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Research article

An analysis of two degenerate double-Hopf bifurcations

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Abstract: The generic double-Hopf bifurcation is presented in detail in literature in textbooks like references. In this paper we complete the study of the double-Hopf bifurcation with two degenerate (or nongeneric) cases. In each case one of the generic conditions is not satisfied. The normal form and the corresponding bifurcation diagrams in each case are obtained. New possibilities of behavior which do not appear in the generic case were found.

Keywords: double Hopf bifurcation; degenerate double Hopf bifurcation; asymptotic analysis; bifurcation diagrams; normal forms

1. Introduction

The theory of bifurcations is crucial in understanding qualitative properties of differential systems depending on one or more parameters. Bifurcations of codimension one and two are well-addressed in the literature [1–3] when they are non-degenerate, but many open problems emerge when they become degenerate [4] or the codimension is beyond 2.

The classical normal form of the double-Hopf bifurcation in differential systems of dimension 4 is based on six generic conditions (HH.0)–(HH.5) [1]. Some of the pioneering papers reporting results on non-degenerate double-Hopf bifurcation are references [5–8].

If one or more generic conditions fail to be satisfied, a degeneracy arises and, thus, the generic results are not valid anymore. In this work we study two degenerate double-Hopf bifurcations. The degeneracy we refer at is related to generic conditions which are necessary in obtaining normal forms.

Another added value of our results is that we discovered new bifurcation diagrams and new phase portraits and, thus, new properties of double-Hopf bifurcation which, to our knowledge, were not previously reported in the literature. The results emerged from the analysis of a new system which appeared due to degeneracy.

Our study is performed in a general framework and produces generic results which can be applied in particular systems exhibiting this bifurcation. The double-Hopf bifurcation (degenerate or not) can be often met in practical models based on differential systems and having a large number of equations (at least four) [4, 9–12]. Interesting results on discrete time systems undergoing double-Hopf bifurcations are presented in [11]. Thus, understanding better this bifurcation and obtaining new generic properties of it is desirable and important for particular models describing real-world phenomena. This is the purpose of the present work.

The paper is organized as follows. In Section 2 the classical normal form obtained in [1] using six generic conditions (HH.0)–(HH.5) is presented and a new normal form up to order 3 in polar coordinates is obtained as two of the generic conditions, namely (HH.1) and (HH.3) fail. In Section 3, the dynamics and bifurcation of this normal form, around the origin, is analyzed as the condition (HH.3) is not satisfied, but (HH.1) is valid, while in Section 4 a similar study is done when the condition (HH.1) is not fulfilled, but (HH.3) is satisfied. In Section 5 some conclusions and the relationship between the dynamics of the 2D amplitude system and the dynamics of the 4D truncated normal form are given.

2. About the double-Hopf bifurcation

Consider a differential system of the form

$$\dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^4, \quad \alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2, \tag{2.1}$$

with f smooth, $\dot{x} = \frac{dx}{dt}$, and assume that $x = 0$ is an equilibrium point of the system for all α with $|\alpha| = \sqrt{\alpha_1^2 + \alpha_1^2}$ small enough, that is, $f(0, \alpha) \equiv 0$; $x = 0$ stands for $x = (0, 0, 0, 0)$ and $\alpha = 0$ for $\alpha = (0, 0)$. The system (2.1) can be written as

$$\dot{x} = A(\alpha)x + F(x, \alpha) \tag{2.2}$$

where $F(x, \alpha) = O\left(|x|^2\right)$ is a smooth function denoting Taylor rest with terms of order at least 2.

Assume the matrix $A(\alpha)$ has two pairs of simple complex-conjugate eigenvalues λ_1, $\overline{\lambda}_1$, λ_2, $\overline{\lambda}_2$,

$$\lambda_1(\alpha) = \mu_1(\alpha) + i\omega_1(\alpha), \quad \lambda_2(\alpha) = \mu_2(\alpha) + i\omega_2(\alpha)$$

for all sufficiently small $|\alpha|$, where $\mu_{1,2}(\alpha)$ and $\omega_{1,2}(\alpha)$ are smooth functions of α such that $\mu_1(0) = \mu_2(0) = 0$, respectively, $\omega_1(0) = \omega_{10} > 0$ and $\omega_2(0) = \omega_{20} > 0$. When these conditions are satisfied, the origin of system (2.1) is called a double-Hopf or Hopf-Hopf singularity, and, under supplementary conditions, a double-Hopf bifurcation occurs at $\alpha = 0$.

In complex coordinates, the system (2.2) can be further reduced to the form

$$\begin{cases} \dot{z}_1 = \lambda_1(\alpha)z_1 + g(z_1, \overline{z}_1, z_2, \overline{z}_2, \alpha) \\ \dot{z}_2 = \lambda_2(\alpha)z_2 + h(z_1, \overline{z}_1, z_2, \overline{z}_2, \alpha) \end{cases} \tag{2.3}$$

where

$$g(z_1, \overline{z}_1, z_2, \overline{z}_2, \alpha) = \sum_{j+k+l+m \geq 2} g_{jklm}(\alpha) z_1^j \overline{z}_1^k z_2^l \overline{z}_2^m$$

and

$$h(z_1, \overline{z}_1, z_2, \overline{z}_2, \alpha) = \sum_{j+k+l+m\geq 2} h_{jklm}(\alpha) z_1^j \overline{z}_1^k z_2^l \overline{z}_2^m.$$

Other two equations are the conjugates of system (2.3) and are omitted in what follows.

A first normal form corresponding to system(2.3) is described in the following lemma, reported in [1].

Lemma 2.1. *Assume that the following conditions take place [1]*

$$\text{(HH.0)} \qquad k\omega_{10} \neq l\omega_{20}, \quad k, l > 0, \quad k + l \leq 5.$$

Then, there exists a locally defined, smooth and smoothly parameter-dependent, invertible transformation of the complex variables that reduces system (2.3) *for all sufficiently small* $|\alpha|$ *into the following form:*

$$\begin{cases} \dot{w}_1 = \lambda_1 w_1 + w_1 |w_1|^2 (a_1 + ib_1) + w_1 |w_2|^2 (a_2 + ib_2) + w_1 |w_1|^4 (a_3 + ib_3) \\ \quad + w_1 |w_1|^2 |w_2|^2 (a_4 + ib_4) + w_1 |w_2|^4 (a_5 + ib_5) + O\left(\|(w_1, \overline{w}_1, w_2, \overline{w}_2)\|^6\right) \\ \dot{w}_2 = \lambda_2 w_2 + w_2 |w_1|^2 (c_1 + id_1) + w_2 |w_2|^2 (c_2 + id_2) + w_2 |w_1|^4 (c_3 + id_3) \\ \quad + w_2 |w_1|^2 |w_2|^2 (c_4 + id_4) + w_2 |w_2|^4 (c_5 + id_5) + O\left(\|(w_1, \overline{w}_1, w_2, \overline{w}_2)\|^6\right) \end{cases} \tag{2.4}$$

where $w_{1,2}$ *are complex-valued functions,* $\|(w_1, \overline{w}_1, w_2, \overline{w}_2)\|^2 = |w_1|^2 + |w_2|^2$.

In system (2.4) we used the following notations: $a_1 + ib_1 = G_{2100}(\alpha)$, $a_2 + ib_2 = G_{1011}(\alpha)$, $a_3 + ib_3 = G_{3200}(\alpha)$, $a_4 + ib_4 = G_{2111}(\alpha)$ and $a_5 + ib_5 = G_{1022}(\alpha)$, respectively, $c_1 + id_1 = H_{1110}(\alpha)$, $c_2 + id_2 = H_{0021}(\alpha)$, $c_3 + id_3 = H_{2210}(\alpha)$, $c_4 + id_4 = H_{1121}(\alpha)$ and $c_5 + id_5 = H_{0032}(\alpha)$, where $a_i = a_i(\alpha)$, $b_i = b_i(\alpha)$, $c_i = c_i(\alpha)$ and $d_i = d_i(\alpha)$, $i = \overline{1,5}$, are real smooth functions of α which depend on the coefficients $g_{ijkl}(\alpha)$ and $h_{ijkl}(\alpha)$. The coefficients of third order terms in system (2.4) are given explicitly in [1].

The system (2.4) has been transformed further in [1] to a normal form based on six generic conditions.

In polar coordinates $w_1 = r_1 e^{i\varphi_1}$ and $w_2 = r_2 e^{i\varphi_2}$, the system (2.4) becomes

$$\begin{cases} \dot{r}_1 = r_1\left(\mu_1 + a_1 r_1^2 + a_2 r_2^2 + a_3 r_1^4 + a_4 r_1^2 r_2^2 + a_5 r_2^4\right) + \Phi_1(r_1, r_2, \varphi_1, \varphi_2, \alpha) \\ \dot{r}_2 = r_2\left(\mu_2 + c_1 r_1^2 + c_2 r_2^2 + c_3 r_1^4 + c_4 r_1^2 r_2^2 + c_5 r_2^4\right) + \Phi_2(r_1, r_2, \varphi_1, \varphi_2, \alpha) \\ \dot{\varphi}_1 = \omega_1 + \Psi_1(r_1, r_2, \varphi_1, \varphi_2, \alpha) \\ \dot{\varphi}_2 = \omega_2 + \Psi_2(r_1, r_2, \varphi_1, \varphi_2, \alpha) \end{cases}, \tag{2.5}$$

where the real functions Φ_k, Ψ_k are smooth functions of their arguments and are 2π-periodic in φ_j, $\Phi_k = O\left(\left(r_1^2 + r_2^2\right)^3\right)$, $\Psi_k(0, 0, \varphi_1, \varphi_2) = 0$.

