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Research article

Long-time dynamics of an epidemic model with nonlocal diffusion and free boundaries

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Abstract: In this paper, we consider a reaction-diffusion epidemic model with nonlocal diffusion and free boundaries, which generalises the free-boundary epidemic model by Zhao et al. [1] by including spatial mobility of the infective host population. We obtain a rather complete description of the long-time dynamics of the model. For the reproduction number R_0 arising from the corresponding ODE model, we establish its relationship to the spreading-vanishing dichotomy via an associated eigenvalue problem. If $R_0 \leq 1$, we prove that the epidemic vanishes eventually. On the other hand, if $R_0 > 1$, we show that either spreading or vanishing may occur depending on its initial size. In the case of spreading, we make use of recent general results by Du and Ni [2] to show that finite speed or accelerated spreading occurs depending on whether a threshold condition is satisfied by the kernel functions in the nonlocal diffusion operators. In particular, the rate of accelerated spreading is determined for a general class of kernel functions. Our results indicate that, with all other factors fixed, the chance of successful spreading of the disease is increased when the mobility of the infective host is decreased, reaching a maximum when such mobility is 0 (which is the situation considered by Zhao et al. [1]).

Keywords: nonlocal diffusion; free boundary; spreading speed; accelerated spreading

1. Introduction

To model the 1973 cholera epidemic in the European Mediterranean region, Capasso and Paveri-Fontana [3] proposed the following ODE system

$$u'(t) = -au(t) + cv(t), \quad v'(t) = -bv(t) + G(u(t)), \quad t > 0, \tag{1.1}$$

where

- $u(t)$ and $v(t)$ represent respectively the average population concentration of the infectious agents and the infective humans in the infected area at time t,

- a, b, c are all positive constants such that $1/a$ represented the mean lifetime of the agents in the environment, $1/b$ the mean infectious period of the infective humans, c the multiplicative factor of the infectious agents due to the infective humans, and
- the function $G(u)$ is the infection rate of the human population, assuming that the total number of susceptible humans remain constant during the epidemic. The function G is assumed to satisfy the following:

(G1) $G \in C^1([0,\infty]), G(0) = 0, G'(z) > 0$ for all $z \geq 0$;
(G2) $\left(\frac{G(z)}{z}\right)' < 0$ for $z > 0$ and $\lim_{z\to+\infty} \frac{G(z)}{z} < \frac{ab}{c}$.

A simple example of such a function is given by $G(z) = \frac{\alpha z}{1+z}$ with $\alpha \in (0, ab/c)$.

They were able to establish the following result for the long-time dynamics of (1.1): Let $R_0 := \frac{cG'(0)}{ab}$; then regardless of the positive initial values $u(0)$ and $v(0)$,

(i) the epidemic tends to extinction if $R_0 < 1$, namely $\lim_{t\to\infty}(u(t), v(t)) \to (0,0)$ if $R_0 < 1$,
(ii) the epidemic tends to a positive equilibrium state if $R_0 > 1$, namely $\lim_{t\to\infty}(u(t), v(t)) \to (u^*, v^*)$ if $R_0 > 1$, where u^*, v^* are uniquely determined by

$$\frac{G(u^*)}{u^*} = \frac{ab}{c} \quad \text{and} \quad v^* = \frac{a}{c}u^*. \tag{1.2}$$

To include the mobility of the infectious agents (assuming the mobility of the infective human population is small and thus ignored), Capasso and Maddalena [4] proposed the following spatial reaction-diffusion model with Robin (or Neumann) boundary conditions

$$\begin{cases} u'(t) = d\Delta u - au + cv, & t > 0, x \in \Omega, \\ v'(t) = -bv + G(u), & t > 0, x \in \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} + \alpha u = 0, & t > 0, x \in \partial\Omega, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), & x \in \overline{\Omega}, \end{cases} \tag{1.3}$$

where d denotes the diffusion rate of u, the epidemic region $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain and $\alpha \geq 0$. They proved that the long-time behaviour of (1.3) is similar to the ODE model (1.1) with R_0 there replaced by $\tilde{R}_0 := \frac{cG'(0)}{(a+d\lambda_1)b}$, where λ_1 is the first eigenvalue of the eigenvalue problem

$$\begin{cases} \Delta\phi = \lambda\phi & \text{in } \Omega, \\ \dfrac{\partial\phi}{\partial \mathbf{n}} + \alpha\phi = 0 & \text{on } \partial\Omega. \end{cases}$$

In the literature, R_0 and $\tilde{R}_0$ are often called the reproduction number of the epidemic being modelled.

To describe the spatial spreading of an epidemic, it is important to know how the front of the epidemic propagates. In [5], Ahn, Baek, and Lin regarded the epidemic region as a changing interval and used the following free boundary problem to model the evolution and spreading of the epidemic:

$$\begin{cases} u_t = du_{xx} - au + cv, & t > 0,\ x \in (g(t), h(t)), \\ v_t = -bv + G(u), & t > 0,\ x \in (g(t), h(t)), \\ u(x,t) = v(x,t) = 0, & t > 0,\ x \in \{g(t), h(t)\}, \\ h'(t) = -\mu u_x(h(t), t), & t > 0, \\ g'(t) = -\mu u_x(g(t), t), & t > 0, \\ h(0) = -g(0) = h_0, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), & x \in [-h_0, h_0], \end{cases} \tag{1.4}$$

where $h(t)$ and $g(t)$ are the moving boundaries of the infected region, μ is a positive constant and the initial data (u_0, v_0) satisfy

$$u_0, v_0 \in C([-h_0, h_0]), \quad u_0(\pm h_0) = v_0(\pm h_0) = 0, \quad \text{and } u_0, v_0 > 0 \text{ in } (-h_0, h_0). \tag{1.5}$$

The equations for $h'(t)$ and $g'(t)$ mean that the expanding rate of the infected region is proportional to the spatial gradient of u at the front. This is known as the Stefan condition which was first used to describe the melting of ice (see, e.g., [6]). It has been extensively used in the study of the spread of population since Du and Lin [7].

The long-time dynamics of (1.4) can be described by a spreading-vanishing dichotomy; more precisely, Ahn *et al.* showed that the unique solution (u, v, g, h) to (1.4) satisfies either

(i) **(vanishing)**

$$\begin{cases} \lim_{t\to\infty}(g(t), h(t)) = (g_\infty, h_\infty) \text{ is a finite interval, and} \\ \lim_{t\to\infty}(u(x,t), v(x,t)) = (0,0) \text{ uniformly for } x \in [g(t), h(t)], \end{cases} \qquad \text{or}$$

(ii) **(spreading)**

$$\begin{cases} \lim_{t\to\infty}(g(t), h(t)) = \mathbb{R}, \text{ and} \\ \lim_{t\to\infty}(u(x,t), v(x,t)) = (u^*, v^*) \text{ locally uniformly for } x \in \mathbb{R}. \end{cases}$$

Furthermore, using the reproduction number of (1.1), namely

$$R_0 := \frac{cG'(0)}{ab}, \tag{1.6}$$

the dichotomy is determined as follows: If $R_0 \le 1$, then vanishing always happens; in the case where $R_0 > 1$, there exists a critical length, $l^* := \frac{\pi}{2}\sqrt{\frac{d_1}{a(R_0-1)}}$, such that

- if $h_0 \ge l^*$ (i.e., the initial size of the infected region is no less than $2l^*$), then spreading always happens, and
- if $h_0 < l^*$, then vanishing (resp. spreading) happens if the initial functions (u_0, v_0) are sufficiently small (resp. large).

In the case of an epidemic spreading predicted by (1.4), it was shown by Zhao, Li, and Ni [8] that there exists a uniquely determined $c_0 > 0$ such that

$$\lim_{t\to\infty} \frac{h(t)}{t} = \lim_{t\to\infty} \frac{g(t)}{-t} = c_0,$$

which means the epidemic region $[g(t), h(t)]$ expands with asymptotic speed c_0.

In (1.4) (as well as in (1.3)), the spatial dispersal of the infectious agents is assumed to follow the rules of random walk, which ignores any nonlocal effect in the dispersal process. Such nonlocal effect can be included if the local diffusion operator is replaced by a nonlocal diffusion operator of the form

$$d \int_{\mathbb{R}} J(x-y)[u(y,t) - u(x,t)]dy$$

with an appropriate kernel function J. Here $J(x)$ can be interpreted as the probability that an individual of the species moves from location 0 to x. A widely used class of kernel functions consists of $J : \mathbb{R} \to \mathbb{R}$ satisfying

$$\textbf{(J):} \quad J \in C(\mathbb{R}) \cap L^\infty(\mathbb{R}), J \text{ is even and nonnegative, } J(0) > 0, \int_{\mathbb{R}} J(x)dx = 1.$$

One recent paper by Cao, Du, Li, and Li [9] extended many basic results of [7] to the corresponding nonlocal model with the above kernel. Following the fashion of [9], Zhao, Zhang, Li, and Du [1] considered the corresponding nonlocal version of (1.4), which has the following form

$$\begin{cases} u_t = d \displaystyle\int_{\mathbb{R}} J(x-y)[u(y,t) - u(x,t)]dy - au + cv, & t > 0,\ x \in (g(t), h(t)), \\ v_t = -bv + G(u), & t > 0,\ x \in (g(t), h(t)), \\ u(x,t) = v(x,t) = 0, & t > 0,\ x \notin (g(t), h(t)), \\ h'(t) = \mu \displaystyle\int_{g(t)}^{h(t)} \int_{h(t)}^{\infty} J(x-y)u(x,t)dydx, & t > 0, \\ g'(t) = -\mu \displaystyle\int_{g(t)}^{h(t)} \int_{-\infty}^{g(t)} J(x-y)u(x,t)dydx, & t > 0, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), \quad h(0) = -g(0) = h_0, & |x| \le h_0, \end{cases} \tag{1.7}$$

It was shown in [1] that (1.7) has a unique solution defined for all $t > 0$, and its long-time dynamics is determined by a spreading-vanishing dichotomy, in a similar fashion to (1.4) (with some subtle differences though). A striking difference of (1.7) to (1.4) is revealed by [2], which shows that the spreading determined by (1.7) may have infinite asymptotic spreading speed, a phenomenon known as “accelerated spreading”. More precisely, if the kernel function J satisfies

$$\int_0^\infty xJ(x) = \infty,$$

and spreading happens, then

$$\lim_{t\to\infty} \frac{h(t)}{t} = \lim_{t\to\infty} \frac{g(t)}{-t} = \infty.$$

Moreover, if $J(x) \simeq |x|^{-\gamma}$ for some $\gamma \in (1,2]$ and all large $|x| > 0$, then for all large $t > 0$,

$$\begin{cases} -g(t), h(t) \simeq t \ln t & \text{if } \gamma = 2, \\ -g(t), h(t) \simeq t^{1/(\gamma-1)} & \text{if } \gamma \in (1,2). \end{cases}$$

Here, and in what follows, $\eta(t) \simeq \xi(t)$ means $C_1\xi(t) \le \eta(t) \le C_2\xi(t)$ for some positive constants $C_1 \le C_2$ and all t in the specified range.

