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ERA, 30(1): 52–65.
DOI: 10.3934/era.2022003
Received: 20 October 2021
Revised: 23 November 2021
Accepted: 23 November 2021
Published: 09 December 2021

Research article

On second order mock theta function $B(q)$

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Abstract: In this paper, we present some arithmetic properties for the second order mock theta function $B(q)$ given by McIntosh as:

$$B(q) = \sum_{n=0}^{\infty} \frac{q^n(-q;q^2)_n}{(q;q^2)_{n+1}}.$$

Keywords: mock theta function; congruences

1. Introduction

Ramanujan's last letter to Hardy is one of the most mysterious and important mathematical letters in the history of mathematics. He introduced a class of functions that he called mock theta functions in his letter. For nearly a century, properties of these functions have been widely studied by different mathematicians. The important direction involves the arithmetic properties (see [1, 2]), combinatorics (see [3, 4]), identities between these functions, and generalized Lambert series (see [5, 6]). For the interested reader, regarding the history and new developments in the study of mock theta functions, we refer to [7].

In 2007, McIntosh studied two second order mock theta functions in reference [8]; more details are given in reference [9]. These mock theta functions are:

$$A(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q;q^2)_n}{(q;q^2)^2_{(n+1)}} = \sum_{n=0}^{\infty} \frac{q^{n+1}(-q^2;q^2)_n}{(q;q^2)_{n+1}}, \tag{1.1}$$

$$B(q) = \sum_{n=0}^{\infty} \frac{q^n(-q;q^2)_n}{(q;q^2)_{n+1}} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q^2;q^2)_n}{(q;q^2)^2_{n+1}}, \tag{1.2}$$

where

$$(a;q)_n = \prod_{i=0}^{n-1}(1-aq^i), \qquad (a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i),$$

$$(a_1, a_2, \cdots, a_m; q)_\infty = (a_1; q)_\infty (a_2; q)_\infty \cdots (a_m; q)_\infty,$$

for $|q| < 1$.

The functions $A(q)$ and $B(q)$ have been combinatorially interpreted in terms of overpartitions in [3] using the odd Ferrers diagram. In this paper, we study some arithmetic properties of one of the second order mock theta functions $B(q)$. We start by noting, Bringmann, Ono and Rhoades [10] obtained the following identity:

$$\frac{B(q) + B(-q)}{2} = \frac{f_4^5}{f_2^4}, \tag{1.3}$$

where

$$f_m^k := (q^m; q^m)_\infty^k,$$

for positive integers m and k. We consider the function

$$B(q) := \sum_{n=0}^{\infty} b(n) q^n. \tag{1.4}$$

Followed by Eq (1.3), the even part of $B(q)$ is given by:

$$\sum_{n=0}^{\infty} b(2n) q^n = \frac{f_2^5}{f_1^4}. \tag{1.5}$$

In 2012, applying the theory of (mock) modular forms and Zwegers' results, Chan and Mao [5] established two identities for $b(n)$, shown as:

$$\sum_{n=0}^{\infty} b(4n+1) q^n = 2\frac{f_2^8}{f_1^7}, \tag{1.6}$$

$$\sum_{n=0}^{\infty} b(4n+2) q^n = 4\frac{f_2^2 f_4^4}{f_1^5}. \tag{1.7}$$

In a sequel, Qu, Wang and Yao [6] found that all the coefficients for odd powers of q in $B(q)$ are even. Recently, Mao [11] gave analogues of Eqs (1.6) and (1.7) modulo 6

$$\sum_{n=0}^{\infty} b(6n+2) q^n = 4\frac{f_2^{10} f_3^2}{f_1^{10} f_6}, \tag{1.8}$$

$$\sum_{n=0}^{\infty} b(6n+4) q^n = 9\frac{f_2^4 f_3^4 f_6}{f_1^8}, \tag{1.9}$$

and proved several congruences for the coefficients of $B(q)$. Motivated from this, we prove similar results for $b(n)$ by applying identities on the coefficients in arithmetic progressions. We present some congruence relations for the coefficients of $B(q)$ modulo certain numbers of the form $2^\alpha \cdot 3^\beta$, $2^\alpha \cdot 5^\beta$, $2^\alpha \cdot 7^\beta$ where $\alpha, \beta \geq 0$. Our main theorems are given below:

Theorem 1.1. *For $n \geq 0$, we have*

$$\sum_{n=0}^{\infty} b(12n+9)q^n = 18\left[\frac{f_2^9 f_3^{12}}{f_1^{17} f_6^3} + 2\frac{f_2^5 f_3^4 f_6}{f_1^9} + 28\frac{f_2^6 f_3^3 f_6^6}{f_1^{14}}\right], \tag{1.10}$$

$$\sum_{n=0}^{\infty} b(12n+10)q^n = 36\left[2\frac{f_2^{16} f_6^{10}}{f_1^{20} f_3 f_{12}^4} - q\frac{f_2^{28} f_3^3 f_{12}^2}{f_1^{24} f_4^8 f_6^2} - 16q^2\frac{f_2^2 f_3^3 f_4^8 f_{12}^2}{f_1^{16} f_6^2}\right]. \tag{1.11}$$

In particular, $b(12n+9) \equiv 0 \pmod{18}, b(12n+10) \equiv 0 \pmod{36}$.

