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Research article

Pullback $\mathcal{D}$-attractors of the three-dimensional non-autonomous micropolar equations with damping

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Abstract: In this paper, we consider the three-dimensional non-autonomous micropolar equations with damping term in periodic domain $\mathbb{T}^3$. By assuming external forces satisfy certain condtions, the existence of pullback $\mathcal{D}$-attractors for the three-dimensional non-autonomous micropolar equations with damping term is proved in $V_1 \times V_2$ and $H^2 \times H^2$ with $3 < \beta < 5$.

Keywords: micropolar equations; damping; pullback $\mathcal{D}$-attractors; non-autonomous; periodic domain

1. Introduction

In this paper, we study the following 3D non-autonomous micropolar equations with a damping term

$$\begin{cases} u_t + (u \cdot \nabla)u - (\nu + \kappa)\Delta u + \sigma |u|^{\beta-1} u + \nabla p = 2\kappa \nabla \times w + f(x,t), \\ w_t + (u \cdot \nabla)w + 4\kappa w - \gamma \Delta w - \mu \nabla \nabla \cdot w = 2\kappa \nabla \times u + g(x,t), \\ \nabla \cdot u = 0, \\ u(x,t)|_{t=\tau} = u_\tau, w(x,t)|_{t=\tau} = w_\tau, \end{cases} \tag{1.1}$$

where $(x, t) \in \mathbb{T}^3 \times [\tau, +\infty)$ and $\tau \in \mathbb{R}$. $\mathbb{T}^3 \subset \mathbb{R}^3$ is a periodic domain. In system (1.1), the fluid velocity and the micro-rotational velocity are represented by $u = u(x, t)$ and $w = w(x, t)$, respectively. $p = p(x, t)$ is the scalar pressure. $f = f(x, t)$ and $g = g(x, t)$ denote the given external forces. ν, κ and σ denote kinematic viscosity, micro-rotational viscosity and damping coefficient, respectively, which are all positive constants. $\beta \geq 1$ is a constant. γ and μ, representing the angular viscosities, are also positive constants. For convenience, let $\nu = \kappa = \gamma = \mu = \sigma = 1$.

Eringen firstly introduced microfluids in [4] and showed a complete theory for micropolar fluids in [5]. For physical background and mathematical theory, we can refer [12] and [18]. Galdi and Rionero proved the existence and uniqueness of weak solutions for the micropolar equations in [6]. In [24], the existence of strong solution for 3D micropolar equations was proved for $\beta = 3$ and $4\sigma(\nu + \kappa) > 1$ or $\beta > 3$. And there were also many works with magneto-micropolar equations, we can refer [9–11, 20]. In [20], global well-posedness of a 3D MHD system was studied in porous media.

As $w = 0$, Eqs (1.1) are Navier-Stokes equations. Cai and Jiu [1] firstly considered 3D incompressible Navier-Stokes equations with damping $\alpha|u|^{\beta-1}u$ and they proved the existence of weak solution with $\beta \geq 1$ as well as strong solution with $\beta \geq \frac{7}{2}$, and uniqueness for strong solution with $\frac{7}{2} \leq \beta \leq 5$. In [25], the regularity and uniqueness for three-dimensional incompressible Navier-Stokes system with damping term were studied. The generalized Navier-Stokes equations with damping were researched in [13] and the existence of weak solution was proved in R^n, $n \geq 2$. The uniform global attractor and trajectory attractor for 3D Navier-Stokes equations were considered in [3]. In [8], L_2 decay of weak solutions for $\beta > 2$ with $\alpha > 0$ and the asymptotic stability of the solution to incompressible Navier-Stokes equations with damping for $\beta > 3$ with $\alpha > 0$ or $\beta = 3$ with $\alpha \geq \frac{3}{2}$ were proved.

In this paper, we devote to research pullback attractors for three-dimensional non-autonomous micropolar equations (1.1) with damping term. Recently, attractors have been interested many authors [2, 7, 10, 17, 19, 21–23]. Caraballo, Lukaszewicz and Real considered pullback attractors of two-dimensional Navier-Stokes system in [2]. In [7], the existence of pullback attractors for nonautonomous reaction-diffusion equation was proved in R^n, $n \geq 3$. In [17], the existence of pullback attractors for 3D Navier-Stokes problem with damping was proved in V and H^2 for $3 < \beta \leq 5$. In [19], Sun and Li have studied global pullback attractors and pullback exponential attractors for the 2D non-autonomous micropolar fluid system. Global attractor of the 3D magnetohydrodynamics equations with damping was considered in [10]. However, the existence of pullback $\mathcal{D}$-attractors for Eqs (1.1) is not obtained in $V_1 \times V_2$ and $H^2 \times H^2$.

To obtain our main results, we need to deal with nonlinear terms $(u \cdot \nabla)u$, $(u \cdot \nabla)w$ and $\sigma|u|^{\beta-1}u$. Hence, we should show some estimates by using Sobolev and uniform Gronwall inequalities. To prove the existence of pullback $\mathcal{D}$-attractors, we should restrict $3 < \beta < 5$ from (3.20) and (3.47).

In this paper, the structure is organized as follows. In section 2, some definitions as well as notions are recalled and our main results are given. In section 3, we show some estimates to overcome the difficulties of nonlinear terms. In section 4, the existence of pullback $\mathcal{D}$-attractors for Eqs (1.1) is proved in $V_1 \times V_2$ and $H^2 \times H^2$ with $3 < \beta < 5$.

2. Preliminaries

In this section, we will show some definitions and lemmas. We also give some notions and assumptions which we will use in the following. Finally, we give our main results.

Firstly, assumed X is a complete metric space, $\mathcal{P}(X)$ represents the family of all nonempty subsets of X and $\mathcal{D}$ is a nonempty class of parameterized sets $\hat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$. In the following, we give the definition of pullback $\mathcal{D}$-attractor which we can refer [17] to get.

Definition 2.1. A family $\hat{A} = \{A(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is called a pullback $\mathcal{D}$-attractor for the process $\{U(t, \tau)\}_{t \geq \tau}$ in X, if
(1) $A(t)$ is compact for every $t \in \mathbb{R}$,

(2) $\hat{A}$ is invariant, that is, $U(t,\tau)A(\tau) = A(t)$, for $-\infty < \tau \le t < +\infty$,
(3) $\hat{A}$ is pullback $\mathcal{D}$-attracting, that is,

$$\lim_{\tau\to-\infty} \operatorname{dist}(U(t,\tau)D(\tau), A(t)) = 0,\ \forall\hat{D} \in \mathcal{D} \text{ and } \forall t \in \mathbb{R}.$$

And $\hat{A}$ is said to be minimal if $A(t) \subset C(t)$ for any family $\hat{C} = \{C(t); t \in \mathbb{R}\} \subset \mathcal{P}(X)$ of closed sets such that for any $\hat{B} \in \mathcal{D}$,

$$\lim_{\tau\to-\infty} \operatorname{dist}(U(t,\tau)B(\tau), C(t)) = 0.$$

In following, we give lemmas and definitions which we can refer [17], and these play important roles in the proof of our main results.

Definition 2.2. Assumed X is a complete metric space, a two-parameter family $\{U(t,\tau) : -\infty < \tau \le t < +\infty\}$ of mapping $U(t,\tau) : X \to X, t \ge \tau, \tau \in \mathbb{R}$ is called evolutionary process if
(1) $U(t,s)U(s,\tau) = U(t,\tau)$, for all $\tau \le s \le t$,
(2) $U(\tau,\tau) = Id$ is identity operator for all $\tau \in \mathbb{R}$.

Lemma 2.3. *Assume $\{U(t,\tau)\}_{t\ge\tau}$ is a process in X satisfying the following conditions*
(1) $\{U(t,\tau)\}_{t\ge\tau}$ is norm-to-weak continuous in X,
(2) there exists a family $\hat{B}$ of pullback $\mathcal{D}$-absorbing sets $\{B(t); t \in \mathbb{R}\}$ in X,
(3) $\{U(t,\tau)\}_{t\ge\tau}$ is pullback $\mathcal{D}$-asymptotically compact.
Then there exists a minimal pullback $\mathcal{D}$-attractor $\hat{A} = \{A(t) : t \in \mathbb{R}\}$ in X given by

$$A(t) = \bigcap_{s\le t} \overline{\bigcup_{\tau\le s} U(t,\tau)B(\tau)}.$$

Definition 2.4. It is said that a process $U(t,\tau)$ is norm-to-weak continuous on X if for all $t,\tau \in \mathbb{R}$ with $t \ge \tau$ and for any sequence $x_n \in X$,

$$x_n \to x \text{ strongly in } X \Rightarrow U(t,\tau)x_n \rightharpoonup U(t,\tau)x \text{ weakly in } X,$$

where $\to$ and $\rightharpoonup$ represent strong convergence and weak convergence, respectively.

And we can easily get that it is norm-to-weak continuous process as it is a continuous process.

Next, we give the following lemma which can help us complete the proof of norm-to-weak continuous.