In this paper, to understand the effect of the mobility of the infective host on the epidemic spreading, we examine a full version of (1.7) *, where *the dispersal of infective host* is included. Before giving this full version, let us note that, in (1.7), since $u(x,t) = 0$ for $x \notin (g(t), h(t))$ and $\int_{\mathbb{R}} J(x)dx = 1$,

$$\int_{\mathbb{R}} J(x-y)[u(y,t) - u(x,t)]dy = \int_{g(t)}^{h(t)} J(x-y)u(y,t)dy - u(x,t) \text{ for } x \in (g(t), h(t)).$$

For $i = 1, 2$, suppose $J_i : \mathbb{R} \to \mathbb{R}$ satisfy **(J)**. Let $a, b, c, d_1, d_2, \mu_1, \mu_2$ and h_0 be constants, all positive except μ_1 and μ_2, which are assumed to be nonnegative with $\mu_1 + \mu_2 > 0$, and let the initial functions $u_0(x)$ and $v_0(x)$ satisfy (1.5). Then the full version of (1.7) can be written in the following form

$$\begin{cases} u_t = d_1 \displaystyle\int_{g(t)}^{h(t)} J_1(x-y)u(y,t)dy - d_1 u - au + cv, & t > 0, x \in (g(t), h(t)), \\ v_t = d_2 \displaystyle\int_{g(t)}^{h(t)} J_2(x-y)v(y,t)dy - d_2 v - bv + G(u), & t > 0, x \in (g(t), h(t)), \\ u(x,t) = v(x,t) = 0, & t > 0, x \in \{g(t), h(t)\}, \\ h'(t) = \displaystyle\int_{g(t)}^{h(t)} \int_{h(t)}^{\infty} [\mu_1 J_1(x-y)u(x,t) + \mu_2 J_2(x-y)v(x,t)]dydx, & t > 0, \\ g'(t) = -\displaystyle\int_{g(t)}^{h(t)} \int_{-\infty}^{g(t)} [\mu_1 J_1(x-y)u(x,t) + \mu_2 J_2(x-y)v(x,t)]dydx, & t > 0, \\ h(0) = -g(0) = h_0, \;\; u(x,0) = u_0(x), v(x,0) = v_0(x), & |x| \le h_0, \end{cases} \tag{1.8}$$

We will prove the following results.

Theorem 1.1 (Existence and Uniqueness)**.** *The problem* (1.8) *admits a unique positive solution* (u, v, g, h) *defined for* $t \ge 0$.

Theorem 1.2 (Spreading-Vanishing Dichotomy)**.** *Let* (u, v, g, h) *be the solution to* (1.8) *and denote* $h_\infty := \lim_{t\to\infty} h(t)$ *and* $g_\infty := \lim_{t\to\infty} g(t)$*. Then either*

(i) **(vanishing)** (g_∞, h_∞) *is a finite interval and*

$$\lim_{t\to\infty}(u(x,t), v(x,t)) = (0,0) \textit{ uniformly for } x \in [g(t), h(t)], \textit{ or} \tag{1.9}$$

(ii) **(spreading)** $(g_\infty, h_\infty) = \mathbb{R}$ *and*

$$\lim_{t\to\infty}(u(x,t), v(x,t)) = (u^*, v^*) \textit{ locally uniformly for } x \in \mathbb{R}.$$

*A full version of the local diffusion model (1.4) was recently investigated in [10], which showed that its long-time dynamics is similar to that of (1.4) though some differences occur in the criteria governing the spreading-vanishing dichotomy. In particular, when spreading happens, there exists a finite asymptotic spreading speed.

Theorem 1.3 (Spreading-Vanishing Criteria)**.** *Let* (u, v, g, h) *be the solution of* (1.8) *and* R_0 *be given by* (1.6).

(a) *If* $R_0 \le 1$*, then vanishing always occurs.*
(b) *If* $R_0 > 1$*, then spreading always occurs if one of the following holds:*

$$\text{(I)}\ \frac{cG'(0)}{(d_1 + a)(d_2 + b)} \ge 1,$$
$$\text{(II)}\ \frac{cG'(0)}{(d_1 + a)(d_2 + b)} < 1 \text{ and } h_0 \ge L^*,$$

where $L^* > 0$ *is a certain critical length depending on* $a, b, c, d_1, d_2, J_1, J_2$ *but independent of the initial data* (u_0, v_0) *.*
(c) *If* $R_0 > 1$ *and*

$$\frac{cG'(0)}{(d_1 + a)(d_2 + b)} < 1 \text{ and } h_0 < L^*,$$

then
  (i) *for any given initial datum* (u_0, v_0) *satisfying* (1.5)*, and any given constants* σ_1^0, σ_2^0 *nonnegative satisfying* $\sigma_1^0 + \sigma_2^0 > 0$*, there exists* $\mu^* > 0$ *such that*
    (α) *if* $(\mu_1, \mu_2) = (\mu\sigma_1^0, \mu\sigma_2^0)$ *and* $0 < \mu \le \mu^*$*, then vanishing occurs, and*
    (β) *if* $(\mu_1, \mu_2) = (\mu\sigma_1^0, \mu\sigma_2^0)$ *and* $\mu > \mu^*$*, then spreading occurs.*
  (ii) *for fixed* (μ_1, μ_2) *and sufficiently small initial datum* (u_0, v_0)*, vanishing occurs.*

Remark 1.4. (i) The constant L^* in Theorem 1.3 is uniquely determined by an eigenvalue problem; see Proposition 3.4(iii) below.

(ii) In the case of (1.7) considered in [1], which is equivalent to (1.8) with $d_2 = \mu_2 = 0$, the long-time dynamics is also governed by a spreading-vanishing dichotomy, and the spreading-vanishing criteria coincide with those in Theorem 1.3 but with d_2 and μ_2 replaced by 0.

(iii) If $\mu_2 = 0$ and all the other parameters in (1.8) are positive and fixed except d_2, which is allowed to vary in $[0, \infty)$, then from Lemmas 3.2 and 3.3 below it is easily seen that $L^* = L^*(d_2)$ is strictly increasing in d_2. Therefore part (b) of Theorem 1.3 and the result in [1] indicate that the range of parameters (a, b, c, d_1, h_0) for which spreading happens regardless of the size of the initial function pare (u_0, v_0), i.e., (I) or (II) above holds, is enlarged as d_2 is decreased, and such a range is maximized when $d_2 = 0$. Biologically, this means that reducing the mobility of the infective host increases the chance of successful spreading of the disease, which appears counter-intuitive at first look. However, such a phenomenon is not new; it arises in the local diffusion models considered in [4, 10] already.

When spreading happens, the spreading profile of (1.8) can be determined by using the general results in [2]. For this purpose, we will need the following condition

(J1): $\displaystyle\int_0^\infty xJ_i(x)dx < \infty$ for $i = 1, 2$ with $\mu_i > 0$.

Theorem 1.5 (Spreading Speed)**.** *In Theorem 1.2, if spreading happens, then*

$$\lim_{t\to\infty} \frac{g(t)}{-t} = \lim_{t\to\infty} \frac{h(t)}{t} = \begin{cases} c_0 & \text{if } \textbf{(J1)} \text{ holds,} \\ \infty & \text{if } \textbf{(J1)} \text{ does not hold,} \end{cases} \tag{1.10}$$

where $c_0 > 0$ *is uniquely determined by the associated semi-wave problem to* (1.8) (see [2, Section 1.2]).

Furthermore, in the case of accelerated spreading, we can determine the rate of accelerated spreading for a rather general class of kernel functions.

Theorem 1.6 (Rate of Accelerated Spreading)**.** *In Theorem 1.5 suppose additionally that for* $i = 1, 2$ *with* $\mu_i > 0$*, the kernel function* J_i *satisfies* $J_i(x) \simeq |x|^{-\gamma}$ *for some* $\gamma \in (1, 2]$ *and* $|x| \gg 1$*. Then for* $t \gg 1$*, we have*

$$\begin{aligned} &-g(t), h(t) \simeq t \ln t \qquad && \text{if } \gamma = 2,\\ &-g(t), h(t) \simeq t^{1/(\gamma-1)} \qquad && \text{if } \gamma \in (1, 2). \end{aligned} \tag{1.11}$$

Let us note that when J_i satisfies $J_i(x) \simeq |x|^{-\gamma}$ for $|x| \gg 1$ and for $i \in \{1, 2\}$ such that $\mu_i > 0$, **(J1)** holds if and only if $\gamma > 2$. Thus Theorem 1.6 covers the exact range of γ such that accelerated spreading is possible. Note also that in condition **(J1)** as well as in Theorem 1.6, the condition only applies to the kernel function J_i where $\mu_i > 0$. For example, if $\mu_2 = 0$, then no extra condition on J_2 is needed apart from satisfying **(J)**.

Problem (1.8) has an entire space version where no free boundary is involved, which has the form

$$\begin{cases} u_t = d_1 \displaystyle\int_{\mathbb{R}} J_1(x-y)u(y,t)dy - d_1 u - au + cv, & t > 0,\ x \in \mathbb{R},\\ v_t = d_2 \displaystyle\int_{\mathbb{R}} J_2(x-y)v(y,t)dy - d_2 v - bv + G(u), & t > 0,\ x \in \mathbb{R},\\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.12}$$

Problem (1.12) has been successfully used to determine the spreading speed of the epidemic; see [11] and the references therein for many interesting results on this and related problems. For the entire space version of (1.7), see [2, 12] and the references therein for more details. The local diffusion counterparts of these entire space problems have been studied more extensively; see, for example, [13–15]. As mentioned above, the corresponding free boundary models have the advantage of providing the exact location of the spreading front of the concerned epidemics.

The rest of the paper is organized as follows. In Section 2, we introduce the preparatory results relating to (1.8) and use them to prove Theorem 1.1. In Section 3, we gather the necessary results associated with the corresponding fixed boundary problems, which will be used to determine the long-time dynamical behavior of (1.8). In Section 4, we use the results of the previous sections to establish the vanishing-spreading dichotomy as related to the reproduction number R_0, proving Theorems 1.2 and 1.3. Finally, Section 5 is devoted to proving the assumptions required in [2] for Theorems 1.5 and 1.6.

We would like to point out that, although (1.8) has some significantly different features from the West Nile virus model studied in [16], for example, the nature of the reaction terms in (1.8) makes any nonnegative initial function admissible while the model in [16] only allows initial functions taken from a certain bounded order interval, but many techniques of [16] can be adapted to treat (1.8), which has helped to considerably reducing the length of the current paper. Here we only provide the details of the proofs when they are very different from [16].

2. Existence and uniqueness, and comparison principle

In this section, we prove the well-posedness of (1.8) and some associated comparison principles.

We first recall a maximum principle from [16, Lemma 3.1], which is more general than needed in this paper, but in view of possible applications elsewhere, we state it in the general form as in [16].

