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Research article

Periodic measures of reaction-diffusion lattice systems driven by superlinear noise

Yusen Lin*

School of Mathematics, Southwest Jiaotong University, Chengdu, 610031, China; National Engineering Laboratory of Integrated Transportation Big Data Application Technology, Chengdu, 610031, China

* **Correspondence:** Email: linyusen@my.swjtu.edu.cn.

Abstract: The periodic measures are investigated for a class of reaction-diffusion lattice systems driven by superlinear noise defined on $\mathbb{Z}^k$. The existence of periodic measures for the lattice systems is established in l^2 by Krylov-Bogolyubov's method. The idea of uniform estimates on the tails of solutions is employed to establish the tightness of a family of distribution laws of the solutions.

Keywords: superlinear noise; periodic measure; reaction-diffusion lattice system

1. Introduction

This paper deals with periodic measures of the following reaction-diffusion lattice systems driven by superlinear noise defined on the integer set $\mathbb{Z}^k$:

$$\begin{aligned} du_i(t) + \lambda(t)u_i(t)dt - \nu(t)(&u_{(i_1-1,i_2,\dots,i_k)}(t) + u_{i_1,i_2-1,\dots,i_k}(t) + \dots + u_{i_1,i_2,\dots,i_k-1}(t) \\ &- 2ku_{(i_1,i_2,\dots,i_k)}(t) + u_{(i_1+1,i_2,\dots,i_k)}(t) + u_{(i_1,i_2+1,\dots,i_k)}(t) + \dots + u_{(i_1,i_2,\dots,i_k+1)}(t))dt \\ =&f_i(t,u_i(t))dt + g_i(t)dt + \sum_{j=1}^{\infty}(h_{i,j}(t) + \delta_{i,j}\hat{\sigma}_{i,j}(t,u_i(t)))dW_j(t), \end{aligned} \tag{1.1}$$

along with initial conditions:

$$u_i(0) = u_{0,i}, \tag{1.2}$$

where $i = (i_1, i_2, \dots, i_k) \in \mathbb{Z}^k$, $\lambda(t), \nu(t)$ are continuous functions, $\lambda(t) > 0$, $(f_i)_{i\in\mathbb{Z}^k}$ and $(\hat{\sigma}_{i,j})_{i\in\mathbb{Z}^k, j\in\mathbb{N}}$ are two sequences of continuously differentiable nonlinearities with arbitrary and superlinear growth rate from $\mathbb{R}\times\mathbb{R} \to \mathbb{R}$, respectively, $g = (g_i)_{i\in\mathbb{Z}^k}$ and $h = (h_{i,j})_{i\in\mathbb{Z}^k, j\in\mathbb{N}}$ are two time-dependent random sequences, and $\delta = (\delta_{i,j})_{i\in\mathbb{Z}^k, j\in\mathbb{N}}$ is a sequence of real numbers. The sequence of independent two-sided real-valued Wiener processes $(W_j)_{j\in\mathbb{N}}$ is defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$. Furthermore, we assume that system (1.1) is a time periodic system; more precisely, there exists $T > 0$ such that the time-dependent functions $\lambda, \nu, f_i, g, h, \sigma_{i,j}(i \in \mathbb{Z}^k, j \in \mathbb{N})$ in (1.1) are all T-periodic in time.

Lattice systems are gradually becoming a large and evolving interdisciplinary research field, due to wide range of applications in physics, biology and engineering such as pattern recognition, propagation of nerve pulses, electric circuits, and so on, see [1–6] and the references therein for more details. The well-posedness and the dynamics of these equations have been studied by many authors, [7–10] for deterministic systems and [11–19] for stochastic systems where the existence of random attractors and probability measures have been examined. Especially, the authors research the limiting behavior of periodic measures of lattice systems in [15].

Nonlinear noise was proposed and studied for the first time in [19], the authors researches the long-term behavior of lattice systems driven by nonlinear noise in terms of random attractors and invariant measures. Before that, the research on noise was limited to additive noise and linear multiplicative noise, which can be transformed into a deterministic system. However, if the diffusion coefficients are nonlinear, then one cannot convert the stochastic system into a pathwise deterministic one, and thereby this problem cannot be studied under the frameworks of deterministic systems aforementioned. As an extension of [19], a class of reaction-diffusion lattice systems driven by superlinear noise, where the noise has a superlinear growth order $q \in [2, p)$, is studied by taking advantage of the dissipativeness of the nonlinear drift function f_i in (1.1) to control the superlinear noise in [20].