Lemma 2.5. *Assume X, Y are two Banach spaces. Let X^*, Y^* be dual spaces of X and Y, respectively. Suppose that X is dense in Y, the injection $i : X \to Y$ is continuous, its adjoint $i^* : Y^* \to X^*$ is dense, and $\{U(t,\tau)\}_{t\ge\tau}$ is a norm-to-weak continuous process on Y. Then $\{U(t,\tau)\}_{t\ge\tau}$ is a norm-to-weak continuous process on X if and only if $U(t,\tau)$ maps compact sets of X into bounded sets of X, for every $\tau \in \mathbb{R}$ and $t \ge \tau$.*

Definition 2.6. $\hat{B} \in \mathcal{D}$ is said to be pullback $\mathcal{D}$-absorbing for the process $\{U(t,\tau)\}_{t\ge\tau}$, if for every $t \in \mathbb{R}$ and $\hat{D} \in \mathcal{D}$, there exists a $\tau_0(t,\hat{D}) \le t$ such that $U(t,\tau)D(\tau) \subset B(t)$ for every $\tau \le \tau_0(t,\hat{D})$.

Definition 2.7. It is said that the process $\{U(t,\tau)\}_{t\geq\tau}$ is a pullback $\mathcal{D}$-asymptotically compact, if sequence $\{U(t,\tau_n)x_n\}_{n=1}^{\infty}$ is relatively compact in X for all $t\in\mathbb{R}$, $\hat{D}\in\mathcal{D}$ and every sequence $\tau_n\to-\infty$ as well as $x_n\in D(\tau_n)$.

In the following, we give notions of function spaces

$$
\begin{aligned}
&\mathcal{V}_1=\{u\in(C^{\infty}(\mathbb{T}^3))^3:\operatorname{div}u=0,\int_{\mathbb{T}^3}udx=0\},\\
&\mathcal{V}_2=\{w\in(C^{\infty}(\mathbb{T}^3))^3:\int_{\mathbb{T}^3}wdx=0\},\\
&H_1=\text{the closure of }\mathcal{V}_1\text{ in }(L^2(\mathbb{T}^3))^3,\\
&H_2=\text{the closure of }\mathcal{V}_2\text{ in }(L^2(\mathbb{T}^3))^3,\\
&V_1=\text{the closure of }\mathcal{V}_1\text{ in }(H^1(\mathbb{T}^3))^3,\\
&V_2=\text{the closure of }\mathcal{V}_2\text{ in }(H^1(\mathbb{T}^3))^3.
\end{aligned}
$$

Let the norm of the space $(L^p(\mathbb{T}^3))^3$ be represented by $\|\cdot\|_p$, particularly, $\|\cdot\|$ represents the norm of the space H_1 and the space H_2. H^s means the usual Sobolev space and its norm $\|\cdot\|_{H^s}=\|A^{\frac{s}{2}}\cdot\|$, particularly, as $s=2$, $\|\cdot\|_{H^2}=\|A\cdot\|$.

In the periodic space, we recall that

Lemma 2.8. *[14] The Leray projector $\mathbb{P}$ on the torus and on the whole space commutes with any derivative:*

$$
\mathbb{P}(\partial_j u)=\partial_j\mathbb{P}u,\ j=1,2,3, \tag{2.1}
$$

for all $u\in\dot{H}^1$.

In the following, let

$$
\begin{aligned}
&Au=-\mathbb{P}\Delta u=-\Delta u,\ Aw=-\Delta w,\ \forall(u,w)\in H^2\times H^2,\\
&B(u)=B(u,u)=\mathbb{P}((u\cdot\nabla)u),\ B(u,w)=(u\cdot\nabla)w,\ \forall(u,w)\in V_1\times V_2,
\end{aligned}
$$

where $\mathbb{P}$ represents the Helmholtz-Leray orthogonal projection from $(L^2(\mathbb{T}^3))^3$ onto H_1 and $\mathbb{P}u=u$ on the torus. Then we can rewrite the Eqs (1.1) as following

$$
\begin{cases}
u_t+B(u)+2Au+G(u)=2\nabla\times w+f,\\
w_t+B(u,w)+4w+Aw-\nabla\nabla\cdot w=2\nabla\times u+g,\\
\nabla\cdot u=0,\\
u(x,t)|_{t=\tau}=u_\tau,w(x,t)|_{t=\tau}=w_\tau,
\end{cases} \tag{2.2}
$$

where let $G(u)=\mathbb{P}|u|^{\beta-1}u$.

In this paper, to complete our proof, we should assume

$$
f\in L^2_{loc}(\mathbb{R};H_1),\ g\in L^2_{loc}(\mathbb{R};H_2),
$$

and

$$
\frac{\partial f}{\partial t}=f_t\in L^2_b(\mathbb{R};H_1),\ \frac{\partial g}{\partial t}=g_t\in L^2_b(\mathbb{R};H_2),
$$

where $L_b^2(\mathbb{R}; H_1)$ represents the collection of functions that are translation bounded in $L_{loc}^2(\mathbb{R}; H_1)$. Note that function $f(t)$ is translation bounded in $L_{loc}^2(\mathbb{R}; H_1)$ if

$$\|f\|_{L_b^2}^2 = \|f\|_{L_b^2(\mathbb{R};H_1)}^2 = \sup_{t\in\mathbb{R}} \int_t^{t+1} \|f(s)\|^2 ds < \infty.$$

And for $L_b^2(\mathbb{R}; H_2)$, we have similar definition.

We also assume $f(x,t)$ is uniformly bounded in H_1, $g(x,t)$ is uniformly bounded in H_2, that is, there exists a positive constant C such that

$$\sup_{t\in\mathbb{R}}(\|f(t)\|^2 + \|g(t)\|^2) \leq C.$$

Please note that C is a positive constant and it could mean different numbers in different places.

Further suppose that $f(x,t)$ and $g(x,t)$ satisfy following inequalities

$$G_1(t) := \int_{-\infty}^{t} e^{\lambda s}(\|f(s)\|^2 + \|g(s)\|^2)ds < \infty, \tag{2.3}$$

$$G_2(t) := \int_{-\infty}^{t}\int_{-\infty}^{s} e^{\lambda r}(\|f(r)\|^2 + \|g(r)\|^2)drds < \infty, \tag{2.4}$$

for any $t \in \mathbb{R}$, where λ is given in the following.

Next, let $\mathcal{D}$ be the class of all families $\hat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}((H^1(\mathbb{T}^3))^3)$ such that

$$\lim_{t\to-\infty} e^{\lambda t}[D(t)] = 0, \tag{2.5}$$

where $[D(t)] = \sup\{\|u(t)\|_{V_1}^2 + \|w(t)\|_{V_2}^2 : u, w \in D(t)\}$ and $\lambda > 0$ is given in the following.

And we can use the Poincaré inequality, i.e., there exists a constant $\lambda > 0$ such that

$$\lambda(\|u(t)\|^2 + \|w(t)\|^2) \leq \|\nabla u(t)\|^2 + \|\nabla w(t)\|^2, \ \ \forall(u,w) \in V_1 \times V_2, \tag{2.6}$$

where λ is the minimum of the first eigenvalues of Stokes operators Au and Aw.

Then, we give our main theorems.

Theorem 2.9. *Suppose (2.3)-(2.5) hold,* $f \in L_{loc}^2(\mathbb{R}; H_1)$*,* $g \in L_{loc}^2(\mathbb{R}; H_2)$*,* $f_t \in L_b^2(\mathbb{R}; H_1)$ *and* $g_t \in L_b^2(\mathbb{R}; H_2)$*. Let* $3 < \beta < 5$ *and* $\tau \in \mathbb{R}$*, then there exists a pullback* $\mathcal{D}$*-attractor* $\mathcal{A}_1$ *of the process* $\{U(t,\tau)\}_{t\geq\tau}$ *for system (1.1) in* $V_1 \times V_2$*.*

Theorem 2.10. *Suppose (2.3)-(2.5) hold,* $f \in L_{loc}^2(\mathbb{R}; H_1)$*,* $g \in L_{loc}^2(\mathbb{R}; H_2)$*,* $f_t \in L_b^2(\mathbb{R}; H_1)$ *and* $g_t \in L_b^2(\mathbb{R}; H_2)$*. Let* $3 < \beta < 5$ *and* $\tau \in \mathbb{R}$*, then* $\{U(t,\tau)\}_{t\geq\tau}$ *for system (1.1) has a pullback* $\mathcal{D}$*-attractor* $\mathcal{A}_2$ *in* $H^2 \times H^2$*.*

Now, let recall the existence of weak and strong solutions for Eqs (1.1).