Theorem 1.2. *For $n \geq 0$, we have*

$$\sum_{n=0}^{\infty} b(18n+10)q^n = 72\left[\frac{f_2^{16} f_3^{21}}{f_1^{27} f_6^9} + 38q\frac{f_2^{13} f_3^{12}}{f_1^{24}} + 64q^2\frac{f_2^{10} f_3^3 f_6^9}{f_1^{21}}\right], \tag{1.12}$$

$$\sum_{n=0}^{\infty} b(18n+16)q^n = 72\left[5\frac{f_2^{15} f_3^{18}}{f_1^{26} f_6^6} + 64q\frac{f_2^{12} f_3^9 f_6^3}{f_1^{23}} + 32q^2\frac{f_2^9 f_6^{12}}{f_1^{20}}\right]. \tag{1.13}$$

In particular, $b(18n+10) \equiv 0 \pmod{72}, b(18n+16) \equiv 0 \pmod{72}$.

Apart from these congruences, we find some relations between $b(n)$ and restricted partition functions. Here we recall, *Partition of a positive integer ν, is a representation of ν as a sum of non-increasing sequence of positive integers $\mu_1, \mu_2, \cdots, \mu_n$.* The number of partitions of ν is denoted by $p(\nu)$ which is called the partition function. If certain conditions are imposed on parts of the partition, is called the restricted partition and corresponding partition function is named as restricted partition function. Euler proved the following recurrence for $p(n)$ [12] [p. 12, Cor. 1.8]:

$$\begin{aligned} p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) - p(n-12) - p(n-15) + \cdots \\ + (-1)^k p(n - k(3k-1)/2) + (-1)^k p(n - k(3k+1)/2) + \cdots = \begin{cases} 1, & \text{if } n = 0, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

The numbers $k(3k \pm 1)/2$ are pentagonal numbers. Following the same idea, different recurrence relations have been found by some researchers for restricted partition functions. For instance, Ewell [13] presented the recurrence for $p(n)$ involving the triangular numbers. For more study of recurrences, see [14–16]. Under the influence of these efforts, we express the coefficients of mock theta function $B(q)$ which are in arithmetic progression in terms of recurrence of some restricted partition functions.

This paper is organized as follows: Section 2, here we recall some preliminary lemmas and present the proof of Theorems 1.1 and 1.2. Section 3 includes some more congruences based on the above results. Section 4 depicts the links between $b(n)$ and some of the restricted partition functions.

2. Proof of Theorems 1.1 and 1.2

Before proving the results, we recall Ramanujan's theta function:

$$j(a,b) = \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, \text{ for } |ab| < 1.$$

Some special cases of $j(a,b)$ are:

$$\phi(q) := j(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{f_2^5}{f_1^2 f_4^2},$$
$$\psi(q) := j(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{f_2^2}{f_1}.$$

Also,

$$\phi(-q) = \frac{f_1^2}{f_2}.$$

The above function satisfy the following properties (see Entries 19, 20 in [17]).

$$j(a,b) = (-a,-b,ab;ab)_\infty, \quad \text{(Jacobi's triple product identity)},$$

$$j(-q,-q^2) = (q;q)_\infty, \quad \text{(Euler's pentagonal number theorem)}.$$

We note the following identities which will be used below.

Lemma 2.1. [[18], Eq (3.1)] *We have*

$$\frac{f_2^3}{f_1^3} = \frac{f_6}{f_3} + 3q\frac{f_6^4 f_9^5}{f_3^8 f_{18}} + 6q^2\frac{f_6^3 f_9^2 f_{18}^2}{f_3^7} + 12q^3\frac{f_6^2 f_{18}^5}{f_3^6 f_9}. \tag{2.1}$$

Lemma 2.2. *We have*

$$\frac{f_2^2}{f_1} = \frac{f_6 f_9^2}{f_3 f_{18}} + q\frac{f_{18}^2}{f_9}, \tag{2.2}$$
$$\frac{f_2}{f_1^2} = \frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q\frac{f_6^3 f_9^3}{f_3^7} + 4q^2\frac{f_6^2 f_{18}^3}{f_3^6}. \tag{2.3}$$