Let $T > 0$ and $\xi \in C([0,T])$. We define the set of strict local semi-maximum points of ξ by

$$\Sigma^{\xi}_{\max} := \{t \in (0,T] : \exists \epsilon > 0 \text{ such that } \xi(t) > \xi(s) \text{ for } s \in [t-\epsilon, t)\}.$$

Similarly, the set of strict local semi-minimum points of ξ is given by

$$\Sigma^{\xi}_{\min} := \{t \in (0,T] : \exists \epsilon > 0 \text{ such that } \xi(t) < \xi(s) \text{ for } s \in [t-\epsilon, t)\}.$$

If ξ is strictly increasing, then $\Sigma^{\xi}_{\max} = (0,T]$. If ξ is nondecreasing, then $\Sigma^{\xi}_{\min} = \emptyset$.

Lemma 2.1 (Maximum Principle)**.** *Let $T, h_0 > 0, g, h \in C([0,T])$ satisfy $g(t) < h(t)$ and $-g(0) = h(0) = h_0$. Denote $D_T := \{(x,t) : t \in (0,T], g(t) < x < h(t)\}$ and suppose that for $i, j \in \{1,2,...,n\}$, $\phi_i, \partial_t\phi_i \in C(\overline{D_T}), d_i, c_{ij} \in L^\infty(D_T)$, $d_i \geq 0$, and*

$$\begin{cases} (\phi_i)_t \geq d_i \displaystyle\int_{g(t)}^{h(t)} J_i(x-y)\phi_i(y,t)dy - d_i\phi_i + \sum_{j=1}^{n} c_{ij}\phi_j, & (x,t) \in D_T, \\ \phi_i(g(t),t) \geq 0, & t \in \Sigma^g_{\min}, \\ \phi_i(h(t),t) \geq 0, & t \in \Sigma^h_{\max}, \\ \phi_i(x,0) \geq 0, & |x| \leq h_0, \end{cases}$$

where J_i satisfies **(J)**. *Then the following holds:*

(i) *If $c_{ij} \geq 0$ on D_T for $i, j \in \{1,2,...n\}$ and $i \neq j$, then $\phi_i \geq 0$ on $\overline{D_T}$ for $i \in \{1,2,...,n\}$.*

(ii) *If for some $i_0 \in \{1,...,n\}$ we assume additionally that $d_{i_0} > 0$ in D_T and $\phi_{i_0}(x,0) \not\equiv 0$ in $[-h_0, h_0]$, then $\phi_{i_0}(x,t) > 0$ in D_T.*

Lemma 2.2 (Comparison Principle I). *For $T \in (0,+\infty)$, suppose that $\overline{g}, \overline{h} \in C([0,T])$, $D = \{(x,t) : t \in (0,T], \overline{g}(t) < x < \overline{h}(t)\}$, $\overline{u}, \overline{v} \in C(\overline{D})$, $\overline{u}, \overline{v} \geq 0$. If $(\overline{u}, \overline{v}, \overline{g}, \overline{h})$ satisfies*

$$\begin{cases} \overline{u}_t \geq d_1 \displaystyle\int_{g(t)}^{h(t)} J_1(x-y)\overline{u}(y,t)dy - d_1\overline{u} - a\overline{u} + c\overline{v}, & t > 0, x \in (\overline{g}(t), \overline{h}(t)), \\ \overline{v}_t \geq d_2 \displaystyle\int_{g(t)}^{h(t)} J_2(x-y)\overline{v}(y,t)dy - d_2\overline{v} - b\overline{v} + G(\overline{u}), & t > 0, x \in (\overline{g}(t), \overline{h}(t)), \\ \overline{h}'(t) \geq \displaystyle\int_{g(t)}^{h(t)} \int_{h(t)}^{\infty} [\mu_1 J_1(x-y)\overline{u}(x,t) + \mu_2 J_2(x-y)\overline{v}(x,t)]dydx, & t > 0, \\ \overline{g}'(t) \leq -\displaystyle\int_{g(t)}^{h(t)} \int_{-\infty}^{g(t)} [\mu_1 J_1(x-y)\overline{u}(x,t) + \mu_2 J_2(x-y)\overline{v}(x,t)]dydx, & t > 0, \\ \overline{g}(0) \leq -h_0, \overline{h}(0) \geq h_0,\ \overline{u}(x,0) \geq u_0(x), \overline{v}(x,0) \geq v_0(x), & |x| \leq h_0, \end{cases} \tag{2.1}$$

then the unique solution (u,v,g,h) of (1.8) satisfies

$$u(x,t) \leq \overline{u}(x,t), \quad v(x,t) \leq \overline{v}(x,t), \quad g(t) \geq \overline{g}(t), \quad h(t) \leq \overline{h}(t), \tag{2.2}$$

for $0 < t \leq T$ and $g(t) \leq x \leq h(t)$.

Proof. By **(G1)**, we have that $G(\overline{u}) = G'(\xi)\overline{u}$ with $\xi = \xi(x,t) \in (0, \overline{u}(x,t)]$. From (1.5) and Lemma 2.1, we infer that $\overline{u}, \overline{v} > 0$, for $0 < t \le T$ and $\overline{g}(t) < x < \overline{h}(t)$. Therefore, $-\overline{g}$ and $\overline{h}$ are strictly increasing.

For small $\epsilon > 0$, let $(u_\epsilon, v_\epsilon, g_\epsilon, h_\epsilon)$ denote the unique solution of (1.8) with h_0 replaced by $h_0^\epsilon := h_0(1-\epsilon)$, μ_i by $\mu_i^\epsilon := \mu_i(1-\epsilon)$ for $i = 1, 2$, and (u_0, v_0) by $(u_0^\epsilon, v_0^\epsilon)$ satisfying

$$\begin{cases} 0 < u_0^\epsilon(x) < u_0(x), 0 < v_0^\epsilon(x) < v_0(x) \quad \text{in } (-h_0^\epsilon, h_0^\epsilon), \\ u_0^\epsilon(\pm h_0^\epsilon) = v_0^\epsilon(\pm h_0^\epsilon) = 0, \text{ and} \\ (u_0^\epsilon(\frac{h_0}{h_0^\epsilon}x), v_0^\epsilon(\frac{h_0}{h_0^\epsilon}x)) \to (u_0(x), v_0(x)) \quad \text{as } \epsilon \to 0 \text{ in the } C([-h_0, h_0]) \text{ norm.} \end{cases} \tag{2.3}$$

We claim that $h_\epsilon(t) < \overline{h}(t)$ and $g_\epsilon(t) > \overline{g}(t)$ for all $t \in (0, T]$. It is clear that these hold for small $t > 0$. Suppose that there exists $t_1 \le T$ such that

$$h_\epsilon(t) < \overline{h}(t),\ g_\epsilon(t) > \overline{g}(t) \text{ for } t \in (0, t_1) \text{ and } [h_\epsilon(t_1) - \overline{h}(t_1)][g_\epsilon(t_1) - \overline{g}(t_1)] = 0.$$

Without loss of generality, assume that $h_\epsilon(t_1) = \overline{h}(t_1)$ and $g_\epsilon(t_1) \ge \overline{g}(t_1)$. Let $w := \overline{u} - u_\epsilon$ and $z := \overline{v} - v_\epsilon$; then (w, z) satisfies

$$\begin{cases} w_t \ge d_1 \displaystyle\int_{g_\epsilon(t)}^{h_\epsilon(t)} J_1(x-y)w(y,t)dy - d_1 w - aw + cz, & 0 < t \le t_1,\ x \in (g_\epsilon(t), h_\epsilon(t)), \\ z_t \ge d_2 \displaystyle\int_{g_\epsilon(t)}^{h_\epsilon(t)} J_2(x-y)z(y,t)dy - d_2 z - bz + G'(\eta)w, & 0 < t \le t_1,\ x \in (g_\epsilon(t), h_\epsilon(t)), \\ w(x,t) \ge 0,\ z(x,t) \ge 0, & 0 < t \le t_1,\ x = g_\epsilon(t) \text{ or } h_\epsilon(t), \\ w(x,0) > 0,\ z(x,0) > 0, & x \in [g_\epsilon(0), h_\epsilon(0)], \end{cases} \tag{2.4}$$

where $\eta = \eta(x,t)$ is between $\overline{u}(x,t)$ and $u_\epsilon(x,t)$. Therefore we can apply Lemma 2.1 to conclude that $w(x,t) > 0$ and $z(x,t) > 0$ for $0 < t \le t_1$ and $g_\epsilon(t) < x < h_\epsilon(t)$.

However, by definition of t_1, we have $h_\epsilon'(t_1) \ge \overline{h}'(t_1)$, giving us that

$$\begin{aligned} 0 \ge\ & \overline{h}'(t_1) - h_\epsilon'(t_1) \\ \ge & \int_{\overline{g}(t_1)}^{\overline{h}(t_1)} \int_{\overline{h}(t_1)}^{\infty} [\mu_1 J_1(x-y)\overline{u}(x,t_1) + \mu_2 J_2(x-y)\overline{v}(x,t_1)]dydx \\ & - \int_{g_\epsilon(t_1)}^{h_\epsilon(t_1)} \int_{h_\epsilon(t_1)}^{\infty} [\mu_1^\epsilon J_1(x-y)u_\epsilon(x,t_1) + \mu_2^\epsilon J_2(x-y)v_\epsilon(x,t_1)]dydx \\ \ge & \int_{g_\epsilon(t_1)}^{h_\epsilon(t_1)} \int_{h_\epsilon(t_1)}^{\infty} [\mu_1^\epsilon J_1(x-y)w(x,t_1) + \mu_2^\epsilon J_2(x-y)z(x,t_1)]dydx > 0. \end{aligned}$$

This contradiction proves our claim, namely, $h_\epsilon(t) < \overline{h}(t)$ and $g_\epsilon(t) > \overline{g}(t)$ for all $t \in (0, T]$. Hence (2.4) holds with t_1 replaced by T, which yields that $\overline{u}(x,t) > u_\epsilon(x,t)$ and $\overline{v}(x,t) > v_\epsilon(x,t)$ for $0 < t \le T$ and $g_\epsilon(t) < x < h_\epsilon(t)$. Letting $\epsilon \to 0$, we obtain the desired result from the continuous dependence of $(u_\epsilon, v_\epsilon, g_\epsilon, h_\epsilon)$ on ϵ. □

We introduce a second comparison principle where the boundaries are regarded as given.