Theorem 2.11. *Suppose* $f \in L_b^2(\mathbb{R}; H_1)$*,* $g \in L_b^2(\mathbb{R}; H_2)$*,* $u_\tau \in H_1$*,* $w_\tau \in H_2$ *and* $\beta \geq 1$*. Then for every given* $T > \tau$*, there exist at least one solution* (u, w) *of (2.2),*

$$u \in L^\infty(\tau, T; H_1) \cap L^2(\tau, T; V_1) \cap L^{\beta+1}(\tau, T; (L^{\beta+1}(\mathbb{T}^3))^3), \tag{2.7}$$

$$w \in L^\infty(\tau, T; H_2) \cap L^2(\tau, T; V_2). \tag{2.8}$$

Proof. In [1], the existence of weak solution for Navier-Stoke equations with damping has been proved, we can use the similar proof to get the existence of weak solutions for Eqs (2.2) and omit it. □

We say that (u, w) is a strong solution of (1.1), if it is a weak solution of (1.1), and satisfies

$$u \in L^\infty(\tau, T; V_1) \cap L^2(\tau, T; H^2) \cap L^\infty(\tau, T; (L^{\beta+1}(\mathbb{T}^3))^3), \tag{2.9}$$

$$w \in L^\infty(\tau, T; V_2) \cap L^2(\tau, T; H^2). \tag{2.10}$$

Theorem 2.12. *Suppose $\beta > 3$, $f \in L^2_b(\mathbb{R}; H_1)$, $g \in L^2_b(\mathbb{R}; H_2)$, $u_\tau \in V_1 \cap (L^{\beta+1}(\mathbb{T}^3))^3$ and $w_\tau \in V_2$. Then there exists a strong solution (u, w) of Eqs (1.1),*

$$u \in L^\infty(\tau, T; V_1) \cap L^2(\tau, T; H^2) \cap L^\infty(\tau, T; (L^{\beta+1}(\mathbb{T}^3))^3), \tag{2.11}$$

$$w \in L^\infty(\tau, T; V_2) \cap L^2(\tau, T; H^2), \tag{2.12}$$

$$\nabla u|u|^{\frac{\beta-1}{2}} \in L^2(\tau, T; H_1),\ u_t \in L^2(\tau, T; H_1),\ w_t \in L^2(\tau, T; H_2). \tag{2.13}$$

Proof. Due to [24], we can take similar method to prove the existence of strong solution for Eqs (1.1) and omit it. □

3. Estimates of solutions

In this section, some estimates will be given. These estimates play an important role in the proof of our main results. In the following, we give some lemmas we will use.

Lemma 3.1. *Assume (2.3)-(2.5) hold, $f \in L^2_{loc}(\mathbb{R}; H_1)$ and $g \in L^2_{loc}(\mathbb{R}; H_2)$. Let $3 < \beta < 5$ and $\tau \in \mathbb{R}$, and (u, w) be the solution of system (1.1). For every $t \in \mathbb{R}$ and $\hat{D} \in \mathcal{D}$, there exists a constant $\tau_0 = \tau_0(t, \hat{D}) < t$ such that*

$$\|u(t)\|^2 + \|w(t)\|^2 \le Ce^{-\lambda t} G_1(t), \tag{3.1}$$

and

$$\int_\tau^t e^{\lambda s}(\|\nabla u(s)\|^2 + \|\nabla w(s)\|^2 + \|\nabla \cdot w(s)\|^2 + \|u(s)\|^{\beta+1}_{\beta+1}) ds \le C(G_1(t) + G_2(t)), \tag{3.2}$$

for every $u_\tau \in D(\tau)$, $w_\tau \in D(\tau)$ and $\tau \le \tau_0(t, \hat{D})$.

Proof. Multiplying the first equation and the second equation of (2.2) by u and w, respectively, and integrating over $\mathbb{T}^3$, we can have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|u(t)\|^2 + \|w(t)\|^2) + 2\|\nabla u(t)\|^2 + \|\nabla w(t)\|^2 + 4\|w(t)\|^2 \\
&+ \|\nabla \cdot w(t)\|^2 + \|u(t)\|^{\beta+1}_{\beta+1}) \\
\le& \frac{3}{2}\|\nabla u(t)\|^2 + \frac{1}{2}\|\nabla w(t)\|^2 + 4\|w(t)\|^2 + \frac{1}{2\lambda}(\|f(t)\|^2 + \|g(t)\|^2).
\end{aligned} \tag{3.3}$$

So, we easily get

$$\begin{aligned}
&\frac{d}{dt}(\|u(t)\|^2 + \|w(t)\|^2) + \|\nabla u(t)\|^2 + \|\nabla w(t)\|^2 + 2\|\nabla \cdot w(t)\|^2 + 2\|u(t)\|^{\beta+1}_{\beta+1} \\
\le& \frac{1}{\lambda}(\|f(t)\|^2 + \|g(t)\|^2),
\end{aligned} \tag{3.4}$$

and

$$\frac{d}{dt}(\|u(t)\|^2 + \|w(t)\|^2) + \lambda(\|u(t)\|^2 + \|w(t)\|^2) \le \frac{1}{\lambda}(\|f(t)\|^2 + \|g(t)\|^2). \tag{3.5}$$

Multiplying (3.5) by $e^{\lambda t}$ and integrating over $[\tau, t]$, then we obtain

$$e^{\lambda t}(\|u(t)\|^2 + \|w(t)\|^2) \le e^{\lambda\tau}(\|u_\tau\|^2 + \|w_\tau\|^2) + \frac{1}{\lambda}\int_{-\infty}^{t} e^{\lambda s}(\|f(s)\|^2 + \|g(s)\|^2)ds. \tag{3.6}$$

Due to $u_\tau \in D(\tau)$ and $w_\tau \in D(\tau)$, for any $t \in \mathbb{R}$, we can have there exists a constant $\tau_0 \le t$ such that

$$e^{\lambda\tau}(\|u_\tau\|^2 + \|w_\tau\|^2) \le \frac{1}{\lambda}G_1(t), \quad \forall \tau \le \tau_0, \tag{3.7}$$

where $\tau_0 = \frac{1}{\lambda}\ln\frac{\int_{-\infty}^{t} e^{\lambda s}(\|f(s)\|^2+\|g(s)\|^2)ds}{\lambda(\|u_\tau\|^2+\|w_\tau\|^2)}$.

So we easily get

$$\|u(t)\|^2 + \|w(t)\|^2 \le \frac{2}{\lambda}e^{-\lambda t}G_1(t). \tag{3.8}$$

Integrating (3.6) over $[\tau, t]$, we have

$$\int_{\tau}^{t} e^{\lambda s}(\|u(s)\|^2 + \|w(s)\|^2)ds \le \frac{2}{\lambda}G_2(t). \tag{3.9}$$

Multiplying (3.4) by $e^{\lambda t}$ and integrating over $[\tau, t]$, then using (3.9), we can get

$$\int_{\tau}^{t} e^{\lambda s}(\|\nabla u(s)\|^2 + \|\nabla w(s)\|^2 + \|\nabla\cdot w(s)\|^2 + \|u(s)\|_{\beta+1}^{\beta+1})ds \le C(G_1(t) + G_2(t)). \tag{3.10}$$

By (3.8) and (3.10), the proof of Lemma 3.1 is completed. □

Lemma 3.2. *Under the assumption of Lemma 3.1. For every $t \in \mathbb{R}$ and $\hat{D} \in \mathcal{D}$, then there exists a constant $\tau_1 = \tau_1(t, \hat{D})$ such that for every $\tau \le \tau_1$ and $u_\tau \in D(\tau)$, $w_\tau \in D(\tau)$,*

$$\int_{t-1}^{t} e^{\lambda s}(\|u(s)\|^2 + \|w(s)\|^2)ds \le CG_2(t), \tag{3.11}$$

and

$$\int_{t-1}^{t} e^{\lambda s}(\|\nabla u(s)\|^2 + \|\nabla w(s)\|^2 + \|\nabla\cdot w(s)\|^2 + \|u(s)\|_{\beta+1}^{\beta+1})ds \le C(G_1(t) + G_2(t)). \tag{3.12}$$

Proof. Multiplying (3.5) by $e^{\lambda t}$ and integrating over $[\tau, s]$, then we can have for any $s \in [t-1, t]$, there exists a constant $\tau_1 \equiv \tau_0 - 1 < t - 1$, such that for every $\tau \le \tau_1$,

$$\begin{aligned}
&e^{\lambda s}(\|u(s)\|^2 + \|w(s)\|^2) \\
\le& e^{\lambda\tau}(\|u_\tau\|^2 + \|w_\tau\|^2) + \frac{1}{\lambda}\int_{\tau}^{s} e^{\lambda r}(\|f(r)\|^2 + \|g(r)\|^2)dr \\
\le& e^{\lambda\tau}(\|u_\tau\|^2 + \|w_\tau\|^2) + \frac{1}{\lambda}\int_{-\infty}^{t} e^{\lambda s}(\|f(s)\|^2 + \|g(s)\|^2)ds \\
\le& \frac{2}{\lambda}G_1(t).
\end{aligned} \tag{3.13}$$

Integrating (3.13) over $[t-1, t]$ with respect to s, we can obtain

$$\int_{t-1}^{t} e^{\lambda s}(\|u(s)\|^2 + \|w(s)\|^2)ds \leq \frac{2}{\lambda}G_2(t). \tag{3.14}$$

Multiplying (3.4) by $e^{\lambda t}$ and integrating over $[t-1, t]$, then using (3.14), we can have for every $\tau \leq \tau_1$,

$$\begin{aligned}&\int_{t-1}^{t} e^{\lambda s}(\|\nabla u(s)\|^2 + \|\nabla w(s)\|^2 + \|\nabla \cdot w(s)\|^2 + \|u(s)\|_{\beta+1}^{\beta+1})ds \\ \leq& C(G_1(t) + G_2(t)).\end{aligned} \tag{3.15}$$

This completes the proof of Lemma 3.2. □

Lemma 3.3. *Under the hypothesis of Lemma 3.2, for every $t \in \mathbb{R}$ and $\hat{D} \in \mathcal{D}$, we have*

$$\int_{t-1}^{t} (\|u(s)\|^2 + \|w(s)\|^2)ds \leq Ce^{-\lambda t}G_2(t), \tag{3.16}$$

and

$$\begin{aligned}&\int_{t-1}^{t} (\|\nabla u(s)\|^2 + \|\nabla w(s)\|^2 + \|\nabla \cdot w(s)\|^2 + \|u(s)\|_{\beta+1}^{\beta+1})ds \\ \leq& Ce^{-\lambda t}(G_1(t) + G_2(t)),\end{aligned} \tag{3.17}$$

for $\tau \leq \tau_1$ and $u_\tau \in D(\tau)$, $w_\tau \in D(\tau)$.

