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Research article

A degenerate bifurcation from simple eigenvalue theorem

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Abstract: A new bifurcation from simple eigenvalue theorem is proved for general nonlinear functional equations. It is shown that in this bifurcation scenario, the bifurcating solutions are on a curve which is tangent to the line of trivial solutions, while in typical bifurcations the curve of bifurcating solutions is transversal to the line of trivial ones. The stability of bifurcating solutions can be determined, and examples from partial differential equations are shown to demonstrate such bifurcations.

Keywords: degenerate bifurcation; simple eigenvalue; tangential bifurcation

1. Introduction

In many mathematical models, it is required to finding the solutions of a stationary problem, which can be formulated as an equation

$$F(\lambda, u) = 0, \tag{1.1}$$

where F is a nonlinear smooth mapping defined on $(\lambda, u) \in \mathbb{R} \times X$ and mapped to Y, λ is a parameter, and X, Y are Banach spaces. Often the system Eq (1.1) has a trivial state $u = u_0$ for all parameter values λ, and it may have other nontrivial solutions near (λ_0, u_0) for some λ_0. Such λ_0 is called a bifurcation point for Eq (1.1), and the bifurcating nontrivial solutions near a bifurcation point are often with significance for the models as they represent states breaking from the symmetric or uniform ones.

If the Fréchet derivative $F_u(\lambda_0, u_0)$ of F at (λ_0, u_0) is invertible, then (λ_0, u_0) is not a bifurcation point from the Implicit Function Theorem [1–3]. Hence a necessary condition for the bifurcation to occur is that $F_u(\lambda_0, u_0)$ is not invertible. The most useful bifurcation occurs when that 0 is a simple eigenvalue of the linearized operator $F_u(\lambda_0, u_0)$, that is

(F1) $\dim N(F_u(\lambda_0, u_0)) = \operatorname{codim} R(F_u(\lambda_0, u_0)) = 1$, and $N(F_u(\lambda_0, u_0)) = \operatorname{span}\{w_0\}$, where $N(F_u)$ and $R(F_u)$ are the null space and the range of linear operator F_u. Crandall and Rabinowitz [2] prove the following celebrated "bifurcation from a simple eigenvalue" theorem (see [2, Theorem 1.7]). Here is an expanded version of the theorem for our purpose:

Theorem 1.1. *Let U be a neighborhood of (λ_0, u_0) in $\mathbb{R}\times X$, and let $F : U \to Y$ be a twice continuously differentiable mapping. Assume that $F(\lambda, u_0) = 0$ for $(\lambda, u_0) \in U$. At (λ_0, u_0), F satisfies* **(F1)** *and*

(F3) $F_{\lambda u}(\lambda_0, u_0)[w_0] \notin R(F_u(\lambda_0, u_0))$.

Let Z be any complement of $\mathrm{span}\{w_0\}$ *in X. Then the solution set of* (1.1) *near (λ_0, u_0) consists precisely of the curves $\Gamma_0 = \{(\lambda, u_0)\}$ and $\Gamma_1 = \{(\lambda(s), u(s)) : s \in |s| < \delta\}$, where $\lambda : I \to \mathbb{R}$, $z : I \to Z$ are C^1 functions such that $u(s) = u_0 + sw_0 + sz(s)$, $\lambda(0) = \lambda_0$, $z(0) = 0$, and*

$$\lambda'(0) = -\frac{\langle l, F_{uu}(\lambda_0, u_0)[w_0]^2\rangle}{2\langle l, F_{\lambda u}(\lambda_0, u_0)[w_0]\rangle}, \tag{1.2}$$

where $l \in Y^*$ satisfying $N(l) = R(F_u(\lambda_0, u_0))$. If F also satisfies

(F4) $F_{uu}(\lambda_0, u_0)[w_0]^2 \notin R(F_u(\lambda_0, u_0))$,

then we have $\lambda'(0) \neq 0$, and it is called a transcritical bifurcation; If F satisfies

(F4′) $F_{uu}(\lambda_0, u_0)[w_0]^2 \in R(F_u(\lambda_0, u_0))$,

and in addition $F \in C^3$, then $\lambda'(0) = 0$ and

$$\lambda''(0) = -\frac{\langle l, F_{uuu}(\lambda_0, u_0)[w_0]^3\rangle + 3\langle l, F_{uu}(\lambda_0, u_0)[w_0, \theta_1]\rangle}{3\langle l, F_{\lambda u}(\lambda_0, u_0)[w_0]\rangle}, \tag{1.3}$$

where θ_1 satisfies

$$F_{uu}(\lambda_0, u_0)[w_0]^2 + F_u(\lambda_0, u_0)[\theta_1] = 0. \tag{1.4}$$

If $\lambda'(0) = 0$ and $\lambda''(0) \neq 0$, then it is called a pitchfork bifurcation.