Proof. The first identity follows from [[19] Eq (14.3.3)]. The proof of second identity can be seen from [20]. □

Lemma 2.3. *We have*

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}, \tag{2.4}$$
$$f_1^4 = \frac{f_4^{10}}{f_2^2 f_8^4} - 4q\frac{f_2^2 f_8^4}{f_4^2}. \tag{2.5}$$

Proof. Identity (2.4) is Eq (1.10.1) from [19]. To obtain (2.5), replacing q by $-q$ and then using

$$(-q;-q)_\infty = \frac{f_2^3}{f_1 f_4}.$$

□

Now, we present the proof of Theorems 1.1 and 1.2.

Proof of Theorems 1.1 and 1.2. From Eq (1.6), we have

$$\sum_{n=0}^{\infty} b(4n+1)q^n = 2\left(\frac{f_2^3}{f_1^3}\right)^3 \cdot \frac{f_2^2}{f_1}.$$

Substituting the values from Eqs (2.1) and (2.2) in above, we get

$$\begin{aligned}\sum_{n=0}^{\infty} b(4n+1)q^n \quad = \quad & 2\frac{f_6^3 f_9^2}{f_3^3 f_{18}} + 2q\frac{f_6^2 f_{18}^2}{f_3^2 f_9} + 12q\frac{f_6^6 f_9^7}{f_3^{10} f_{18}^2} + 18q^2\frac{f_6^9 f_9^{12}}{f_3^{17} f_{18}^3} + 36q^2\frac{f_6^5 f_9^4 f_{18}}{f_3^9} + \\ & 90q^3\frac{f_6^8 f_9^9}{f_3^{16}} + 72q^3\frac{f_6^4 f_9 f_{18}^4}{f_3^8} + 48q^4\frac{f_6^3 f_{18}^7}{f_3^7 f_9^2} + 288q^4\frac{f_6^7 f_9^6 f_{18}^3}{f_3^{15}} + \\ & 504q^5\frac{f_6^6 f_9^3 f_{18}^6}{f_3^{14}} + 576q^6\frac{f_6^5 f_{18}^9}{f_3^{13}}. \end{aligned} \tag{2.6}$$

Bringing out the terms involving q^{3n+2}, dividing by q^2 and replacing q^3 by q, we get (1.10). Considering Eq (1.5), we have

$$\sum_{n=0}^{\infty} b(2n)q^n = \frac{f_2^3}{f_1^3} \cdot \frac{f_2^2}{f_1}.$$

Substituting the values from Eqs (2.1) and (2.2), we obtain

$$\sum_{n=0}^{\infty} b(2n)q^n = \left(\frac{f_6}{f_3} + 3q\frac{f_6^4 f_9^5}{f_3^8 f_{18}} + 6q^2\frac{f_6^3 f_9^2 f_{18}^2}{f_3^7} + 12q^3\frac{f_6^2 f_{18}^5}{f_3^6 f_9}\right)\left(\frac{f_6 f_9^2}{f_3 f_{18}} + q\frac{f_{18}^2}{f_9}\right).$$

Extracting the terms involving $q^{3n}, q^{3n+1}, q^{3n+2}$ from the above equation, we have

$$\sum_{n=0}^{\infty} b(6n)q^n = \frac{f_2^2 f_3^2}{f_1^2 f_6} + 18q\frac{f_2^3 f_3 f_6^4}{f_1^7}, \tag{2.7}$$

$$\sum_{n=0}^{\infty} b(6n+2)q^n = \frac{f_2 f_6^2}{f_1 f_3} + 3\frac{f_2^5 f_3^7}{f_1^9 f_6^2} + 12q\frac{f_2^2 f_6^7}{f_1^6 f_3^2}, \tag{2.8}$$

$$\sum_{n=0}^{\infty} b(6n+4)q^n = 9\frac{f_2^4 f_3^4 f_6}{f_1^8}. \tag{2.9}$$

Using Eqs (2.4) and (2.5) in Eq (2.9), we get

$$\sum_{n=0}^{\infty} b(6n+4)q^n = 9f_2^4 f_6\left(\frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}\right)^2\left(\frac{f_{12}^{10}}{f_6^2 f_{24}^4} - 4q^3\frac{f_6^2 f_{24}^4}{f_{12}^2}\right).$$