Lemma 2.3 (Comparison Principle II)**.** *Assume* **(J)** *holds,* $T > 0$*,* $g, h \in C([0, T])$ *satisfying* $g(t) < h(t)$*, and* D_T *defined as in Lemma 2.1. If for* $i = 1, 2$*,* $u_i, \tilde{u}_i \in C(\overline{D_T})$ *satisfy the following conditions:*

(i) $\phi_t \in C(\overline{D_T})$ *for* $\phi \in \{u_1, u_2, \tilde{u}_1, \tilde{u}_2\}$*,*

(ii) *for* $(x, t) \in D_T$*,*

$$\begin{cases} (\tilde{u}_1)_t \geq d_1 \displaystyle\int_{g(t)}^{h(t)} J_1(x-y)\tilde{u}_1(y,t)dy - d_1\tilde{u}_1 - a\tilde{u}_1 + c\tilde{u}_2, \\ (\tilde{u}_2)_t \geq d_2 \displaystyle\int_{g(t)}^{h(t)} J_2(x-y)\tilde{u}_2(y,t)dy - d_2\tilde{u}_2 - b\tilde{u}_2 + G(\tilde{u}_1), \end{cases} \tag{2.5}$$

(iii) *for* $(x, t) \in D_T$*,* (u_1, u_2) *satisfies* (2.5) *but with the inequalities reversed,*

(iv) *at the boundary,*

$$\begin{cases} u_i(g(t), t) \leq \tilde{u}_i(g(t), t) & \text{for } t \in \Sigma^g_{\min}, \\ u_i(h(t), t) \leq \tilde{u}_i(h(t), t) & \text{for } t \in \Sigma^h_{\min}, \end{cases}$$

(v) *and at the initial time,* $u_i(x, 0) \leq \tilde{u}_i(x, 0)$ *for* $x \in [g(0), h(0)]$ *and* $i = 1, 2$*.*

Then for $i = 1, 2$*, we must have*

$$u_i(x, t) \leq \tilde{u}_i(x, t) \quad \text{for } (x, t) \in D_T.$$

Proof. For $i = 1, 2$, define $\phi_i := \tilde{u}_i - u_i$ and

$$c_{11} := -a, c_{12} := c, c_{21} := \frac{G(\tilde{u}_1) - G(u_1)}{\tilde{u}_1 - u_1}, c_{22} := -b.$$

Then by the maximum principle in Lemma 2.1, we obtain that $\phi_i \geq 0$ in D_T for $i = 1, 2$. □

Lemma 2.4 (A Priori Bound)**.** *For* $T \in (0, +\infty)$*, let* (u, v, g, h) *be a solution of* (1.8) *for* $t \in (0, T]$*. Then there exists constants* C_1 *and* C_2 *independent of* T *such that*

$$u(x, t) \leq C_1 \text{ and } v(x, t) \leq C_2 \quad \text{for } g(t) < x < h(t), t \in (0, T].$$

Proof. By assumption **(G2)**, there exist C_1 and C_2 such that

$$\frac{G(C_1)}{C_1} < \frac{ab}{c} \quad \text{and} \quad C_1 \geq u_0(x) \text{ in } [-h_0, h_0] \tag{2.6}$$

and

$$\frac{G(C_1)}{b} < C_2 < \frac{a}{c}C_1 \quad \text{and} \quad C_2 \geq v_0(x) \text{ in } [-h_0, h_0]. \tag{2.7}$$

Let $(U_1(x, t), U_2(x, t)) \equiv (C_1, C_2)$. For $i = 1, 2$, let us denote

$$\mathcal{L}_i[w](x, t) := \int_{g(t)}^{h(t)} J_i(x-y)w(y, t)dy - w(x, t).$$

Clearly $\mathcal{L}_i[U_i](x, t) \leq 0$ and $(U_i)_t = 0$. It now follows from (2.6) and (2.7) that

$$\begin{cases} (U_1)_t > d_1\mathcal{L}_1[U_1] - aU_1 + cU_2, & t > 0, x \in (g(t), h(t)), \\ (U_2)_t > d_2\mathcal{L}_2[U_2] - bU_2 + G(U_1), & t > 0, x \in (g(t), h(t)), \\ U_1(x, t) > u(x, t), \quad U_2(x, t) > v(x, t), & t > 0, x \in \{g(t), h(t)\}, \\ U_1(x, 0) \geq u_0(x), \ U_2(x, 0) \geq v_0(x), & |x| \leq h_0. \end{cases}$$

By Lemma 2.3, we obtain $u(x, t) \leq C_1$ and $v(x, t) \leq C_2$ for all $0 < t \leq T$ and $g(t) \leq x \leq h(t)$. □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let us first consider the following slightly modified problem, with C_2 taken from Lemma 2.4:

$$\begin{cases} u_t = d_1 \displaystyle\int_{g(t)}^{h(t)} J_1(x-y)u(y,t)dy - d_1 u - au + c\min\{v, C_2\}, & t>0, x\in(g(t),h(t)),\\ v_t = d_2 \displaystyle\int_{g(t)}^{h(t)} J_2(x-y)v(y,t)dy - d_2 v - bv + G(u), & t>0, x\in(g(t),h(t)),\\ u(x,t)=v(x,t)=0, & t>0, x\in\{g(t),h(t)\},\\ h'(t) = \displaystyle\int_{g(t)}^{h(t)}\int_{h(t)}^{\infty} [\mu_1 J_1(x-y)u(x,t)+\mu_2 J_2(x-y)v(x,t)]dydx, & t>0,\\ g'(t) = -\displaystyle\int_{g(t)}^{h(t)}\int_{-\infty}^{g(t)} [\mu_1 J_1(x-y)u(x,t)+\mu_2 J_2(x-y)v(x,t)]dydx, & t>0,\\ u(x,0)=u_0(x), v(x,0)=v_0(x), \quad h(0)=-g(0)=h_0, & |x|\le h_0. \end{cases} \tag{2.8}$$

By taking $f_1(u,v) = -au + c\min\{v, C_2\}$ and $f_2(u,v) = -bv + G(u)$, in view of the conditions on G, we see that they satisfy the conditions of Theorem 4.1 in [16]. Therefore (2.8) has a unique solution (u, v, g, h) defined for all $t > 0$.

Since $f_1(u,v) \le -au + cv$, from the proof of Lemma 2.4 we see that $u(x,t) \le C_1$ and $v(x,t) \le C_2$, and therefore $f_1(u(x,t), v(x,t)) = -au(x,t) + cv(x,t)$. Thus (u, v, g, h) solves (1.8).

Conversely, by Lemma 2.4 any solution of (1.8) satisfies $v(x,t) \le C_2$ and hence it solves (2.8). Thus global existence and uniqueness holds for (1.8). □

3. Some associated fixed boundary problems

3.1. Eigenvalue problem

For any $L > 0$, we consider the eigenvalue problem

$$\begin{cases} \lambda\phi = -d_1 \displaystyle\int_{-L}^{L} J_1(x-y)\phi(y)dy + d_1\phi + a\phi - c\psi, & x\in(-L,L),\\ \lambda\psi = -d_2 \displaystyle\int_{-L}^{L} J_2(x-y)\psi(y)dy + d_2\psi + b\psi - G'(0)\phi, & x\in(-L,L). \end{cases} \tag{3.1}$$

By Theorems 2.2 and 2.3 of [17], we have the following result:

Proposition 3.1. *The eigenvalue problem* (3.1) *has a principal eigenvalue* $\lambda = \lambda_1(L)$ *with positive eigenfunction pair* $(\phi, \psi) = (\phi_1, \psi_1) \in C([-L, L]) \times C([-L, L])$.

Then by Lemma 2.2 and Proposition 2.3 of [16], we have the following two results on the properties of the eigenvalue $\lambda_1(L)$.

Lemma 3.2. *Let $\lambda_1(L)$ be the principal eigenvalue of* (3.1). *Let $\Phi, \Psi \in C([-L, L])$ be two functions such that $\Phi, \Psi \geq 0$ and $\Phi, \Psi \not\equiv 0$ in $[-L, L]$, and $\tilde{\lambda}$ be a constant such that*

$$\begin{cases} -d_1 \displaystyle\int_{-L}^{L} J_1(x-y)\Phi(y)dy + d_1\Phi + a\Phi - c\Psi \geq (\leq)\tilde{\lambda}\Phi, & x \in (-L, L), \\ -d_2 \displaystyle\int_{-L}^{L} J_2(x-y)\Psi(y)dy + d_2\Psi + b\Psi - G'(0)\Phi \geq (\leq)\tilde{\lambda}\Psi, & x \in (-L, L), \end{cases} \tag{3.2}$$

then $\lambda_1(L) \geq (\leq)\tilde{\lambda}$. Moreover, $\lambda_1(L) = \tilde{\lambda}$ only if equalities hold in (3.2).

Lemma 3.3. *Let $\lambda_1(L)$ be the principal eigenvalue of* (3.1). *Then*

(i) *$\lambda_1(L)$ is strictly decreasing with respect to $L \in (0, \infty)$,*
(ii) *$\lambda_1(L)$ is continuous for $L \in (0, \infty)$.*

The following proposition is essential for establishing the spreading and vanishing criteria in Theorem 1.3.

Proposition 3.4. *The principal eigenvalue $\lambda_1(L)$ of* (3.1) *has the following properties:*

(i) *If $R_0 \leq 1$, then $\lambda_1(L) > 0$ for any $L > 0$.*
(ii) *If $R_0 > 1$ and*

$$\frac{cG'(0)}{(d_1 + a)(d_2 + b)} \geq 1, \tag{3.3}$$

then $\lambda_1(L) < 0$ for any $L > 0$.

(iii) *If $R_0 > 1$ and*

$$\frac{cG'(0)}{(d_1 + a)(d_2 + b)} < 1, \tag{3.4}$$

then there exists L^* such that

$$\lambda_1(L^*) = 0 \text{ and } (L - L^*)\lambda_1(L) < 0 \quad \text{for } L \in (0, L^*) \cup (L^*, \infty).$$

Proof. The proof follows that of [16, Proposition 2.4], and the details are omitted. □

Corollary 3.5. *Let $l_1 < l_2$ and $\lambda_1(l_1, l_2)$ be the principal eigenvalue of* (3.1) *with $[-L, L]$ replaced by $[l_1, l_2]$. Then*

(i) *$\lambda_1(l_1, l_2)$ is strictly decreasing with respect to $l_2 - l_1$ and is continuous in l_1 and l_2.*
(ii) *If $R_0 \leq 1$, then $\lambda_1(l_1, l_2) > 0$ for any l_1 and l_2.*
(iii) *If $R_0 > 1$ and* (3.3) *holds, then $\lambda_1(l_1, l_2) < 0$ for any l_1 and l_2.*
(iv) *If $R_0 > 1$ and* (3.4) *holds, then $\lambda_1(l_1, l_2) = 0$ for $l_2 - l_1 = 2L^*$ and*

$$\lambda_1(l_1, l_2) > 0 \text{ for } l_2 - l_1 < 2L^*, \quad \lambda_1(l_1, l_2) < 0 \text{ for } l_2 - l_1 > 2L^*, \tag{3.5}$$

where $L^* > 0$ is given by Proposition 3.4(iii).

3.2. Fixed boundary problem

For $L > 0$, we define $Q_L = (-L, L) \times (0, \infty)$ and consider the corresponding fixed boundary problem of (1.8):

$$\begin{cases} u_t = d_1 \displaystyle\int_{-L}^{L} J_1(x-y)u(y,t)dy - d_1 u - au + cv, & (x,t) \in Q_L, \\ v_t = d_2 \displaystyle\int_{-L}^{L} J_2(x-y)v(y,t)dy - d_2 v - bv + G(u), & (x,t) \in Q_L, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), & x \in [-L, L], \end{cases} \tag{3.6}$$

where $u_0, v_0 \in C([-L, L])$ are nonnegative and not identically 0 simultaneously. It is well-known that fixed boundary problems such as (3.6) has a unique positive solution which is defined for all $t > 0$ (see, for example, Remark 4.3 in [16]).