Note that the classification of transcritical and pitchfork bifurcations using **(F4)** and **(F4′)** was first used in reference [4], although this has been widely used in finite dimensional dynamical systems [5].

The transversality condition **(F3)** holds in most practical situations, but there are also important exceptional cases for which **(F3)** fail. In reference [6], we considered a degenerate bifurcation scenario in which **(F3)** is not satisfied. In this case, we prove that, under some higher order transversality conditions on F, the local solution set of Eq (1.1) near the bifurcation point (λ_0, u_0) consists of the line of trivial solutions, and two other solution curves. First we recall a degenerate version of Theorem 1.1, which can be used to obtain more than two intersecting solution curves near the bifurcation point.

Theorem 1.2. ([6, Theorem 2.3]) *Let U be a neighborhood of (λ_0, u_0) in $\mathbb{R} \times X$, and let $F \in C^3(U, Y)$. Assume that $F(\lambda, u_0) = 0$ for $(\lambda, u_0) \in U$ and at (λ_0, u_0), F satisfies* **(F1)**, **(F4′)**, *and*

(F3′) $F_{\lambda u}(\lambda_0, u_0)[w_0] \in R(F_u(\lambda_0, u_0))$.

Let $X = N(F_u(\lambda_0, u_0)) \oplus Z$ be a fixed splitting of X, and let $l \in Y^*$ such that $R(F_u(\lambda_0, u_0)) = \{v \in Y : \langle l, v\rangle = 0\}$. Denote by $\theta_2 \in Z$ the unique solution of

$$F_{\lambda u}(\lambda_0, u_0)[w_0] + F_u(\lambda_0, u_0)[\theta_2] = 0, \tag{1.5}$$

and recall θ_1 to be the unique solution of Eq (1.4). *We assume that the matrix (all derivatives are evaluated at (λ_0, u_0))*

$$H = H(\lambda_0, u_0) = \begin{pmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{pmatrix} \tag{1.6}$$

is non-degenerate, i.e., $\det(H) \neq 0$*, where H_{ij} is given by*

$$H_{11} = \langle l, F_{\lambda\lambda u}[w_0] + 2F_{\lambda u}[\theta_2]\rangle, \tag{1.7}$$

$$H_{12} = \frac{1}{2}\langle l, F_{\lambda uu}[w_0]^2 + F_{\lambda u}[\theta_1] + 2F_{uu}[w_0, \theta_2]\rangle, \tag{1.8}$$

$$H_{22} = \frac{1}{3}\langle l, F_{uuu}[w_0]^3 + 3F_{uu}[w_0, \theta_1]\rangle. \tag{1.9}$$

1) *If H is definite, i.e.,* $\det(H) > 0$*, then the solution set of Eq* (1.1) *near* $(\lambda, u) = (\lambda_0, u_0)$ *is the line* $\Gamma_0 = \{(\lambda, u_0)\}$.
2) *If H is indefinite, i.e.,* $\det(H) < 0$*, then the solution set of Eq* (1.1) *near* $(\lambda, u) = (\lambda_0, u_0)$ *is the union of C^1 curves intersecting at (λ_0, u_0), including the line of trivial solutions $\Gamma_0 = \{(\lambda, u_0)\}$ and two other curves $\Gamma_i = \{(\lambda_i(s), u_i(s)) : |s| < \delta\}$ $(i = 1, 2)$ for some $\delta > 0$, with*

$$\lambda_i(s) = \lambda_0 + \mu_i s + s\alpha_i(s), \quad u_i(s) = u_0 + \eta_i s w_0 + s\beta_i(s),$$

where (μ_1, η_1) and (μ_2, η_2) are non-zero linear independent solutions of the equation

$$H_{11}\mu^2 + 2H_{12}\mu\eta + H_{22}\eta^2 = 0, \tag{1.10}$$

$\alpha_i(0) = \alpha_i'(0) = 0$*, $\beta_i(s) \in Z$, and $\beta_i(0) = \beta_i'(0) = 0$, $i = 1, 2$.*

In this paper, we prove another bifurcation result when the transversality condition **(F3)** fails. In this case, under the complement **(F3′)** of **(F3)**, and as well as **(F4)**, we show that the solution set of Eq (1.1) near the bifurcation point (λ_0, u_0) consists of the line of trivial solution, and another curve of nontrivial solutions which is tangent to the line of trivial ones.

