \|v\|_{W_T}^{2\beta}, \text{for some } 0 < \nu < 1. \tag{4.63}$$

Thus, K is a contraction. The Banach fixed point theorem implies the existence of a unique $u \in B$ satisfying $K(u) = u$. Thus, u is a local solution of (1.1).

Uniqueness: Suppose that problem (1.1) has two weak solutions (u, v). Taking, $w = u - v$, that satisfies the following equation, for all $t \in (0, T)$,

$$\begin{cases} w_{tt} - \Delta^2 w + \alpha(t)g(u_t) - \alpha(t)g(v_t) = u|u|^\beta - v|v|^\beta \\ w(0,y,t) = w_{xx}(0,y,t) = w(\pi,y,t) = w_{xx}(\pi,y,t) = 0, \ (y,t) \in (-d,d) \times (0,T), \\ w(x,0) = w_t(x,0) = 0, \ \text{ in } \ \Omega. \end{cases} \tag{4.64}$$

Multiplying (4.64) by w_t and integrating over $\Omega \times (0,t)$, we obtain

$$\begin{aligned} &\frac{1}{2}\int_\Omega w_t^2 dx + \frac{1}{2}\int_\Omega |\Delta w|^2 dx + \int_0^t \int_\Omega (\alpha(t)g(u_t) - \alpha(t)g(v_t))(u_t - v_t)\, dxds \\ &= \int_0^t \int_\Omega \left(u|u|^\beta - v|v|^\beta\right) w_t dxds. \end{aligned} \tag{4.65}$$

Using (3.5) and (H2) implies that

$$\frac{1}{2}\int_\Omega w_t^2 dx + \frac{1}{2}\int_\Omega |\Delta w|^2 dx \le \int_0^t \int_\Omega \left(u|u|^\beta - v|v|^\beta\right) w_t dxds. \tag{4.66}$$

By repeating the same above estimates, we obtain

$$\int_\Omega \left(w_t^2 dx + |\Delta w|^2\right) dx = 0. \tag{4.67}$$

This gives $w \equiv 0$. The proof of the uniqueness is completed.

5. Global existence

In this section, we prove that problem (1.1) has a global solution. For this purpose, we introduce the following functionals. The energy functional associated with problem (1.1) is

$$E(t) = \frac{1}{2}\left(\|u_t\|_2^2 + \|u\|^2_{H^2_*(\Omega)}\right) - \frac{1}{\beta+2}\|u\|^{\beta+2}_{\beta+2}. \tag{5.1}$$

Direct differentiation of (5.1), using (1.1), leads to

$$E'(t) = -\alpha(t)\int_\Omega u_t g(u_t) dx \le 0. \tag{5.2}$$

$$J(t) = \frac{1}{2}\|u\|^2_{H^2_*(\Omega)} - \frac{1}{\beta+2}\|u\|^{\beta+2}_{\beta+2} \tag{5.3}$$

and

$$I(t) = \|u\|^2_{H^2_*(\Omega)} - \|u\|^{\beta+2}_{\beta+2}. \tag{5.4}$$

Clearly, we have

$$E(t) = J(t) + \frac{1}{2}\|u_t\|_2^2. \tag{5.5}$$

Lemma 5.1. *Suppose that (H1) and (H2) hold and* $(u_0, u_1) \in H^2_*(\Omega) \times L^2(\Omega)$*, such that*

$$0 < \gamma = C_e^{\beta+2}\left(\frac{2(\beta+2)}{\beta}E(0)\right)^{\frac{\beta}{2}} < 1, \quad I(u_0) > 0, \tag{5.6}$$

then $I(u(t)) > 0, \forall t > 0$.

Proof. Since $I(u_0) > 0$, then there exists (by continuity) $T_m < T$ such that $I(u(t) \geq 0$, $\forall t \in [0, T_m]$; which gives

$$\begin{aligned} J(t) &= \frac{1}{2}\|u\|^2_{H^2_*(\Omega)} - \frac{1}{\beta+2}\|u\|^{\beta+2}_{\beta+2} \\ &= \frac{\beta}{2(\beta+2)}\|u\|^2_{H^2_*(\Omega)} + \frac{1}{\beta+2}I(t) \\ &\geq \frac{\beta}{2(\beta+2)}\|u\|^2_{H^2_*(\Omega)}. \end{aligned} \tag{5.7}$$

By using (5.2), (5.5) and (5.7), we have

$$\|u\|^2_{H^2_*(\Omega)} \leq \frac{2(\beta+2)}{\beta}J(t) \leq \frac{2(\beta+2)}{\beta}E(t) \leq \frac{2(\beta+2)}{\beta}E(0), \quad \forall t \in [0, T_m]. \tag{5.8}$$

The embedding theorem, (5.6) and (5.8) give, $\forall t \in [0, T_m]$,

$$\|u\|^{\beta+2}_{\beta+2} \leq C_e^{\beta+2}\|u\|^{\beta+2}_{H^2_*(\Omega)} \leq C_e^{\beta+2}\|u\|^{\beta}_{H^2_*(\Omega)}\|u\|^2_{H^2_*(\Omega)} \leq \gamma\|u\|^2_{H^2_*(\Omega)} < \|u\|^2_{H^2_*(\Omega)}. \tag{5.9}$$

Therefore,

$$I(t) = \|u\|^2_{H^2_*(\Omega)} - \|u\|^{\beta+2}_{\beta+2} > 0, \quad \forall t \in [0, T_m].$$

By repeating this procedure, and using the fact that

$$\lim_{t \to T_m} C_e^{\beta+2}\left(\frac{2(\beta+2)}{\beta}E(t)\right)^{\frac{\beta}{2}} \leq \gamma < 1,$$

T_m is extended to T.

Remark 5.2. *The restriction* (5.6) *on the initial data will guarantee the nonnegativeness of* $E(t)$.