In the paper, we will study the existence of periodic measures of reaction-diffusion lattice systems drive by superlinear noise. One of the main tasks in our analysis is to solve the superlinear noise terms. We remark that if the noise grows linearly, then the estimates we need can be obtained by applying the standard methods available in the literature. We adopt the ideas that take advantage of the nonlinear drift terms' the polynomical growth rate p ($p \geq 2$) to control the noise polynomical rate $q \in [2, p)$. Furthermore, notice that l^2 is an infinite-dimensional phase space and problem (1.1)–(1.2) is defined on the unbounded set $\mathbb{Z}^k$. The unboundedness of $\mathbb{Z}^k$ as well as the infinite-dimensionalness of l^2 introduce a major difficulty, because of the non-compactness of usual Sobolev embeddings on unbounded domains. We will employ the dissipativeness of the drift function in (1.1) as well as a cutoff technique to prove that the tails of solutions are uniformly small in $L^2(\Omega, l^2)$. Based upon this fact we obtain the tightness of distribution laws of solutions, and then the existence of periodic measures.

In the next section, we discuss the well-poseness of solutions of (1.1) and (1.2). Section 3 is devoted to the uniform estimates of solutions including the uniform estimates on the tails of solutions. In Section 4, we show the existence of periodic measures of (1.1) and (1.2).

2. Global well-posedness of reaction-diffusion lattice systems with superlinear noise

In this section, we prove the existence and uniqueness of solutions to system (1.1) and (1.2). We first discuss the assumptions on the nonlinear drift and diffusion terms in (1.1).

We begin with the following Banach space:

$$l^r = \{u = (u_i)_{i\in\mathbb{Z}^k} : \sum_{i\in\mathbb{Z}^k} |u_i|^r < +\infty\} \text{ with norm } \|u\|_r = \Big(\sum_{i\in\mathbb{Z}^k} |u_i|^r\Big)^{\frac{1}{r}}, \forall r \geq 1.$$

The norm and inner product of l^2 are denoted by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively. For the nonlinear drift function $f_i \in C^1(\mathbb{R}\times\mathbb{R},\mathbb{R})$ in the equation we assume that for all $s \in \mathbb{R}$ and $i \in \mathbb{Z}^k$,

$$f_i(t,s)s \le -\gamma_1|s|^p + \phi_{1,i},\ \phi_1 = \{\phi_{1,i}\}_{i\in\mathbb{Z}^k} \in l^1, \tag{2.1}$$

$$|f_i(t,s)| \le \phi_{2,i}|s|^{p-1} + \phi_{3,i},\ \phi_2 = \{\phi_{2,i}\}_{i\in\mathbb{Z}^k} \in l^\infty,\ \phi_3 = \{\phi_{3,i}\}_{i\in\mathbb{Z}^k} \in l^2, \tag{2.2}$$

$$|f_i^{'}(t,s)| \le \phi_{4,i}|s|^{p-2} + \phi_{5,i},\ \phi_4 = \{\phi_{4,i}\}_{i\in\mathbb{Z}^k} \in l^\infty,\ \phi_5 = \{\phi_{5,i}\}_{i\in\mathbb{Z}^k} \in l^\infty, \tag{2.3}$$

where $p > 2$ and $\gamma_1 > 0$ are constants. For the sequence of continuously differentiable diffusion functions $\hat{\sigma} = (\hat{\sigma}_{i,j})_{i\in\mathbb{Z}^k, j\in\mathbb{N}}$, we assume, for all $s \in \mathbb{R}$ and $j \in \mathbb{N}$,

$$|\hat{\sigma}_{i,j}(t,s)| \le \varphi_{1,i}|s|^{\frac{q}{2}} + \varphi_{2,i},\ \varphi_1 = \{\varphi_{1,i}\}_{i\in\mathbb{Z}^k} \in l^{\frac{2p}{p-q}},\ \varphi_2 = \{\varphi_{2,i}\}_{i\in\mathbb{Z}^k} \in l^2, \tag{2.4}$$