Theorem 1.3. *Let U be a neighborhood of (λ_0, u_0) in $\mathbb{R} \times X$, and let $F \in C^2(U, Y)$. Assume that $F(\lambda, u_0) = 0$ for any $\lambda \in \mathbb{R}$. At (λ_0, u_0), F satisfies* **(F1)**, **(F3′)** *and* **(F4)**. *Then the solution set of Eq* (1.1) *near* $(\lambda, u) = (\lambda_0, u_0)$ *is the union of two C^2 curves which are tangent to each other at (λ_0, u_0), including the line of trivial solutions $\Gamma_0 = \{(\lambda, u_0)\}$ and $\Gamma_1 = \{(\lambda, u(\lambda)) : \lambda \in I\}$ for some $\delta > 0$, where $I = (\lambda_0 - \delta, \lambda_0 + \delta)$, $u(\lambda) = u_0 + t(\lambda)w_0 + g(\lambda, t(\lambda))$, $t : I \to V$ and $g : I \times V \to Z$ are continuously differentiable functions, $t(\lambda_0) = t'(\lambda_0) = 0$, and $g(\lambda, 0) = g_\lambda(\lambda_0, 0) = g_t(\lambda_0, 0) = 0$, where $V \subset \mathbb{R}$ is a neighborhood of $t = 0$. Moreover if $F \in C^4(U, Y)$, then $t(\lambda)$ is C^3 and*

$$t''(\lambda_0) = -2\frac{\langle l, F_{\lambda\lambda u}(\lambda_0, u_0)[w_0] + 2F_{\lambda u}(\lambda_0, u_0)[\theta_2]\rangle}{\langle l, F_{uu}(\lambda_0, u_0)[w_0]^2\rangle}, \tag{1.11}$$

where θ_2 is defined by Eq (1.5).

If F satisfies $F_{\lambda\lambda u}(\lambda_0, u_0)[w_0] + 3F_{\lambda u}(\lambda_0, u_0)[\theta_2] \notin R(F_u(\lambda_0, u_0))$, then the solution set of Eq (1.1) near (λ_0, u_0) in Theorem 1.3 is the union of a line and a parabola-like curve which is tangent to the line. The simplest example for Theorem 1.3 is the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(\lambda, u) = u(u - \lambda^2)$, and its solution set of $f(\lambda, u) = 0$ near $(\lambda, u) = (0, 0)$ is the union of the line $u = 0$ and the curve of nontrivial solutions $u = \lambda^2$ which are tangent to each other at $(0, 0)$ (see the lower left panel of Figure 1).

Theorem 1.1, Theorem 1.2 and Theorem 1.3 together provide a complete classification of bifurcation scenarios for Eq (1.1) under the conditions **(F3)** or **(F3′)**, and **(F4)** or **(F4′)**, while $F(\lambda, u_0) \equiv 0$ and **(F1)** are assumed:

(a) Transcritical: **(F3)** and **(F4)**, a crossing curve of nontrivial solutions (Theorem 1.1);
(b) Pitchfork: **(F3)** and **(F4′)**, a crossing curve of nontrivial solutions bending leftward or rightward (Theorem 1.1);
(c) Tangential: **(F3′)** and **(F4)**, a tangential curve of nontrivial solutions bending upward or downward (Theorem 1.3); and
(d) Double transcritical: **(F3′)** and **(F4′)**, two crossing curves of nontrivial solutions (Theorem 1.2).

Because of the condition **(F3′)**, the tangential and the double transcritical bifurcations are called degenerate ones. Figure 1 shows examples of each types of bifurcations using simple mappings $F_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and these mappings can be regarded as normal forms of these bifurcations.