Proof. By using Lemma 3.2, we can directly get the result. □

Lemma 3.4. *Assume (2.3)-(2.5) hold, $f \in L^2_{loc}(\mathbb{R}; H_1)$ and $g \in L^2_{loc}(\mathbb{R}; H_2)$. Let $3 < \beta < 5$ and $\tau \in \mathbb{R}$, and (u, w) be the solution of system (1.1). For every $t \in \mathbb{R}$ and $\hat{D} \in \mathcal{D}$, then there exists a constant $\tau_3 = \tau_3(t, \hat{D})$, such that for any $\tau \leq \tau_3$ and $u_\tau \in D(\tau)$, $w_\tau \in D(\tau)$,*

$$\begin{aligned}&\|\nabla u(t)\|^2 + \int_{t-1}^{t} (\|Au(s)\|^2 + \||u|^{\frac{\beta-1}{2}}\nabla u\|^2 + \|\nabla |u|^{\frac{\beta+1}{2}}\|^2)ds \\ \leq& Ce^{-\lambda t}(G_1(t) + G_2(t)),\end{aligned} \tag{3.18}$$

and

$$\|\nabla w(t)\|^2 + \int_{t-1}^{t} (\|Aw(s)\|^2 + \|\nabla\nabla \cdot w(s)\|^2)ds \leq Ce^{-\lambda t}(G_1(t) + G_2(t)). \tag{3.19}$$

Proof. Inspired by [24], we can get for $\beta > 3$,

$$\begin{aligned}&\frac{d}{dt}\|\nabla u(t)\|^2 + \|Au(t)\|^2 + \||u|^{\frac{\beta-1}{2}}\nabla u\|^2 + \|\nabla |u|^{\frac{\beta+1}{2}}\|^2 \\ \leq& C(\|\nabla u(t)\|^2 + \|\nabla w(t)\|^2 + \|f(t)\|^2),\end{aligned} \tag{3.20}$$

then by using uniform Gronwall Lemma on $[t-1,t]$, we can obtain that there exists a constant $\tau_2 \equiv \tau_1 - 1$, for any $\tau \le \tau_2$,

$$\begin{aligned}&\|\nabla u(t)\|^2 + \int_{t-1}^{t}(\|Au(s)\|^2 + \||u|^{\frac{\beta-1}{2}}\nabla u\|^2 + \|\nabla |u|^{\frac{\beta+1}{2}}\|^2)ds\\ \le& Ce^{-\lambda t}(G_1(t)+G_2(t)),\end{aligned} \tag{3.21}$$

where we use the following inequality

$$\begin{aligned}\int_{t-1}^{t}\|f(s)\|^2 ds &= e^{-\lambda(t-1)}\int_{t-1}^{t} e^{\lambda(t-1)}\|f(s)\|^2 ds\\ &\le Ce^{-\lambda(t-1)}\int_{t-1}^{t} e^{\lambda s}\|f(s)\|^2 ds\\ &\le Ce^{-\lambda t}G_1(t).\end{aligned}$$

Multiplying the second equation of (2.2) by Aw and integrating over $\mathbb{T}^3$, then we can get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla w(t)\|^2 + \|Aw(t)\|^2 + 4\|\nabla w(t)\|^2 + \|\nabla\nabla\cdot w(t)\|^2\\ \le&\left|\int_{\mathbb{T}^3} u\cdot\nabla w Aw dx\right| + 2\left|\int_{\mathbb{T}^3}\nabla\times u Aw dx\right| + (g(t), Aw(t))\\ \le&\frac{1}{2}\|Aw(t)\|^2 + C(\|B(u,w)\|^2 + \|\nabla u(t)\|^2 + \|g(t)\|^2).\end{aligned} \tag{3.22}$$

Since

$$\begin{aligned}C\|B(u,w)\|^2 &\le C\|u(t)\|_\infty^2\|\nabla w(t)\|^2\\ &\le C\|\nabla u(t)\|\|\Delta u(t)\|\|\nabla w(t)\|^2\\ &\le C(\|\nabla u(t)\|^2 + \|Au(t)\|^2)\|\nabla w(t)\|^2.\end{aligned} \tag{3.23}$$

By using above inequalities, we easily get

$$\begin{aligned}&\frac{d}{dt}\|\nabla w(t)\|^2 + \|Aw(t)\|^2 + 2\|\nabla\nabla\cdot w(t)\|^2\\ \le& C(\|\nabla u(t)\|^2 + \|Au(t)\|^2)\|\nabla w(t)\|^2 + C(\|\nabla u(t)\|^2 + \|g(t)\|^2).\end{aligned} \tag{3.24}$$

Then by using uniform Gronwall inequality on $[t-1,t]$, we can obtain there has a constant $\tau_3 \equiv \tau_2 - 1$, for any $\tau \le \tau_3$,

$$\|\nabla w(t)\|^2 + \int_{t-1}^{t}(\|Aw(s)\|^2 + \|\nabla\nabla\cdot w(s)\|^2)ds \le Ce^{-\lambda t}(G_1(t)+G_2(t)). \tag{3.25}$$

Hence, we complete the Lemma 3.4. □

Lemma 3.5. *Under the hypothesis of Lemma 3.4. Then for any $t\in\mathbb{R}$ and $\hat{D}\in\mathcal{D}$,*

$$\|u(t)\|_{\beta+1}^{\beta+1} \le Ce^{-\lambda t}(G_1(t)+G_2(t)), \tag{3.26}$$

and

$$\|\nabla\cdot w(t)\|^2 \le Ce^{-\lambda t}(G_1(t)+G_2(t)), \tag{3.27}$$

for every $\tau\le\tau_3$ and $u_\tau\in D(\tau)$, $w_\tau\in D(\tau)$.

Proof. Multiplying the first equation of (2.2) by u_t and integrating over $\mathbb{T}^3$, then we can obtain

$$\begin{aligned}&\|u_t(t)\|^2+\frac{d}{dt}(\|\nabla u(t)\|^2+\frac{1}{\beta+1}\|u(t)\|_{\beta+1}^{\beta+1})\\ \le&\frac{1}{2}\|u_t(t)\|^2+C(\|B(u)\|^2+\|\nabla w(t)\|^2+\|f(t)\|^2).\end{aligned}\tag{3.28}$$

For $3<\beta<5$, we can have the following inequality

$$\begin{aligned}C\|B(u)\|^2\le C&\int_{\mathbb{T}^3}|u|^2|\nabla u|^{\frac{4}{\beta-1}}|\nabla u|^{2-\frac{4}{\beta-1}}dx\\ &\le C(\||u|^{\frac{\beta-1}{2}}\nabla u\|^2+\|\nabla u(t)\|^2).\end{aligned}\tag{3.29}$$

Taking (3.29) into (3.28), we can have

$$\begin{aligned}&\frac{d}{dt}(\|\nabla u(t)\|^2+\frac{1}{\beta+1}\|u(t)\|_{\beta+1}^{\beta+1})\\ \le&C(\||u|^{\frac{\beta-1}{2}}\nabla u\|^2+\|\nabla u(t)\|^2+\|\nabla w(t)\|^2+\|f(t)\|^2).\end{aligned}\tag{3.30}$$

For (3.30), using uniform Gronwall Lemma, we get

$$\begin{aligned}&\|\nabla u(t)\|^2+\frac{\sigma}{\beta+1}\|u(t)\|_{\beta+1}^{\beta+1}\\ \le&C[\int_{t-1}^{t}(\|\nabla u(s)\|^2+\|u(s)\|_{\beta+1}^{\beta+1})ds\\ &+\int_{t-1}^{t}(\||u|^{\frac{\beta-1}{2}}\nabla u\|^2+\|\nabla w(s)\|^2+\|f(s)\|^2)ds]\\ \le&Ce^{-\lambda t}(G_1(t)+G_2(t)).\end{aligned}\tag{3.31}$$

Next, multiplying the second equation of (2.2) by w_t and integrating over $\mathbb{T}^3$, we have

$$\begin{aligned}&\|w_t(t)\|^2+\frac{d}{dt}(2\|w(t)\|^2+\frac{1}{2}\|\nabla w(t)\|^2+\frac{1}{2}\|\nabla\cdot w(t)\|^2)\\ \le&\frac{1}{2}\|w_t(t)\|^2+C(\|B(u,w)\|^2+\|\nabla u(t)\|^2+\|g(t)\|^2)\\ \le&\frac{1}{2}\|w_t(t)\|^2+C(\|\nabla u(t)\|^2+\|Au(t)\|^2)\|\nabla w(t)\|^2\\ &+C(\|\nabla u(t)\|^2+\|g(t)\|^2),\end{aligned}\tag{3.32}$$

where we use inequality (3.23). Then applying uniform Gronwall Lemma, we obtain

$$\begin{aligned}&2\|w(t)\|^2+\frac{1}{2}\|\nabla w(t)\|^2+\frac{1}{2}\|\nabla\cdot w(t)\|^2\\ \le&e^{C\int_{t-1}^{t}(\|\nabla u(s)\|^2+\|Au(s)\|^2)ds}[\int_{t-1}^{t}(\|w(s)\|^2+\|\nabla w(s)\|^2\\ &+\|\nabla\cdot w(s)\|^2)ds+C\int_{t-1}^{t}(\|\nabla u(s)\|^2+\|g(s)\|^2)ds]\\ \le&Ce^{-\lambda t}(G_1(t)+G_2(t)).\end{aligned}$$

So, the proof of Lemma 3.5 is finished. □

Lemma 3.6. *Assume (2.3)-(2.5) hold, $f \in L^2_{loc}(\mathbb{R}; H_1)$, $g \in L^2_{loc}(\mathbb{R}; H_2)$, $f_t \in L^2_b(\mathbb{R}; H_1)$ and $g_t \in L^2_b(\mathbb{R}; H_2)$. Let $3 < \beta < 5$ and $\tau \in \mathbb{R}$ and (u, w) be the solution of Eqs (1.1). Then for every $t \in \mathbb{R}$ and $\hat{D} \in \mathcal{D}$, there exists a constant $\tau_4 = \tau_4(t, \hat{D})$, such that for any $\tau \leq \tau_4$ and $u_\tau \in D(\tau)$, $w_\tau \in D(\tau)$,*

$$\|u_t(t)\|^2 + \|w_t(t)\|^2 \leq r_1(t), \tag{3.33}$$

where $r_1(t)$ is a positive constant which is independent of the initial data.