The main dynamics of the system (2.5) is given by its first two equations

$$\begin{cases} \dot{r}_1 = r_1\left(\mu_1 + a_1 r_1^2 + a_2 r_2^2 + a_3 r_1^4 + a_4 r_1^2 r_2^2 + a_5 r_2^4\right) + O\left(\left(r_1^2 + r_2^2\right)^3\right) \\ \dot{r}_2 = r_2\left(\mu_2 + c_1 r_1^2 + c_2 r_2^2 + c_3 r_1^4 + c_4 r_1^2 r_2^2 + c_5 r_2^4\right) + O\left(\left(r_1^2 + r_2^2\right)^3\right) \end{cases}, \tag{2.6}$$

the other two describing only the rate of rotation of an orbit, and, because $\omega_{1,2}(0) \neq 0$, they can be approximated by

$$\dot{\varphi}_1 = \omega_{10}, \quad \dot{\varphi}_2 = \omega_{20}.$$

Changing to $\rho_i = r_i^2$, $i = 1,2$, and truncating the higher order terms, system (2.6) leads to the planar system

$$\begin{cases} \dot{\rho}_1 = 2\rho_1\left(\mu_1 + a_1\rho_1 + a_2\rho_2 + a_3\rho_1^2 + a_4\rho_1\rho_2 + a_5\rho_2^2\right) \\ \dot{\rho}_2 = 2\rho_2\left(\mu_2 + c_1\rho_1 + c_2\rho_2 + c_3\rho_1^2 + c_4\rho_1\rho_2 + c_5\rho_2^2\right) \end{cases}, \tag{2.7}$$

referred as the amplitude equations; $\dot{\rho}_i = \frac{d\rho_i}{dt}$, $i = 1,2$.

In the hypotheses

$$\begin{array}{ll} \text{(HH.0)} & k\omega_1(0) \neq l\omega_2(0), \quad k,l > 0, \quad k+l \leq 5; \\ \text{(HH.1)} & p_{11}(0) = Re\,(G_{2100}(0)) \neq 0; \\ \text{(HH.2)} & p_{12}(0) = Re\,(G_{1011}(0)) \neq 0; \\ \text{(HH.3)} & p_{21}(0) = Re\,(H_{1110}(0)) \neq 0; \\ \text{(HH.4)} & p_{22}(0) = Re\,(H_{0021}(0)) \neq 0; \\ \text{(HH.5)} & \text{the map } \alpha \longrightarrow \mu(\alpha) \text{ is regular at } \alpha = 0, \end{array} \tag{2.8}$$

in [1], the system (2.4) has been transformed further into the following normal form

$$\begin{cases} \dot{r}_1 = r_1\left(\mu_1 + p_{11}r_1^2 + p_{12}r_2^2 + s_1r_2^4\right) + \Phi_1(r_1,r_2,\varphi_1,\varphi_2,\alpha) \\ \dot{r}_2 = r_2\left(\mu_2 + p_{21}r_1^2 + p_{22}r_2^2 + s_2r_1^4\right) + \Phi_2(r_1,r_2,\varphi_1,\varphi_2,\alpha) \\ \dot{\varphi}_1 = \omega_1 + \Psi_1(r_1,r_2,\varphi_1,\varphi_2,\alpha) \\ \dot{\varphi}_2 = \omega_2 + \Psi_2(r_1,r_2,\varphi_1,\varphi_2,\alpha) \end{cases}, \tag{2.9}$$

where the real functions Φ_k, Ψ_k are smooth functions of their arguments and are 2π-periodic in φ_j, $\Phi_k = O\left(\left(r_1^2 + r_2^2\right)^3\right)$, $\Psi_k(0,0,\varphi_1,\varphi_2) = 0$.

Consequently, the truncated amplitude system obtained in [1] reads:

$$\begin{cases} \dot{\rho}_1 = 2\rho_1\left(\mu_1 + p_{11}\rho_1 + p_{12}\rho_2 + s_1\rho_2^2\right) \\ \dot{\rho}_2 = 2\rho_2\left(\mu_2 + p_{21}\rho_1 + p_{22}\rho_2 + s_2\rho_1^2\right) \end{cases}. \tag{2.10}$$

This system is analyzed in two cases, the "simple case" as $p_{11}(0)p_{22}(0) > 0$ and the "difficult case" as $p_{11}(0)p_{22}(0) < 0$.

The study of the generic Hopf-Hopf bifurcation is also done in [2], starting with the amplitude system (2.6), truncated up to third order terms, in the nondegeneracy hypotheses

$$a_1(0) \neq 0, a_2(0) \neq 0, c_1(0) \neq 0, c_2(0) \neq 0, \tag{2.11}$$

and

$$a_1(0)c_2(0) - a_2(0)c_1(0) \neq 0. \tag{2.12}$$

In this paper we study the degenerate Hopf-Hopf bifurcation when conditions (HH.1) and (HH.3) are not satisfied. In order to do this, we derive a new normal form for the amplitude system.

Theorem 2.2. *Assume that the following three generic conditions are satisfied:*

(HH.2) $a_2(0) = Re[G_{1011}(0)] \neq 0$,

(HH.4) $c_2(0) = Re[H_{0021}(0)] \neq 0$,

(HH.5) *the map* $\alpha \mapsto (\mu_1(\alpha), \mu_2(\alpha))$ *is regular at* $\alpha = 0$.

Then, system (2.6) *is locally topologically equivalent around the origin O, for all sufficiently small* $|\alpha|$, *to the following system*

$$\begin{cases} \frac{d\rho_1}{d\tau} = 2\rho_1\left[\mu_1 + p_{11}\rho_1 + p_{12}\rho_2 + p_{13}\rho_1^2 + p_{14}\rho_1\rho_2\right] + \sqrt{\rho_1}O\left((\rho_1+\rho_2)^3\right) \\ \frac{d\rho_2}{d\tau} = 2\rho_2\left[\mu_2 + p_{21}\rho_1 + p_{22}\rho_2 + p_{23}\rho_1^2 + p_{25}\rho_2^2\right] + \sqrt{\rho_2}O\left((\rho_1+\rho_2)^3\right) \end{cases}, \tag{2.13}$$

where

$$p_{11} = a_1 + \mu_1 n_1, p_{12} = a_2 + \mu_1 n_2, p_{13} = a_3 + a_1 n_1, p_{14} = a_4 + a_1 n_2 + a_2 n_1$$

and

$$p_{21} = c_1 + \mu_2 n_1, p_{22} = c_2 + \mu_2 n_2, p_{23} = c_3 + c_1 n_1, p_{25} = c_5 + c_2 n_2,$$

with $n_2(\mu) = -\frac{a_5(\mu)}{a_2(\mu)}$ *and* $n_1(\mu) = -\frac{c_4(\mu)+c_1(\mu)n_2(\mu)}{c_2(\mu)}$, *where* $p_{ij} = p_{ij}(\mu)$ *for* $i = 1,2$ *and* $j = \overline{1,5}$, *are well-defined and smooth functions for all* $|\mu|$ *small enough.*

Proof. We rescale the time by

$$dt = (1 + n_1\rho_1 + n_2\rho_2)\, d\tau$$

where $n_i = n_i(\alpha)$, $i = \overline{1,2}$, are smooth functions which will be determined later. Then, system (2.6) is orbitally equivalent to the system

$$\begin{cases} \frac{d\rho_1}{d\tau} = 2\rho_1\left[\mu_1 + p_{11}\rho_1 + p_{12}\rho_2 + p_{13}\rho_1^2 + p_{14}\rho_1\rho_2 + p_{15}\rho_2^2\right] + \sqrt{\rho_1}O\left(|\rho|^3\right) \\ \frac{d\rho_2}{d\tau} = 2\rho_2\left[\mu_2 + p_{21}\rho_1 + p_{22}\rho_2 + p_{23}\rho_1^2 + p_{24}\rho_1\rho_2 + p_{25}\rho_2^2\right] + \sqrt{\rho_2}O\left(|\rho|^3\right) \end{cases},$$

where $p_{11} = a_1 + \mu_1 n_1$, $p_{12} = a_2 + \mu_1 n_2$, $p_{13} = a_3 + a_1 n_1$, $p_{14} = a_4 + a_1 n_2 + a_2 n_1$, and $p_{15} = a_5 + a_2 n_2$, respectively $p_{21} = c_1 + \mu_2 n_1$, $p_{22} = c_2 + \mu_2 n_2$, $p_{23} = c_3 + c_1 n_1$, $p_{24} = c_4 + c_1 n_2 + c_2 n_1$ and $p_{25} = c_5 + c_2 n_2$.

Since $a_2(0) \neq 0$ and $c_2(0) \neq 0$, one can nullify the coefficients p_{15} and p_{24} by taking $n_2(\alpha) = -\frac{a_5(\alpha)}{a_2(\alpha)}$ and $n_1(\alpha) = -\frac{c_4(\alpha)+c_1(\alpha)n_2(\alpha)}{c_2(\alpha)}$. Notice that $n_1(\alpha)$ and $n_2(\alpha)$ are well-defined for all $|\alpha|$ small enough, and so are $p_{ij}(\alpha)$.