$$|\hat{\sigma}_{i,j}^{'}(t,s)| \le \varphi_{3,i}|s|^{\frac{q}{2}-1} + \varphi_{4,i},\ \varphi_3 = \{\varphi_{3,i}\}_{i\in\mathbb{Z}^k} \in l^q,\ \varphi_4 = \{\varphi_{4,i}\}_{i\in\mathbb{Z}^k} \in l^\infty, \tag{2.5}$$

where $q \in [2,p)$ is a constant. For processes $g(t) = (g_i(t))_{i\in\mathbb{Z}^k}$ and $h(t) = (h_{i,j})_{i\in\mathbb{Z}^k, j\in\mathbb{N}}$ are both continuous in $t \in \mathbb{R}$, which implies that for all $t \in \mathbb{R}$,

$$\|g(t)\|^2 = \sum_{i\in\mathbb{Z}^k} |g_i(t)|^2 < \infty \text{ and } \|h(t)\|^2 = \sum_{i\in\mathbb{Z}^k}\sum_{j\in\mathbb{N}} |h_{i,j}(t)|^2 < \infty. \tag{2.6}$$

In addition, we assume $\delta = (\delta_{i,j})_{i\in\mathbb{Z}^k, j\in\mathbb{N}}$ satisfies

$$c_\delta := \sum_{j\in\mathbb{N}}\sum_{i\in\mathbb{Z}^k} |\delta_{i,j}|^2 < \infty. \tag{2.7}$$

We will investigate the periodic measures of system (1.1)–(1.2) for which we assume that all given time-dependent functions are T-periodic in $t \in \mathbb{R}$ for some $T > 0$; that is, for all $t \in \mathbb{R}$, $i \in \mathbb{Z}^k$ and $k \in \mathbb{N}$.

$$\begin{aligned} &\lambda(t+T) = \lambda(t), \qquad \nu(t+T) = \nu(t), \qquad h(t+T) = h(t),\\ &g(t+T) = g(t), \qquad f(t+T,\cdot) = f(t,\cdot), \qquad \sigma(t+T,\cdot) = \sigma(t,\cdot). \end{aligned}$$

If $m : \mathbb{R} \to \mathbb{R}$ is a continuous T-periodic function, we denote

$$\overline{m} = \max_{0\le t\le T} m(t), \qquad \underline{m} = \min_{0\le t\le T} m(t).$$

We want to reformulate problem (1.1)–(1.2) as an abstract one in l^2. Given $1 \le j \le k$, $u = (u_i)_{i\in\mathbb{Z}^k} \in l^2$ and $i = (i_1, i_2, \ldots, i_k) \in \mathbb{Z}^k$. Let us define the operators from l^2 to l^2 by

$$\begin{aligned} (B_j u)_i &= u_{(i_1,\ldots,i_j+1,\ldots,i_k)} - u_{(i_1,\ldots,i_j,\ldots,i_k)},\\ (B_j^* u)_i &= u_{(i_1,\ldots,i_j-1,\ldots,i_k)} - u_{(i_1,\ldots,i_j,\ldots,i_k)},\\ (A_j u)_i &= -u_{(i_1,\ldots,i_j+1,\ldots,i_k)} + 2u_{(i_1,\ldots,i_j,\ldots,i_k)} - u_{(i_1,\ldots,i_j-1,\ldots,i_k)}, \end{aligned}$$

and

$$\begin{aligned} (A_k u)_i =& - u_{(i_1-1,i_2,\ldots,i_k)} - u_{(i_1,i_2-1,\ldots,i_k)} - \ldots - u_{(i_1,i_2,\ldots,i_k-1)}\\ &+ 2k u_{(i_1,i_2,\ldots,i_k)} - u_{(i_1+1,i_2,\ldots,i_k)} - u_{(i_1,i_2+1,\ldots,i_k)} - \ldots - u_{(i_1,i_2,\ldots,i_k+1)}. \end{aligned}$$

For all $1 \leq j \leq k$, $u = (u_i)_{i\in\mathbb{Z}^k} \in l^2$ and $v = (v_i)_{i\in\mathbb{Z}^k} \in l^2$ we see

$$\|B_j u\| \leq 2\|u\|, (B_j^* u, v) = (u, B_j v), A_j = B_j B_j^* \text{ and } A_k = \sum_{j=1}^{k} A_j. \tag{2.8}$$