Proposition 5.3. *Suppose that (H1) and (H2) hold. Let* $(u_0, u_1) \in H^2_*(\Omega) \times L^2(\Omega)$ *be given, satisfying* (5.6). *Then the solution of* (1.1) *is global and bounded.*

Proof. It suffices to show that $\|u\|^2_{H^2_*(\Omega)} + \|u_t\|^2_2$ is bounded independently of t. To achieve this, we use (5.2), (5.4) and (5.5) to get

$$\begin{aligned} E(0) \geq E(t) &= J(t) + \frac{1}{2}\|u_t\|^2_2 \\ &\geq \frac{\beta-2}{2\beta}\|u\|^2_{H^2_*(\Omega)} + \frac{1}{2}\|u_t\|^2_2 + \frac{1}{\beta}I(t) \\ &\geq \frac{\beta-2}{2\beta}\|u\|^2_{H^2_*(\Omega)} + \frac{1}{2}\|u_t\|^2_2, \end{aligned} \tag{5.10}$$

since $I(t)$ is positive. Therefore

$$\|u\|^2_{H^2_*(\Omega)} + \|u_t\|^2_2 \leq CE(0),$$

where C is a positive constant, which depends only on β.

6. Stability result

In this section, we state and prove our stability result. For this purpose, we establish some lemmas.

Lemma 6.1. ***(Case:*** *G* ***is linear)*** *Let u be the solution of* (1.1)*. Then, for* $T > S \geq 0$*, the energy functional satisfies*

$$\int_S^T \alpha(t)E(t)dt \leq cE(S). \tag{6.1}$$

Proof. We multiply (1.1) by αu and integrate over $\Omega \times (S, T)$ to get

$$\begin{aligned}
0 &= \int_S^T \alpha(t) \int_\Omega \left(uu_{tt} + u\Delta^2 u + \alpha(t) ug(u_t) - |u|^{\beta+2} \right) dxdt \\
&= \int_S^T \alpha(t) \int_\Omega \left((uu_t)_t - u_t^2 + \alpha(t) ug(u_t) - |u|^{\beta+2} \right) dxdt + \int_S^T \alpha(t) \|u\|^2_{H^2_*(\Omega)} dt \\
&= \int_S^T \alpha(t) \frac{d}{dt} \left(\int_\Omega uu_t dx \right) dt + \int_S^T \alpha(t) \int_\Omega u_t^2 dxdt \\
&\quad + \int_S^T \alpha(t) \|u\|^2_{H^2_*(\Omega)} dt - 2 \int_S^T \alpha(t) \int_\Omega u_t^2 dxdt \\
&\quad + \int_S^T \alpha^2(t) \int_\Omega ug(u_t) dxdt - \int_S^T \alpha(t) \|u\|^{\beta+2}_{\beta+2} dt.
\end{aligned} \tag{6.2}$$

Adding and subtracting the following terms

$$\gamma \int_S^T \alpha(t) \|u\|^2_{H^2_*(\Omega)} dt + (1+\gamma) \int_S^T \alpha(t) \|u_t\|_2^2 dt, \text{ where } \gamma \text{ is defined in (5.6),}$$

to (6.2), and recalling (5.9), we arrive at

$$\begin{aligned}
&\int_S^T \alpha(t) \frac{d}{dt} \left(\int_\Omega uu_t dx \right) dt + (1-\gamma) \int_S^T \alpha(t) \left(\|u\|^2_{H^2_*(\Omega)} + \|u_t\|_2^2 \right) dt \\
&\quad - (2-\gamma) \int_S^T \alpha(t) \int_\Omega u_t^2 dxdt + \int_S^T \alpha^2(t) \int_\Omega ug(u_t) dxdt \\
&\quad = - \int_S^T \alpha(t) \left(\gamma \|u\|^2_{H^2_*(\Omega)} - \|u\|^{\beta+2}_{\beta+2} \right) dt \leq 0.
\end{aligned} \tag{6.3}$$

Integrating the first term of (6.3) by parts and using (5.1), then (6.3) becomes

$$\begin{aligned}
(1-\gamma) \int_S^T \alpha E dt &\leq (1-\gamma) \int_S^T \alpha \left(\|u\|^2_{H^2_*(\Omega)} + \|u_t\|_2^2 \right) dt \\
&\leq - \left[\alpha \int_\Omega uu_t dx \right]_S^T + \int_S^T \alpha' \int_\Omega uu_t dxdt \\
&\quad + (2-\gamma) \int_S^T \alpha \int_\Omega u_t^2 dxdt - \int_S^T \alpha^2 \int_\Omega ug(u_t) dxdt.
\end{aligned} \tag{6.4}$$

Now, we estimate the terms in the right-hand side of (6.4) as follows:

1) **Estimate for** $-\left[\alpha \int_\Omega uu_t dx\right]_S^T$.
Using Lemma 3.1 and Young's inequality, we obtain

$$\int_\Omega uu_t dx \leq \frac{1}{2}\int_\Omega \left(u^2 + u_t^2\right) dx \leq c\|u\|_{H_*^2(\Omega)} + \|u_t\|_2^2 \leq cE(t), \tag{6.5}$$

which implies that

$$-\left[\alpha \int_\Omega uu_t dx\right]_S^T \leq c[-\alpha(T)E(T) + \alpha(S)E(S)] \leq c\alpha(S)E(S) \leq cE(S). \tag{6.6}$$

2) **Estimate for** $\int_S^T \alpha' \int_\Omega uu_t dxdt$.
The use of (6.5) and (H2) leads to

$$\int_S^T \alpha' \int_\Omega uu_t dxdt \leq c\left|\int_S^T \alpha' E dt\right| \leq cE(S)\left|\int_S^T \alpha' dt\right| \leq cE(S). \tag{6.7}$$

3) **Estimate for** $\int_S^T \alpha\left(\int_\Omega u_t^2 dx\right) dt$.
Using (H1), (5.2) and recalling that G is linear, we have

$$\begin{aligned}\int_S^T \alpha\left(\int_\Omega u_t^2 dx\right) dt &\leq \frac{1}{c_1}\int_S^T \alpha(t)\int_\Omega u_t g(u_t) dxdt\\ &\leq -\int_S^T cE'(t)dt\\ &\leq cE(S).\end{aligned} \tag{6.8}$$

4) **Estimate for** $-\int_S^T \alpha^2(t)\int_\Omega ug(u_t)dxdt$.