Assume the map $\alpha \mapsto (\mu_1(\alpha), \mu_2(\alpha))$ is regular at $\alpha = 0$, that is,

$$\left.\frac{\partial\mu_1}{\partial\alpha_1}\frac{\partial\mu_2}{\partial\alpha_2} - \frac{\partial\mu_1}{\partial\alpha_2}\frac{\partial\mu_2}{\partial\alpha_1}\right|_{\alpha=0} \neq 0.$$

From the Inverse Function Theorem this condition ensures that for any $\mu_1, \mu_2 \in \mathbb{R}$ with $|\mu| = \sqrt{\mu_1^2 + \mu_2^2}$ small enough, there exist $\alpha_1 = \alpha_1(\mu)$ and $\alpha_2 = \alpha_2(\mu)$ which are obtained locally from the system

$$\mu_{1,2} = \frac{\partial\mu_{1,2}}{\partial\alpha_1}(0,0)\,\alpha_1 + \frac{\partial\mu_{1,2}}{\partial\alpha_2}(0,0)\,\alpha_2.$$

This implies that, one can consider further $\mu = (\mu_1, \mu_2)$ as the parameter of the system. Thus, $\omega_{1,2} = \omega_{1,2}(\mu)$, $a_i = a_i(\mu)$, $b_i = b_i(\mu)$, $c_i = c_i(\mu)$, $d_i = d_i(\mu)$, $p_{ij} = p_{ij}(\mu)$, $i = 1,2$, $j = \overline{1,5}$. Since $\mu_{1,2}(0) = 0$, the properties $\omega_1(0) = \omega_{10} > 0$ and $\omega_2(0) = \omega_{20} > 0$ are preserved. The theorem is proved. □

Remark that the coefficients p_{ij} in system (2.13) are not the same as the ones in system (2.10), but the values $p_{11}(0), p_{12}(0), p_{21}(0), p_{22}(0)$ are the same as in system (2.8).

The truncated 4D system of system (2.13) reads

$$\begin{cases} \frac{d\rho_1}{d\tau} = 2\rho_1\left[\mu_1 + p_{11}\rho_1 + p_{12}\rho_2 + p_{13}\rho_1^2 + p_{14}\rho_1\rho_2\right], \\ \frac{d\rho_2}{d\tau} = 2\rho_2\left[\mu_2 + p_{21}\rho_1 + p_{22}\rho_2 + p_{23}\rho_1^2 + p_{25}\rho_2^2\right], \\ \frac{d\varphi_1}{d\tau} = \omega_{10}, \\ \frac{d\varphi_2}{d\tau} = \omega_{20}, \end{cases} \tag{2.14}$$

where $p_{12}(0)p_{22}(0) \neq 0$, because $p_{12}(0) = a_2(0) \neq 0$, from (HH.2) and $p_{22}(0) = c_2(0) \neq 0$, from (HH.4).

We aim to tackle in this work the normal form system (2.13), truncated up to third order terms, when $a_2(0)c_2(0) > 0$.

Assume $p_{12}(0) = a_2(0) < 0$ and $p_{22}(0) = c_2(0) < 0$. Make the changes

$$\xi_1 = -p_{12}(\mu)\rho_1, \xi_2 = -p_{22}(\mu)\rho_2, t = 2\tau. \tag{2.15}$$

The transformation $(\rho_1, \rho_2) \longmapsto (\xi_1, \xi_2)$ is well defined for all $|\mu|$ small enough, because $p_{12}(0) \neq 0$ and $p_{22}(0) \neq 0$ are satisfied from (HH.2) and (HH.4), and it is nonsingular as $a_2(0)c_2(0) \neq 0$. Since $\frac{d\xi_1}{dt} = -p_{12}(\mu)\frac{d\rho_1}{d\tau}\frac{d\tau}{dt}$ and $\frac{d\xi_2}{dt} = -p_{22}(\mu)\frac{d\rho_2}{d\tau}\frac{d\tau}{dt}$, the form system (2.13) in its lowest terms in (ξ_1, ξ_2), known as the truncated form, becomes

$$\begin{cases} \dot{\xi}_1 = \xi_1\left[\mu_1 - \theta(\mu)\xi_1 - \gamma(\mu)\xi_2 + M(\mu)\xi_1\xi_2 + N(\mu)\xi_1^2\right] \\ \dot{\xi}_2 = \xi_2\left[\mu_2 - \delta(\mu)\xi_1 - \xi_2 + S(\mu)\xi_1^2 + P(\mu)\xi_2^2\right] \end{cases}, \tag{2.16}$$

where $\theta(\mu) = \frac{p_{11}}{p_{12}}(\mu)$, $\gamma(\mu) = \frac{p_{12}}{p_{22}}(\mu)$, $M(\mu) = \frac{p_{14}}{p_{12}p_{22}}(\mu)$, $N(\mu) = \frac{p_{13}}{p_{12}^2}(\mu)$, $\delta(\mu) = \frac{p_{21}}{p_{12}}(\mu)$, $S(\mu) = \frac{p_{23}}{p_{12}^2}(\mu)$ and $P(\mu) = \frac{p_{25}}{p_{22}^2}(\mu)$. However, in what follows, some of these expressions are needed only at $\mu = 0$.

The dot over quantities stands now for the derivatives with respect to the new time. Denote also by f the smooth vector field associated with system (2.16).

Remark 1. Due to the transformation system (2.15), the stability of an equilibrium point in the 2D system (2.16) is preserved in the 4D system (2.14); see also [1].

Remark 2. 1) Notice that $\theta(0) = \frac{a_1(0)}{a_2(0)}$, $\gamma(0) = \frac{a_2(0)}{c_2(0)}$ and $\delta(0) = \frac{c_1(0)}{a_2(0)}$ are well-defined from (HH.2), (HH.4). Moreover, $\gamma(0) > 0$ while $\theta(0)\delta(0)$ can be 0.

2) When $p_{12}(0) > 0$ and $p_{22}(0) > 0$, the change $\xi_1 = p_{12}(\mu)\rho_1, \xi_2 = p_{22}(\mu)\rho_2, t = -2\tau$ lead to

$$\begin{cases} \dot{\xi}_1 = \xi_1\left[-\mu_1 - \theta(\mu)\xi_1 - \gamma(\mu)\xi_2 - M(\mu)\xi_1\xi_2 - N(\mu)\xi_1^2\right] \\ \dot{\xi}_2 = \xi_2\left[-\mu_2 - \delta(\mu)\xi_1 - \xi_2 - S(\mu)\xi_1^2 - P(\mu)\xi_2^2\right] \end{cases}, \tag{2.17}$$

that can be reduced to system (2.16).

As $\rho_i = r_i^2 \geq 0$, $i = 1, 2$, taking into account system (2.15) and the fact that $p_{12}(0) < 0$, $p_{22}(0) < 0$, it follows that the system (2.16) must be studied only on the set (the first quadrant)

$$D = \left\{(\xi_1, \xi_2) \in \mathbb{R}^2, \xi_1 \geq 0, \xi_2 \geq 0\right\}.$$

Thus, only the equilibria of system (2.16) situated in D will be analyzed.

Remark 3. The lines $\xi_1 = 0$ and $\xi_2 = 0$ are invariant curves for system (2.16), thus the set D is invariant with respect to the dynamical system associated to system (2.16).

In the following we analyze the truncated amplitude system (2.16) in two cases when the Hopf-Hopf bifurcation degenerates, namely (i) $\delta(0) = 0$, $\theta(0) \neq 0$; (ii) $\theta(0) = 0$, $\delta(0) \neq 0$.

Remind that system (2.16) which is analyzed in the next two sections was obtained in the hypotheses (HH.0), (HH.2), (HH.4), (HH.5), and, as $p_{12}(0) < 0$, $p_{22}(0) < 0$, we have $\gamma(0) > 0$.

In addition, our study is done for μ in a neighborhood of the origin V_ε given by $|\mu| < \varepsilon$, for $\varepsilon > 0$ sufficiently small.

As all the coefficients are smooth functions depending on the parameter μ, for $\mu \in V_\varepsilon$ we can write

$$\theta(\mu) = \theta(0) + \frac{\partial\theta}{\partial\mu_1}(0)\,\mu_1 + \frac{\partial\theta}{\partial\mu_2}(0)\,\mu_2 + O(|\mu|^2),$$

and similar for the other coefficients. In all the expressions below only the significant lower order terms in μ_1, μ_2 will be considered, as necessary. Also, in order to save symbols, if, for instance we have $N(0) \neq 0$ we shall denote $N = N(0)$, thus, close to $\mu = 0$, we have $N(\mu) = N + O(|\mu|)$, and so on for the other coefficients.

3. Analysis of system (2.16) when $\delta(0) = 0$ and $\theta(0) \neq 0$

Assume that $\delta(0) = 0$, $\theta(0) \neq 0$. This means that condition (HH.3) is not satisfied, but (HH.1) is valid.

Then $\delta(\mu) = \delta_1\mu_1 + \delta_2\mu_2 + O(|\mu|^2)$, with $\delta_1 = \frac{\partial\delta}{\partial\mu_1}(0)$, $\delta_2 = \frac{\partial\delta}{\partial\mu_2}(0)$. One of the equilibria of system (2.16) is $E_0 = (0,0)$. Two more equilibria

$$E_1 = \left(\frac{1}{\theta}\mu_1 + O\left(\mu_1^2\right), 0\right) \text{ and } E_2 = \left(0, \mu_2 + O\left(\mu_2^2\right)\right)$$

where $\theta = \theta(0)$, bifurcate from E_0 as soon as $\mu_1 \neq 0$, respectively, $\mu_2 \neq 0$, i.e. at the bifurcation lines

$$Y_- = \{(\mu_1, \mu_2), \mu_1 = 0, \mu_2 < 0\}, \quad Y_+ = \{(\mu_1, \mu_2), \mu_1 = 0, \mu_2 > 0\},$$

and

$$X_- = \{(\mu_1, \mu_2), \mu_2 = 0, \mu_1 < 0\}, \quad X_+ = \{(\mu_1, \mu_2), \mu_2 = 0, \mu_1 > 0\},$$

respectively. Their stability is described by Lemma 3.1.