Again, define the operators $f, \sigma_j : \mathbb{R} \times l^2 \to l^2$ by

$$f(t,u) = (f_i(t,u_i))_{i\in\mathbb{Z}^k} \text{ and } \sigma_j(t,u) = (\delta_{i,j}\hat{\sigma}_{i,j}(t,u_i))_{i\in\mathbb{Z}^k}, \forall t \in \mathbb{R}, \forall u = (u_i)_{i\in\mathbb{Z}^k} \in l^2.$$

It follows from (2.3) that there exists $\theta \in (0,1)$ such that for $p > 2$ and $u, v \in l^2$,

$$\begin{aligned}
\sum_{i\in\mathbb{Z}^k} |f_i(t,u_i) - f_i(t,v_i)|^2 &= \sum_{i\in\mathbb{Z}^k} |f_i'(\theta u_i + (1-\theta)v_i)|^2 |u_i - v_i|^2 \\
&\leq \sum_{i\in\mathbb{Z}^k} (|\phi_{4,i}||\theta u_i + (1-\theta)v_i|^{p-2} + |\phi_{5,i}|)^2 |u_i - v_i|^2 \\
&\leq \sum_{i\in\mathbb{Z}^k} (2^{2p-4}|\phi_{4,i}|^2(|u_i|^{2p-4} + |v_i|^{2p-4}) + 2|\phi_{5,i}|^2)|u_i - v_i|^2 \\
&\leq (2^{2p-4}\|\phi_4\|^2_{l^\infty}(\|u\|^{2p-4} + \|v\|^{2p-4}) + 2\|\phi_5\|^2_{l^\infty})\|u - v\|^2.
\end{aligned} \tag{2.9}$$

This together with $f(t,0) \in l^2$ by (2.2) yields $f(t,u) \in l^2$ for all $u \in l^2$, and thereby $f : \mathbb{R} \times l^2 \to l^2$ is well-defined. In addition, we deduce from (2.9) that $f : \mathbb{R} \times l^2 \to l^2$ is a locally Lipschitz continuous function, that is, for every $n \in \mathbb{N}$, we can find a constant $c_1(n) > 0$ satisfying, for all $u, v \in l^2$ with $\|u\| \leq n$ and $\|v\| \leq n$,

$$\|f(u) - f(v)\| \leq c_1(n)\|u - v\|. \tag{2.10}$$

For $q \in [2, p)$ and $u \in l^2$, one can deduce from(2.4), (2.7) and Young's inequality that for all $\varpi > 0$,

$$\begin{aligned}
\varpi \sum_{j\in\mathbb{N}} \|\sigma_j(t,u)\|^2 &= \varpi \sum_{j\in\mathbb{N}} \sum_{i\in\mathbb{Z}^k} |\delta_{i,j}\hat{\sigma}_{i,j}(t,u_i)|^2 \\
&\leq 2\varpi \sum_{j\in\mathbb{N}} \sum_{i\in\mathbb{Z}^k} |\delta_{i,j}|^2(|\varphi_{1,i}|^2|u_i|^q + |\varphi_{2,i}|^2) \leq 2\varpi c_\delta \sum_{i\in\mathbb{Z}^k} (|\varphi_{1,i}|^2|u_i|^q + |\varphi_{2,i}|^2) \\
&\leq \frac{\gamma_1}{2} \sum_{i\in\mathbb{Z}^k} |u_i|^p + \frac{p-q}{p}\Big(\frac{p\gamma_1}{2q}\Big)^{-\frac{q}{p-q}} (2\varpi c_\delta)^{\frac{p}{p-q}} \sum_{i\in\mathbb{Z}^k} |\varphi_{1,i}|^{\frac{2p}{p-q}} + 2\varpi c_\delta \sum_{i\in\mathbb{Z}^k} |\varphi_{2,i}|^2 \\
&\leq \frac{\gamma_1}{2}\|u\|_p^p + \frac{p-q}{p}\Big(\frac{p\gamma_1}{2q}\Big)^{-\frac{q}{p-q}} (2\varpi c_\delta)^{\frac{p}{p-q}} \|\varphi_1\|^{\frac{2p}{p-q}}_{\frac{2p}{p-q}} + 2\varpi c_\delta \|\varphi_2\|^2,
\end{aligned} \tag{2.11}$$

where γ_1 is the same number as in (2.1). From (2.11) and $l^2 \subseteq l^p$ for $p > 2$, we find that $\sigma_j(t,u) \in l^2$ for all $u \in l^2$. Then $\sigma_j : \mathbb{R} \times l^2 \to l^2$ is also well-defined. In addition, it yields from (2.5) and (2.7) that there exists $\eta \in (0,1)$ such that for $q \in [2,p)$ and $u, v \in l^2$,