Using (H1), Lemma 3.1, Holder's inequality and recalling G is linear, we obtain

$$\begin{aligned}\alpha^2(t)\int_\Omega ug(u_t)dx &\leq \alpha^2(t)\left(\int_\Omega |u|^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega |g(u_t)|^2 dx\right)^{\frac{1}{2}}\\ &\leq \alpha^{\frac{3}{2}}(t)\|u\|_{H_*^2(\Omega)}\left(\alpha(t)\int_\Omega u_t g(u_t)dx\right)^{\frac{1}{2}}\\ &\leq c\alpha(t)E^{\frac{1}{2}}(t)\left(-E'(t)\right)^{\frac{1}{2}}.\end{aligned} \tag{6.9}$$

Applying Young's inequality to $E^{\frac{1}{2}}(t)(-E'(t))^{\frac{1}{2}}$ with $p = 2$ and $p^* = 2$, to get

$$\begin{aligned}\alpha^2(t)\int_\Omega ug(u_t)dx &\leq c\alpha(t)\left(\varepsilon E(t) - C_\varepsilon E'(t)\right)\\ &\leq c\varepsilon\alpha E(t) - C_\varepsilon E'(t),\end{aligned} \tag{6.10}$$

which implies that

$$\begin{aligned}&\int_S^T \alpha^2(t)\left(\int_\Omega (-ug(u_t))dx\right) dt\\ &\leq c\varepsilon\int_S^T \alpha(t)E(t)dt + C_\varepsilon E(S).\end{aligned} \tag{6.11}$$

Combining the above estimates and taking ε small enough, we get (6.1).

Lemma 6.2. ***(Case: G is nonlinear)*** *Let u be the solution of* (1.1)*. Then, for* $T > S \geq 0$*, the energy functional satisfies*

$$\int_S^T \alpha(t)\tilde{\phi}(E(t))\,dt \leq c\tilde{\phi}(E(S)) + c\int_S^T \alpha(t)\frac{\tilde{\phi}(E)}{E}\int_\Omega \Big(|u_t|^2 + |ug(u_t)|\Big)\,dxdt, \tag{6.12}$$

where $\tilde{\phi} : \mathbb{R}^+ \to \mathbb{R}^+$ *is any convex, increasing and of class* $C^1[0,\infty)$ *function such that* $\tilde{\phi}(0) = 0$.

Proof. We multiply (1.1) by $\alpha(t)\frac{\tilde{\phi}(E)}{E}u$ and integrate over $\Omega \times (S,T)$ to get

$$\begin{aligned}
0 &= \int_S^T \alpha(t)\frac{\tilde{\phi}(E)}{E}\int_\Omega \Big((uu_t)_t - u_t^2 + \alpha(t)ug(u_t) - |u|^{\beta+2}\Big)\,dxdt \\
&\quad + \int_S^T \alpha(t)\frac{\tilde{\phi}(E)}{E}\|u\|^2_{H^2_*(\Omega)}dt \\
&= \int_S^T \alpha(t)\frac{\tilde{\phi}(E)}{E}\frac{d}{dt}\left(\int_\Omega uu_t dx\right)dt + \int_S^T \alpha(t)\frac{\tilde{\phi}(E)}{E}\|u\|^2_{H^2_*(\Omega)}dt \\
&\quad + \int_S^T \alpha(t)\frac{\tilde{\phi}(E)}{E}\int_\Omega u_t^2 dxdt - 2\int_S^T \alpha(t)\frac{\tilde{\phi}(E)}{E}\int_\Omega u_t^2 dxdt \\
&\quad + \int_S^T \alpha^2(t)\frac{\tilde{\phi}(E)}{E}\int_\Omega ug(u_t)dxdt - \int_S^T \alpha(t)\frac{\tilde{\phi}(E)}{E}\|u\|^{\beta+2}_{\beta+2}.
\end{aligned} \tag{6.13}$$

Adding and subtracting to (6.13) the following terms

$$\gamma\int_S^T \alpha(t)\frac{\tilde{\phi}(E)}{E}\|u\|^2_{H^2_*(\Omega)}dt + (1+\gamma)\int_S^T \alpha(t)\frac{\tilde{\phi}(E)}{E}\|u_t\|_2^2 dt, \text{ where } \gamma \text{ is defined in (5.6),}$$

we arrive at

$$\begin{aligned}
(1-\gamma)\int_S^T \alpha(t)\tilde{\phi}(E)dt &\leq -\int_S^T \alpha(t)\frac{\tilde{\phi}(E)}{E}\frac{d}{dt}\left(\int_\Omega uu_t dx\right)dt \\
&\quad + (2-\gamma)\int_S^T \alpha(t)\frac{\tilde{\phi}(E)}{E}\int_\Omega u_t^2 dxdt - \int_S^T \alpha^2(t)\frac{\tilde{\phi}(E)}{E}\int_\Omega ug(u_t)dxdt \\
&\quad - \int_S^T \alpha\frac{\tilde{\phi}(E)}{E}\left(\gamma\|u\|^2_{H^2_*(\Omega)} - \|u\|^{\beta+2}_{\beta+2}\right).
\end{aligned} \tag{6.14}$$

Using (5.9), it is easy to deduce that $-\int_S^T \alpha\frac{\tilde{\phi}(E)}{E}\left(\gamma\|u\|^2_{H^2_*(\Omega)} - \|u\|^{\beta+2}_{\beta+2}\right)dt \leq 0$.