Extracting the terms involving q^{2n}, q^{2n+1} from above, we arrive at

$$\sum_{n=0}^{\infty} b(12n+4)q^n = 9\left(\frac{f_2^{28} f_6^{10}}{f_1^{24} f_3 f_4^8 f_{12}^4} + 16q\frac{f_2^4 f_4^8 f_6^{10}}{f_1^{16} f_3 f_{12}^4} - 32q^2\frac{f_2^{16} f_3^3 f_{12}^4}{f_1^{20} f_6^2}\right), \tag{2.10}$$

$$\sum_{n=0}^{\infty} b(12n+10)q^n = 9\left(8\frac{f_2^{16}f_6^{10}}{f_1^{20}f_3f_{12}^4} - 4q\frac{f_2^{28}f_3^3f_{12}^4}{f_1^{24}f_4^8f_6^2} - 16q^2\frac{f_2^4f_3^3f_4^8f_{12}^4}{f_1^{16}f_6^2}\right). \tag{2.11}$$

From Eq (2.11), we ultimately arrive at Eq (1.11). To prove Theorem 1.2, consider Eq (2.9) as:

$$\sum_{n=0}^{\infty} b(6n+4)q^n = 9f_3^4f_6\left(\frac{f_2}{f_1^2}\right)^4.$$

Using Eq (2.3) in above, we get

$$\begin{aligned}\sum_{n=0}^{\infty} b(6n+4)q^n \;=\;& 9\frac{f_6^{17}f_9^{24}}{f_3^{28}f_{18}^{12}} + 72q\frac{f_6^{16}f_9^{21}}{f_3^{27}f_{18}^9} + 360q^2\frac{f_6^{15}f_9^{18}}{f_3^{26}f_{18}^6} + 288q^3\frac{f_6^{14}f_9^{15}}{f_3^{25}f_{18}^3} + \\ & 864q^3\frac{f_6^{12}f_9^{15}}{f_3^{19}f_{18}^6} + 2736q^4\frac{f_6^{13}f_9^{12}}{f_3^{24}} + 4608q^5\frac{f_6^{12}f_9^9f_{18}^3}{f_3^{23}} + \\ & 5760q^6\frac{f_6^{11}f_9^6f_{18}^6}{f_3^{22}} + 4608q^7\frac{f_6^{10}f_9^3f_{18}^9}{f_3^{21}} + 2304q^8\frac{f_6^9f_{18}^{12}}{f_3^{20}}.\end{aligned} \tag{2.12}$$

Bringing out the terms involving q^{3n+1} and q^{3n+2} from Eq (2.12), we get Eqs (1.12) and (1.13), respectively. □

3. Congruences

This segment of the paper contains some more interesting congruence relations for $b(n)$.

Theorem 3.1. *For $n \geq 0$, we have*

$$b(12n+1) \equiv \begin{cases} 2(-1)^k \pmod 6, & \text{if } n = 3k(3k+1)/2, \\ 0 \pmod 6, & \text{otherwise.} \end{cases} \tag{3.1}$$

Theorem 3.2. *For $n \geq 0$, we have*

$$b(2n) \equiv \begin{cases} (-1)^k(2k+1) \pmod 4, & \text{if } n = k(k+1), \\ 0 \pmod 4, & \text{otherwise.} \end{cases} \tag{3.2}$$

Theorem 3.3. *For $n \geq 0$, we have*

$$b(36n+10) \equiv 0 \pmod{72}, \tag{3.3}$$
$$b(36n+13) \equiv 0 \pmod{6}, \tag{3.4}$$
$$b(36n+25) \equiv 0 \pmod{12}, \tag{3.5}$$
$$b(36n+34) \equiv 0 \pmod{144}, \tag{3.6}$$
$$b(108n+t) \equiv 0 \pmod{18}, \quad \text{for } t \in \{49, 85\}. \tag{3.7}$$

Theorem 3.4. *For $n \geq 0$, we have*

$$b(20n+t) \equiv 0 \pmod{5}, \quad \text{for } t \in \{8, 16\} \tag{3.8}$$
$$b(20n+t) \equiv 0 \pmod{20}, \quad \text{for } t \in \{6, 18\} \tag{3.9}$$
$$b(20n+17) \equiv 0 \pmod{10}, \tag{3.10}$$
$$b(28n+t) \equiv 0 \pmod{14}, \quad \text{for } t \in \{5, 21, 25\}. \tag{3.11}$$

Proof of Theorem 3.1. From Eq (2.6), picking out the terms involving q^{3n} and replacing q^3 by q, we have

$$\sum_{n=0}^{\infty} b(12n+1)q^n = 2\frac{f_2^3 f_3^2}{f_1^3 f_6} + 90q\frac{f_2^8 f_3^9}{f_1^{16}} + 72q\frac{f_2^4 f_3 f_6^4}{f_1^8} + 576q^2\frac{f_2^5 f_6^9}{f_1^3}. \tag{3.12}$$