The corresponding steady state problem of (3.6) is

$$\begin{cases} -d_1 \displaystyle\int_{-L}^{L} J_1(x-y)\tilde{u}(y)dy + d_1\tilde{u} = -a\tilde{u} + c\tilde{v}, & x \in (-L, L), \\ -d_2 \displaystyle\int_{-L}^{L} J_2(x-y)\tilde{v}(y)dy + d_2\tilde{v} = -b\tilde{v} + G(\tilde{u}), & x \in (-L, L). \end{cases} \tag{3.7}$$

Definition 3.6. A function pair $(\phi, \psi) \in C([-L, L]) \times C([-L, L])$ is said to be an upper solution of (3.7) if

$$\begin{cases} -d_1 \displaystyle\int_{-L}^{L} J_1(x-y)\phi(y)dy + d_1\phi \geq -a\phi + c\psi, & x \in (-L, L), \\ -d_2 \displaystyle\int_{-L}^{L} J_2(x-y)\psi(y)dy + d_2\psi \geq -b\psi + G(\phi), & x \in (-L, L). \end{cases}$$

It is called a lower solution of (3.7) if these inequalities are reversed.

Proposition 3.7. *Suppose that $R_0 > 1$ and (u, v) is the unique positive solution of* (3.6). *Let $\lambda_1(L)$ be the principal eigenvalue of* (3.1) *and (u^*, v^*) be as defined in* (1.2). *Then the following conclusions hold:*

(i) *The fixed boundary problem* (3.6) *has a unique positive steady state solution $(\tilde{u}, \tilde{v}) \in C([-L, L]) \times C([-L, L])$ if $\lambda_1(L) < 0$, and $(0, 0)$ is the only nonnegative steady state when $\lambda_1(L) \geq 0$. Moreover, $0 < \tilde{u}(x) \leq u^*$ and $0 < \tilde{v}(x) \leq v^*$ in $[-L, L]$ when $\lambda_1(L) < 0$.*
(ii) *If $\lambda_1(L) \geq 0$, then $(u(x,t), v(x,t))$ converges to $(0, 0)$ as $t \to \infty$ uniformly for $x \in [-L, L]$.*
(iii) *If $\lambda_1(L) < 0$, then $(u(x,t), v(x,t))$ converges to $(\tilde{u}, \tilde{v})$ as $t \to \infty$ uniformly for $x \in [-L, L]$.*

Proof. Due to the different nature of the reaction terms in (1.8) from the model in [16], our proof here uses rather different techniques. In particular, we will make use of the monotonicity (in time t) of the to-be-constructed lower and upper solutions and Dini's theorem.

(i) Suppose that $\lambda_1(L) < 0$. Then we easily see that (Mu^*, Mv^*) and $(\epsilon\phi_1, \epsilon\psi_1)$ are respectively upper and lower solutions of (3.7) for small enough $\epsilon > 0$ and any $M \geq 1$, where (ϕ_1, ψ_1) is a positive eigenfunction pair corresponding to $\lambda_1(L)$.

Let $(\underline{u}, \underline{v})$ be the unique positive solution of (3.6) with initial function pair $(\epsilon\phi_1, \epsilon\psi_1)$. Using $(\epsilon\phi_1, \epsilon\psi_1)$ as a lower solution of (3.6), we can use the comparison principle in Lemma 2.3 with

$(g(t), h(t)) \equiv (-L, L)$ to conclude that $(\epsilon\phi_1(x), \epsilon\psi_1(x)) \le (\underline{u}(x,t), \underline{v}(x,t)) \le (u^*, v^*)$ for $(x,t) \in Q_L$. In particular, for any fixed $s > 0$,

$$(\epsilon\phi_1(x), \epsilon\psi_1(x)) = (\underline{u}(x,0), \underline{v}(x,0)) \le (\underline{u}(x,s), \underline{v}(x,s)) \text{ for } x \in [-L, L].$$

It is easily seen that $(\hat{u}(x,t), \hat{v}(x,t) := (\underline{u}(x, s+t), \underline{v}(x, s+t))$ is a solution of (3.6) with initial data $(\underline{u}(x,s), \underline{v}(x,s))$. Therefore we can use the comparison principle to deduce

$$(\underline{u}(x,t), \underline{v}(x,t)) \le (\hat{u}(x,t), \hat{v}(x,t)) = (\underline{u}(x,s+t), \underline{v}(x,s+t)) \text{ for } (x,t) \in Q_L.$$

Since $s > 0$ is arbitrary, this implies that $(\underline{u}(x,t), \underline{v}(x,t))$ is nondecreasing in t and hence

$$(U(x), V(x)) := \lim_{t\to\infty} (\underline{u}(x,t), \underline{v}(x,t)) \text{ exists,}$$

and $(\epsilon\phi_1(x), \epsilon\psi_1(x)) \le (U(x), V(x)) \le (u^*, v^*)$ in $[-L, L]$. Moreover, it is easily seen that (U, V) solves (3.7). Thus there exists at least one positive steady state solution.

We show next that (U, V) as well as any other positive solution of (3.7) are continuous in $[-L, L]$. Indeed, from the continuity of J_1 and J_2, we easily see that

$$G_1(x) := d_1 \int_{-L}^{L} J_1(x-y)U(y)dy, \;\; G_2(x) := d_2 \int_{-L}^{L} J_2(x-y)V(y)dy$$

are continuous in $[-L, L]$. From (3.7), we obtain

$$\begin{cases} V(x) = \frac{a+d_1}{c} U(x) - \frac{G_1(x)}{c}, \\ \frac{(a+d_1)(b+d_2)}{c} U(x) - G(U(x)) = G_2(x) + \frac{b+d_2}{c} G_1(x). \end{cases}$$

From the conditions on G, we see that $F(z) := \frac{(a+d_1)(b+d_2)}{c} z - G(z)$ satisfies

$$F'(z) = \frac{(a+d_1)(b+d_2)}{c} - G'(z) \ge \frac{(a+d_1)(b+d_2)}{c} - \frac{ab}{c} > 0 \text{ for } z > 0.$$

Thus from $F(U(x)) = G_2(x) + \frac{b+d_2}{c} G_1(x)$, $F'(z) > 0$ and the fact that $G_1(x)$ and $G_2(x)$ are continuous, we obtain $U(x)$ is continuous, which in turn implies that $V(x) = \frac{a+d_1}{c} U(x) - \frac{G_1(x)}{c}$ is continuous.

To prove uniqueness, let $(\hat{U}, \hat{V})$ be another positive solution of (3.7). By choosing $\epsilon > 0$ sufficiently small, we may assume that $(\epsilon\phi_1(x), \epsilon\psi_1(x)) \le (\hat{U}(x), \hat{V}(x))$ in $[-L, L]$. Thus the above obtained (U, V) satisfies $(U, V) \le (\hat{U}, \hat{V})$.

We define

$$k^* := \inf\{k > 0 : k(U, V) \ge (\hat{U}, \hat{V}) \text{ in } [-L, L]\}.$$

From $(U, V) \le (\hat{U}, \hat{V})$ and the definition of k^*, we have immediately $k^*(U, V) \ge (\hat{U}, \hat{V})$ and $k^* \ge 1$. If $k^* = 1$, then we immediately obtain $(U, V) = (\hat{U}, \hat{V})$ and the uniqueness is proved. If $k^* > 1$, we show that a contradiction arises. So suppose $k^* > 1$. Since $G(z)/z$ is decreasing by **(G2)**, it is easily checked that (k^*U, k^*V) is an upper solution of (3.7). We now consider $(\Phi(x), \Psi(x)) := (k^*U(x) - \hat{U}(x), k^*V(x) - \hat{V}(x))$. It can be shown that it satisfies a pair of inequalities of the form in Lemma 2.1, and $\Phi(x) \not\equiv 0$, $\Psi(x) \not\equiv 0$ (due to $k^* > 1$ and $(U, V) \le (\hat{U}, \hat{V})$). Thus, by Lemma 2.1 we deduce $\Phi(x) > 0$ and $\Psi(x) > 0$ in $[-L, L]$. Since they are continuous functions on $[-L, L]$, this implies that $k(U, V) \ge (\hat{U}, \hat{V})$ in $[-L, L]$

for some $k < k^*$, which is a contradiction to the definition of k^*. Thus we must have that $k^* = 1$, and uniqueness is proved.

Now let us assume that $(\tilde{u}, \tilde{v})$ is a positive steady state solution of (3.6) and $\lambda_1(L) \geq 0$. By **(G2)**, we see that $(\tilde{u}, \tilde{v})$ satisfies

$$\begin{cases} -d_1 \displaystyle\int_{-L}^{L} J_1(x-y)\tilde{u}(y,t)dy + d_1\tilde{u} + a\tilde{u} - c\tilde{v} = 0, & x \in (-L, L), \\ -d_2 \displaystyle\int_{-L}^{L} J_2(x-y)\tilde{v}(y,t)dy + d_2\tilde{v} + b\tilde{v} - G'(0)\tilde{u} < 0 & x \in (-L, L). \end{cases} \tag{3.8}$$

By applying Lemma 3.2 with $\tilde{\lambda} = 0$, we obtain that $\lambda_1(L) < 0$. This contradicts our assumption, proving the nonexistence.

Next we prove (ii) and (iii) simultaneously. By choosing $\epsilon > 0$ sufficiently small and $M > 1$ sufficiently large, we can ensure that $(\epsilon\phi_1(x), \epsilon\psi_1(x)) \leq (u(x,1), v(x,1)) \leq (Mu^*, Mv^*)$ in $[-L, L]$. Let $(\overline{u}(x,t), \overline{v}(x,t))$ be the unique solution of (3.6) with initial data (Mu^*, Mv^*). Then an analogous reasoning to that for $(\underline{u}(x,t), \underline{v}(x,t))$ shows that $(\overline{u}(x,t), \overline{v}(x,t))$ is non-increasing in t and hence

$$(\tilde{U}(x), \tilde{V}(x)) := \lim_{t\to\infty}(\overline{u}(x,t), \overline{v}(x,t)) \text{ exists and is a nonnegative solution of (3.7).}$$

If $\lambda_1(L) \geq 0$, then by (i) we know that necessarily $(\tilde{U}, \tilde{V}) = (0, 0)$. By Dini's theorem, the monotonicity in t implies that the convergence in the above limit is uniform in $x \in [-L, L]$. The comparison principle implies that $(0, 0) \leq (u(x, t+1), v(x, t+1)) \leq (\overline{u}(x,t), \overline{v}(x,t))$. Letting $t \to \infty$, we thus obtain $\lim_{t\to\infty}(u(x,t), v(x,t)) = (0,0)$ uniformly in $[-L, L]$ when $\lambda_1(L) \geq 0$. This proves (ii).