Integrating by parts in the first term, in the right-hand side of (6.14), we get

$$\begin{aligned}
(1-\gamma)\int_S^T \alpha(t)\tilde{\phi}(E)dt &\leq -\left[\alpha(t)\frac{\tilde{\phi}(E)}{E}\int_\Omega uu_t dx\right]_S^T \\
&\quad + \int_S^T\int_\Omega u_t\left(\alpha'(t)\frac{\tilde{\phi}(E)}{E}u + \alpha(t)\left(\frac{\tilde{\phi}(E)}{E}\right)' u\right)dxdt \\
&\quad + (2-\gamma)\int_S^T \alpha(t)\frac{\tilde{\phi}(E)}{E}\int_\Omega u_t^2 dxdt \\
&\quad - \int_S^T \alpha^2(t)\frac{\tilde{\phi}(E)}{E}\int_\Omega ug(u_t)dxdt.
\end{aligned} \tag{6.15}$$

Using Cauchy Schwarz' inequality, Lemmas 3.1 and 5.1, we obtain

$$\begin{aligned}\int_\Omega uu_t dx &\leq \left(\int_\Omega |u|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |u_t|^2 dx\right)^{\frac{1}{2}} \\ &\leq c\|u\|_{H^2_*(\Omega)}\|u_t\|_2 \leq cE(t).\end{aligned} \tag{6.16}$$

Using (6.16), the properties of $\alpha(t)$ and the fact that the function $s \to \frac{\tilde{\phi}(s)}{s}$ is non-decreasing and E is non-increasing, we have

$$\begin{aligned}\int_S^T \alpha'(t)\frac{\tilde{\phi}(E)}{E}\int_\Omega uu_t dxdt &\leq c\int_S^T \alpha'(t)\frac{\tilde{\phi}(E)}{E}Edt \\ &\leq c\tilde{\phi}(E(S))\int_S^T \alpha'(t)dt \leq c\tilde{\phi}(E(S)).\end{aligned} \tag{6.17}$$

Similarly, we get

$$\begin{aligned}\int_S^T \alpha(t)\left(\frac{\tilde{\phi}(E)}{E}\right)'\int_\Omega uu_t dxdt &\leq E(S)\int_S^T \alpha(t)\left(\frac{\tilde{\phi}(E)}{E}\right)' dt \\ &\leq E(S)\left[\alpha(t)\frac{\tilde{\phi}(E)}{E}\right]_S^T - E(S)\int_S^T \alpha'(t)\frac{\tilde{\phi}(E)}{E}dt \\ &\leq E(S)\left(\alpha(T)\frac{\tilde{\phi}(E(T))}{E(T)} - \alpha(S)\frac{\tilde{\phi}(E(S))}{E(S)}\right) \\ &\quad - E(S)\frac{\tilde{\phi}(E(S))}{E(S)}\int_S^T \alpha'(t)dt \\ &\leq E(S)\alpha(T)\frac{\tilde{\phi}(E(T))}{E(T)} - \tilde{\phi}(E(S))\left(\alpha(T)-\alpha(S)\right) \\ &\leq E(S)\alpha(S)\frac{\tilde{\phi}(E(S))}{E(S)} + \tilde{\phi}(E(S))\alpha(S) \leq c\tilde{\phi}(E(S)).\end{aligned} \tag{6.18}$$

A combination of (6.15)–(6.18) leads to (6.12).

In order to finalize the proof of our result, we let

$$\tilde{\phi}(s) = 2\varepsilon_0 sG'(\varepsilon_0^2 s), \text{ and } \quad G_1(s) = G(s^2),$$

where $\varepsilon_0 > 0$ is small enough and G^* and G_1^* denote the dual functions of the convex functions G and G_1 respectively in the sense of Young (see, Arnold [33], pp. 64).

Lemma 6.3. *Suppose G is nonlinear, then the following estimates*

$$G^*\left(\frac{\tilde{\phi}(s)}{s}\right) \leq \frac{\tilde{\phi}(s)}{s}(G')^{-1}\left(\frac{\tilde{\phi}(s)}{s}\right) \tag{6.19}$$

and

$$G_1^*\left(\frac{\tilde{\phi}(s)}{\sqrt{s}}\right) \leq \varepsilon_0\tilde{\phi}(\sqrt{s}). \tag{6.20}$$

hold, where $\tilde{\phi}$ is defined earlier in Lemma 6.2.

Proof. Since G^* and G_1^* are the dual functions of the convex functions G and G_1 respectively, then

$$G^*(s) = s(G')^{-1}(s) - G\left[(G')^{-1}(s)\right] \leq s(G')^{-1}(s) \tag{6.21}$$

and

$$G_1^*(s) = s(G_1')^{-1}(s) - G_1\left[(G_1')^{-1}(s)\right] \leq s(G_1')^{-1}(s). \tag{6.22}$$

Using (6.21) and the definition of $\tilde{\phi}$, we obtain (6.19). For the proof of (6.20), we use (6.22) and the definitions of G_1 and $\tilde{\phi}$ to obtain

$$\begin{aligned}\frac{\tilde{\phi}(s)}{\sqrt{s}}(G_1')^{-1}\left(\frac{\tilde{\phi}(s)}{\sqrt{s}}\right) &\leq 2\varepsilon_0\sqrt{s}G'(\varepsilon_0^2 s)(G_1')^{-1}\left(2\varepsilon_0\sqrt{s}G'(\varepsilon_0^2 s)\right)\\ &= 2\varepsilon_0\sqrt{s}G'(\varepsilon_0^2 s)(G_1')^{-1}\left(G_1'(\varepsilon_0\sqrt{s})\right)\\ &= 2\varepsilon_0^2 sG'(\varepsilon_0^2 s)\\ &= \varepsilon_0\tilde{\phi}(\sqrt{s}).\end{aligned} \tag{6.23}$$

Now, we state and prove our main decay results.