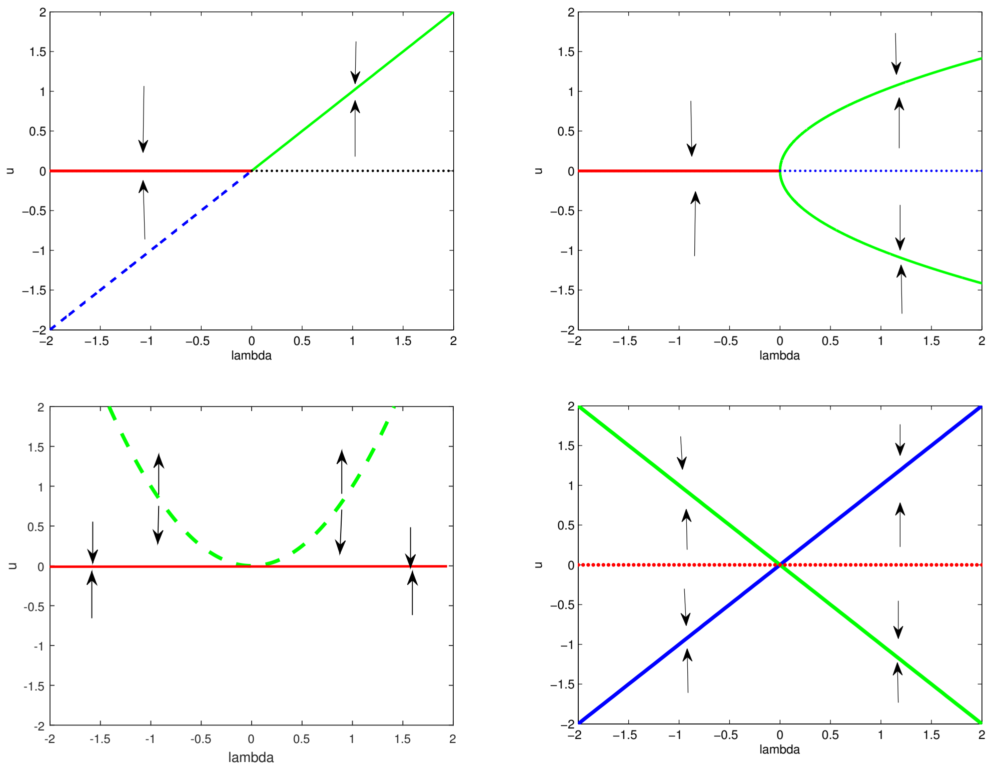

Figure 1. Bifurcation diagrams for Eq (1.1) with a simple eigenvalue. (Upper left): **(F3)** and **(F4)**, $F_1(\lambda, u) = \lambda u - u^2$; (Upper Right): **(F3)** and **(F4′)**, $F_2(\lambda, u) = \lambda u - u^3$; (Lower left): **(F3′)** and **(F4)**, $F_3(\lambda, u) = u^2 - \lambda^2 u$; (Lower right): **(F3′)** and **(F4′)**, $F_4(\lambda, u) = \lambda^2 u - u^3$.

Another bifurcation result with solution set being two tangential curves was proved in reference [7] but it is under the assumption that the kernel $N(F_u(\lambda_0, u_0))$ is of two-dimensional and it is of saddle-node bifurcation type. It is applied to a nonlinear Schrödinger system with quadratic nonlinearity [8], where one of the two tangential curves is indeed vertical in a form $\{(\lambda_0, k\varphi_1) : k \in \mathbb{R}\}$. Here one of the two tangential curves is horizontal in a form $\{(\lambda, u_0) : \lambda \in \mathbb{R}\}$.

We also remark that transversality conditions like **(F3)** or **(F4)** are not needed for global bifurcation theorems concerning the topological structure of the solution continuum. Indeed in the celebrated Rabinowitz global bifurcation theorem [9] (see also extensions in [10, 11]), only the odd algebraic multiplicity was assumed, thus all four scenarios shown in Figure 1 can occur as local pictures for the global bifurcation diagrams in reference [9–11].

We prove Theorem 1.3 in Section 2, and we consider the stability of the bifurcating solutions obtained in Theorem 1.3 in Section 3. Finally in Section 4 we show some examples to apply Theorem 1.3. Throughout the paper, we use the same labeling of conditions such as **(F1)** and (**F2**) on F as in our previous work [4, 6, 12], and we use the convention that **(Fi′)** stands for the negation of **(Fi)** for i$\in \mathbb{N}$. We use $\|\cdot\|$ as the norm of Banach space X, $\langle\cdot,\cdot\rangle$ as the duality pair of a Banach space X and its dual space X^*. For a linear operator L, we use $N(L)$ as the null space of L and $R(L)$ as the range space of L, and we use $L[w]$ to denote the image of w under the linear mapping L. For a multilinear operator L, we use $L[w_1, w_2, \cdots, w_k]$ to denote the image of $(w_1, w_2, \cdots, w_k)$ under L, and when $w_1 = w_2 = \cdots = w_k$, we use $L[w_1]^k$ instead of $L[w_1, w_1, \cdots, w_1]$. For a nonlinear operator F, we use F_u as the partial derivative of F with respect to argument u.

2. Proof of Theorem 1.3

First we recall an important lemma from our previous work [12]. First is the well-known Lyapunov-Schmidt reduction under the condition **(F1)** which is standard from many textbooks in nonlinear analysis (see for example [1, 3, 13]).

Lemma 2.1. *Suppose that $F : \mathbb{R} \times X \to Y$ is a C^p ($p \geq 1$) mapping such that $F(\lambda_0, u_0) = 0$, and F satisfies* **(F1)** *at (λ_0, u_0). Then the equation $F(\lambda, u) = 0$ for (λ, u) near (λ_0, u_0) can be reduced to*

$$\langle l, F(\lambda, u_0 + tw_0 + g(\lambda, t))\rangle = 0,$$

where $t \in (-\delta, \delta)$, $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$ where δ is a small constant, $l \in Y^*$ such that $\langle l, v\rangle = 0$ if and only if $v \in R(F_u(\lambda_0, u_0))$, and g is a C^p function into Z such that $g(\lambda_0, 0) = 0$ and Z is a complement of $N(F_u(\lambda_0, u_0))$ in X.