Reducing modulo 6, we obtain

$$\sum_{n=0}^{\infty} b(12n+1)q^n \equiv 2f_3 \pmod{6}. \tag{3.13}$$

With the help of Euler's pentagonal number theorem,

$$\sum_{n=0}^{\infty} b(12n+1)q^n \equiv 2\sum_{k=-\infty}^{\infty}(-1)^k q^{\frac{3k(3k+1)}{2}} \pmod{6},$$

which completes the proof of Theorem 3.1. □

Proof of Theorem 3.2. Reducing Eq (1.5) modulo 4, we get

$$\sum_{n=0}^{\infty} b(2n)q^n \equiv f_2^3 \pmod{4}. \tag{3.14}$$

From Jacobi's triple product identity, we obtain

$$\sum_{n=0}^{\infty} b(2n)q^n \equiv \sum_{k=0}^{\infty}(-1)^k(2k+1)q^{k(k+1)} \pmod{4},$$

which completes the proof of Theorem 3.2. □

Proof of Theorem 3.3. Consider Eq (1.11), reducing modulo 72

$$\sum_{n=0}^{\infty} b(12n+10)q^n \equiv 36q\frac{f_2^{28} f_3^3 f_{12}^4}{f_1^{24} f_4^8 f_6^2} \pmod{72},$$

$$\sum_{n=0}^{\infty} b(12n+10)q^n \equiv 36q\frac{f_2^{28} f_3^3 f_{12}^4}{f_2^{12} f_4^8 f_{12}} = 36q\frac{f_2^{16} f_3^3 f_{12}^3}{f_4^8} \pmod{72}$$

or

$$\sum_{n=0}^{\infty} b(12n+10)q^n \equiv 36q f_3^3 f_{12}^3 \pmod{72}. \tag{3.15}$$

Extracting the terms involving q^{3n}, replacing q^3 by q in Eq (3.15), we arrive at Eq (3.3). Similarly, consider Eq (1.13) and reducing modulo 144, we have

$$\sum_{n=0}^{\infty} b(18n+16)q^n \equiv 72\cdot 5\frac{f_2^{15} f_3^{18}}{f_1^{26} f_6^6} \pmod{144},$$

$$\equiv 72\frac{f_2^{15}f_6^9}{f_2^{13}f_6^6} = 72f_2^2f_6^3 \pmod{144}.$$

Extracting the terms involving q^{2n+1}, dividing both sides by q and replacing q^2 by q, we get Eq (3.6). From Eq (3.20), we get

$$\sum_{n=0}^{\infty} b(12n+1)q^n \equiv 2f_3 \pmod{6}.$$

Bringing out the terms containing q^{3n+1}, dividing both sides by q and replacing q^3 by q, we have $b(36n+13) \equiv 0 \pmod{6}$. Reducing Eq (3.12) modulo 12, we have

$$\sum_{n=0}^{\infty} b(12n+1)q^n \equiv 2\frac{f_2^3f_3^2}{f_1^3f_6} + 90q\frac{f_2^8f_3^9}{f_1^{16}} \pmod{12},$$

$$\sum_{n=0}^{\infty} b(12n+1)q^n \equiv 2\frac{f_3^2}{f_6}\left(\frac{f_6}{f_3} + 3q\frac{f_6^4f_9^5}{f_3^8f_{18}} + 6q^2\frac{f_6^3f_9^2f_{18}^2}{f_3^7} + 12q^3\frac{f_6^2f_{18}^5}{f_3^6f_9}\right) + 6q\frac{f_2^8f_3^9}{f_2^8}.$$

Extracting the terms containing q^{3n+2}, dividing by q^2 and replacing q^3 by q, we obtain Eq (3.5). Reducing Eq (3.12) modulo 18,

$$\sum_{n=0}^{\infty} b(12n+1)q^n \equiv 2\frac{f_2^3f_3^2}{f_1^3f_6} \pmod{18},$$

$$= 2\frac{f_3^2}{f_6}\left(\frac{f_6}{f_3} + 3q\frac{f_6^4f_9^5}{f_3^8f_{18}} + 6q^2\frac{f_6^3f_9^2f_{18}^2}{f_3^7} + 12q^3\frac{f_6^2f_{18}^5}{f_3^6f_9}\right).$$

Extracting the terms involving q^{3n+1}, dividing both sides by q and replacing q^3 by q, we have

$$\sum_{n=0}^{\infty} b(36n+13)q^n \equiv 6\frac{f_2^3f_3^5}{f_1^6f_6} \equiv 6\frac{f_6f_3^5}{f_3^2f_6} \pmod{18}$$

or

$$\sum_{n=0}^{\infty} b(36n+13)q^n \equiv 6f_3^3 \pmod{18}.$$

Extracting the terms containing q^{3n+1}, q^{3n+2} from above to get Eq (3.7). □

Proof of Theorem 3.4. From Eqs (1.5) and (2.4), we have

$$\sum_{n=0}^{\infty} b(2n)q^n = f_2^5\left(\frac{f_4^{14}}{f_2^{14}f_8^4} + 4q\frac{f_4^2f_8^4}{f_2^{10}}\right).$$