Lemma 3.1. *(1) The eigenvalues of E_0 are μ_1 and μ_2.*
(2) The eigenvalues of E_1 are $-\mu_1 + O\left(\mu_1^2\right)$ and $\left(\mu_2 + \frac{S-\delta_1\theta}{\theta^2}\mu_1^2\right)(1 + O(\mu_1))$, thus, whenever E_1 lies in D, E_1 is either (i) a saddle as $\mu_1\left(\mu_2 + \frac{S-\delta_1\theta}{\theta^2}\mu_1^2\right) > 0$, (ii) an unstable node as $\mu_2 + \frac{S-\delta_1\theta}{\theta^2}\mu_1^2 > 0, \mu_1 < 0$, (iii) a stable node as $\mu_2 + \frac{S-\delta_1\theta}{\theta^2}\mu_1^2 < 0, \mu_1 > 0$.
(3) The eigenvalues of E_2 are $\mu_1 - \gamma\mu_2 + O(\mu_2^2)$ and $-\mu_2 + O(\mu_2^2)$. Therefore, for $|\mu|$ sufficiently small, E_2 is (i) a saddle as $\mu_1 - \gamma\mu_2 > 0$, (ii) a stable node as $\mu_1 - \gamma\mu_2 < 0$, whenever it exists in D.

On the other hand, system (2.16) has the nontrivial equilibrium $E_3 = \left(\xi_1^*, \xi_2^*\right)$, with

$$\begin{cases} \xi_1^* = \frac{1}{\theta}\mu_1\,(1 + O(|\mu|) - \frac{\gamma}{\theta}\mu_2\,(1 + O(|\mu|) \\ \xi_2^* = \mu_2\,(1 + O(|\mu|) + \frac{S-\delta_1\theta}{\theta^2}\mu_1^2\,(1 + O(|\mu|)) \end{cases}.$$

The associated eigenvalues satisfy

$$\begin{aligned} \lambda_1\lambda_2 &= \xi_1^*\xi_2^*\left[\theta + O\left(|\mu|\right)\right], \\ \lambda_1 + \lambda_2 &= -\mu_1 + (\gamma - 1)\mu_2 + O\left(|\mu|^2\right). \end{aligned} \tag{3.1}$$

When $\theta > 0$, E_3 lies in D for parameters in the region

$$R_1 = \left\{(\mu_1, \mu_2) \in V_\varepsilon, \mu_1 - \gamma\mu_2 > 0, \mu_2 + \frac{S - \delta_1\theta}{\theta^2}\mu_1^2 > 0\right\}.$$

It is easy to see that for $\mu \in R_1$, with $|\mu|$ sufficiently small, we have $\lambda_1\lambda_2 > 0$ and the curve $\lambda_1 + \lambda_2 = 0$ does not cross region R_1, thus $\lambda_1 + \lambda_2 < 0$, and E_3 is stable.

When $\theta < 0$, E_3 lies in D for parameters in region

$$R_2 = \left\{(\mu_1, \mu_2) \in V_\varepsilon, \mu_1 - \gamma\mu_2 < 0, \mu_2 + \frac{S - \delta_1\theta}{\theta^2}\mu_1^2 > 0\right\}.$$

By system (3.1), it follows $\lambda_1\lambda_2 < 0$, for sufficiently small $|\mu|$. Thus, E_3 is a saddle. The following result is proved.

Lemma 3.2. *If $\theta < 0$, then E_3 is a saddle, while if $\theta > 0$, the equilibrium E_3 is a hyperbolic attractor, for all sufficiently small $|\mu|$, for which E_3 lies in D.*

Consequently there can be no Hopf bifurcation at E_3.

As the term δ_2 does not influence the topological type of equilibria, we may restrict our attention only to the (θ, δ_1) – plane.

Regions R_1 and R_2 are delimitated by the curves

$$T_1 = \left\{(\mu_1, \mu_2) \in V_\varepsilon, \mu_1 = \gamma\mu_2 + O\left(\mu_2^2\right), \mu_2 > 0\right\}, \tag{3.2}$$

$$T_2 = \left\{(\mu_1, \mu_2) \in V_\varepsilon, \mu_2 = -\frac{S - \delta_1\theta}{\theta^2}\mu_1^2 + O\left(\mu_1^3\right)\right\}. \tag{3.3}$$

We notice that E_3 collides with E_2 on T_1, and it collides with E_1 on T_2.

Proposition 3.3. *System* (2.16) *experiences the following transcritical bifurcations:*
(i) at the point E_0 as the parameter μ_1 varies through the bifurcation value $\mu_1 = 0$, for a fixed $\mu_2 \neq 0$ (when $E_0 = E_1$);
(ii) at the point E_0 as the parameter μ_2 varies through the bifurcation value $\mu_2 = 0$, for a fixed $\mu_1 \neq 0$ (when $E_0 = E_2$);
(iii) at the point E_1 as the parameter (μ_1, μ_2) crosses the curve T_2 (when $E_1 = E_3$);
(iv) at the point E_2 as the parameter (μ_1, μ_2) crosses the curve T_1 (when $E_2 = E_3$).

Proof. We apply Sotomayor Theorem ([13, 14], to prove these statements.

(i) The Jacobian matrix $Df(E_0, \mu_0)$ at $\mu_0 = (0, \mu_2)$, $\mu_2 \neq 0$, has a zero eigenvalue with the right eigenvector $v = (1, 0)^T$ and the left eigenvector $w = (1, 0)^T$. It follows

$$w^T f_{\mu_1}(E_0, \mu_0) = 0, \quad w^T Df_{\mu_1}(E_0, \mu_0) = 1 \neq 0, \quad w^T[D^2 f(E_0, \mu_0)(v, v)] = -2\theta \neq 0,$$

thus the transcritical bifurcation conditions are satisfied.

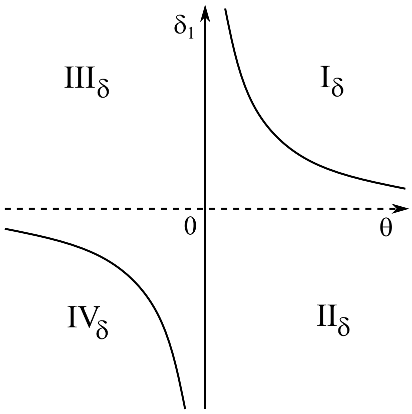

Figure 1. Four regions in the (θ, δ_1) plane, $\gamma > 0$, and $S > 0$.

(ii) The Jacobian matrix $Df(E_0, \mu_0)$ at $\mu_0 = (\mu_1, 0)$, $\mu_1 \neq 0$, has a zero eigenvalue with the right eigenvector $v = (0, 1)^T$ and the left eigenvector $w = (0, 1)^T$. It follows

$$w^T f_{\mu_2}(E_0, \mu_0) = 0, \quad w^T Df_{\mu_2}(E_0, \mu_0) = 1 \neq 0, \quad w^T[D^2 f(E_0, \mu_0)(v, v)] = -2 \neq 0,$$

ensuring the existence of a transcritical bifurcation.

(iii) Consider $\mu_0 \in T_2$, $\mu_1 \neq 0$, and μ_2 as a bifurcation parameter. We find that $v = (-\gamma, \theta)^T$ and $w = (0, 1)^T$ are right and left eigenvectors of the Jacobian matrix $Df(E_1, \mu_0)$, respectively, corresponding to the zero eigenvalue, and

$$\begin{aligned} &w^T f_{\mu_2}(E_1, \mu_0) = 0, \quad w^T Df_{\mu_2}(E_1, \mu_0) = \theta \neq 0, \\ &w^T[D^2 f(E_1, \mu_0)(v, v)] = 2\theta(\delta\gamma - \theta) - 4S\gamma\mu_1 \neq 0, \end{aligned}$$

consequently, for sufficiently small $|\mu|$, the conditions are satisfied.

(iv) Finally, consider $\mu_0 \in T_1$, $\mu_2 \neq 0$, and μ_1 as a bifurcation parameter, thus $\mu_0 = (\gamma\mu_2, \mu_2)$. We find the eigenvectors $v = (1, -\delta)^T$ and $w = (1, 0)^T$, and

$$\begin{aligned} &w^T f_{\mu_1}(E_2, \mu_0) = 0, \quad w^T Df_{\mu_1}(E_2, \mu_0) = 2(\delta\gamma - \theta) + 2M\mu_2 \neq 0, \\ &w^T[D^2 f(E_1, \mu_0)(v, v)] = 1 \neq 0, \end{aligned}$$

for sufficiently small $|\mu|$. □

For a fixed $\gamma > 0$, and $S > 0$, the curves $S - \delta_1\theta = 0$, $\theta = 0$, determine four regions in the $(\theta, \delta_1)-$ plane, illustrated in Figure 1, corresponding to the following cases:

I_δ: $\theta > 0$, $S - \delta_1\theta < 0$;
II_δ: $\theta > 0$, $S - \delta_1\theta > 0$;
III_δ: $\theta < 0$, $S - \delta_1\theta > 0$;
IV_δ: $\theta < 0$, $S - \delta_1\theta < 0$.

For each region I_δ-IV_δ, in the parametric portraits in the (μ_1, μ_2)- plane, the parameter strata are determined by the origin and the bifurcation curves X_-, X_+, Y_-, Y_+, T_1, and T_2. Consequently, the following result is obtained.