Proof. According to (3.28), (3.29) and (3.32), we have

$$\begin{aligned}
&\|u_t(t)\|^2 + \|w_t(t)\|^2 + \frac{d}{dt}(2\|\nabla u(t)\|^2 + \|\nabla w(t)\|^2 + \frac{2}{\beta+1}\|u(t)\|^{\beta+1}_{\beta+1} \\
&+ 4\|w(t)\|^2 + \|\nabla \cdot w(t)\|^2) \\
\leq &C(1 + \|\nabla u(t)\|^2 + \|Au(t)\|^2)(\|\nabla u(t)\|^2 + \|\nabla w(t)\|^2) \\
&+ C(\||u|^{\frac{\beta-1}{2}} \nabla u\|^2 + \|f(t)\|^2 + \|g(t)\|^2),
\end{aligned} \tag{3.34}$$

then integrating over $[t-1, t]$, we can get there has a constant $\tau_4 \equiv \tau_3 - 1$, for every $\tau \leq \tau_4$,

$$\begin{aligned}
&\int_{t-1}^{t} (\|u_s(s)\|^2 + \|w_s(s)\|^2) ds \\
\leq &2\|\nabla u(t-1)\|^2 + \|\nabla w(t-1)\|^2 + \frac{2}{\beta+1}\|u(t-1)\|^{\beta+1}_{\beta+1} \\
&+ 4\|w(t-1)\|^2 + \|\nabla \cdot w(t-1)\|^2 + C\int_{t-1}^{t} (1 + \|\nabla u(s)\|^2 \\
&+ \|Au(s)\|^2)(\|\nabla u(s)\|^2 + \|\nabla w(s)\|^2) ds \\
&+ C\int_{t-1}^{t} (\||u|^{\frac{\beta-1}{2}} \nabla u\|^2 + \|f(s)\|^2 + \|g(s)\|^2) ds \\
\leq &r_0^2(t) + r_0(t),
\end{aligned} \tag{3.35}$$

where $r_0(t) = Ce^{-\lambda t}(G_1(t) + G_2(t))$.

Applying ∂_t to the first and the second equations of system (2.2), and multiplying u_t and w_t, respectively, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|u_t\|^2 + \|w_t\|^2) + 2\|\nabla u_t\|^2 + \|\nabla w_t\|^2 + 4\|w_t\|^2 + \|\nabla \cdot w_t\|^2 \\
= &-\int_{\mathbb{T}^3} G'(u)u_t u_t dx - \int_{\mathbb{T}^3} u_t \cdot \nabla u u_t dx - \int_{\mathbb{T}^3} u_t \cdot \nabla w w_t dx \\
&+ 2\int_{\mathbb{T}^3} \nabla \times w_t u_t dx + 2\int_{\mathbb{T}^3} \nabla \times u_t w_t dx + (f_t, u_t) + (g_t, w_t) \\
:= &\sum_{i=1}^{7} I_i.
\end{aligned} \tag{3.36}$$

For I_1, according to the Lemma 2.4 of [16], we obtain that $I_1 \leq 0$.

For I_2, by using Hölder and Young inequalities, we easily get

$$I_2 \leq C\|u_t\|^{\frac{1}{2}}\|\nabla u_t\|^{\frac{3}{2}}\|\nabla u(t)\| \leq \frac{1}{2}\|\nabla u_t\|^2 + C\|u_t\|^2\|\nabla u(t)\|^4. \tag{3.37}$$

Similarly, we have

$$\begin{aligned} I_3 &\leq C\|u_t\|_4\|w_t\|_4\|\nabla w(t)\| \\ &\leq C\|u_t\|^{\frac{1}{4}}\|\nabla u_t\|^{\frac{3}{4}}\|w_t\|^{\frac{1}{4}}\|\nabla w_t\|^{\frac{3}{4}}\|\nabla w(t)\| \\ &\leq \frac{1}{4}(\|\nabla u_t\|^2 + \|\nabla w_t\|^2) + C(\|u_t\|^2 + \|w_t\|^2)\|\nabla w(t)\|^4, \end{aligned} \tag{3.38}$$

and

$$I_4 + I_5 \leq \frac{1}{4}(\|\nabla u_t\|^2 + \|\nabla w_t\|^2) + C(\|u_t\|^2 + \|w_t\|^2), \tag{3.39}$$

$$I_6 + I_7 \leq C(\|u_t\|^2 + \|w_t\|^2 + \|f_t\|^2 + \|g_t\|^2). \tag{3.40}$$

By using above inequalities, we can get

$$\begin{aligned} &\frac{d}{dt}(\|u_t\|^2 + \|w_t\|^2) + \|\nabla u_t\|^2 + \|\nabla w_t\|^2 + \|\nabla \cdot w_t\|^2 \\ \leq &C(1 + \|\nabla u(t)\|^4 + \|\nabla w(t)\|^4)(\|u_t\|^2 + \|w_t\|^2) + C(\|f_t\|^2 + \|g_t\|^2). \end{aligned} \tag{3.41}$$

Then using uniform Gronwall Lemma, we can obtain

$$\begin{aligned} &\|u_t(t)\|^2 + \|w_t(t)\|^2 \\ \leq &e^{C(t)}[\int_{t-1}^{t}(\|u_s(s)\|^2 + \|w_s(s)\|^2)ds + C\int_{t-1}^{t}(\|f_s(s)\|^2 + \|g_s(s)\|^2)ds] \\ \leq &C[r_0^2(t) + r_0(t) + \int_{t-1}^{t}(\|f_s(s)\|^2 + \|g_s(s)\|^2)ds] \\ := &r_1(t), \end{aligned} \tag{3.42}$$

where we let $C(t) = C\int_{t-1}^{t}(1 + \|\nabla u(s)\|^4 + \|\nabla w(s)\|^4)ds$.

By inequality (3.42), the proof of Lemma 3.6 is completed. □

Lemma 3.7. *Under the assumption of Lemma 3.6. Then for every $t \in \mathbb{R}$ and $\hat{D} \in \mathcal{D}$, we obtain that for any $\tau \leq \tau_4$ and $u_\tau \in D(\tau)$, $w_\tau \in D(\tau)$,*

$$\|Au(t)\|^2 \leq r_2(t), \tag{3.43}$$

and

$$\|Aw(t)\|^2 \leq r_3(t), \tag{3.44}$$

where $r_2(t)$ and $r_3(t)$ are all positive constants which are independent of the initial data.

Proof. Here, applying Minkowshi inequality to the first equation of system (2.2), then we can obtain

$$\|Au(t)\| \leq C(\|u_t(t)\| + \|B(u)\| + \||u|^{\beta-1}u\| + \|\nabla w(t)\| + \|f(t)\|). \tag{3.45}$$

For term $\|B(u)\|$, using similar method of (3.23), we have

$$C\|B(u)\| \leq C\|\nabla u(t)\|^{\frac{3}{2}}\|Au(t)\|^{\frac{1}{2}} \leq \frac{1}{4}\|Au(t)\| + C\|\nabla u(t)\|^3. \tag{3.46}$$

For term $\||u|^{\beta-1}u\|$, using Sobolev Lemma, we get for $3 < \beta < 5$,

$$C\||u|^{\beta-1}u\| \leq C\|\Delta u(t)\|^{\frac{\beta-3}{2}}\|\nabla u(t)\|^{\frac{\beta+3}{2}} \leq \frac{1}{4}\|Au(t)\| + C\|\nabla u(t)\|^{\frac{\beta+3}{5-\beta}}. \tag{3.47}$$

Taking (3.46) and (3.47) into (3.45), we obtain

$$\begin{aligned}\|Au(t)\|^2 &\leq C(\|u_t(t)\|^2 + \|\nabla u(t)\|^6 + \|\nabla u(t)\|^{\frac{2(\beta+3)}{5-\beta}} + \|\nabla w(t)\|^2 + \|f(t)\|^2)\\ &\leq C(r_1(t) + r_0^3(t) + r_0^{\frac{\beta+3}{5-\beta}}(t) + r_0(t) + \|f(t)\|^2)\\ &:= r_2(t).\end{aligned} \tag{3.48}$$