$$\begin{aligned}
\sum_{j\in\mathbb{N}}\sum_{i\in\mathbb{Z}^k}|\delta_{i,j}\hat{\sigma}_{i,j}(t,u_i)-\delta_{i,j}\hat{\sigma}_{i,j}(t,v_i)|^2 &= \sum_{i\in\mathbb{Z}^k}\sum_{j\in\mathbb{N}}|\delta_{i,j}|^2|\hat{\sigma}_{i,j}(t,u_i)-\hat{\sigma}_{i,j}(t,v_i)|^2\\
&= \sum_{i\in\mathbb{Z}^k}\sum_{j\in\mathbb{N}}|\delta_{i,j}|^2|\hat{\sigma}'_{i,j}(\eta u_i+(1-\eta)v_i)|^2|u_i-v_i|^2\\
&\le c_\delta\sum_{i\in\mathbb{Z}^k}(|\varphi_{3,i}||\eta u_i+(1-\eta)v_i|^{\frac{q}{2}-1}+|\varphi_{4,i}|)^2|u_i-v_i|^2\\
&\le c_\delta\sum_{i\in\mathbb{Z}^k}(2^{q-2}|\varphi_{3,i}|^2(|u_i|^{q-2}+|v_i|^{q-2})+2|\varphi_{4,i}|^2)|u_i-v_i|^2\\
&\le c_\delta\sum_{i\in\mathbb{Z}^k}\Big(2^{q-2}\Big(\frac{4}{q}|\varphi_{3,i}|^q+\frac{q-2}{q}|u_i|^q+\frac{q-2}{q}|v_i|^q\Big)\\
&\qquad+2|\varphi_{4,i}|^2\Big)|u_i-v_i|^2\\
&\le c_\delta(2^{q-1}(\|\varphi_3\|_q^q+\|u\|^q+\|v\|^q)+2\|\varphi_4\|_{l^\infty}^2)\|u-v\|^2.
\end{aligned} \tag{2.12}$$

This implies that $\sigma_j : \mathbb{R}\times l^2 \to l^2$ is also locally Lipschitz continuous, more precisely, for every $n \in \mathbb{N}$, one can find a constant $c_2(n) > 0$ satisfying, for all $u, v \in l^2$ with $\|u\| \le n$ and $\|v\| \le n$,

$$\sum_{j\in\mathbb{N}}\|\sigma_j(u)\|^2 \le c_2^2(n). \tag{2.13}$$

and

$$\sum_{j\in\mathbb{N}}\|\sigma_j(u)-\sigma_j(v)\|^2 \le c_2^2(n)\|u-v\|^2. \tag{2.14}$$

By above notations one is able to rewrite (1.1)–(1.2) as the following system in l^2 for $t > 0$:

$$du(t)+\nu(t)A_ku(t)dt+\lambda(t)u(t)dt = f(t,u(t))dt+g(t)dt+\sum_{j=1}^{\infty}(h_j(t)+\sigma_j(t,u(t)))dW_j(t), \tag{2.15}$$

with initial condition:

$$u(0) = u_0 \in l^2, \tag{2.16}$$

in the present article, the solutions of system (2.15)–(2.16) are interpreted in the following sense.

Definition 2.1. Suppose $u_0 \in L^2(\Omega, l^2)$ is $\mathcal{F}_0$-measurable, a continuous l^2-valued $\mathcal{F}_t$-adapted stochastic process u is called a solution of equations (2.15) and (2.16) if $u \in L^2(\Omega, C([0,T], l^2)) \cap L^p(\Omega, L^p(0,T; l^p))$ for all $T > 0$, and the following equation holds for all $t \ge 0$ and almost all $\omega \in \Omega$:

$$\begin{aligned}
u(t) =&u_0+\int_0^t(-\nu(s)A_ku(s)-\lambda(s)u(s)+f(s,u(s))+g(s))ds\\
&+\sum_{j=1}^{\infty}\int_0^t(h_j(s)+\sigma_j(s,u(s)))dW_j(s) \text{ in } l^2.
\end{aligned} \tag{2.17}$$

Similar to Ref. [20], we can get (2.15) and (2.16) exist global solutions in the sense of Definition 2.1.