Proof of Theorem 1.3. We denote the projection from Y into $R(F_u(\lambda_0, u_0))$ by Q. Applying Lemma 2.1 to F in Theorem 1.3 at (λ_0, u_0), we have that the function $g(\lambda, t)$ in Lemma 2.1 is obtained from (see [12]),

$$f_1(\lambda, t) \equiv Q \circ F(\lambda, u_0 + tw_0 + g(\lambda, t)) = 0. \tag{2.1}$$

Since u_0 is a trivial solution for all λ near λ_0, that is, $F(\lambda, u_0) \equiv 0$, then by Lemma 2.1 we have $g(\lambda, 0) \equiv 0$, hence $g_\lambda(\lambda_0, 0) = g_{\lambda\lambda}(\lambda_0, 0) = 0$. Differentiating f_1 and evaluating at $(\lambda, t) = (\lambda_0, 0)$, we obtain

$$0 = \nabla f_1 = (Q \circ (F_\lambda + F_u[g_\lambda]), Q \circ F_u[w_0 + g_t]). \tag{2.2}$$

Since $F_u[w_0] = 0$ and $g_t \in Z$, and $F_u(\lambda_0, c_0 u_*)|_Z$ is an isomorphism, then $g_t(\lambda_0, 0) = 0$.

Next we calculate the second derivatives of f_1:

$$\begin{aligned}&\frac{\partial^2 f_1}{\partial\lambda\partial t}(\lambda_0,0)\\ =&Q\circ(F_{\lambda u}(\lambda_0,u_0)[w_0+g_t(\lambda_0,0)]+F_{uu}(\lambda_0,u_0)[w_0+g_t(\lambda_0,0),g_\lambda(\lambda_0,0)]\\ &+F_u(\lambda_0,u_0)[g_{\lambda t}(\lambda_0,0)])\\ =&F_{\lambda u}(\lambda_0,u_0)[w_0]+F_u(\lambda_0,u_0)[g_{\lambda t}(\lambda_0,0)]=0,\end{aligned}$$

thus $g_{\lambda t}(\lambda_0,0)=\theta_2$ from **(F3′)**, where θ_2 is defined as in Eq (1.5). We define the bifurcation function

$$f(\lambda,t)=\langle l,F(\lambda,u_0+tw_0+g(\lambda,t))\rangle. \tag{2.3}$$

From the assumptions, f is C^2 in U and $f(\lambda,0)=0$. Next we apply the Implicit Function Theorem to the equation $h(\lambda,t)=0$ where the function $h(\lambda,t)$ is defined by

$$h(\lambda,t)=\begin{cases}\frac{1}{t}f(\lambda,t), & \text{if } t\neq 0,\\ f_t(\lambda,0), & \text{if } t=0.\end{cases} \tag{2.4}$$

Then $h(\lambda,0)=0$ from the assumption that $F(\lambda,u_0)=0$, and from **(F4)**, we have

$$\begin{aligned}h_t(\lambda_0,0)&=\lim_{t\to0}\frac{1}{t}(h(\lambda_0,t)-h(\lambda_0,0))=\lim_{t\to0}\frac{1}{t}\left(\frac{1}{t}f(\lambda_0,t)-f_t(\lambda_0,0)\right)\\ &=\lim_{t\to0}\frac{1}{t^2}(f(\lambda_0,t)-f(\lambda_0,0)-f_t(\lambda_0,0)t)=\frac{1}{2}f_{tt}(\lambda_0,0)\\ &=\frac{1}{2}\langle l,F_{uu}(\lambda_0,u_0)[w_0+g_t(\lambda_0,0)]^2+F_u(\lambda_0,u_0)[g_{tt}(\lambda_0,0)]\rangle\\ &=\frac{1}{2}\langle l,F_{uu}(\lambda_0,u_0)[w_0]^2\rangle\neq0.\end{aligned}$$

By the Implicit Function Theorem, there exists a unique continuously differentiable function $t=t(\lambda)\in\mathbb{R}$ satisfying $h(\lambda,t(\lambda))=0$ and $t(\lambda_0)=0$, and

$$F(\lambda,u_0+t(\lambda)w_0+g(\lambda,t(\lambda)))=0. \tag{2.5}$$

Now we assume that $F\in C^4(U,Y)$. Let $u(\lambda)=u_0+t(\lambda)w_0+g(\lambda,t(\lambda))$. Then we have

$$F(\lambda,u(\lambda))=0. \tag{2.6}$$

Differentiating Eq (2.6) with respect to λ twice and evaluating at $\lambda=\lambda_0$, we obtain that

$$F_{\lambda\lambda}+2F_{\lambda u}[u_\lambda]+F_{uu}[u_\lambda]^2+F_u[u_{\lambda\lambda}]=0. \tag{2.7}$$