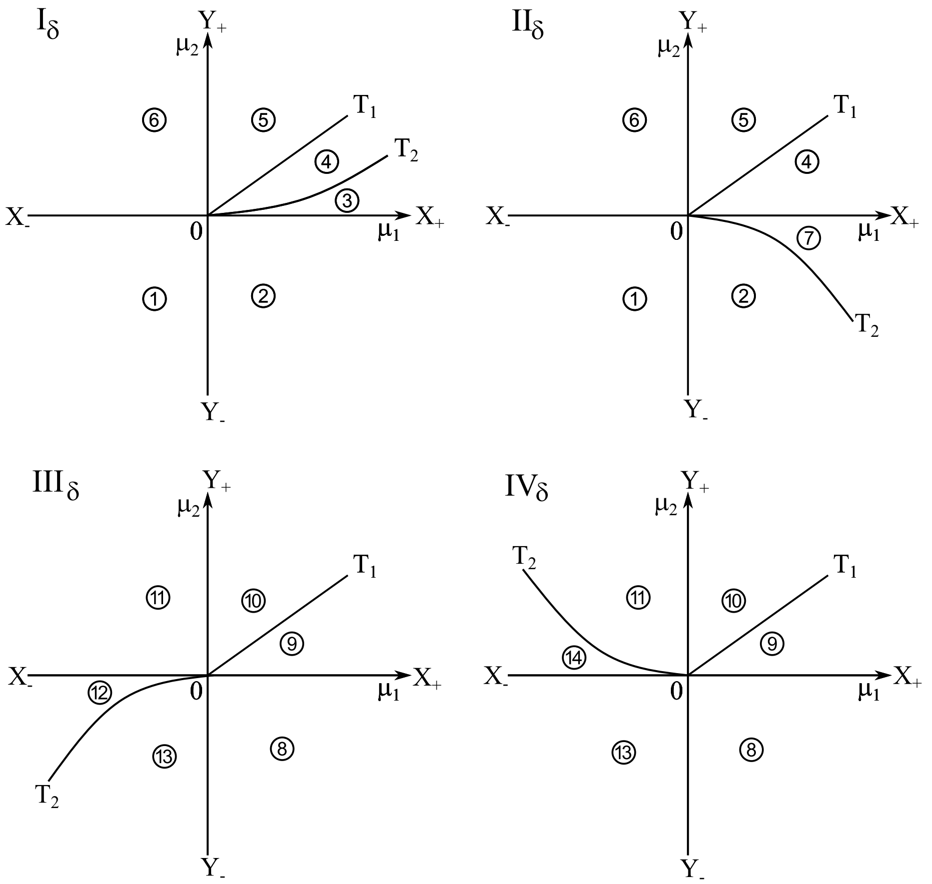

Figure 2. Parametric portraits in the case $\delta = 0$.

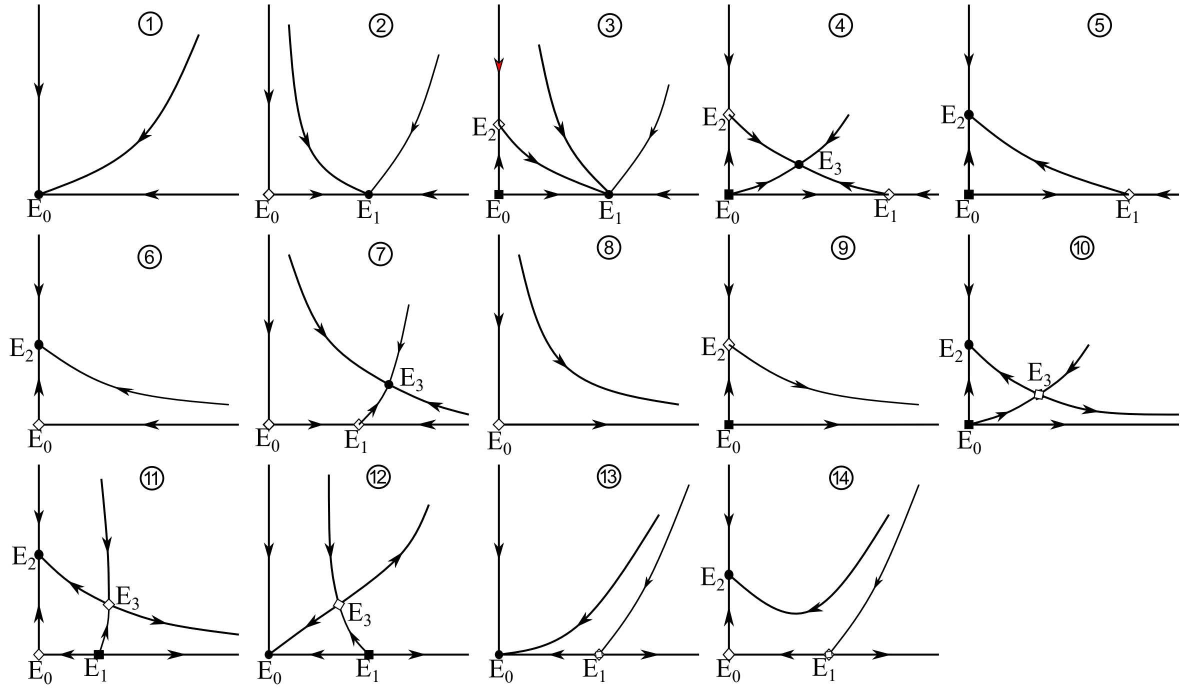

Figure 3. Generic phase portraits in the case $\delta = 0$. Legend: a black disc for an attractor, a black square for a repeller, a diamond for a saddle equilibrium.

Theorem 3.4. *For all* $\gamma > 0, S > 0$, *in the* (θ, δ_1) – *plane, the bifurcation curves consist of*

$$O \cup T_1 \cup T_2 \cup X_- \cup X_+ \cup Y_- \cup Y_+.$$

The four parameter portraits for (θ, δ_1) *in regions* I_δ, II_δ, III_δ, IV_δ *are shown in Figure 2. The 14 generic phase portraits are given in Figure 3.*

In Figure 3 we used the following markers to emphasize the topological type of the equilibria: a black disc for an attractor, a black square for a repeller and a diamond for a saddle point. All of these phase portraits are also found in the nondegenerate double-Hopf bifurcation case [1, 2]. As $S \leq 0$, a similar study can be done.

Remark 4. As proved in Proposition 3.3, each of the curves X_-, X_+, Y_-, Y_+, T_1, T_2 consists of values of transcritical bifurcation (for which two of the four equilibria collide and change topological type), separating two parameter strata, one where both equilibria involved in the transcritical bifurcation lie on D and one where only one of them lies on D. Although for parameters on a bifurcation curve the corresponding equilibrium is a saddle-node, the generic phase portrait is equivalent to the one of the stratum where only one of the two equilibria is in D.

4. Analysis of system (2.16) when $\theta(0) = 0$ and $\delta(0) \neq 0$

Assume $\theta(0) = 0$, $\delta(0) \neq 0$. In this case the condition (HH.1) is not satisfied, and (HH.3) is valid. Then $\theta(\mu) = \theta_1\mu_1 + \theta_2\mu_2 + O(|\mu|^2)$.

In this case, two trivial equilibria of system (2.16) are $E_0 = (0, 0)$, with the eigenvalues μ_1 and μ_2, and $E_2 = \left(0, \mu_2 + O\left(\mu_2^2\right)\right)$ with eigenvalues $\mu_1 - \gamma\mu_2 + O(|\mu|^2)$ and $-\mu_2 + P\mu_2^2(1 + O(|\mu|))$. Remark that these equilibria keep the same form as in the case treated in the previous section, and their stability remains as described by Lemma 3.1.

As $\xi_2 = 0$, the equation giving the equilibrium points reads:

$$\mu_1 - \theta(\mu)\xi_1 + N(\mu)\xi_1^2 = 0. \tag{4.1}$$

Assume that $N(0) \neq 0$, thus $N(\mu) \neq 0$ for sufficiently small $|\mu|$. If $\Delta(\mu) = \theta^2(\mu) - 4N(\mu)\mu_1 \geq 0$, Eq (4.1) has two real solutions

$$\xi_{11}(\mu) = \frac{\theta(\mu) - \sqrt{\Delta(\mu)}}{2N(\mu)}, \quad \xi_{12}(\mu) = \frac{\theta(\mu) + \sqrt{\Delta(\mu)}}{2N(\mu)}, \quad \xi_{11} \leq \xi_{12}.$$

Thus, as $\mu_1 N > 0$, two equilibrium points, $E_{11} = (\xi_{11}, 0)$ and $E_{12} = (\xi_{12}, 0)$, are lying on the positive ξ_1−axis, provided that $(\mu_1, \mu_2) \in R_3$, where R_3 is a region in the parametric plane given by

$$R_3 = \{(\mu_1, \mu_2) \in V_\varepsilon \,|\Delta(\mu) > 0, \mu_1 N > 0, \theta N > 0\}. \tag{4.2}$$

Obviously, as $\Delta(\mu) = 0$, we have $E_{11} = E_{12} = \frac{\theta}{2N}$.

As $\mu_1 N < 0$, we have $\xi_{11} < 0 < \xi_{12}$, thus, only the equilibrium E_{12} lies in D for parameters in region

$$R_4 = \{(\mu_1, \mu_2) \in V_\varepsilon \,|\Delta(\mu) > 0, \mu_1 N < 0\}. \tag{4.3}$$

As $\mu_1 = 0$, we have $E_{11} = E_0$ if $\theta_2\mu_2 N > 0$, while $E_{12} = E_0$ if $\theta_2\mu_2 N < 0$. Using the Implicit Functions Theorem, we find that, for sufficiently small $|\mu|$, $\Delta(\mu) = 0$ for parameters on the curve, denoted also by Δ, given as

$$\Delta = \left\{(\mu_1, \mu_2) \in V_\varepsilon, \mu_1 = \frac{\theta_2^2}{4N}\mu_2^2 + O(\mu_2^3), \theta_2\mu_2 N > 0\right\}.$$

The eigenvalues of $E_{11} = (\xi_{11}, 0)$ satisfy $\lambda_1^{E_{11}} = -\xi_{11}\sqrt{\Delta(\mu)} \leq 0$, and

$$\lambda_2^{E_{11}} = \mu_2 - \frac{S}{N}\mu_1 + \xi_{11}\left(\frac{\theta S}{N} - \delta\right).$$

For $E_{12} = (\xi_{12}, 0)$ we have $\lambda_1^{E_{12}} = \xi_{12}\sqrt{\Delta(\mu)} \geq 0$ and

$$\lambda_2^{E_{12}} = \mu_2 - \frac{S}{N}\mu_1 + \xi_{12}\left(\frac{\theta S}{N} - \delta\right).$$

An important relation for studying the behavior of E_{11} and E_{12} is

$$\lambda_2^{E_{11}}\lambda_2^{E_{12}} = \frac{1}{N}\delta^2\mu_1(1 + O(|\mu|)) + \frac{1}{N}(N - \theta_2\delta)\mu_2^2(1 + O(|\mu|). \tag{4.4}$$

Note that as $\lambda_2^{E_{11}}\lambda_2^{E_{12}} = 0$, we get

$$\lambda_2^{E_{11}} + \lambda_2^{E_{12}} = \frac{1}{N}(2N - \theta_2\delta)\mu_2 + O(\mu_2^2).$$

Lemma 4.1. *As the parameter μ crosses the curve Δ, a saddle-node bifurcation takes place. In addition, if $N > 0$, for parameter sufficiently small, close to the curve Δ, we have:*
(i) E_{11} is an attractor and E_{12} is a saddle as $\mu_2\,(2N - \delta\theta_2) < 0$;
(ii) E_{11} is a saddle and E_{12} is a repeller as $\mu_2\,(2N - \delta\theta_2) > 0$.