Bringing out the terms containing even powers of q, we obtain

$$\sum_{n=0}^{\infty} b(4n)q^n = \frac{f_2^{14}}{f_1^9f_4^4},$$

which can be written as:

$$\sum_{n=0}^{\infty} b(4n)q^n = \frac{f_2^{15}}{f_1^{10} f_4^5} . \frac{f_1 f_4}{f_2} \equiv \frac{f_{10}^3}{f_5^2 f_{20}} . \frac{f_1 f_4}{f_2} \pmod 5.$$

Here

$$\begin{aligned} \frac{f_1 f_4}{f_2} &= \frac{(q;q)_\infty (q^4;q^4)_\infty}{(q^2;q^2)_\infty}, \\ &= \frac{(q;q^2)_\infty (q^2;q^2)_\infty (q^4;q^4)_\infty}{(q^2;q^2)_\infty}, \end{aligned}$$

$$\frac{f_1 f_4}{f_2} = (q, q^3, q^4; q^4)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{2n^2-n}, \tag{3.16}$$

where the last equality follows from Jacobi's triple product identity. Using the above identity, we have

$$\sum_{n=0}^{\infty} b(4n)q^n \equiv \frac{f_{10}^3}{f_5^2 f_{20}} \sum_{n=-\infty}^{\infty} (-1)^n q^{2n^2-n} \pmod 5. \tag{3.17}$$

Since $2n^2 - n \not\equiv 2, 4 \pmod 5$, it follows that the coefficients of q^{5n+2}, q^{5n+4} in $\sum_{n=0}^{\infty} b(4n)q^n$ are congruent to $0 \pmod 5$, which proves that $b(20n + t) \equiv 0 \pmod 5$, for $t \in \{8, 16\}$.
Consider Eq (1.7)

$$\sum_{n=0}^{\infty} b(4n+2)q^n = 4\frac{f_4^5}{f_1^5} \frac{f_2^2}{f_4} \equiv 4\frac{f_{20}}{f_5} \frac{f_2^2}{f_4} \pmod{20}.$$

Now

$$\begin{aligned} \frac{f_2^2}{f_4} &= \frac{(q^2;q^2)_\infty^2}{(q^4;q^4)_\infty}, \\ &= \frac{(q^2;q^2)_\infty (q^2;q^4)_\infty (q^4;q^4)_\infty}{(q^4;q^4)_\infty}, \end{aligned}$$

$$\frac{f_2^2}{f_4} = (q^2, q^2, q^4; q^4)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{2n^2}.$$

Using the above identity, we get

$$\sum_{n=0}^{\infty} b(4n+2)q^n \equiv 4\frac{f_{20}}{f_5} \sum_{n=-\infty}^{\infty} (-1)^n q^{2n^2} \pmod{20}. \tag{3.18}$$

Since $2n^2 \not\equiv 1, 4 \pmod 5$, it follows that the coefficients of q^{5n+1}, q^{5n+4} in $\sum_{n=0}^{\infty} b(4n+2)q^n$ are congruent to $0 \pmod{20}$, which proves Eq (3.9). For the proof of next part, consider Eq (1.6) as:

$$\sum_{n=0}^{\infty} b(4n+1)q^n = 2\frac{f_2^5}{f_1^{10}} f_1^3 f_2^3 \equiv 2\frac{f_{10}}{f_5^2} f_1^3 f_2^3 \pmod{10},$$

$$\sum_{n=0}^{\infty} b(4n+1)q^n \equiv 2\frac{f_{10}}{f_5^2}\sum_{k=0}^{\infty}(-1)^k(2k+1)q^{\frac{k(k+1)}{2}}\sum_{m=0}^{\infty}(-1)^m(2m+1)q^{m(m+1)} \pmod{10}. \tag{3.19}$$

Therefore, to contribute the coefficient of q^{5n+4}, $(k,m) \equiv (2,2) \pmod 5$ and thus the contribution towards the coefficient of q^{5n+4} is a multiple of 5.
Consider Eq (1.6) as:

$$\sum_{n=0}^{\infty} b(4n+1)q^n = 2\frac{f_2^7}{f_1^7}f_2 \equiv 2\frac{f_{14}}{f_7}f_2 \pmod{14}.$$