Theorem 6.4. *Let* $(u_0, u_1) \in H_*^2(\Omega)\times L^2(\Omega)$. *Assume that (H1) and (H2) hold. Then there exist positive constants k and c such that, for t large, the solution of* (1.1) *satisfies*

$$E(t) \leq ke^{-c\int_0^t \alpha(s)ds}, \qquad \text{if } G \text{ is linear,} \tag{6.24}$$

$$E(t) \leq \psi^{-1}\left(h(\tilde{\alpha}(t)) + \psi\left(E(0)\right)\right), \qquad \text{if } G \text{ is nonlinear,} \tag{6.25}$$

where

$$\tilde{\alpha}(t) = \int_0^t \alpha(t)dt,\ \psi(t) = \int_t^1 \frac{1}{\chi(s)}ds,\ \text{and}\ \chi(s) = 2\varepsilon_0 csG'(\varepsilon_0^2 s)$$

and

$$\begin{cases} h(t) = 0, \quad 0 \leq t \leq \frac{E(0)}{\chi(E(0))},\\ h^{-1}(t) = t + \frac{\psi^{-1}(t+\psi(E(0)))}{\chi\left(\psi^{-1}(t+\psi(E(0)))\right)}, \quad t > 0.\end{cases}$$

Proof. To establish (6.24), we use (6.1) and Lemma 3.5 for $\gamma(t) = \int_0^t \alpha(s)ds$. Consequently the result follows. For the proof of (6.25), we re-estimate the terms of (6.12) as follows: we consider the following partition of Ω:

$$\Omega_1 = \{x \in \Omega : |u_t| \geq \varepsilon_1\}, \quad \Omega_2 = \{x \in \Omega : |u_t| \leq \varepsilon_1\}.$$

So,

$$\begin{aligned}&\int_S^T \alpha(t)\frac{\tilde{\phi}(E)}{E}\int_{\Omega_1}\left(|u_t|^2 + |ug(u_t)|\right)dxdt\\ &\quad= \int_S^T \alpha(t)\frac{\tilde{\phi}(E)}{E}\int_{\Omega_1}|u_t|^2dxdt + \int_S^T \alpha(t)\frac{\tilde{\phi}(E)}{E}\int_{\Omega_1}|ug(u_t)|dxdt\\ &\quad:= I_1 + I_2.\end{aligned}$$

Using the definition of Ω_1, (3.4) and (5.2), we have

$$\begin{aligned} I_1 &\le c \int_S^T \alpha(t) \frac{\tilde{\phi}(E)}{E} \int_{\Omega_1} u_t g(u_t) dx dt \\ &\le c \int_S^T \frac{\tilde{\phi}(E)}{E} (-E'(t))\, dt \le c\tilde{\phi}(E(S)). \end{aligned} \tag{6.26}$$

After applying Hölder's and Young's inequalities and Lemma 3.1, we obtain for some $\varepsilon > 0$,

$$\begin{aligned} I_2 &\le \int_S^T \alpha(t) \frac{\tilde{\phi}(E)}{E} \left(\int_{\Omega_1} |u|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega_1} |g(u_t)|^2 \right)^{\frac{1}{2}} dt \\ &\le \varepsilon \int_S^T \alpha(t) \frac{\tilde{\phi}^2(E)}{E^2} \|u\|_{H^2_*(\Omega)} dt + c(\varepsilon) \int_S^T \alpha(t) \int_{\Omega_1} |g(u_t)|^2 dt. \end{aligned} \tag{6.27}$$

The definition of Ω_1, (3.4), (5.1), (5.2) and (6.27) lead to

$$\begin{aligned} I_2 &\le \varepsilon \int_S^T \alpha(t) \frac{\tilde{\phi}^2(E)}{E} dt + c(\varepsilon) \int_S^T \alpha(t) \int_{\Omega_1} u_t g(u_t) dx dt \\ &\le \varepsilon \int_S^T \alpha(t) \frac{\tilde{\phi}^2(E)}{E} dt + c(\varepsilon) E(S). \end{aligned} \tag{6.28}$$

Using the definition of $\tilde{\phi}$ and the convexity of G, then (6.28) becomes

$$\begin{aligned} I_2 &\le \varepsilon \int_S^T \alpha(t) \frac{\tilde{\phi}^2(E)}{E} dt + cE(S) \\ &= 2\varepsilon\varepsilon_0 \int_S^T \alpha(t)\tilde{\phi}(E) G'\left(\varepsilon_0^2 E(t)\right) dt + cE(S) \\ &\le 2\varepsilon\varepsilon_0 \int_S^T \alpha(t)\tilde{\phi}(E) G'\left(\varepsilon_0^2 E(0)\right) dt + cE(S) \\ &\le 2c\varepsilon\varepsilon_0 \int_S^T \alpha(t)\tilde{\phi}(E) dt + cE(S). \end{aligned} \tag{6.29}$$

Combining (6.12), (6.26) and (6.29) and choosing ε small enough, we obtain

$$\int_S^T \alpha(t)\tilde{\phi}(E) dt \le c\tilde{\phi}(E) + c \int_S^T \alpha(t) \frac{\tilde{\phi}(E)}{E} \int_{\Omega_2} \left(|u_t|^2 + |ug(u_t)| \right) dx dt. \tag{6.30}$$

Using Young's inequality and Jensen's inequality (Eq 3.3), (Eq 3.4) and (Eq 5.1), we get

$$\begin{aligned} \int_S^T \alpha(t) \frac{\tilde{\phi}(E)}{E} \int_{\Omega_2} \left(|u_t|^2 + |ug(u_t)| \right) dx dt &\le \int_S^T \alpha(t) \frac{\tilde{\phi}(E)}{E} \int_{\Omega_2} G^{-1}\left(u_t g(u_t)\right) dx dt \\ &\quad + \int_S^T \alpha(t) \frac{\tilde{\phi}(E)}{E} \|u\|^{\frac{1}{2}}_{H^2_*(\Omega)} \left(\int_{\Omega_2} G^{-1}(u_t g(u_t)) dx \right)^{\frac{1}{2}} dx dt \\ &\le |\Omega| \int_S^T \alpha(t) \frac{\tilde{\phi}(E)}{E} G^{-1}\left(\frac{1}{|\Omega|} \int_\Omega u_t g(u_t) dx \right) dt \\ &\quad + \int_S^T \alpha(t) \frac{\tilde{\phi}(E)}{E} \sqrt{E} \sqrt{|\Omega| G^{-1}\left(\frac{1}{|\Omega|} \int_\Omega u_t g(u_t) dx \right)} dt. \end{aligned} \tag{6.31}$$