Proof. Consider $\mu_0 \in \Delta$. Then $\xi_{11} = \xi_{12} = \frac{\theta_2}{2N}\mu_2 + O(\mu_2^2)$, and the eigenvalues of equilibrium E_{11} are $\lambda_1 = 0$, $\lambda_2 = \frac{2N-\delta\theta_2}{2N}\mu_2 + O(\mu_2^2)$. The Jacobian matrix $Df(E_{11},\mu_0)$, has for the zero eigenvalue the right eigenvector $v = (1,0)^T$ and the left eigenvector $w = (2N - \delta\theta_2, \gamma\theta_2)^T$. It follows

$$w^T f_{\mu_1}(E_0,\mu_0) = \frac{\theta_2}{2N}\mu_2 \neq 0, \quad w^T[D^2 f(E_0,\mu_0)(v,v)] = \theta_2\mu_2 + O(\mu_2^2) \neq 0,$$

for sufficiently small $|\mu|$. Thus, according to Sotomayor Theorem, a saddle-node bifurcation takes place.

For parameters close to the bifurcation curve we have $sign(\lambda_1^{E_{11}}) = -sign(N)$, $sign(\lambda_1^{E_{12}}) = sign(N)$, and $sign(\lambda_2^{E_{11}}) = sign(\lambda_2^{E_{12}}) = sign(\frac{2N-\delta\theta_2}{2N}\mu_2)$. Consequently, if $N > 0$, the equilibrium E_{11} is an attractor and E_{12} is a saddle as $\mu_2\,(2N - \delta\theta_2) < 0$, while as $\mu_2\,(2N - \delta\theta_2) > 0$, the equilibrium E_{11} is a saddle and E_{12} is a repeller. □

Denote by

$$\Delta_+ = \{(\mu_1,\mu_2) \in \Delta, \mu_2\,(2N - \delta\theta_2) > 0\}$$

and

$$\Delta_- = \{(\mu_1,\mu_2) \in \Delta, \mu_2\,(2N - \delta\theta_2) < 0\}.$$

As a consequence of Lemma 4.1, for a parameter μ in Δ_+, sufficiently small, the equilibrium $E_{11}(= E_{12})$ is a saddle-node, with one attractive direction and three repelling directions, while for a parameter μ in Δ_-, the equilibrium E_{11} is a saddle-node, with one repelling direction and three attractive ones.

Consider the curves

$$\begin{aligned} Y'_- &= \{(\mu_1,\mu_2) \in V_\varepsilon, \mu_1 = 0, \theta_2 N\mu_2 < 0\}, \\ Y'_+ &= \{(\mu_1,\mu_2) \in V_\varepsilon, \mu_1 = 0, \theta_2 N\mu_2 > 0\}. \end{aligned}$$

Proposition 4.2. *The following statements hold:*
(i) for $\mu \in Y'_+$, equilibria E_0 and E_{11} coincide; when parameters cross this curve, a transcritical bifurcation takes place, and E_0 and E_{11} interchange the topological type;
(ii) for $\mu \in Y'_-$, equilibria E_0 and E_{12} coincide and are saddle-nodes. A transcritical bifurcation takes place when parameters are crossing the curve Y'_-, and E_0 and E_{12} interchange the topological type.
(iii) for $\mu \in \{(\mu_1,\mu_2), \mu_2 = 0, \mu_1 \neq 0\}$, equilibria E_0 and E_2 coincide and a transcritical bifurcation takes place when parameters are crossing this curve.

Proof. As in Proposition 3.3, we apply Sotomayor Theorem ([13]) to prove these statements.

(i) Consider $\mu_0 \in Y'_+$. As $\mu_1 = 0$ and $\theta_2\mu_2 N > 0$, it follows that $E_{11} = E_0$. The Jacobian matrix $Df(E_0,\mu_0)$ at $\mu_0 = (0,\mu_2)$, $\mu_2 \neq 0$, has a zero eigenvalue with the right eigenvector $v = (1,0)^T$ and the left eigenvector $w = (1,0)^T$. It follows

$$w^T f_{\mu_1}(E_0,\mu_0) = 0, \quad w^T Df_{\mu_1}(E_0,\mu_0) = 1 \neq 0,$$

$$w^T[D^2 f(E_0,\mu_0)(v,v)] = -2\theta_2\mu_2 \neq 0,$$

for sufficiently small $|\mu|$, thus the transcritical bifurcation conditions are satisfied.

(ii) Consider $\mu_0 \in Y'_-$. As $\mu_1 = 0$ and $\theta_2\mu_2 N < 0$, it follows that $\xi_{12} = 0$, thus $E_{12} = E_0$. We obtain the same values for the quantities involved in the transcritical bifurcation conditions as in case (i) .

(iii) The Jacobian matrix $Df(E_0,\mu_0)$ at $\mu_0 = (\mu_1, 0)$, $\mu_1 \neq 0$, has a zero eigenvalue with the right eigenvector $v = (0,1)^T$ and the left eigenvector $w = (0,1)^T$. It follows

$$w^T f_{\mu_2}(E_0,\mu_0) = 0, \quad w^T D f_{\mu_2}(E_0,\mu_0) = 1 \neq 0, \quad w^T[D^2 f(E_0,\mu_0)(v,v)] = -2 \neq 0,$$

ensuring the existence of a transcritical bifurcation. □

For $E_3 = \left(\xi_1^*, \xi_2^*\right)$, solution of system

$$\begin{cases} \mu_1 - (\theta_1\mu_1 + \theta_2\mu_2)\xi_1 - \gamma(\mu)\xi_2 + M(\mu)\xi_1\xi_2 + N(\mu)\xi_1^2 = 0, \\ \mu_2 - \delta(\mu)\xi_1 - \xi_2 + S(\mu)\xi_1^2 + P(\mu)\xi_2^2 = 0, \end{cases}$$

we obtain

$$\begin{aligned} \xi_1^* &= -\frac{1}{\gamma\delta}(\mu_1 - \gamma\mu_2) + O(|\mu|^2), \\ \xi_2^* &= \frac{1}{\gamma}\mu_1(1 + O(|\mu|)) + \frac{N - \theta_2\delta}{\delta^2\gamma}\mu_2^2(1 + O(|\mu|)). \end{aligned}$$

Thus, E_3 is a nontrivial equilibrium for system (2.16) provided

$$\mu \in R_5 = \left\{(\mu_1,\mu_2) \in V_\varepsilon, \delta(\mu_1 - \gamma\mu_2) < 0, \delta^2\mu_1 + (N - \theta_2\delta)\mu_2^2 > 0\right\}.$$

Denote by

$$T_3 = \left\{(\mu_1,\mu_2) \in V_\varepsilon, \mu_1 = \frac{\theta_2\delta - N}{\delta^2}\mu_2^2 + O(\mu_2^3)\right\}$$

the parameter curve for which $\xi_2^* = 0$. Thus R_5 is delimited by T_1 and T_3.

The eigenvalues of E_3 satisfy

$$\lambda_1\lambda_2 = \xi_1^*\xi_2^*(-\gamma\delta + O(|\mu|)) \text{ and } \lambda_1 + \lambda_2 = \frac{1}{\gamma\delta^2}p(\mu), \tag{4.5}$$

where $p(\mu) = -\delta^2\mu_1(1 + O(|\mu|)) + k_1\mu_2^2(1 + O(|\mu|))$ and $k_1 = (\theta_2\delta - N) + \gamma(2N - \theta_2\delta)$.

Applying Implicit Functions Theorem to the implicit equation $p(\mu_1,\mu_2) = 0$, the following lemma is obtained.

Lemma 4.3. *If* $(\theta_2\delta - N) + \gamma(2N - \theta_2\delta) \neq 0$, *there exists a neighborhood* U *of* 0 *in the parameter plane such that* $p(\mu) = 0$ *in* U *iff* $\mu \in U \cap H$, *where*

$$H = \left\{(\mu_1,\mu_2), \mu_1 = \frac{(\theta_2\delta - N) + \gamma(2N - \theta_2\delta)}{\delta^2}\mu_2^2 + O\left(\mu_2^3\right)\right\} \tag{4.6}$$

for $|\mu|$ *sufficiently small.*

As a consequence we obtain the following result concerning the topological type of the nontrivial equilibrium E_3.