Inspired by [15], we let

$$Lw(t) = Aw(t) - \nabla\nabla \cdot w(t), \tag{3.49}$$

and we have

$$(Lw(t), Aw(t)) \geq \|Aw(t)\|^2 - C\|\nabla w(t)\|^2. \tag{3.50}$$

Hence, multiplying the second equation of (2.2) by Aw, we have

$$\|Aw(t)\|^2 \leq C(\|w_t(t)\|^2 + \|B(u,w)\|^2 + \|w(t)\|^2 + \|\nabla w(t)\|^2 + \|\nabla u(t)\|^2 + \|g(t)\|^2). \tag{3.51}$$

By using (3.23), we get

$$\begin{aligned}\|Aw(t)\|^2 \leq &C[\|w_t(t)\|^2 + \|w(t)\|^2 + \|\nabla u(t)\|^2 + \|g(t)\|^2\\ &+ (1 + \|\nabla u(t)\|^2 + \|Au(t)\|^2)\|\nabla w(t)\|^2]\\ \leq &C(r_0(t) + r_0^2(t) + r_0(t)r_2(t) + r_1(t) + \|g(t)\|^2)\\ := &r_3(t).\end{aligned}$$

The proof of Lemma 3.7 is finished. □

Lemma 3.8. *Under the hypothesis of Lemma 3.7, for any $t \in \mathbb{R}$ and $\hat{D} \in \mathcal{D}$, then we get for every $\tau \leq \tau_4$ and $u_\tau \in D(\tau)$, $w_\tau \in D(\tau)$,*

$$\|\nabla u_t(t+1)\|^2 + \|\nabla w_t(t+1)\|^2 \leq r_5(t), \tag{3.52}$$

where $r_5(t)$ is a positive constant which is independent of the initial data.

Proof. Integrating (3.41) over $[t, t+1]$, we have

$$\begin{aligned}
&\int_t^{t+1} (\|\nabla u_s(s)\|^2 + \|\nabla w_s(s)\|^2 + \|\nabla \cdot w_s(s)\|^2) ds \\
\le & \|u_t(t)\|^2 + \|w_t(t)\|^2 + C \int_t^{t+1} (1 + \|\nabla u(s)\|^4 + \|\nabla w(s)\|^4)(\|u_s(s)\|^2 \\
&+ \|w_s(s)\|^2) ds + C \int_t^{t+1} (\|f_s(s)\|^2 + \|g_s(s)\|^2) ds \\
\le & r_1(t) + (1 + r_0^2(t+1))(r_0(t+1) + r_0^2(t+1)) + C \int_t^{t+1} (\|f_s(s)\|^2 + \|g_s(s)\|^2) ds \\
:= & r_4(t).
\end{aligned} \tag{3.53}$$

Then using Sobolev inequality and Lemma 3.7, we get

$$\|u(t)\|_\infty^2 \le C\|\nabla u(t)\|\|Au(t)\| \le C\|Au(t)\|^2 \le r_2(t), \tag{3.54}$$

similarly,

$$\|w(t)\|_\infty^2 \le C\|\nabla w(t)\|\|Aw(t)\| \le C\|Aw(t)\|^2 \le r_3(t). \tag{3.55}$$

Applying ∂_t to the first and second equations of (2.2), then multiplying them by Au_t and Aw_t, respectively, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla u_t\|^2 + \|\nabla w_t\|^2) + 2\|Au_t\|^2 + \|Aw_t\|^2 + 4\|\nabla w_t\|^2 + \|\nabla\nabla \cdot w_t\|^2 \\
\le & \left|\int_{\mathbb{T}^3} u_t \cdot \nabla u A u_t dx\right| + \left|\int_{\mathbb{T}^3} u \cdot \nabla u_t A u_t dx\right| + \left|\int_{\mathbb{T}^3} u_t \cdot \nabla w A w_t dx\right| \\
&+ \left|\int_{\mathbb{T}^3} u \cdot \nabla w_t A w_t dx\right| + 2\left|\int_{\mathbb{T}^3} \nabla \times w_t A u_t dx\right| + 2\left|\int_{\mathbb{T}^3} \nabla \times u_t A w_t dx\right| \\
&+ (f_t, Au_t) + (g_t, Aw_t) + \left|\int_{\mathbb{T}^3} G'(u) u_t A u_t dx\right| \\
:= & \sum_{i=1}^{9} J_i.
\end{aligned} \tag{3.56}$$

Next, we estimate right terms of inequality.

For J_1, using Sobolev embedding Lemma, we have

$$\begin{aligned}
J_1 &\le C\|u_t\|_\infty \|\nabla u(t)\|\|Au_t\| \\
&\le C\|\nabla u_t\|^{\frac{1}{2}} \|Au_t\|^{\frac{3}{2}} \|\nabla u(t)\| \\
&\le \frac{1}{2}\|Au_t\|^2 + C\|\nabla u(t)\|^4 \|\nabla u_t\|^2.
\end{aligned} \tag{3.57}$$

For J_3, we can apply the similar method and obtain

$$\begin{aligned}
J_3 &\le C\|\nabla u_t\|^{\frac{1}{2}} \|Au_t\|^{\frac{1}{2}} \|\nabla w(t)\|\|Aw_t\| \\
&\le \frac{1}{8}(\|Au_t\|^2 + \|Aw_t\|^2) + C\|\nabla w(t)\|^4 \|\nabla u_t\|^2.
\end{aligned} \tag{3.58}$$

For others, using Hölder and Young inequalities, we get

$$J_2 \le C\|u(t)\|_\infty \|\nabla u_t\| \|Au_t\| \le \frac{1}{2}\|Au_t\|^2 + C\|u(t)\|_\infty^2 \|\nabla u_t\|^2, \tag{3.59}$$

$$J_4 \le C\|u(t)\|_\infty \|\nabla w_t\| \|Aw_t\| \le \frac{1}{8}\|Aw_t\|^2 + C\|u(t)\|_\infty^2 \|\nabla w_t\|^2, \tag{3.60}$$

$$J_5 + J_6 \le \frac{1}{8}(\|Au_t\|^2 + \|Aw_t\|^2) + C(\|\nabla u_t\|^2 + \|\nabla w_t\|^2), \tag{3.61}$$

$$J_7 + J_8 \le \frac{1}{8}(\|Au_t\|^2 + \|Aw_t\|^2) + C(\|f_t\|^2 + \|g_t\|^2), \tag{3.62}$$

and

$$J_9 \le C\|u(t)\|_\infty^{\beta-1} \|u_t\| \|Au_t\| \le \frac{1}{8}\|Au_t\|^2 + C\|u(t)\|_\infty^{2(\beta-1)} \|u_t\|^2. \tag{3.63}$$

Taking (3.57)-(3.63) into (3.56), we deduce

$$\begin{aligned}
&\frac{d}{dt}(\|\nabla u_t\|^2 + \|\nabla w_t\|^2) + \|Au_t\|^2 + \|Aw_t\|^2 \\
\le& C(1 + \|\nabla u(t)\|^4 + \|\nabla w(t)\|^4 + \|u(t)\|_\infty^2)(\|\nabla u_t\|^2 + \|\nabla w_t\|^2) \\
&+ C(\|u(t)\|_\infty^{2(\beta-1)} \|u_t\|^2 + \|f_t\|^2 + \|g_t\|^2).
\end{aligned}$$

Then by using uniform Gronwall Lemma over $[t, t+1]$, we obtain

$$\begin{aligned}
&\|\nabla u_t(t+1)\|^2 + \|\nabla w_t(t+1)\|^2 \\
\le& Ce^{C_2(t)}[\int_t^{t+1} (\|\nabla u_s(s)\|^2 + \|\nabla w_s(s)\|^2) ds \\
&+ C\int_t^{t+1} (\|u(s)\|_\infty^{2(\beta-1)} \|u_s(s)\|^2 + \|f_s(s)\|^2 + \|g_s(s)\|^2) ds] \\
\le& r_4(t) + r_2^{\beta-1}(t+1)(r_0(t+1) + r_0^2(t+1)) + C\int_t^{t+1} (\|f_s(s)\|^2 + \|g_s(s)\|^2) ds \\
:=& r_5(t),
\end{aligned}$$

where let $C_2(t) = \int_t^{t+1} (1 + \|\nabla u(s)\|^4 + \|\nabla w(s)\|^4 + \|u(s)\|_\infty^2) ds$.

Hence, the proof of Lemma 3.8 is finished. □

4. Existence of pullback $\mathcal{D}$-attractors

In this section, we devote to prove the existence of pullback $\mathcal{D}$-attractors in $V_1 \times V_2$ and $H^2 \times H^2$ for Eqs (2.2).

Firstly, we give the following lemma.