Applying the generalized Young inequality

$$AB \le G^*(A) + G(B)$$

to the first term of (6.31), with $A = \frac{\tilde{\phi}(E)}{E}$ and $B = G^{-1}\left(\frac{1}{|\Omega|}\int_\Omega u_t g(u_t)dx\right)$, we easily see that

$$\frac{\tilde{\phi}(E)}{E} G^{-1}\left(\frac{1}{|\Omega|}\int_\Omega u_t g(u_t)dx\right) \le G^*\left(\frac{\tilde{\phi}(E)}{E}\right) + \frac{1}{|\Omega|}\int_\Omega u_t g(u_t)dx. \tag{6.32}$$

Then we apply it to the second term of (6.31), with $A = \frac{\tilde{\phi}(E)}{E}\sqrt{E}$ and $B = \sqrt{|\Omega| G^{-1}\left(\frac{1}{|\Omega|}\int_\Omega u_t g(u_t)dx\right)}$ to obtain

$$\frac{\tilde{\phi}(E)}{E}\sqrt{E}\sqrt{|\Omega| G^{-1}\left(\frac{1}{|\Omega|}\int_\Omega u_t g(u_t)dx\right)} \le G_1^*\left(\frac{\tilde{\phi}(E)}{E}\sqrt{E}\right) + |\Omega| G^{-1}\left(\frac{1}{|\Omega|}\int_\Omega u_t g(u_t)dx\right). \tag{6.33}$$

Combining (6.31)–(6.33) and using (6.19) and (6.20), we arrive at

$$\begin{aligned} &\int_S^T \alpha(t)\frac{\tilde{\phi}(E)}{E}\int_{\Omega_2}\left(|u_t|^2 + |ug(u_t)|\right)dxdt \\ &\quad\le c\int_S^T \alpha(t)\left(G_1^*\left(\frac{\tilde{\phi}(E)}{E}\sqrt{E}\right) + G^*\left(\frac{\tilde{\phi}(E)}{E}\right)\right)dt + c\int_S^T \alpha(t)\int_\Omega u_t g(u_t)dxdt \\ &\quad\le c\int_S^T \alpha(t)\left(\varepsilon_0 + \frac{(G')^{-1}\left(\frac{\tilde{\phi}(E)}{E}\right)}{E}\right)\tilde{\phi}(E)dt + cE(S). \end{aligned} \tag{6.34}$$

Using the definition of $\tilde{\phi}$ and the fact that $s \to (G')^{-1}(s)$ is non-decreasing, we deduce that, for $0 < \varepsilon_0 \le \frac{1}{2}$,

$$\frac{(G')^{-1}\left(\frac{\tilde{\phi}(E)}{E}\right)}{E} = \frac{(G')^{-1}\left(2\varepsilon_0 G'(\varepsilon_0^2 E)\right)}{E} \le \varepsilon_0^2. \tag{6.35}$$

Combining (6.34) and (6.35) leads to

$$\int_S^T \alpha(t)\frac{\tilde{\phi}(E)}{E}\int_{\Omega_2}\left(|u_t|^2 + |ug(u_t)|\right)dxdt \le c\varepsilon_0\int_S^T \alpha(t)\tilde{\phi}(E)dt + cE(S). \tag{6.36}$$

Then, choosing ε_0 small enough, we deduce from (6.30) and (6.36) that

$$\int_S^T \alpha(t)\tilde{\phi}(E(t))dt \le c\left(1 + \frac{\tilde{\phi}(E(S))}{E(S)}\right)E(S).$$

Using the facts that E is non-increasing and $s \to \frac{\tilde{\phi}(s)}{s}$ is non-decreasing, we obtain

$$\int_S^{+\infty} \alpha(t)\tilde{\phi}(E(t))dt \le cE(S). \tag{6.37}$$

Let $\tilde{E} = E \circ \tilde{\alpha}^{-1}$, where $\tilde{\alpha}(t) = \int_0^t \alpha(s)ds$. Then we deduce from (6.37) that

$$\begin{aligned}\int_S^\infty \tilde{\phi}(\tilde{E}(t))dt &= \int_S^\infty \tilde{\phi}(E(\tilde{\alpha}^{-1}(t)))dt \\ &= \int_{\tilde{\alpha}^{-1}(S)}^\infty \alpha(\eta)\tilde{\phi}(E(\eta))d\eta \\ &\le cE\left(\tilde{\alpha}^{-1}(S)\right) \le c\tilde{E}(S).\end{aligned}$$

Using Lemma 3.6 for $\tilde{E}$ and $\chi(s) = \frac{1}{c}\tilde{\phi}(s)$, we deduce from (3.6) the following estimate

$$\tilde{E}(t) \le \psi^{-1}\left(h(t) + \psi\left(E(0)\right)\right),$$

which gives (6.25), by using the definition of $\tilde{E}$ and the change of variables.

Remark 6.5. *The stability result* (6.25) *is a decay result. Indeed,*

$$\begin{aligned}h^{-1}(t) &= t + \frac{\psi^{-1}\left(t + \psi(E(0))\right)}{\chi\left(\psi^{-1}(t + \psi(E(0)))\right)} \\ &= t + \frac{c}{2\varepsilon_0 c G'\left(\varepsilon_0^2 \psi^{-1}(t+r)\right)} \\ &\ge t + \frac{c}{2\varepsilon_0 c G'\left(\varepsilon_0^2 \psi^{-1}(r)\right)} \\ &\ge t + \tilde{c}.\end{aligned}$$

Hence, $\lim_{t\to\infty} h^{-1}(t) = \infty$*, which implies that* $\lim_{t\to\infty} h(t) = \infty$*. Using the convexity of G, we have*

$$\psi(t) = \int_t^1 \frac{1}{\chi(s)}ds = \int_t^1 \frac{c}{2\varepsilon_0 s G'\left(\varepsilon_0^2 s\right)} \ge \int_t^1 \frac{c}{sG'\left(\varepsilon_0^2\right)} \ge c\left[\ln|s|\right]_t^1 = -c\ln t.$$

Therefore, $\lim_{t\to 0^+} \psi(t) = \infty$ *which leads to* $\lim_{t\to\infty} \psi^{-1}(t) = 0$.