Lemma 4.4. *Assume $N > 0$. For sufficiently small $|\mu|$, the following statements hold:*
(1) If $\delta > 0$, then E_3 is a saddle.
(2) If $\delta < 0$, $\theta_2\delta - 2N \geq 0$, then E_3 is a hyperbolic attractor.
(3) If $\delta < 0$, $\theta_2\delta - 2N < 0$, then the equilibrium E_3 is a hyperbolic repeller as $\delta^2\mu_1 < [\gamma(2N - \theta_2\delta) + (\theta_2\delta - N)]\mu_2^2$, and a hyperbolic attractor for the other parameters in $R_5\backslash H$,

A similar result can be obtained if $N < 0$. We explore next the local bifurcations at the equilibrium E_3.

Proposition 4.5. *The following statements hold:*
(i) For $\mu_0 \in T_1$, equilibria E_2 and E_3 coincide. When parameter μ crosses T_1 a transcritical bifurcation takes place at E_3.
(ii) For $\mu_0 \in T_3$, E_3 coincides either with E_{11} if $\frac{2N-\theta_2\delta}{2N\delta}\mu_2 < 0$, or with E_{12} if $\frac{2N-\theta_2\delta}{2N\delta}\mu_2 > 0$.
(iii) Assume $2N - \delta\theta_2 \neq 0$, and $\frac{\partial\delta}{\partial\mu_1} \neq 0$. Then, when parameter μ crosses T_3, a transcritical bifurcation takes place at E_3.

Proof. (i) Consider $\mu_0 \in T_1$, $\mu_0 = (\gamma\mu_2 + O(\mu_2^2), \mu_2)$, with $\mu_2 > 0$. Applying Sotomayor theorem, we obtain the right eigenvector $v = (1, -\delta)^T$ and the left eigenvector $w = (1, 0)^T$, and

$$w^T f_{\mu_1}(E_3, \mu_0) = 0, \quad w^T Df_{\mu_1}(E_3, \mu_0) = 2\gamma\delta + O(\mu_2) \neq 0,$$
$$w^T[D^2 f(E_0, \mu_0)(v, v)] = 1 \neq 0,$$

for $|\mu_0|$ sufficiently small. Thus the transcritical bifurcation conditions are satisfied.

(ii) Consider $\mu_0 \in T_3$, sufficiently small. Then $\mu_0 = \left(\frac{\theta_2\delta-N}{\delta^2}\mu_2^2 + O(\mu_2^3), \mu_2\right)$, $\mu_2 \neq 0$. We obtain $\xi_1^* = \frac{1}{\delta}\mu_2 + O(\mu_2^2) \neq 0$, $\xi_2^* = 0$, and

$$\xi_1^* - \frac{\theta_1\mu_1 + \theta_2\mu_2}{2N} = \frac{2N - \theta_2\delta}{2N\delta}\mu_2 + O(\mu_2^2),$$

consequently, $E_3 = E_{12}$ if $\frac{2N-\theta_2\delta}{2N\delta}\mu_2 > 0$, and $E_3 = E_{11}$ if $\frac{2N-\theta_2\delta}{2N\delta}\mu_2 < 0$.

(iii) Consider $\mu_0 \in T_3$, sufficiently small. In order to prove the existence of a transcritical bifurcation, we choose μ_1 as the bifurcation parameter. Write δ as

$$\delta(\mu) = \delta(0) + \delta_1\mu_1 + \delta_2\mu_2 + O(|\mu|^2).$$

The Jacobian matrix $Df(E_3, \mu_0)$ has a zero eigenvalue with the right eigenvector

$$v = \left(\gamma\delta - M\mu_2 + O(\mu_2^2), (2N - \theta_2\delta)\mu_2 + O(\mu_2^2)\right)^T$$

and the left eigenvector $w = (0, 1)^T$. It follows

$$w^T f_{\mu_1}(E_3, \mu_0) = 0, \quad w^T Df_{\mu_1}(E_3, \mu_0) = \frac{\delta_1}{\delta}(2N - \delta\theta_2)\mu_2^2(1 + O(\mu_2)) \neq 0,$$

$$w^T[D^2 f(E_3, \mu_0)(v, v)] = 2\gamma\delta^2(2N - \delta\theta_2)\mu_2(1 + O(\mu_2)) \neq 0,$$

for sufficiently small $|\mu|$, ensuring the existence of a transcritical bifurcation. □

From Lemma 4.4 it follows that system (2.16) may exhibit a Hopf bifurcation at E_3.

Remark 5. The above analysis shows that the term θ_1 does not influence the topological type of the equilibria, and only the coefficients γ, δ, θ_2 and N are significant, for sufficiently small $|\mu|$.

For a fixed $\gamma > 0$, and $N > 0$, the curves $\delta = 0$, $\theta_2 = 0, \theta_2\delta - N = 0$, $\theta_2\delta - 2N = 0$ determine eight regions in the (θ_2, δ) – plane (see Figure 4), corresponding to the following cases:

I_θ: $\theta_2 > 0, \delta > 0$, $\theta_2\delta - 2N > 0$;

II_θ: $\theta_2 > 0, \delta > 0$, $\theta_2\delta - N > 0, \theta_2\delta - 2N < 0$;

III_θ: $\theta_2 > 0, \delta > 0$, $\theta_2\delta - N < 0$;

IV_θ: $\theta_2 < 0, \delta > 0$, $\theta_2\delta - N < 0$;

V_θ: $\theta_2 < 0, \delta < 0$, $\theta_2\delta - 2N > 0$;

VI_θ: $\theta_2 < 0, \delta < 0$, $\theta_2\delta - N > 0, \theta_2\delta - 2N < 0$;

VII_θ: $\theta_2 < 0, \delta < 0$, $\theta_2\delta - N < 0$;

$VIII_\theta$: $\theta_2 > 0, \delta < 0$, $\theta_2\delta - N < 0$.

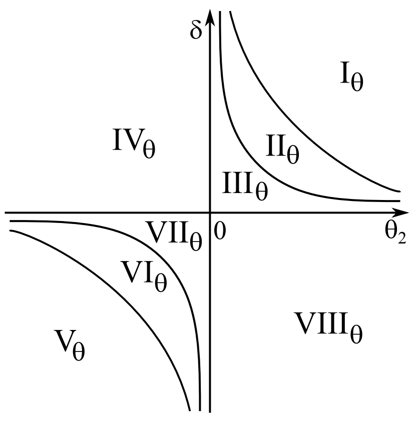

Figure 4. Eight regions in the (θ_2, δ) plane, $\gamma > 0$, and $N > 0$.

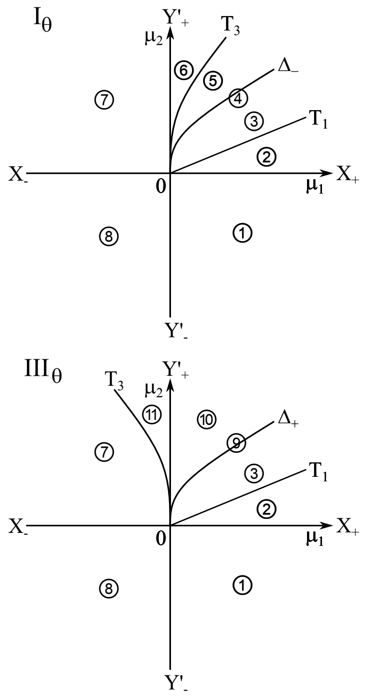

Figure 5. Parametric portraits in the case $\theta = 0$, regions corresponding to $\delta > 0$.

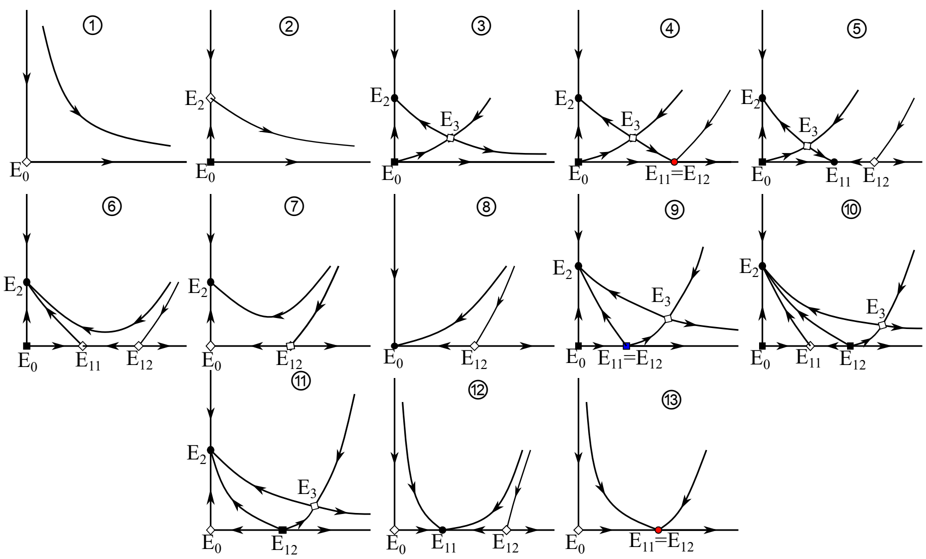

Figure 6. Generic phase portraits in the case $\theta = 0$, regions corresponding to $\delta > 0$.