With the help of Euler's pentagonal number theorem,

$$\sum_{n=0}^{\infty} b(4n+1)q^n \equiv 2\frac{f_{14}}{f_7}\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n+1)} \pmod{14}. \tag{3.20}$$

As $n(3n+1) \not\equiv 1,5,6 \pmod 7$, it readily proves Eq (3.11). □

4. Recurrence relations

In this section, we find some recurrence relations connecting $b(n)$ and restricted partition functions. First we define some notations. Let $\overline{p}_l(n)$ denotes the number of overpartitions of n with l copies. Then

$$\sum_{n=0}^{\infty} \overline{p}_l(n)q^n = \left(\frac{f_2}{f_1^2}\right)^l.$$

Let $p_{ld}(n)$ denotes the number of partitions of n into distinct parts with l copies. Then

$$\sum_{n=0}^{\infty} p_{ld}(n)q^n = \left(\frac{f_2}{f_1}\right)^l.$$

Theorem 4.1. *We have*

$$b(2n) = \overline{p}_2(n) - 3\overline{p}_2(n) + 5\overline{p}_2(n) + \cdots + (-1)^k(2k+1)\overline{p}_2(n-k(k+1)) + \cdots, \tag{4.1}$$

$$b(2n) = p_{4d}(n) - p_{4d}(n-2) - p_{4d}(n-4) + p_{4d}(n-10) + p_{4d}(n-14) + \cdots + (-1)^k p_{4d}(n-k(3k-1)) + (-1)^k p_{4d}(n-k(3k+1)) + \cdots. \tag{4.2}$$

Theorem 4.2.

$$b(4n+1) = 2p_{8d}(n) - 2p_{8d}(n-1) - 2p_{8d}(n-2) + 2p_{8d}(n-5) + 2p_{8d}(n-7) + \cdots + (-1)^k 2p_{8d}\left(n - \frac{k(3k-1)}{2}\right) + (-1)^k 2p_{8d}\left(n - \frac{k(3k+1)}{2}\right) + \cdots, \tag{4.3}$$

$$b(4n+1) = 2\sum_{c=0}^{n} b(2c)p_{3d}(n-c). \tag{4.4}$$

Theorem 4.3.

$$b(6n+2) = 4p_{10d}(n) - 8p_{10d}(n-3) + 8p_{10d}(n-12) + 8p_{10d}(n-27) + \cdots + 8(-1)^k p_{10d}(n-3k^2) + \cdots . \quad (4.5)$$

Proof of Theorem 4.1. Consider (1.5) as:

$$\sum_{n=0}^{\infty} b(2n)q^n = \left(\frac{f_2}{f_1^2}\right)^2 \cdot f_2^3.$$

Then

$$\begin{aligned}\sum_{n=0}^{\infty} b(2n)q^n &= \left(\sum_{n=0}^{\infty} \overline{p}_2(n)q^n\right)\left(\sum_{k=0}^{\infty}(-1)^k(2k+1)q^{k(k+1)}\right),\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-1)^k(2k+1)\overline{p}_2(n)q^{n+k(k+1)},\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}(-1)^k(2k+1)\overline{p}_2(n-k(k+1))\right)q^n.\end{aligned}$$

From the last equality, we readily arrive at (4.1). To prove (4.2), consider (1.5) as:

$$\begin{aligned}\sum_{n=0}^{\infty} b(2n)q^n &= \left(\frac{f_2}{f_1}\right)^4 \cdot f_2,\\ &= \left(\sum_{n=0}^{\infty} p_{4d}(n)q^n\right)\left(\sum_{k=-\infty}^{\infty}(-1)^k q^{k(3k+1)}\right),\\ &= \left(\sum_{n=0}^{\infty} p_{4d}(n)q^n\right)\left(1 + \sum_{k=1}^{\infty}(-1)^k q^{k(3k-1)} + \sum_{k=1}^{\infty}(-1)^k q^{k(3k+1)}\right),\end{aligned}$$

$$\sum_{n=0}^{\infty} b(2n)q^n = \sum_{n=0}^{\infty} p_{4d}(n)q^n + \sum_{n=0}^{\infty}\left(\sum_{k=1}^{\infty}(-1)^k p_{4d}(n)q^{k(3k-1)+n}\right) + \sum_{n=0}^{\infty}\left(\sum_{k=1}^{\infty}(-1)^k p_{4d}(n)q^{k(3k+1)+n}\right),$$

$$\sum_{n=0}^{\infty} b(2n)q^n = \sum_{n=0}^{\infty} p_{4d}(n)q^n + \sum_{n=0}^{\infty}\left(\sum_{k=1}^{\infty}(-1)^k p_{4d}(n-k(3k-1))q^n\right) + \sum_{n=0}^{\infty}\left(\sum_{k=1}^{\infty}(-1)^k p_{4d}(n-k(3k+1))q^n\right),$$