Note that the comparison principle implies

$$(\underline{u}(x,t), \underline{v}(x,t)) \leq (u(x,t+1), v(x,t+1)) \leq (\overline{u}(x,t), \overline{v}(x,t)).$$

When $\lambda_1(L) < 0$, letting $t \to \infty$, we deduce $(U, V) \leq (\tilde{U}, \tilde{V})$, and the uniqueness result obtained in (i) implies $(U, V) = (\tilde{U}, \tilde{V})$. This in turn implies, by the above inequalities, $\lim_{t\to\infty}(u(x,t), v(x,t)) = (U(x), V(x))$. Moreover, this convergence is uniform in $[-L, L]$ (by Dini's theorem again) since the convergences of $(\underline{u}(x,t), \underline{v}(x,t))$ and $(\overline{u}(x,t), \overline{v}(x,t))$ to $(U(x), V(x))$ are uniform due to their monotonicity in t and the continuity of $U(x)$ and $V(x)$. □

Remark 3.8. By Proposition 3.4, if $R_0 > 1$, then $\lambda_1(L) < 0$ for large enough L. Thus the steady state problem (3.7) has a unique positive solution for all large L, which we will denote by $(\tilde{u}_L, \tilde{v}_L)$ to stress its dependence on L.

Following the proof of [16, Proposition 3.5], we have the following result.

Proposition 3.9. *Assume* **(J)** *holds and that* $R_0 > 1$. *Then*

$$\lim_{L\to+\infty} (\tilde{u}_L(x), \tilde{v}_L(x)) = (u^*, v^*) \quad \textit{locally uniformly in } \mathbb{R},$$

where (u^*, v^*) *is defined in* (1.2).

4. Spreading-vanishing dichotomy and critera

In this section, we prove Theorems 1.2 and 1.3. It is clear that $h(t)$ and $g(t)$ are respectively monotonically increasing and decreasing. Therefore, their limits

$$\lim_{t\to\infty} h(t) = h_\infty \in (h_0, +\infty] \quad \text{and} \quad \lim_{t\to\infty} g(t) = g_\infty \in [-\infty, -h_0)$$

are well-defined.

4.1. Vanishing

Here we look at cases where vanishing happens: either when the reproduction number $R_0 \le 1$ or for sufficiently small initial data (u_0, v_0).

Lemma 4.1. *If $h_\infty - g_\infty < \infty$, then*

$$\lim_{t\to\infty} \|u\|_{C[g(t),h(t)]} = \lim_{t\to\infty} \|v\|_{C[g(t),h(t)]} = 0 \tag{4.1}$$

Proof. We claim that $\lambda_1(g_\infty, h_\infty) \ge 0$ where $\lambda_1(g_\infty, h_\infty)$ is the principal eigenvalue of (3.1) with $[-L, L]$ replaced by $[g_\infty, h_\infty]$.

Suppose by contradiction that $\lambda_1(g_\infty, h_\infty) < 0$. Then by Lemma 3.3, there exists a constant $T > 0$ large enough such that $\lambda_1(g(T), h(T)) < 0$. We may assume that $g(T)$ and $h(T)$ satisfy $|g(T) - g_\infty| < \epsilon$ and $|h(T) - h_\infty| < \epsilon$ with $\epsilon \in (0, h_0)$ small enough such that $J_1(x), J_2(x) > 0$ for $x \in [-4\epsilon, 4\epsilon]$. Clearly $[h(t) - 2\epsilon, h(t) - \epsilon] \subset [g(T), h(T)]$ for $t \ge T$.

Let $(u_1(x,t), v_1(x,t))$ be the solution of (3.6) with Q_L replaced by $(g(T), h(T)) \times (0, \infty)$, and initial functions $(u_0, v_0) = (u(x,T), v(x,T))$. By the comparison principle, we obtain that

$$u_1(x,t) \le u(x, t+T), \quad v_1(x,t) \le v(x, t+T) \quad \text{for } (x,t) \in (g(T), h(T)) \times (0, \infty).$$

By Proposition 3.7(iii), we obtain uniform convergence of (u_1, v_1) for $x \in [g(T), h(T)]$, giving

$$0 < \tilde{u}_1(x) := \lim_{t\to\infty} u_1(x,t) \le \liminf_{t\to\infty} u(t,x) \text{ and } 0 < \tilde{v}_1(x) := \lim_{t\to\infty} v_1(x,t) \le \liminf_{t\to\infty} v(t,x).$$

Therefore, there exists $T_1 \ge T$ such that

$$0 < \frac{1}{2}\tilde{u}_1(x) < u(x,t) \quad \text{and} \quad 0 < \frac{1}{4}\tilde{v}_1(x) < v(x,t) \quad \text{for } t \ge T_1, x \in [g(T), h(T)].$$

Let c_1, c_2, c_3 and c_4 be constants as defined below

$$\begin{aligned} c_1 &:= \min_{x\in[-4\epsilon,4\epsilon]} J_1(x) > 0, \quad & c_2 &:= \min_{x\in[-4\epsilon,4\epsilon]} J_2(x) > 0, \\ c_3 &:= \min_{x\in[g(T),h(T)]} \tilde{u}_1(x) > 0, \quad & c_4 &:= \min_{x\in[g(T),h(T)]} \tilde{v}_1(x) > 0. \end{aligned} \tag{4.2}$$

Then from the above, we can calculate that

$$\begin{aligned} h'(t) &= \int_{g(t)}^{h(t)} \int_{h(t)}^{\infty} [\mu_1 J_1(x-y)u(x,t) + \mu_2 J_2(x-y)v(x,t)]dydx \\ &\geq \int_{h(t)-2\epsilon}^{h(t)} \int_{h(t)}^{h(t)+2\epsilon} [\mu_1 J_1(x-y)u(x,t) + \mu_2 J_2(x-y)v(x,t)]dydx \\ &\geq \int_{h(t)-2\epsilon}^{h(t)} [2\mu_1 c_1 \epsilon u(x,t) + 2\mu_2 c_2 \epsilon v(x,t)]dx \\ &\geq \int_{h(t)-2\epsilon}^{h(t)-\epsilon} [\mu_1 c_1 \epsilon \tilde{u}(x) + \mu_2 c_2 \epsilon \tilde{v}(x)]dx \\ &\geq \epsilon^2 (\mu_1 c_1 c_3 + \mu_2 c_2 c_4) > 0 \quad \text{for } t \geq T_1, \end{aligned}$$

which implies $h_\infty = \infty$, contradicting to $h_\infty < \infty$. Therefore $\lambda_1(g_\infty, h_\infty) \geq 0$.

Now, let $(u_2(x,t), v_2(x,t))$ be the solution of (3.6) with Q_L replaced by $(0,\infty) \times (g_\infty, h_\infty)$ and with the same initial data (u_0, v_0) as (u, v). By the comparison principle, we have $0 \leq u(x,t) \leq u_2(x,t)$ and $0 \leq v(x,t) \leq v_2(x,t)$ for $t > 0$ and $x \in [g(t), h(t)]$. Since $\lambda_1(g_\infty, h_\infty) \geq 0$, Proposition 3.7(ii) gives that

$$\lim_{t\to\infty}(u_2(x,t), v_2(x,t)) = (0,0) \quad \text{uniformly for } x \in [g_\infty, h_\infty], \tag{4.3}$$

which implies the vanishing result. □

Lemma 4.2. *If* $R_0 \leq 1$, *then*

$$h_\infty - g_\infty \leq 2h_0 + \frac{\mu_1 + \mu_2}{m_0} \int_{-h_0}^{h_0} [u_0(x) + \frac{c}{b} v_0(x)]dx, \tag{4.4}$$

and hence vanishing happens, where $m_0 := \min\{d_1, \frac{d_2 c}{b}\}$.

Proof. Since $\int_{\mathbb{R}} J_i(x)dx = 1$ and $J_i(x)$ is even for $i = 1, 2$, a straightforward calculation shows that

$$\begin{aligned} -[h'(t) - g'(t)] =& \mu_1 \left(\int_{g(t)}^{h(t)} \int_{g(t)}^{h(t)} J_1(x-y)u(x,t)dydx - \int_{g(x)}^{h(t)} u(x,t)dx \right) \\ &+ \mu_2 \left(\int_{g(t)}^{h(t)} \int_{g(t)}^{h(t)} J_2(x-y)v(x,t)dydx - \int_{g(x)}^{h(t)} v(x,t)dx \right). \end{aligned}$$

Moreover, we can calculate that

$$\begin{aligned} &\int_{g(t)}^{h(t)} \int_{g(t)}^{h(t)} J_1(x-y)u(x,t)dydx - \int_{g(x)}^{h(t)} u(x,t)dx \\ &= -\int_{g(t)}^{h(t)} \int_{h(t)}^{\infty} J_1(x-y)u(x,t)dydx - \int_{g(t)}^{h(t)} \int_{-\infty}^{g(t)} J_1(x-y)u(x,t)dydx \leq 0, \end{aligned}$$

and the same holds when (u, J_1) is replaced by (v, J_2). By the above, we obtain that

$$\frac{d}{dt} \int_{g(t)}^{h(t)} \left[u(x,t) + \frac{c}{b} v(x,t) \right] dx$$

$$
\begin{aligned}
&= \int_{g(t)}^{h(t)} [u_t(x,t) + \frac{c}{b}v_t(x,t)]dx + h'(t)[u + \frac{c}{b}v]\Big|_{(h(t),t)} + g'(t)[u + \frac{c}{b}v]\Big|_{(g(t),t)} \\
&= \int_{g(t)}^{h(t)} \left[d_1 \left(\int_{g(t)}^{h(t)} J_1(x-y)u(y,t)dy - u(x,t) \right) - au(x,t) \right. \\
&\quad \left. + \frac{cd_2}{b} \left(\int_{g(t)}^{h(t)} J_2(x-y)v(y,t)dy - v(x,t) \right) + \frac{c}{b}G(u(x,t)) \right] dx \\
&\le -\frac{\min\{d_1, d_2 c/b\}}{\mu_1 + \mu_2}[h'(t) - g'(t)] + \int_{g(t)}^{h(t)} \left[-au(x,t) + \frac{c}{b}G(u(x,t)) \right] dx.
\end{aligned}
$$

Using **(G2)**, it follows from $R_0 \le 1$ that $-au(x,t) + \frac{c}{b}G(u(x,t)) \le 0$ for $x \in [g(t), h(t)]$ and $t \ge 0$. Hence,

$$
\frac{d}{dt}\int_{g(t)}^{h(t)} \left[u(x,t) + \frac{c}{b}v(x,t) \right] dx \le -\frac{m_0}{\mu_1 + \mu_2}[h'(t) - g'(t)] \quad \text{for } t > 0.
$$

Integrating the above from 0 to t gives us (4.4). Then by Lemma 4.1, we obtain the vanishing result. □

Now for initial data (u_0, v_0) small enough, we show that vanishing also occurs.