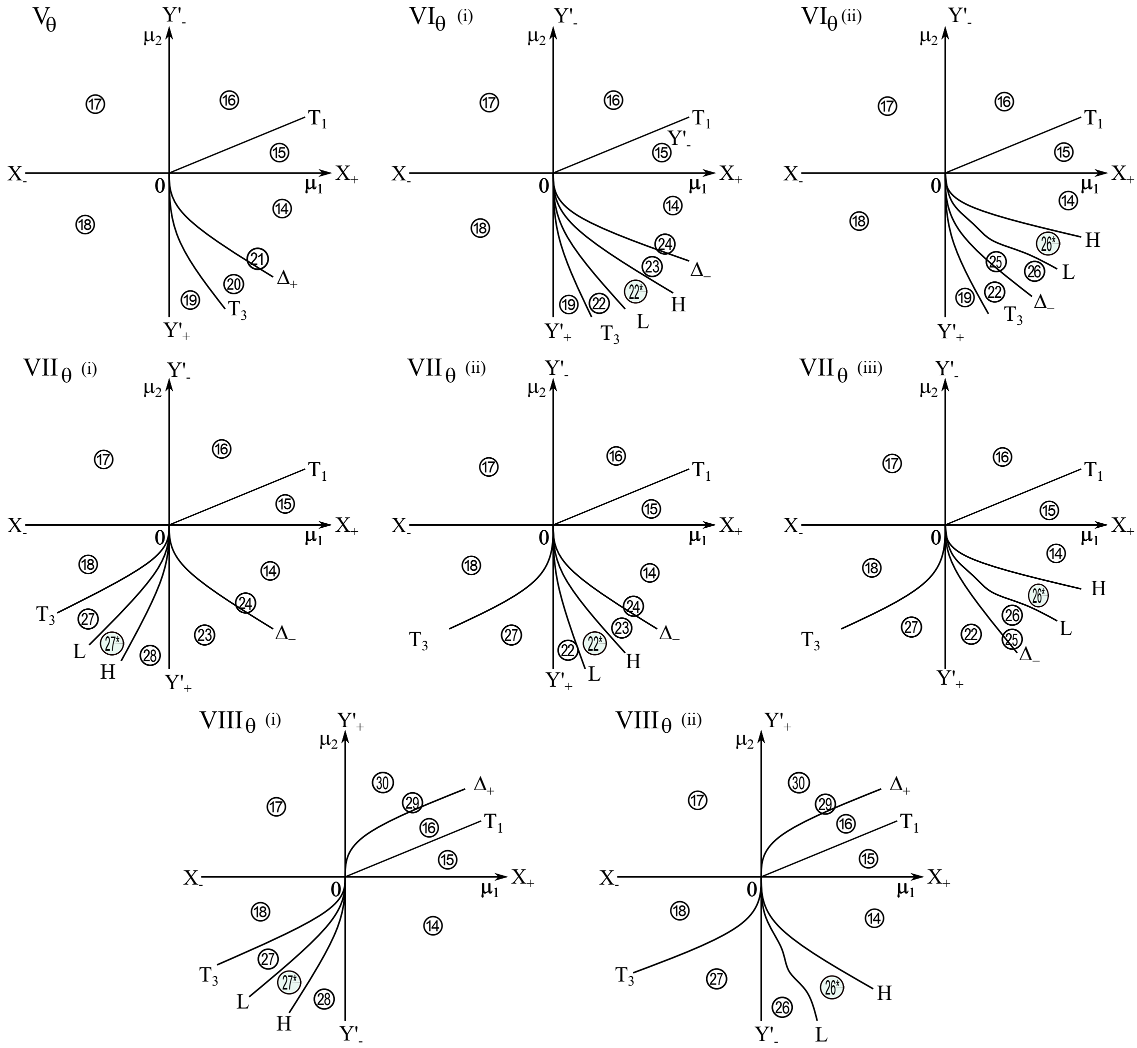

Figure 7. Parametric portraits in the case $\theta = 0$, regions corresponding to $\delta < 0$.

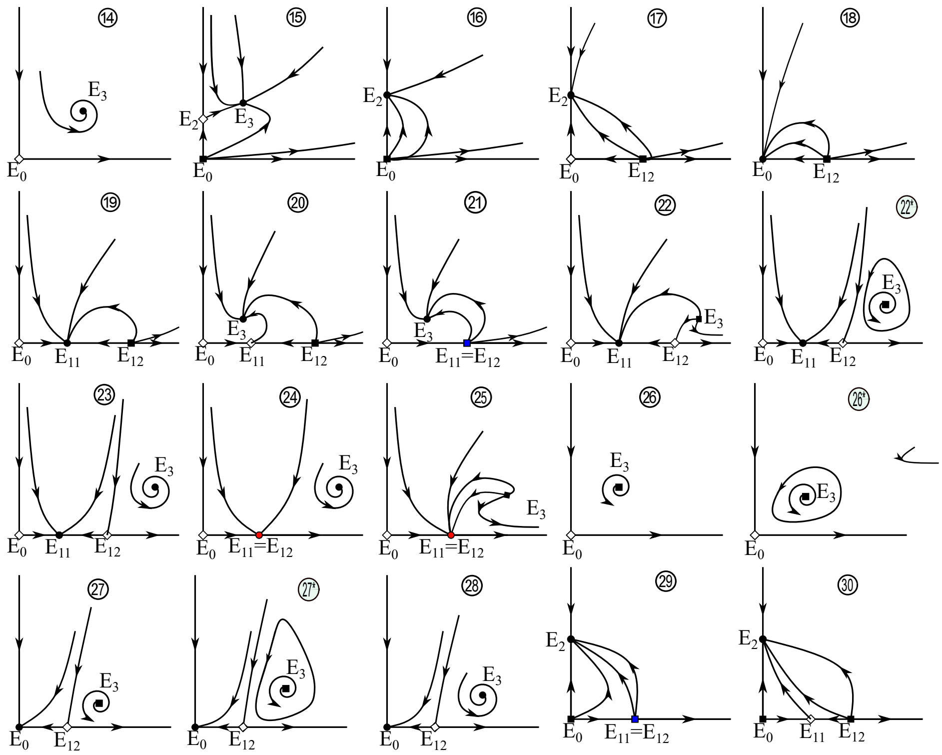

Figure 8. Generic phase portraits in the case $\theta = 0$, regions corresponding to $\delta < 0$.

Theorem 4.6. *For all $\gamma > 0, N > 0$, respectively, (θ_2, δ) in regions I_θ, II_θ, III_θ, IV_θ,V_θ in the $(\theta_2, \delta)-$plane, the bifurcation curves consist of*

$$O \cup T_1 \cup T_3 \cup \Delta_\pm \cup X_- \cup X_+ \cup Y'_- \cup Y'_+.$$

The generic parameter portraits are shown in Figures 5 and 7, and the corresponding generic phase portraits in Figures 6 and 8.

A Hopf bifurcation may occur when (μ_1, μ_2) crosses the curve H and (θ_2, δ) is situated in regions VI_θ, VII_θ, VIII_θ. In the case when the first Lyapunov is non-zero, a limit cycle surrounding the equilibrium E_3 appear. This cycle is stable if the Hopf bifurcation is supercritical (the first Lyapunov coefficient is negative) or is unstable if the Hopf bifurcation is subcritical (i.e., the first Lyapunov coefficient is positive). As the first Lyapunov coefficient is zero, the equilibrium E_3 may be a nonlinear centre.

Remark that, as the parameters move away from H, towards the region containing the limit cycle generated by the Hopf bifurcation, the cycle may disappear, either by transforming into a homoclinic loop or by becoming too large, and no longer inside the visible neighborhood of the origin. In such cases there should exist a bifurcation curve L originating at $\mu = 0$, along which those situations may occur. The position of the curve L depends on the type of the Hopf bifurcation. These situations also appear in the "difficult case" in the nondegenerate double-Hopf bifurcation. The existence and location of the curve L can be proved following the lines in [1, 2].

Theorem 4.7. *For all* $\gamma > 0, N > 0$, *and* (θ_2, δ) *in regions* VI_θ, VII_θ, $VIII_\theta$ *in the* $(\theta_2, \delta)-$ *plane, the bifurcation curves consist of*

$$O \cup T_1 \cup T_3 \cup \Delta_\pm \cup X_- \cup X_+ \cup Y'_- \cup Y'_+ \cup H \cup L.$$

The generic parameter portraits are shown in Figure 7, in the hypothesis that the Lyapunov coefficient is negative. The corresponding generic phase portraits are given in Figure 8. Many of the phase portraits in Figures 6 and 8 cannot be found in the nondegenerate double-Hopf bifurcation.

The effect of adding higher-order terms to the truncated normal form system (2.16) and the correspondence between the properties of the 2D amplitude system (2.16) and the 4D system (2.1), have been described, for example, in [1, 2, 15–18]. However, the study of nonsymmetric general perturbations of the truncated normal forms is far from complete, as pointed out in [1].

Thus, we remark that the study of adding higher order terms to the truncated normal form is a complex open problem which is outside the aim of this article.

5. Conclusions

The classical normal form of the Hopf-Hopf bifurcation, in differential systems of dimension four and having minimum two independent parameters, is based on six generic conditions (2.8).

In this work we have studied two degenerate Hopf-Hopf bifurcations, namely:

1) the case when the generic condition (HH.3) in system (2.8) is not fulfilled, treated in Section 3;

2) the case when the generic condition (HH.1) in system (2.8) is not fulfilled, treated in Section 4.

In our study we have obtained new bifurcation diagrams and new phase portraits which were not previously reported in generic studies on double-Hopf bifurcation. Such an example is a saddle-node bifurcation in the amplitude system which corresponds to a fold bifurcation of cycles in the four dimensional system and which is present only in the degenerate case $\theta = 0$.

The results describing the dynamics, contribute to a better understanding of the behavior of a system presenting a Hopf-Hopf singularity.

Finally, we summarize in the Table 1 the correspondence of equilibria, cycles and bifurcations between the 2D amplitude system (2.16) and the 4D normal form system (2.14):

Table 1. The correspondence of equilibria, cycles and bifurcations.

2D	4D
E_0	origin
E_3	2D torus
cycle	3D torus
$X_+, X_-, Y_+, Y_-, Y'_+, Y'_-$	Hopf bifurcation
T_1, T_2, T_3	Neimark-Sacker bifurcation of cycles
Δ_+, Δ_-	fold bifurcation of cycles
Hopf bifurcation H	branching of 3D torus from a 2D torus
cycle blow up L	blow-up of a 3D torus

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Conflict of interest

The authors declare there is no conflicts of interest.

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