which proves Eq (4.2). □

Proof of Theorem 4.2. Consider Eq (1.6) as:

$$\sum_{n=0}^{\infty} b(4n+1)q^n = 2\left(\frac{f_2}{f_1}\right)^8 f_1,$$

$$= 2\left(\sum_{n=0}^{\infty} p_{8d}(n)q^n\right)\left(\sum_{k=-\infty}^{\infty} (-1)^k q^{\frac{k(3k+1)}{2}}\right),$$

$$= 2\left(\sum_{n=0}^{\infty} p_{8d}(n)q^n\right)\left(1 + \sum_{k=1}^{\infty}(-1)^k q^{k(3k-1)/2} + \sum_{k=1}^{\infty}(-1)^k q^{k(3k+1)/2}\right),$$

$$\sum_{n=0}^{\infty} b(4n+1)q^n = \sum_{n=0}^{\infty} p_{8d}(n)q^n + \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}(-1)^k p_{8d}(n)q^{k(3k-1)/2+n} + \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}(-1)^k p_{8d}(n)q^{k(3k+1)/2+n},$$

$$\sum_{n=0}^{\infty} b(4n+1)q^n = \sum_{n=0}^{\infty} p_{8d}(n)q^n + \sum_{n=0}^{\infty}\left(\sum_{k=1}^{\infty}(-1)^k p_{8d}\left(n - \frac{k(3k-1)}{2}\right)\right)q^n + \sum_{n=0}^{\infty}\left(\sum_{k=1}^{\infty}(-1)^k p_{8d}\left(n - \frac{k(3k+1)}{2}\right)\right)q^n,$$

which proves Eq (4.3). To prove Eq (4.4), consider Eq (1.6) as:

$$\sum_{n=0}^{\infty} b(4n+1)q^n = 2\left(\frac{f_2^5}{f_1^4}\right)\frac{f_2^3}{f_1^3},$$

$$= 2\left(\sum_{n=0}^{\infty} b(2n)q^n\right)\left(\sum_{k=0}^{\infty} p_{3d}(k)q^k\right),$$

$$= 2\sum_{n=0}^{\infty}\left(\sum_{c=0}^{n} b(2c)p_{3d}(n-c)\right)q^n.$$

Comparing the coefficients of q^n, we arrive at Eq (4.4). □

Proof of Theorem 4.3. Consider Eq (1.8) as:

$$\sum_{n=0}^{\infty} b(6n+2)q^n = 4\left(\frac{f_2}{f_1}\right)^{10} \cdot \frac{f_3^2}{f_6},$$

$$= 4\left(\sum_{n=0}^{\infty} p_{10d}(n)q^n\right)\left(\sum_{k=-\infty}^{\infty} (-1)^k q^{3k^2}\right),$$

$$= 4\left(\sum_{n=0}^{\infty} p_{10d}(n)q^n\right)\left(1 + 2\sum_{k=1}^{\infty}(-1)^k q^{3k^2}\right),$$

$$= 4\sum_{n=0}^{\infty} p_{10d}(n)q^n + 8\sum_{n=0}^{\infty}\left(\sum_{k=1}^{\infty}(-1)^k p_{10d}(n)q^{3k^2+n}\right),$$

$$= 4\sum_{n=0}^{\infty} p_{10d}(n)q^n + 8\sum_{n=0}^{\infty}\left(\sum_{k=1}^{\infty}(-1)^k p_{10d}(n-3k^2)\right)q^n.$$

Comparing the coefficients of q^n to obtain Eq (4.5). □

5. Conclusion and open problems

In this paper, we have provided the arithmetic properties of second order mock theta function $B(q)$, introduced by McIntosh. Some congruences are proved for the coefficients of $B(q)$ modulo specific numbers. The questions which arise from this work are:

(i) Are there exist congruences modulo higher primes for $B(q)$?
(ii) Is there exist any other technique (like modular forms) that helps to look for some more arithmetic properties of $B(q)$?
(iii) How can we explore the other second order mock theta function $A(q)$?

Acknowledgments

The first author is supported by University Grants Commission (UGC), under grant Ref No. 971/ (CSIR-UGC NET JUNE 2018) and the the second author is supported by Science and Engineering Research Research Board (SERB-MATRICS) grant MTR/2019/000123. The authors of this paper are thankful to Dr. Rupam Barman, IIT Guwahati, for his valuable insight during establishing Theorems 3.1 and 3.2. We would like to thank the referee for carefully reading our paper and offering corrections and helpful suggestions.

Conflict of interest

The authors declare there is no conflicts of interest.

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