Lemma 4.3. *Let $\lambda_1(h_0)$ be the principal eigenvalue of* (3.1) *with $L = h_0$. If $R_0 > 1$, $\lambda_1(h_0) > 0$ and $\|u_0\|_{C([-h_0,h_0])} + \|v_0\|_{C([-h_0,h_0])}$ is sufficiently small, then vanishing happens.*

Proof. Since $\lambda_1(h_0) > 0$, there exists $h_1 > h_0$ but close to h_0 such that $\lambda_1(h_1) > 0$. Let (ϕ, ψ) be a positive eigenfunction pair corresponding to $\lambda_1(h_1)$ and

$$
\delta := \frac{\lambda_1(h_1)}{2}, \quad c := h_1 - h_0, \quad \text{and } M := \delta c \left(\mu_1 \int_{-h_1}^{h_1} \phi(x)dx + \mu_2 \int_{-h_1}^{h_1} \psi(x)dx \right)^{-1}.
$$

Then define, for $t \ge 0$, $x \in [-h_1, h_1]$,

$$
\begin{aligned}
&\overline{h}(t) := h_0 + c[1 - e^{-\delta t}], \qquad \overline{g}(t) := -\overline{h}(t), \\
&\overline{u}(x,t) := Me^{-\delta t}\phi(x), \qquad \overline{v}(x,t) := Me^{-\delta t}\psi(x).
\end{aligned}
$$

We see that $\overline{h}(t) \in [h_0, h_1)$ for $t \ge 0$ and if we let $\sigma := \min\{\min_{x\in[-h_0,h_0]} \phi(x), \min_{x\in[-h_0,h_0]} \psi(x)\}$ and

$$
\|u_0\|_{C[-h_0,h_0]} + \|v_0\|_{C[-h_0,h_0]} \le \sigma M,
$$

then we have

$$
u_0(x) \le M\phi(x) = \overline{u}(x,0), \quad v_0(x) \le M\psi(x) = \overline{v}(x,0) \quad \text{for } x \in [-h_0, h_0]. \tag{4.5}
$$

Clearly, we can calculate

$$
\begin{aligned}
&\overline{u}_t - d_1 \int_{\overline{g}(t)}^{\overline{h}(t)} J_1(x-y)\overline{u}(y,t)dy + d_1\overline{u} + a\overline{u} - c\overline{v} \\
&\ge -\delta\overline{u} - d_1 \int_{-h_1}^{h_1} J_1(x-y)\overline{u}(y,t)dy + d_1\overline{u} + a\overline{u} - c\overline{v}
\end{aligned}
$$

$$= Me^{-\delta t}[\lambda_1(h_1) - \delta]\phi \geq 0, \qquad \text{for } t > 0,\ x \in [\overline{g}(t), \overline{h}(t)].$$

By **(G2)**, we obtain $G(\overline{u}) \leq G'(0)\overline{u}$ and hence

$$\begin{aligned}
&\overline{v}_t - d_2 \int_{\overline{g}(t)}^{\overline{h}(t)} J_2(x-y)\overline{v}(y,t)dy + d_2\overline{v} + b\overline{v} - G(\overline{u}) \\
&\geq -\delta\overline{v} - d_2 \int_{-h_1}^{h_1} J_2(x-y)\overline{v}(y,t)dy + d_2\overline{v} + b\overline{b} - G'(0)\overline{u} \\
&= Me^{-\delta t}[\lambda_1(h_1) - \delta]\psi \geq 0 \qquad \text{for } t > 0,\ x \in [\overline{g}(t), \overline{h}(t)].
\end{aligned}$$

Moreover, for $x \in \{\overline{g}(t), \overline{h}(t)\}$, we have that $(\overline{u}(x,t), \overline{v}(x,t)) \geq (0,0)$ and

$$\begin{aligned}
&\mu_1 \int_{\overline{g}(t)}^{\overline{h}(t)} \int_{\overline{h}(t)}^{\infty} J_1(x-y)\overline{u}(x,t)dydx + \mu_2 \int_{\overline{g}(t)}^{\overline{h}(t)} \int_{\overline{h}(t)}^{\infty} J_2(x-y)\overline{v}(x,t)dydx \\
&\leq \mu_1 \int_{\overline{g}(t)}^{\overline{h}(t)} \overline{u}(x,t)dx + \mu_2 \int_{\overline{g}(t)}^{\overline{h}(t)} \overline{v}(x,t)dydx \\
&\leq Me^{-\delta t}\left[\mu_1 \int_{-h_1}^{h_1} \phi(x)dx + \mu_2 \int_{-h_1}^{h_1} \psi(x)dx\right] \\
&= \delta c e^{-\delta t} = \overline{h}'(t) \qquad \text{for } t > 0.
\end{aligned}$$

In view of $\overline{g}(t) = -\overline{h}(t)$, we can now use the the comparison principle in Lemma 2.2 to conclude that $h(t) \leq \overline{h}(t) \leq h_1$ for all $t > 0$, and hence vanishing happens. □

Remark 4.4. If $(\mu_1, \mu_2) = (\mu\sigma_1^0, \mu\sigma_2^0)$ with σ_1^0 and σ_2^0 fixed, nonnegative and $\sigma_1^0 + \sigma_2^0 > 0$, then by the proof of Lemma 4.3, we see that $M \to \infty$ when $\mu \to 0$. Thus, for any given initial data (u_0, v_0), there exists $\mu_0 > 0$ such that (4.5) holds for all $\mu \in (0, \mu_0)$. Thus if $0 < \mu \leq \mu_0$, then vanishing must happen for (1.8) for this given initial data (u_0, v_0).

We note that the following lemma implies that if $h_\infty - g_\infty = \infty$ holds, then we must have that $h_\infty = -g_\infty = +\infty$.

Lemma 4.5. *The inequality $h_\infty < +\infty$ if and only if $g_\infty > -\infty$.*

Proof. The proof of this lemma is similar to the proof of Lemma 4.10 in [16]. Since the modifications are obvious, we omit the details. □

4.2. Spreading

In this section, we look at cases where spreading happens.

Lemma 4.6. *If $\lambda_1(g(t_0), h(t_0)) < 0$ for some $t_0 \geq 0$, then $h_\infty = -g_\infty = +\infty$ and*

$$\lim_{t\to\infty} u(x,t) = u^* \quad \textit{and} \quad \lim_{t\to\infty} v(x,t) = v^* \textit{ locally uniformly for } x \in \mathbb{R}, \tag{4.6}$$

where $\lambda_1(g(t_0), h(t_0))$ is the eigenvalue of (3.1) *with $[-L, L]$ replaced by $[g(t_0), h(t_0)]$, and (u^*, v^*) are as defined in* (1.2).

Remark 4.7. If for $R_0 > 1$ and for some $t_0 > 0$, we have $h(t_0) - g(t_0) \geq 2L^*$, then by Proposition 3.4, we obtain that $\lambda_1(h(t_0), g(t_0)) \leq 0$. Hence for any $t_1 > t_0$ we have $\lambda_1(h(t_1), g(t_1)) < 0$. Then by Lemma 4.6, we have that spreading occurs. This implies that when $R_0 > 1$ and vanishing happens, we must have $h(t) - g(t) < 2L^*$ for all $t \geq 0$.

Proof of Lemma 4.6. We see that $h_\infty = -g_\infty = \infty$. Suppose by contradiction that $h_\infty - g_\infty < \infty$. Since $[g(t_0), h(t_0)] \subset [g(t), h(t)]$ for some $t \geq t_0$, by Corollary 3.5, we have $\lambda_1(g(t), h(t)) < 0$ for $t \geq t_0$. Then we derive the contradiction as in Lemma 4.1. By Lemma 4.5, we further obtain that $-g_\infty = h_\infty = \infty$. Then by Lemma 4.2, we must have $R_0 > 1$, which guarantees the existence of the positive equilibrium (u^*, v^*). It remains to show (4.6).

Let us first consider the limit superior of the solution. Let $(\overline{u}, \overline{v})$ be the unique positive solution of the following ODE problem:

$$\begin{cases} \overline{u}\,' = -a\overline{u} + c\overline{v}, & t > 0, \\ \overline{v}\,' = -b\overline{v} + G(\overline{u}), & t > 0, \\ \overline{u}(0) = \|u_0\|_{L^\infty([-h_0,h_0])}, \\ \overline{v}(0) = \|v_0\|_{L^\infty([-h_0,h_0])}. \end{cases} \tag{4.7}$$

Since $R_0 > 1$, we have $\lim_{t\to\infty}(\overline{u}(t), \overline{v}(t)) = (u^*, v^*)$. We then note that

$$d_1 \int_{g(t)}^{h(t)} J_1(x-y)\overline{u}(t)dy - d_1\overline{u}(t) \leq 0,$$
$$d_2 \int_{g(t)}^{h(t)} J_2(x-y)\overline{v}(t)dy - d_2\overline{v}(t) \leq 0,$$

and $\overline{u}(0) \geq u_0(x), \overline{v}(0) \geq v_0(x)$; so by the comparison principle in Lemma 2.3, we have

$$(u(x,t), v(x,t)) \leq (\overline{u}(t), \overline{v}(t)) \quad \text{for } g(t) < x < h(t) \text{ and } t > 0.$$

Thus we have that

$$\limsup_{t\to\infty}(u(x,t), v(x,t)) \leq (u^*, v^*) \quad \text{uniformly for } x \in [g(t), h(t)].$$

Then following [16, Lemma 4.11], we can make use of Propositions 3.7 and 3.9 to show

$$\liminf_{t\to\infty}(u(x,t), v(x,t)) \geq (u^*, v^*) \quad \text{locally uniformly for } x \in \mathbb{R}.$$

Thus (4.6) holds. □

Lemma 4.8. *If $R_0 > 1$, $h_0 < L^*$ and $(\mu_1, \mu_2) = (\mu\sigma_1^0, \mu\sigma_2^0)$ with σ_1^0, σ_2^0 nonnegative and $\sigma_1^0 + \sigma_2^0 > 0$, then there exists $\mu^0 > 0$ depending on the initial data (u_0, v_0) such that spreading happens if $\mu > \mu^0$.*

Proof. Similar to the calculations in the proof of Lemma 4.2, by setting $m^0 := \max\{d_1, \frac{d_2 c}{b}\}$, we obtain for $t > 0$,

$$\int_{g(t)}^{h(t)} \left[u(x,t) + \frac{c}{b}v(x,t)\right] dx \geq \int_{-h_0}^{h_0} \left[u_0(x) + \frac{c}{b}v_0(x)\right] dx + \frac{m^0}{\mu(\sigma_1^0 + \sigma_2^0)}\Big(2h_0 - [h(t) - g(t)]\Big).$$

Suppose that $h_\infty - g_\infty < \infty$; then in view of Lemma 4.1 and Remark 4.7, by letting $t \to \infty$ in the above inequality, we get

$$\int_{-h_0}^{h_0} \left[u_0(x) + \frac{c}{b} v_0(x) \right] dx \le \frac{m^0}{\mu(\sigma_1^0 + \sigma_2^0)} (2L^* - 2h_0).$$

However, this is patently false in the case

$$\mu > \mu_0 := \frac{2m^0(L^* - h_0)}{(\sigma_1^0 + \sigma_2^0) \int_{-h_0}^{h_0} \left[u_0(x) + \frac{c}{b} v_0(x) \right] dx}.$$

This completes the proof. □

4.3. Proof of Theorems 1.2 and 1.3

Proof of Theorem 1.2. If $h_\infty - g_\infty < \infty$, then (1.9) holds by Lemma 4.2. On the other hand, if $h_\infty - g_\infty = \infty$, then $R_0 > 1$ by Lemma 4.2. By Corollary 3.5, we find that $\lambda_1(g(t_0), h(t_0)) < 0$ for some large $t_0 > 0$. Hence (4.6) holds by Lemma 4.6. □

Proof of Theorem 1.3. (a) This follows from Lemma 4.2.