Here all partial derivatives are evaluated as $\lambda=\lambda_0$. By applying $l\in Y^*$ to Eq (2.7), we have $t'(\lambda_0)=0$ from **(F3′)** and **(F4)**, and we also have $u_\lambda=0$ and $u_{\lambda\lambda}=t''(\lambda_0)w_0$. Differentiating Eq (2.6) with respect to λ three times and evaluating at $\lambda=\lambda_0$, we obtain that

$$3F_{\lambda u}[u_{\lambda\lambda}]+F_u[u_{\lambda\lambda\lambda}]=0, \tag{2.8}$$

which implies that $u_{\lambda\lambda\lambda}=3t''(\lambda_0)\theta_2$. Finally differentiating Eq (2.6) with respect to λ four times and evaluating at $\lambda=\lambda_0$, we obtain that

$$6F_{\lambda\lambda u}[u_{\lambda\lambda}]+4F_{\lambda u}[u_{\lambda\lambda\lambda}]+3F_{uu}[u_{\lambda\lambda}]^2+F_u[u_{\lambda\lambda\lambda\lambda}]=0. \tag{2.9}$$

By applying $l\in Y^*$ to Eq (2.9), we can obtain Eq (1.11). □

3. Stability

In this section, we consider the stability of the bifurcating solutions obtained in Theorem 1.3. First similar to [14, Corollary 1.13], we have the

Proposition 3.1. *Let X, Y, U, F, Z, λ_0, w_0, θ_2 be the same as in Theorem 1.3, and let all assumptions in Theorem 1.3 on F be satisfied. In addition we assume that $X \subset Y$, and the inclusion mapping $i : X \to Y$ is continuous. Let $\Gamma_1 = \{(\lambda, u(\lambda)) : |\lambda - \lambda_0| < \delta\}$ be the solution curve in Theorem 1.3. Then there exist $\varepsilon > 0$, C^2 functions $\gamma : (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \to \mathbb{R}$, $\mu : (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \to \mathbb{R}$, $v : (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \to X$ and $\omega : (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \to X$such that*

$$F_u(\lambda, u_0)[v(\lambda)] = \gamma(\lambda)v(\lambda) \quad for \quad \lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon), \tag{3.1}$$

$$F_u(\lambda, u(\lambda))[\omega(\lambda)] = \mu(\lambda)\omega(\lambda) \quad for \quad \lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon), \tag{3.2}$$

where $\gamma(\lambda_0) = \mu(\lambda_0) = 0$, $v(\lambda_0) = \omega(\lambda_0) = w_0$, $v(\lambda) - w_0 \in Z$ and $\omega(\lambda) - w_0 \in Z$.

We have the following result on the stabilities of the bifurcating solution $(\lambda, u(\lambda))$ obtained in Theorem 1.3.

Proposition 3.2. *Let the assumptions of Proposition 3.1 hold, and let $\gamma(\lambda)$ and $\mu(\lambda)$ be the functions defined in Proposition 3.1. In addition, we assume that*

$$w_0 \notin R(F_u(\lambda_0, u_0)), \quad where\ w_0(\neq 0) \in N(F_u(\lambda_0, u_0)). \tag{3.3}$$

Then $\gamma'(\lambda_0) = \mu'(\lambda_0) = 0$ and

$$\gamma''(\lambda_0) = \frac{\langle l, F_{\lambda\lambda u}(\lambda_0, u_0)[w_0] + 2F_{\lambda u}(\lambda_0, u_0)[\theta_2]\rangle}{\langle l, w_0\rangle}, \tag{3.4}$$

$$\mu''(\lambda_0) = -\frac{\langle l, F_{\lambda\lambda u}(\lambda_0, u_0)[w_0] + 2F_{\lambda u}(\lambda_0, u_0)[\theta_2]\rangle}{\langle l, w_0\rangle}. \tag{3.5}$$

Proof. We differentiate Eq (3.1) to obtain

$$F_{\lambda u}(\lambda, u_0)[v(\lambda)] + F_u(\lambda, u_0)[v'(\lambda)] = \gamma'(\lambda)v(\lambda) + \gamma(\lambda)v'(\lambda). \tag{3.6}$$