Examples

1) Let $g(s) = s^m$, where $m \ge 1$. Then the function G is defined in the neighborhood of zero by

$$G(s) = cs^{\frac{m+1}{2}}$$

which gives, near zero

$$\chi(s) = \frac{c(m+1)}{2} s^{\frac{m+1}{2}}.$$

So, we obtain

$$\psi(t) = c\int_t^1 \frac{2}{(m+1)s^{\frac{m+1}{2}}}ds = \begin{cases} \frac{c}{t^{\frac{m-1}{2}}}, & if\ m > 1; \\ -c\ln t, & if\ m = 1, \end{cases}$$

and then, in the neighborhood of ∞

$$\psi^{-1}(t) = \begin{cases} ct^{-\frac{2}{m-1}}, & if\ m > 1; \\ ce^{-t}, & if\ m = 1, \end{cases}$$

Using the fact that $h(t) = t$ as t goes to infinity, we obtain from (6.24) and (6.25)

$$E(t) \leq \begin{cases} c\left(\int_0^t \alpha(s)ds\right)^{-\frac{2}{m-1}}, & if\ m > 1; \\ ce^{-\int_0^t \alpha(s)ds}, & if\ m = 1. \end{cases}$$

2) Let $g(s) = s^m\sqrt{-\ln s}$, where $m \geq 1$. Then the function G is defined in the neighborhood of zero by

$$G(s) = cs^{\frac{m+1}{2}}\sqrt{-\ln\sqrt{s}}$$

which gives, near zero

$$\chi(s) = cs^{\frac{m+1}{2}}\left(-\ln\sqrt{s}\right)^{-\frac{1}{2}}\left(\frac{m+1}{2}\left(-\ln\sqrt{s}\right) - \frac{1}{4}\right).$$

Therefore, we get

$$\begin{aligned}\psi(t) =& c\int_t^1 \frac{1}{s^{\frac{m+1}{2}}\left(-\ln\sqrt{s}\right)^{-\frac{1}{2}}\left(\frac{m+1}{2}\left(-\ln\sqrt{s}\right) - \frac{1}{4}\right)}ds \\ =& c\int_1^{\frac{1}{\sqrt{t}}} \frac{\tau^{m-2}}{(\ln\tau)^{-\frac{1}{2}}\left(\frac{m+1}{2}\ln\tau - \frac{1}{4}\right)}d\tau \\ =& \begin{cases} \frac{c}{t^{\frac{m-1}{2}}\sqrt{-\ln t}}, & if\ m > 1; \\ c\sqrt{-\ln t}, & if\ m = 1, \end{cases}\end{aligned}$$

and then, in the neighborhood of ∞, we have

$$\psi^{-1}(t) = \begin{cases} ct^{-\frac{2}{m-1}}(\ln t)^{-\frac{1}{m-1}}, & if\ m > 1; \\ ce^{-t^2}, & if\ m = 1, \end{cases}$$

Using the fact that $h(t) = t$ as t goes to infinity, we obtain

$$E(t) \leq \begin{cases} c\left(\int_0^t \alpha(s)ds\right)^{-\frac{2}{m-1}}\left(\ln\left(\int_0^t \alpha(s)ds\right)\right)^{-\frac{1}{m-1}}, & if\ m > 1; \\ ce^{-\left(\int_0^t \alpha(s)ds\right)^2}, & if\ m = 1, \end{cases}$$

Acknowledgments

The authors would like to express their profound gratitude to King Fahd University of Petroleum and Minerals (KFUPM) for its continuous support. The authors also thank the referees for their very careful reading and valuable comments. This work was funded by KFUPM under Project #SB201003.

Conflict of interest

The authors declare there is no conflicts of interest.

References

1. F. C. général des ponts, Chaussées, *Rapport a Monsieur Becquey, directeur général des ponts et chaussées et des mines: et Mémoire sur les ponts suspendus*, Impr. Royale, 1823.
2. J. Melan, *Theory of Arches and Suspension Bridges*, MC Clark publishing Company, 1913.
3. O. H. Amman, T. Von Kármán, G. B. Woodruff, *The Failure of the Tacoma Narrows Bridge*, 1941. Available from: https://resolver.caltech.edu/CaltechAUTHORS:20140512-105559175.
4. F. Bleich, *The Mathematical Theory of Vibration in Suspension Bridges: A Contribution to the W ork of the Advisory Board on the Investigation of Suspension Bridges*, US Government Printing Office, 1950.
5. A. C. Lazer, P. McKenna, Large-amplitude periodic oscillations in suspension bridges: some new connections with nonlinear analysis, *SIAM Rev.*, **32** (1990), 537–578. https://doi.org/10.1137/1032120
6. P. J. McKenna, W. Walter, Nonlinear oscillations in a suspension bridge, *Arch. Ration. Mech. Anal.*, **98** (1987), 167–177. https://doi.org/10.1007/BF00251232
7. G. Arioli, F. Gazzola, A new mathematical explanation of what triggered the catastrophic torsional mode of the tacoma narrows bridge, *Appl. Math. Modell.*, **39** (2015), 901–912. https://doi.org/10.1016/j.apm.2014.06.022
8. J. M. W. Brownjohn, Observations on non-linear dynamic characteristics of suspension bridges, *Earthquake Eng. Struct. Dyn.*, **23** (1994), 1351–1367. https://doi.org/10.1002/eqe.4290231206
9. B. Breuer, J. Horák, P. J. McKenna, M. Plum, A computer-assisted existence and multiplicity proof for travelling waves in a nonlinearly supported beam, *J. Differ. Equations*, **224** (2006), 60–97. https://doi.org/10.1016/j.jde.2005.07.016
10. Z. Ding, On nonlinear oscillations in a suspension bridge system, *Trans. Amer. Math. Soc.*, **354** (2002), 265–274. https://doi.org/10.1090/S0002-9947-01-02864-1
11. P. J. McKenna, W. Walter, Travelling waves in a suspension bridge, *SIAM J. Appl. Math.*, **50** (1990), 703–715. https://doi.org/10.1137/0150041
12. Y. Rocard, *Dynamic Instability: Automobiles, Aircraft, Suspension Bridges*, C. Lockwood, 1957.
13. A. Ferrero, F. Gazzola, A partially hinged rectangular plate as a model for suspension bridges, *Discrete Contin. Dyn. Syst.*, **35** (2015), 5879–5908. https://doi.org/10.3934/dcds.2015.35.5879