(b) By Proposition 3.4, we obtain that $\lambda_1(h_0) \le 0$. Since $h(t)$ is strictly increasing in t and $\lambda_1(L)$ is strictly decreasing in L, we obtain $\lambda_1(h(1)) < \lambda_1(h_0) \le 0$. Thus by Lemma 4.6, we have that spreading occurs.

(c) (i) It follows from Remark 4.7 that if vanishing occurs, then $h_\infty - g_\infty \le 2L^*$. Define

$$\Gamma := \{\mu > 0 \,:\, h_\infty - g_\infty \le 2L^*\}.$$

Then by Remark 4.4 and Lemma 4.8, we respectively have that $(0, \mu_0] \subset \Gamma$ and $\Gamma \cap (\mu^0, \infty) = \emptyset$. Denoting by $\mu^* := \sup \Gamma \in [\mu_0, \mu^0]$, we have by definition that $h_\infty - g_\infty > 2L^*$ for $\mu > \mu^*$ and hence spreading happens for $\mu > \mu^*$ by Theorem 1.2.

Suppose that $\mu^* \notin \Gamma$. Then we have $h_\infty - g_\infty = \infty$ when $\mu = \mu^*$ and there exists a $T > 0$ such that $h(T) - g(T) > 2L^*$. Let us emphasis the dependence of the solution (u, v, g, h) of (1.8) on μ by rewriting it as $(u_\mu, v_\mu, g_\mu, h_\mu)$. Then we have $h_{\mu^*}(T) - g_{\mu^*}(T) > 2L^*$. By the continuity of the solution in μ, hence there exists $\epsilon > 0$ such that for $|\mu - \mu^*| < \epsilon$, we have $h_\mu(T) - g_\mu(T) > 2L^*$. Then for every μ such that $|\mu - \mu^*| < \epsilon$, by the monotonicity of $h(t)$ and $-g(t)$ in t, we have that $\lim_{t\to\infty} h_\mu(t) - g_\mu(t) > h_\mu(T) - g_\mu(T) > 2L^*$. Thus we get the contradiction that $\sup \Gamma \le \mu^* - \epsilon$. Hence we must have $\mu^* \in \Gamma$.

It remains to show that vanishing also occurs for $\mu < \mu^*$. For every $\mu \in (0, \mu^*)$, $(u_{\mu^*}, v_{\mu^*}, g_{\mu^*}, h_{\mu^*})$ is an upper solution to (1.8). Thus by the comparison principle, we see that $h_\mu(t) \le h_{\mu^*}(t)$ and $g_\mu(t) \ge g_{\mu^*}(t)$ for $t > 0$. It follows that $\lim_{t\to\infty}(h_\mu(t) - g_\mu(t)) \le \lim_{t\to\infty}(h_{\mu^*}(t) - g_{\mu^*}(t)) \le 2L^*$. This proves our assertion.

(ii) From the assumptions, we obtain that $\lambda_1(h_0) > 0$. Thus the assertion follows directly from Lemma 4.3.

The proof is now complete. □

5. Spreading speed

In this section, we consider the asymptotic spreading speed when spreading happens in our system (1.8). As such, we necessarily have that $R_0 > 1$.

Let $F(u,v) = (f_1(u,v), f_2(u,v))$ with $f_1(u,v) := -au + cv$ and $f_2(u,v) := -bv + G(u)$. We now check that F satisfies the assumptions $(\mathbf{f_1}) - (\mathbf{f_4})$ in [2] with $\hat{\mathbf{u}} = \infty$ and $m = n = 2$, namely

$(\mathbf{f_1})$: $F(u,v) = (0,0)$ has only two nonnegative solutions $(0,0)$ and (u^*, v^*), and the Jacobian matrix of F evaluated at (0,0), denoted by $\nabla F(0,0)$, is irreducible with principal eigenvalue positive.

$(\mathbf{f_2})$: $F(ku, kv) \geq kF(u,v)$ for $k \in [0,1]$ and all $u, v \geq 0$.

$(\mathbf{f_3})$: $\nabla F(u^*, v^*)$ is invertible, $(u^*, v^*)\nabla F(u^*, v^*) \leq (0,0)$ component wise, and for $i \in \{1,2\}$,

$$\text{either } \partial_u f_i(u^*, v^*)u^* + \partial_v f_i(u^*, v^*)v^* < 0$$
$$\text{or } \begin{cases} \partial_u f_i(u^*, v^*)u^* + \partial_v f_i(u^*, v^*)v^* = 0 \text{ and} \\ f_i(u,v) \text{ is linear for } u < u^* \text{ close to } u^* \text{ and } v < v^* \text{ close to } v^*. \end{cases}$$

$(\mathbf{f_4})$: The solution of the corresponding problem (1.12) with initial function pair (u_0, v_0) nonnegative, bounded and not identically $(0,0)$ is positive and globally defined, and as time $t \to \infty$, it converges to (u^*, v^*) locally uniformly for $x \in \mathbb{R}$.

It is straightforward to check that $(\mathbf{f_1})$, $(\mathbf{f_2})$ and $(\mathbf{f_3})$ are satisfied. It now remains for us to prove $(\mathbf{f_4})$, namely the following lemma.

Lemma 5.1. *Let* $(U(x,t), V(x,t))$ *satisfy*

$$\begin{cases} U_t = d_1 \displaystyle\int_{\mathbb{R}} J_1(x-y)U(y,t)dy - dU - aU + cV & \textit{for all } t > 0,\ x \in \mathbb{R}, \\ V_t = d_2 \displaystyle\int_{\mathbb{R}} J_2(x-y)V(y,t)dy - dV - bV + G(U) & \textit{for all } t > 0,\ x \in \mathbb{R}. \end{cases} \tag{5.1}$$

If $(U(\cdot,0), V(\cdot,0)) \in L^\infty(\mathbb{R})^2 \cap C(\mathbb{R})^2$ *is nonnegative, then* $(U(x,t), V(x,t)) \in [0,\infty) \times [0,\infty)$ *for every* $t > 0$ *and* $x \in \mathbb{R}$. *Moreover, it holds that* $\lim_{t\to\infty}(U(x,t), V(x,t)) = (u^*, v^*)$ *in* $L^\infty_{loc}(\mathbb{R})$ *if additionally* $(U(x,0), V(x,0)) \not\equiv (0,0)$.

Proof. Let $(U(\cdot,0), V(\cdot,0)) \in L^\infty(\mathbb{R})^2 \cap C(\mathbb{R})^2$ be nonnegative. If $(U(x,0), V(x,0)) \equiv (0,0)$, then clearly $(U,V) \equiv (0,0)$ is the unique solution of (5.1). In the following we assume that $(U(x,0), V(x,0)) \not\equiv (0,0)$. For $L > 0$, let $(u_L(x,t), v_L(x,t))$ be the solution to (3.6) with initial data $(u_L(x,0), v_L(x,0)) = (U(x,0), V(x,0))|_{[-L,L]}$. By the comparison principle in Lemma 2.3, we get that

$$(0,0) \leq (u_L(x,t), v_L(x,t)) \leq (U(x,t), V(x,t)) \text{ for } (x,t) \in [-L,L] \times (0,\infty).$$

Let $(\overline{u}(t), \overline{v}(t))$ be the unique positive solution of the following system of ordinary differential equations:

$$\begin{cases} \overline{u}\,' = -a\overline{u} + c\overline{v}, & t > 0, \\ \overline{v}\,' = -b\overline{v} + G(\overline{u}), & t > 0, \\ \overline{u}(0) = \|U(\cdot,0)\|_{L^\infty(\mathbb{R})}, \\ \overline{v}(0) = \|V(\cdot,0)\|_{L^\infty(\mathbb{R})}. \end{cases} \tag{5.2}$$

By the comparison principle in Lemma 2.3, we obtain $(U(x,t), V(x,t)) \le (\overline{u}(t), \overline{v}(t))$ for $(x,t) \in [-L, L] \times (0, \infty)$. Since $L \ge L_0$ is arbitrary, this and the earlier estimates imply that

$$(U(x,t), V(x,t)) \in [0,\infty) \times [0,\infty) \text{ for every } t > 0 \text{ and } x \in \mathbb{R}.$$

Moreover, it follows from $R_0 > 1$ that $\lim_{t\to\infty}(\overline{u}(t), \overline{v}(t)) = (u^*, v^*)$. Therefore we must have

$$\limsup_{t\to\infty}(U(x,t), V(x,t)) \le (u^*, v^*) \text{ uniformly for } x \in \mathbb{R}. \tag{5.3}$$

Since $(U(x,0), V(x,0)) \not\equiv (0,0)$ for $x \in \mathbb{R}$, there exists $L_0 > L^*$ large enough such that

$$(U(x,0), V(x,0))|_{[-L,L]} \not\equiv (0,0) \text{ for } x \in [-L, L] \text{ when } L \ge L_0.$$

By Proposition 3.7(iii), we obtain

$$\lim_{t\to\infty}(u_L(x,t), v_L(x,t)) = (\tilde{u}_L(x), \tilde{v}_L(x)) \text{ uniformly for } x \in [-L, L],\ L \ge L_0 > L^*,$$

where $(\tilde{u}_L, \tilde{v}_L)$ is the unique positive steady-state of (3.6). It follows that

$$\liminf_{t\to\infty}(U(x,t), V(x,t)) \ge (\tilde{u}_L(x), \tilde{v}_L(x)) \text{ uniformly for } x \in [-L, L],\ L \ge L_0 > L^*.$$

Letting $L \to \infty$, by Proposition 3.9, we obtain

$$\liminf_{t\to\infty}(U(x,t), V(x,t)) \ge (u^*, v^*) \quad \text{locally uniformly in } \mathbb{R}.$$

This and (5.3) imply

$$\lim_{t\to\infty}(U(x,t), V(x,t)) = (u^*, v^*) \quad \text{locally uniformly in } \mathbb{R}.$$

The proof is complete. □

Since $F = (f_1, f_2)$ satisfies $(\mathbf{f_1}) - (\mathbf{f_4})$ in [2], Theorems 1.3 and 1.5 (as well as two results on the associated semi-wave problem: Theorems 1.1 and 1.2) in [2] can be applied to obtain Theorems 1.5 and 1.6 here.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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