Lemma 4.1. *Assume (u_1, w_1) and (u_2, w_2) are two solutions of Eqs (2.2) with the initial data $(u_{i\tau}, w_{i\tau}) \in V_1 \times V_2$, $i = 1, 2$ and the external forces (f_i, g_i), where $(f_i, g_i) \in L^2_{loc}(\mathbb{R}; H_1) \times L^2_{loc}(\mathbb{R}; H_2)$, $i = 1, 2$. Let $(\tilde{u}, \tilde{w}) = (u_1 - u_2, w_1 - w_2)$, then for every $t \ge \tau$, it holds*

$$\begin{aligned}
\|\nabla \tilde{u}(t)\|^2 + \|\nabla \tilde{w}(t)\|^2 \le& e^{C(\tau,t)}[\|\nabla \tilde{u}_\tau\|^2 + \|\nabla \tilde{w}_\tau\|^2 \\
&+ \int_\tau^t (\|f_1(s) - f_2(s)\|^2 + \|g_1(s) - g_2(s)\|^2) ds].
\end{aligned} \tag{4.1}$$

Proof. By using (2.2), we can get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla\tilde{u}(t)\|^2+\|\nabla\tilde{w}(t)\|^2)+2\|A\tilde{u}(t)\|^2+\|A\tilde{w}(t)\|^2+\|\nabla\nabla\cdot\tilde{w}(t)\|^2+4\|\nabla\tilde{w}(t)\|^2\\
\le&\int_{\mathbb{T}^3}||u_1|^{\beta-1}u_1-|u_2|^{\beta-1}u_2||A\tilde{u}|dx+\int_{\mathbb{T}^3}|\tilde{u}\cdot\nabla u_1\cdot A\tilde{u}|dx\\
&+\int_{\mathbb{T}^3}|u_2\cdot\nabla\tilde{u}\cdot A\tilde{u}|dx+\int_{\mathbb{T}^3}|\tilde{u}\cdot\nabla w_1\cdot A\tilde{w}|dx+\int_{\mathbb{T}^3}|u_2\cdot\nabla\tilde{w}\cdot A\tilde{w}|dx\\
&+2\int_{\mathbb{T}^3}|\nabla\times\tilde{w}\cdot A\tilde{u}|dx+2\int_{\mathbb{T}^3}|\nabla\times\tilde{u}\cdot A\tilde{w}|dx\\
&+|(f_1(t)-f_2(t),A\tilde{u}(t))|+|(g_1(t)-g_2(t),A\tilde{w}(t))|\\
:=&\sum_{i=1}^{9}K_i.
\end{aligned}
\tag{4.2}
$$

Inspired by [16], we can have for $3<\beta<5$,

$$
\int_\tau^t(\|u_1(s)\|_{3(\beta-1)}^{2(\beta-1)}+\|\nabla u_2(s)\|^2(\|u_1(s)\|_{6(\beta-2)}^{2(\beta-2)}+\|u_2(s)\|_{6(\beta-2)}^{2(\beta-2)}))ds\le C(\tau,t).
\tag{4.3}
$$

Then for K_1, we can deduce

$$
\begin{aligned}
K_1\le&\frac{1}{8}\|A\tilde{u}(t)\|^2+C(\|u_1(t)\|_{3(\beta-1)}^{2(\beta-1)}+\|\nabla u_2(t)\|^2(\|u_1(t)\|_{6(\beta-2)}^{2(\beta-2)}\\
&+\|u_2(t)\|_{6(\beta-2)}^{2(\beta-2)}))\|\nabla\tilde{u}(t)\|^2.
\end{aligned}
\tag{4.4}
$$

For other estimates, using Hölder inequality and Young inequality, we get

$$
\begin{aligned}
K_2&\le C\|\nabla\tilde{u}(t)\|^{\frac{1}{2}}\|\Delta\tilde{u}(t)\|^{\frac{1}{2}}\|\nabla u_1(t)\|\|A\tilde{u}(t)\|\\
&\le C\|\nabla\tilde{u}(t)\|^{\frac{1}{2}}\|A\tilde{u}(t)\|^{\frac{3}{2}}\|\nabla u_1(t)\|\\
&\le\frac{1}{8}\|A\tilde{u}(t)\|^2+C\|\nabla u_1(t)\|^4\|\nabla\tilde{u}(t)\|^2, &(4.5)\\
K_3&\le C\|\nabla u_2(t)\|^{\frac{1}{2}}\|Au_2(t)\|^{\frac{1}{2}}\|\nabla\tilde{u}(t)\|\|A\tilde{u}(t)\|\\
&\le\frac{1}{8}\|A\tilde{u}(t)\|^2+C(\|\nabla u_2(t)\|^2+\|Au_2(t)\|^2)\|\nabla\tilde{u}(t)\|^2, &(4.6)\\
K_4&\le C\|\nabla\tilde{u}(t)\|^{\frac{1}{2}}\|A\tilde{u}(t)\|^{\frac{1}{2}}\|\nabla w_1(t)\|\|A\tilde{w}(t)\|\\
&\le\frac{1}{8}(\|A\tilde{u}(t)\|^2+\|A\tilde{w}(t)\|^2)+C\|\nabla w_1(t)\|^4\|\nabla\tilde{u}(t)\|^2, &(4.7)\\
K_5&\le C\|\nabla u_2(t)\|^{\frac{1}{2}}\|Au_2(t)\|^{\frac{1}{2}}\|\nabla\tilde{w}(t)\|\|A\tilde{w}(t)\|\\
&\le\frac{1}{8}\|A\tilde{w}(t)\|^2+C(\|\nabla u_2(t)\|^2+\|Au_2(t)\|^2)\|\nabla\tilde{w}(t)\|^2, &(4.8)\\
K_6+K_7&\le 4\|A\tilde{u}\|\|\nabla\tilde{w}\|\le\|A\tilde{u}\|^2+4\|\nabla\tilde{w}\|^2, &(4.9)\\
K_8+K_9&\le\frac{1}{8}(\|A\tilde{u}(t)\|^2+\|A\tilde{w}(t)\|^2)+C(\|f_1(t)-f_2(t)\|^2+\|g_1(t)-g_2(t)\|^2). &(4.10)
\end{aligned}
$$

Summing above inequalities, we can obtain

$$\begin{aligned}
&\frac{d}{dt}(\|\nabla\tilde{u}(t)\|^2+\|\nabla\tilde{w}(t)\|^2)\\
\le& C[\|u_1(t)\|_{3(\beta-1)}^{2(\beta-1)}+\|\nabla u_2(t)\|^2(\|u_1(t)\|_{6(\beta-2)}^{2(\beta-2)}+\|u_2(t)\|_{6(\beta-2)}^{2(\beta-2)})\\
&+\|\nabla u_1(t)\|^4+\|\nabla w_1(t)\|^4+\|\nabla u_2(t)\|^2+\|Au_2(t)\|^2](\|\nabla\tilde{u}(t)\|^2\\
&+\|\nabla\tilde{w}(t)\|^2)+C(\|f_1(t)-f_2(t)\|^2+\|g_1(t)-g_2(t)\|^2).
\end{aligned} \tag{4.11}$$

Then by using Gronwall Lemma on $[\tau, t]$, we can get

$$\begin{aligned}
\|\nabla\tilde{u}(t)\|^2+\|\nabla\tilde{w}(t)\|^2 \le& e^{C(\tau,t)}[\|\nabla\tilde{u}_\tau\|^2+\|\nabla\tilde{w}_\tau\|^2\\
&+\int_\tau^t(\|f_1(s)-f_2(s)\|^2+\|g_1(s)-g_2(s)\|^2)ds],
\end{aligned} \tag{4.12}$$

where we use Lemma 3.4 and Lemma 3.7. Hence the proof of Lemma 4.1 is finished. □

According to Lemma 4.1, we know that $\{U(t,\tau)\}_{t\ge\tau}$ is continuous in $V_1\times V_2$. So, it is also norm-to-weak continuous in $V_1\times V_2$.

Proof of Theorem 2.9. By using Lemma 3.4 and Lemma 3.7, we obtain there exist pullback $\mathcal{D}$-absorbing sets in $V_1\times V_2$ and $H^2\times H^2$, respectively. According to compact embedding $H^2\times H^2 \hookrightarrow V_1\times V_2$, we can get $\{U(t,\tau)\}_{t\ge\tau}$ is pullback $\mathcal{D}$-asymptotically compact in $V_1\times V_2$. Finally, due to Lemma 2.3 and Lemma 4.1, we obtain that $\{U(t,\tau)\}_{t\ge\tau}$ has a minimal pullback $\mathcal{D}$-attractor $\mathcal{A}_1$ in $V_1\times V_2$. □

Lemma 4.2. *The process $\{U(t,\tau)\}_{t\ge\tau}$ is norm-to-weak continuous in $H^2\times H^2$.*

Proof. Firstly, let $i: H^2\times H^2\hookrightarrow V_1\times V_2$ and $i^*: V_1^*\times V_2^*\hookrightarrow (H^2)^*\times(H^2)^*$. i and i^* are dense. Then, $\{U(t,\tau)\}_{t\ge\tau}: V_1\times V_2\to V_1\times V_2$ is norm-to-weak continuous which we can get from Lemma 4.1. Next, Lemma 3.7 can show that the process $\{U(t,\tau)\}_{t\ge\tau}$ has a pullback $\mathcal{D}$-absorbing set in $H^2\times H^2$. In other words, $\{U(t,\tau)\}_{t\ge\tau}$ maps a bounded set in $V_1\times V_2$ into a bounded set in $H^2\times H^2$. Hence, $\{U(t,\tau)\}_{t\ge\tau}$ maps a compact set in $H^2\times H^2$ into a bounded set in $H^2\times H^2$. By using Lemma 2.5, we can finish Lemma 4.2. □

Proof of Theorem 2.10. Firstly, according to Lemma 3.7, we can assume $B_0=\{B(t): t\in\mathbb{R}\}$ is a pullback $\mathcal{D}$-absorbing set in $H^2\times H^2$. Let

$$u_n(\tau_n)=u(t;\tau_n,u_{0n})=U(t,\tau_n)u_{0n},\ w_n(\tau_n)=w(t;\tau_n,w_{0n})=U(t,\tau_n)w_{0n}. \tag{4.13}$$

Then, we prove that for every $t\in\mathbb{R}$, every $\tau_n\to-\infty$ and $(u_{0n},w_{0n})\in C(\tau_n)$, $\{(u_n(\tau_n),w_n(\tau_n))\}_{n=0}^\infty$ is precompact in $H^2\times H^2$.