14. S. A. Messaoudi, Global existence and nonexistence in a system of petrovsky, *J. Math. Anal. Appl.*, **265** (2002), 296–308. https://doi.org/10.1006/jmaa.2001.7697

15. Y. Wang, Finite time blow-up and global solutions for fourth order damped wave equations, *J. Math. Anal. Appl.*, **418** (2014), 713–733. https://doi.org/10.1016/j.jmaa.2014.04.015

16. S. E. Mukiawa, Existence and general decay estimate for a nonlinear plate problem, *Boundary Value Probl.*, **2018** (2018), 11. https://doi.org/10.1186/s13661-018-0931-0

17. S. A. Messaoudi, S. E. Mukiawa, Existence and stability of fourth-order nonlinear plate problem, *Nonauton. Dyn. Syst.*, **6** (2019), 81–98. https://doi.org/10.1515/msds-2019-0006

18. J. D. Audu, S. E. Mukiawa, D. S. A. Júnior, General decay estimate for a two-dimensional plate equation with time-varying feedback and time-varying coefficient, *Results Appl. Math.*, **12** (2021), 100219. https://doi.org/10.1016/j.rinam.2021.100219

19. R. Xu, W. Lian, X. Kong, Y. Yang, Fourth order wave equation with nonlinear strain and logarithmic nonlinearity, *Appl. Numer. Math.*, **141** (2019), 185–205. https://doi.org/10.1016/j.apnum.2018.06.004

20. W. He, D. Qin, Q. Wu, Existence, multiplicity and nonexistence results for kirchhoff type equations, *Adv. Nonlinear Anal.*, **10** (2021), 616–635. https://doi.org/10.1515/anona-2020-0154

21. J. N. Wang, A. Alsaedi, B. Ahmad, Y. Zhou, Well-posedness and blow-up results for a class of nonlinear fractional rayleigh-stokes problem, *Adv. Nonlinear Anal.*, **11** (2022), 1579–1597. https://doi.org/10.1515/anona-2022-0249

22. M. Al-Gwaiz, V. Benci, F. Gazzola, Bending and stretching energies in a rectangular plate modeling suspension bridges, *Nonlinear Anal. Theory Methods Appl.*, **106** (2014), 18–34. https://doi.org/10.1016/j.na.2014.04.011

23. M. M. Cavalcanti, W. J. Corrêa, R. Fukuoka, Z. Hajjej, Stabilization of a suspension bridge with locally distributed damping, *Math. Control Signals Syst.*, **30** (2018), 20. https://doi.org/10.1007/s00498-018-0226-0

24. A. D. D. Cavalcanti, M. M. Cavalcanti, W. J. Corrêa, Z. Hajjej, M. S. Cortés, R. V. Asem, Uniform decay rates for a suspension bridge with locally distributed nonlinear damping, *J. Franklin Inst.*, **357** (2020), 2388–2419. https://doi.org/10.1016/j.jfranklin.2020.01.004

25. F. Gazzola, *Mathematical Models for Suspension Bridges*, 2015. https://doi.org/10.1007/978-3-319-15434-3

26. W. Liu, H. Zhuang, Global existence, asymptotic behavior and blow-up of solutions for a suspension bridge equation with nonlinear damping and source terms, *Nonlinear Differ. Equations Appl. NoDEA*, **24** (2017), 67. https://doi.org/10.1007/s00030-017-0491-5

27. V. F. Jr, F. Gazzola, E. M. dos Santos, Instability of modes in a partially hinged rectangular plate, *J. Differ. Equations*, **261** (2016), 6302–6340. https://doi.org/10.1007/s00030-017-0491-5

28. I. Lasiecka, D. Tataru, Uniform boundary stabilization of semilinear wave equations with nonlinear boundary damping, *Differ. Integr. Equations*, **6** (1993), 507–533. Available from: file:///C:/Users/97380/Downloads/1370378427.pdf.

29. P. Martinez, A new method to obtain decay rate estimates for dissipative systems, *ESAIM: Control Optim. Calculus Var.*, **4** (1999), 419–444. https://doi.org/10.1051/cocv:1999116

30. A. Guesmia, *Inégalités intégrales et applications à la stabilisation des systèmes distribués non dissipatifs*, Ph.D thesis, Université de Metz, 2006. Available from: https://hal.inria.fr/tel-01283591/.

31. V. Georgiev, G. Todorova, Existence of a solution of the wave equation with nonlinear damping and source terms, *J. Differ. Equations*, **109** (1994), 295–308. https://doi.org/10.1006/jdeq.1994.1051

32. J. Y. Park, T. G. Ha, Existence and asymptotic stability for the semilinear wave equation with boundary damping and source term, *J. Math. Phys.*, **49** (2008), 053511. https://doi.org/10.1063/1.2919886

33. V. I. Arnol'd, *Mathematical Methods of Classical Mechanics*, Springer, 2013.

34. M. T. Lacroix-Sonrier, *Distributions, espaces de Sobolev: Applications*, Ellipses, 1998.

AIMS Press