Setting $\lambda = \lambda_0$ and applying l to the equation, we get $\gamma'(\lambda_0) = 0$ and $v'(\lambda_0) = \theta_2$ by **(F3′)**. Differentiating Eq (3.6) again, we have

$$\begin{aligned} &F_{\lambda\lambda u}(\lambda, u_0)[v(\lambda)] + 2F_{\lambda u}(\lambda, u_0)[v'(\lambda)] + F_u(\lambda, u_0)[v''(\lambda)] \\ =&\gamma''(\lambda)v(\lambda) + 2\gamma'(\lambda)v'(\lambda) + \gamma(\lambda)v''(\lambda). \end{aligned} \tag{3.7}$$

Setting $\lambda = \lambda_0$, we get

$$F_{\lambda\lambda u}(\lambda_0, u_0)[w_0] + 2F_{\lambda u}(\lambda_0, u_0)[\theta_2] + F_u(\lambda_0, u_0)[v''(\lambda_0)] = \gamma''(\lambda_0)v(\lambda_0). \tag{3.8}$$

Thus by applying l to Eq (3.8), we obtain Eq (3.4).

On the other hand, we differentiate Eq (3.2) to obtain

$$\begin{aligned} &F_{\lambda u}(\lambda, u(\lambda))[\omega(\lambda)] + F_{uu}(\lambda, u(\lambda))[u'(\lambda), \omega(\lambda)] + F_u(\lambda, u(\lambda))[\omega'(\lambda)] \\ =&\mu'(\lambda)\omega(\lambda) + \mu(\lambda)\omega'(\lambda). \end{aligned} \tag{3.9}$$

Setting $\lambda = \lambda_0$, we have

$$F_{\lambda u}(\lambda_0, u_0)[w_0] + F_u(\lambda_0, u_0)[\omega'(\lambda_0)] = \mu'(\lambda_0)w_0, \tag{3.10}$$

we get $\mu'(\lambda_0) = 0$ and $\omega'(\lambda_0) = \theta_2$. We differentiate Eq (3.9) again and set $\lambda = \lambda_0$, and we have

$$\begin{aligned} &F_{\lambda\lambda u}(\lambda_0, u_0)[w_0] + 2F_{\lambda u}(\lambda_0, u_0)[\theta_2] + t''(\lambda_0)F_{uu}(\lambda_0, u_0)[w_0]^2 \\ &+F_u(\lambda_0, u_0)[\omega''(\lambda_0)] = \mu''(\lambda_0)w_0, \end{aligned} \tag{3.11}$$

by $u'(\lambda_0) = 0$ and $u''(\lambda_0) = t''(\lambda_0)w_0$. Thus by applying l to Eq (3.11) and using Eq (1.11), we obtain Eq (3.5). □

Proposition 3.2 implies that the trivial solution $u = u_0$ on Γ_0 and the non-trivial solution $u(\lambda)$ on Γ_1 in Theorem 1.3 both have the same stability before and after the bifurcation point. Thus there is no exchange of stability occurring in the tangential bifurcation described in Theorem 1.3. Furthermore, the stability of the trivial solution u_0 on Γ_0 and the one of the non-trivial solution $u(\lambda)$ on Γ_1 in Theorem 1.3 are always opposite: while one is stable, the other is unstable, or vice versa, if $\gamma(\lambda_0) = \mu(\lambda_0) = 0$ is the principal eigenvalue of $F_u(\lambda_0, u_0)$.

4. Examples

We show that the tangential bifurcations described in Theorem 1.3 occurs for the following semi-linear elliptic equations.

Example 4.1.

$$\begin{cases} \Delta u + u(u - \lambda^2) = 0, & x \in \Omega, \\ \dfrac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \end{cases} \tag{4.1}$$

where λ is a positive parameter, Ω is a bounded region with smooth boundary in $\mathbb{R}^n$ for $n \geq 1$.

It is easy to see $(\lambda, 0)$ is a trivial solution of Eq (4.1). Define a nonlinear mapping $F : \mathbb{R} \times X \to Y$ by