By using the fact that $V_1\times V_2\hookrightarrow H_1\times H_2$ and $H^2\times H^2\hookrightarrow V_1\times V_2$ are compact and estimates in section 3, we can have $\{(u_n(\tau_n),w_n(\tau_n))\}_{n=0}^\infty$ and $\{(\frac{\partial}{\partial t}u_n(\tau_n),\frac{\partial}{\partial t}w_n(\tau_n))\}_{n=0}^\infty$ are precompact in $V_1\times V_2$ and $H_1\times H_2$, respectively.

Nextly, we will show that $\{(u_n(\tau_n),w_n(\tau_n))\}_{n=1}^\infty$ is a Cauchy sequence in $H^2\times H^2$. Let

$$Lw_{nk}(\tau_{nk}) := Aw_{nk}(\tau_{nk})-\nabla\nabla\cdot w_{nk}(\tau_{nk}). \tag{4.14}$$

By using (2.2), we get

$$
\begin{aligned}
&2(Au_{nk}(\tau_{nk})-Au_{nj}(\tau_{nj}))\\
=&-\frac{d}{dt}u_{nk}(\tau_{nk})+\frac{d}{dt}u_{nj}(\tau_{nj})-B(u_{nk}(\tau_{nk}))\\
&+B(u_{nj}(\tau_{nj}))-G(u_{nk}(\tau_{nk}))+G(u_{nj}(\tau_{nj}))\\
&-2\nabla\times w_{nk}(\tau_{nk})+2\nabla\times w_{nj}(\tau_{nj}),
\end{aligned}
\tag{4.15}
$$

and

$$
\begin{aligned}
&Lw_{nk}(\tau_{nk})-Lw_{nj}(\tau_{nj})\\
=&-\frac{d}{dt}w_{nk}(\tau_{nk})+\frac{d}{dt}w_{nj}(\tau_{nj})-B(u_{nk}(\tau_{nk}),w_{nk}(\tau_{nk}))\\
&+B(u_{nj}(\tau_{nj}),w_{nj}(\tau_{nj}))-4w_{nk}(\tau_{nk})+4w_{nj}(\tau_{nj})\\
&-2\nabla\times u_{nk}(\tau_{nk})+2\nabla\times u_{nj}(\tau_{nj}).
\end{aligned}
\tag{4.16}
$$

Multiplying (4.15) and (4.16) by $Au_{nk}(\tau_{nk})-Au_{nj}(\tau_{nj})$ and $Aw_{nk}(\tau_{nk})-Aw_{nj}(\tau_{nj})$, respectively, we obtain

$$
\begin{aligned}
&\|Au_{nk}(\tau_{nk})-Au_{nj}(\tau_{nj})\|^2+\|Aw_{nk}(\tau_{nk})-Aw_{nj}(\tau_{nj})\|^2\\
\le& C[\|\nabla w_{nk}(\tau_{nk})-\nabla w_{nj}(\tau_{nj})\|^2+\|\frac{d}{dt}u_{nk}(\tau_{nk})-\frac{d}{dt}u_{nj}(\tau_{nj})\|^2\\
&+\|\frac{d}{dt}w_{nk}(\tau_{nk})-\frac{d}{dt}w_{nj}(\tau_{nj})\|^2+\|B(u_{nk}(\tau_{nk}))-B(u_{nj}(\tau_{nj}))\|^2\\
&+\|B(u_{nk}(\tau_{nk}),w_{nk}(\tau_{nk}))-B(u_{nj}(\tau_{nj}),w_{nj}(\tau_{nj}))\|^2+\|w_{nk}(\tau_{nk})-w_{nj}(\tau_{nj})\|^2\\
&+\|G(u_{nk}(\tau_{nk}))-G(u_{nj}(\tau_{nj}))\|^2+\|\nabla\times w_{nk}(\tau_{nk})-\nabla\times w_{nj}(\tau_{nj})\|^2\\
&+\|\nabla\times u_{nk}(\tau_{nk})-\nabla\times u_{nj}(\tau_{nj})\|^2]\\
\le& C[\|\nabla u_{nk}(\tau_{nk})-\nabla u_{nj}(\tau_{nj})\|^2+\|\nabla w_{nk}(\tau_{nk})-\nabla w_{nj}(\tau_{nj})\|^2\\
&+\|\frac{d}{dt}u_{nk}(\tau_{nk})-\frac{d}{dt}u_{nj}(\tau_{nj})\|^2+\|\frac{d}{dt}w_{nk}(\tau_{nk})-\frac{d}{dt}w_{nj}(\tau_{nj})\|^2\\
&+\|B(u_{nk}(\tau_{nk}))-B(u_{nj}(\tau_{nj}))\|^2+\|G(u_{nk}(\tau_{nk}))-G(u_{nj}(\tau_{nj}))\|^2\\
&+\|B(u_{nk}(\tau_{nk}),w_{nk}(\tau_{nk}))-B(u_{nj}(\tau_{nj}),w_{nj}(\tau_{nj}))\|^2,
\end{aligned}
\tag{4.17}
$$

where we use (3.13) in [15].

By using Lemma 3.7 and Sobolev inequality, we have

$$
\|u_{nk}(\tau_{nk})\|_\infty+\|w_{nk}(\tau_{nk})\|_\infty\le C. \tag{4.18}
$$

Inspired by [16], we get

$$
\|G(u_{nk}(\tau_{nk}))-G(u_{nj}(\tau_{nj}))\|^2\le C\|u_{nk}(\tau_{nk})-u_{nj}(\tau_{nj})\|^2\to 0,\ \text{as } k,j\to+\infty. \tag{4.19}
$$

By using Sobolev inequality, we have

$$
\begin{aligned}
&\|B(u_{nk}(\tau_{nk}))-B(u_{nj}(\tau_{nj}))\|^2\\
\le& C(\|B(u_{nk}(\tau_{nk}),u_{nk}(\tau_{nk})-u_{nj}(\tau_{nj}))\|^2+\|B(u_{nk}(\tau_{nk})-u_{nj}(\tau_{nj}),u_{nj}(\tau_{nj}))\|^2)\\
\le& C(\|\nabla u_{nk}(\tau_{nk})\|^2\|\nabla(u_{nk}(\tau_{nk})-u_{nj}(\tau_{nj}))\|\|A(u_{nk}(\tau_{nk})-u_{nj}(\tau_{nj}))\|\\
&+\|\nabla(u_{nk}(\tau_{nk})-u_{nj}(\tau_{nj}))\|^2\|\nabla u_{nj}(\tau_{nj})\|\|Au_{nj}(\tau_{nj})\|)\\
\to& 0,\ \text{as } k,j\to+\infty,
\end{aligned}
$$

and

$$
\begin{aligned}
&\|B(u_{nk}(\tau_{nk}), w_{nk}(\tau_{nk})) - B(u_{nj}(\tau_{nj}), w_{nj}(\tau_{nj}))\|^2 \\
\leq &C(\|B(u_{nk}(\tau_{nk}), w_{nk}(\tau_{nk}) - w_{nj}(\tau_{nj}))\|^2 + \|B(u_{nk}(\tau_{nk}) - u_{nj}(\tau_{nj}), w_{nj}(\tau_{nj}))\|^2) \\
\leq &C(\|\nabla u_{nk}(\tau_{nk})\|^2 \|\nabla(w_{nk}(\tau_{nk}) - w_{nj}(\tau_{nj}))\| \|A(w_{nk}(\tau_{nk}) - w_{nj}(\tau_{nj}))\| \\
&+ \|\nabla(u_{nk}(\tau_{nk}) - u_{nj}(\tau_{nj}))\|^2 \|\nabla w_{nj}(\tau_{nj})\| \|A w_{nj}(\tau_{nj})\|) \\
\to &0, \text{ as } k, j \to +\infty.
\end{aligned} \tag{4.20}
$$

By using above inequalities, we can obtain

$$
\begin{aligned}
&\|u_{nk}(\tau_{nk}) - u_{nj}(\tau_{nj})\|_{H^2}^2 + \|w_{nk}(\tau_{nk}) - w_{nj}(\tau_{nj})\|_{H^2}^2 \\
\leq &C(\|Au_{nk}(\tau_{nk}) - Au_{nj}(\tau_{nj})\|^2 + \|Aw_{nk}(\tau_{nk}) - Aw_{nj}(\tau_{nj})\|^2) \\
\to &0, \text{ as } k, j \to +\infty.
\end{aligned} \tag{4.21}
$$

Hence, $\{U(t,\tau)\}_{t\geq\tau}$ is pullback $\mathcal{D}$-asymptotically compact in $H^2 \times H^2$. Finally, according to Lemma 2.3, Lemma 3.7 and Lemma 4.2, the proof of Theorem 2.10 is completed. □

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Conflict of interest

The authors declare there is no conflict of interest.

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