$$F(\lambda, u) = \Delta u + u(u - \lambda^2), \tag{4.2}$$

where $X = \left\{u \in W^{2,p}(\Omega) \ : \dfrac{\partial u}{\partial n} = 0\right\}$ and $Y = L^p(\Omega)$. It is easy to verify that $F_u(0,0)[\phi] = \Delta\phi$, we have $N(F_u(0,0)) = \text{span}\{1\}$, $R(F_u(0,0)) = \{y \in Y : \int_\Omega y dx = 0\}$. And $F_{\lambda u}(0,0)[1] = 0$, $\theta_2 = 0$, $F_{uu}(0,0)[1]^2 = 2$, $F_{\lambda\lambda u}(0,0)[1] = -2$, so **(F1)**, **(F3′)**, **(F4)** are satisfied. We can apply Theorem 1.3 to F. Then the solution set of Eq (4.1) near $(\lambda, u) = (0,0)$ is the union of two C^1 curves which are tangent to each other at $(0,0)$, including the line of trivial solutions $\Gamma_0 = \{(\lambda, 0)\}$ and $\Gamma_1 = \{(\lambda, u(\lambda)) : |\lambda| < \delta\}$ for some $\delta > 0$, where $u(\lambda)$ is a continuously differentiable function, $u(\lambda) = t(\lambda)+g(\lambda, t(\lambda))$, $t(0) = t'(0) = 0$ and $t''(0) = 2$. Furthermore from Proposition 3.2, we have $\gamma''(0) = -\mu''(0) = -2$. It implies the trivial solution $(\lambda, 0)$ is stable and the nontrivial solution $(\lambda, u(\lambda))$ is unstable. Note that this example is rather trivial as $u(\lambda) = \lambda^2$ is a constant solution of Eq (4.1).

Example 4.2.

$$\begin{cases}\Delta u + 2\lambda\sqrt{\mu_1}u - \lambda^2 u + \lambda u^2 = 0, & x \in \Omega,\\ u = 0, & x \in \partial\Omega,\end{cases} \tag{4.3}$$

where λ is a positive parameter, Ω is a bounded region with smooth boundary in $\mathbb{R}^n$ for $n \geq 1$, and μ_1 is the principal eigenvalue of $-\Delta$ on $H_0^1(\Omega)$.

For any $\lambda > 0$, $u = 0$ is a trivial solution of (4.3). Define a nonlinear mapping $F : \mathbb{R} \times X \to Y$ by

$$F(\lambda, u) = \Delta u + 2\lambda\sqrt{\mu_1}u - \lambda^2 u + \lambda u^2, \tag{4.4}$$

where $X = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and $Y = L^p(\Omega)$. We show that $\lambda = \sqrt{\mu_1}$ is a bifurcation point for the trivial solution $u = 0$. We can verify that $F_u(\sqrt{\mu_1}, 0)[\phi] = \Delta\phi + \mu_1\phi$, we have $N(F_u(\sqrt{\mu_1}, 0)) = \text{span}\{\varphi_1\}$, $R(F_u(\sqrt{\mu_1}, 0)) = \{y \in Y : \int_\Omega y\varphi_1 dx = 0\}$, where $\varphi_1 > 0$ is the principal eigenfunction of $-\Delta$ on $H_0^1(\Omega)$ corresponding to μ_1. Moreover we can verify that $F_{\lambda u}(\sqrt{\mu_1}, 0)[\varphi_1] = 0$, $\theta_2 = 0$, $F_{uu}(\sqrt{\mu_1}, 0)[\varphi_1]^2 = 2\varphi_1^2$, $F_{\lambda\lambda u}(\sqrt{\mu_1}, 0)[\varphi_1] = -2\varphi_1$. so the conditions **(F1)**, **(F3′)**, **(F4)** are satisfied. We can apply Theorem 1.3 to F at $\lambda = \sqrt{\mu_1}$. The solution set of Eq (4.3) near $(\lambda, u) = (\sqrt{\mu_1}, 0)$ is the union of two C^1 curves which are tangent to each other at $(\sqrt{\mu_1}, 0)$, including the line of trivial solutions $\Gamma_0 = \{(\lambda, 0) : \lambda > 0\}$ and $\Gamma_1 = \{(\lambda, u(\lambda)) : |\lambda - \sqrt{\mu_1}| < \delta\}$ for some $\delta > 0$, where $u(\lambda) = t(\lambda)\varphi_1 + g(\lambda, t(\lambda))$ is smooth, $t(\sqrt{\mu_1}) = t'(\sqrt{\mu_1}) = 0$, $t''(\sqrt{\mu_1}) = 2A > 0$ where $A = \int_\Omega \varphi_1^2 / \int_\Omega \varphi_1^3 > 0$, and $g(\lambda, 0) = g_\lambda(\sqrt{\mu_1}, 0) = g_t(\sqrt{\mu_1}, 0) = 0$. Thus Eq (4.3) has a positive solution $u(\lambda) \approx A(\lambda - \sqrt{\mu_1})^2\varphi_1$ for any $0 < |\lambda - \sqrt{\mu_1}| < \delta$. Furthermore from Proposition 3.2, we have $\gamma''(0) = -\mu''(0) = -2$. It implies the trivial solution $(\lambda, 0)$ is stable and the nontrivial solution $(\lambda, u(\lambda))$ is unstable when $\lambda \neq \sqrt{\mu_1}$.

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Conflict of interest

The authors declare there is no conflicts of interest.

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