3. Uniform estimates

In this section, we derive the uniform estimates of solutions of (2.15)–(2.16). These estimates will be used to establish the tightness of a set of probability distributions of u in l^2.

We assume that

$$\alpha(t) = \lambda(t) - 16k|\nu(t)| > 0. \tag{3.1}$$

Lemma 3.1. *Let* (2.1)–(2.7) *and* (3.1) *hold. Then the solutions* $u(t, 0, u_0)$ *of system* (2.15) *and* (2.16) *with initial data* u_0 *at time* 0 *satisfy, for all* $t \geq 0$,

$$\begin{aligned} &E(\|u(t, 0, u_0)\|^2) + \int_0^t e^{\underline{\alpha}(r-t)} E(\|u(r, 0, u_0)\|_p^p) dr \\ &\leq L_1\Big(E(\|u_0\|^2) + \sum_{j=1}^{\infty} \overline{\|h_j\|}^2 + \overline{\|g\|}^2 + \|\varphi_1\|_{\frac{2p}{p-q}}^{\frac{2p}{p-q}} + \|\varphi_2\|^2 + \|\phi_1\|_1\Big), \end{aligned} \tag{3.2}$$

where $L_1 > 0$ *is a positive constant which depends on* $\underline{\alpha}, p, q, \gamma, c_\delta, t$, *but indepentent of* u_0.

Proof. Applying Ito's formula to (2.15) we get

$$d(\|u(t)\|^2) + 2\nu(t) \sum_{j=1}^{k} \|B_j u(t)\|^2 dt + 2\lambda(t)\|u(t)\|^2 dt = 2(f(t, u(t)), u(t))dt$$

$$+ 2(g(t), u(t))dt + \sum_{j=1}^{\infty} \|h_j(t) + \sigma(t, u(t))\|^2 dt + 2\sum_{j=1}^{\infty} u(t)(h_j(t) + \sigma_j(t, u(t)))dW_j(t).$$

This implies

$$\begin{aligned} &\frac{d}{dt}E(\|u(t)\|^2) + 2\nu(t)\sum_{j=1}^{k} E(\|B_j u(t)\|^2) + 2\lambda(t)E(\|u(t)\|^2) \\ &\leq 2E(f(t, u(t)), u(t)) + 2E(g(t), u(t)) + 2\sum_{j=1}^{\infty} E(\|h_j(t)\|^2) + 2\sum_{j=1}^{\infty} E(\|\sigma(t, u(t))\|^2). \end{aligned} \tag{3.3}$$

For the second term on the left-hand side of (3.3), we have

$$2|\nu(t)| \sum_{j=1}^{k} E(\|B_j u(t)\|^2) \leq 8k|\nu(t)|E(\|u(t)\|^2). \tag{3.4}$$

For the first term on the right-hand side of (3.3), we get from (2.1) that

$$2E(f(t, u(t)), u(t)) \leq -2\gamma_1 E(\|u(t)\|_p^p) + 2\|\phi_1\|_1. \tag{3.5}$$

For the second term on the right-hand side of (3.3), we have

$$2E(g(t), u(t)) \leq \lambda(t)E(\|u(t)\|^2) + \frac{1}{\lambda(t)}E(\|g(t)\|^2). \tag{3.6}$$

For the last term on the right-hand side of (3.3), we infer from (2.11) with $\omega = 2$ that

$$2\sum_{j=1}^{\infty} E(\|\sigma_j(t, u(t))\|^2) \le \frac{\gamma_1}{2} E(\|u(t)\|_p^p) + \frac{p-q}{p}\Big(\frac{p\gamma_1}{2q}\Big)^{-\frac{q}{p-q}} (4c_\delta)^{\frac{p}{p-q}} \|\varphi_1\|_{\frac{2p}{p-q}}^{\frac{2p}{p-q}} + 4c_\delta \|\varphi_2\|^2. \tag{3.7}$$

By (3.3)–(3.7) we get

$$\begin{aligned} &\frac{d}{dt}E(\|u(t)\|^2) + \underline{\alpha} E(\|u(t)\|^2) + \frac{3}{2}\gamma_1 E(\|u(t)\|_p^p) \\ &\quad \le E\bigg(\sum_{j=1}^{\infty} 2\|h_j(t)\|^2 + \frac{1}{\lambda(t)}\|g(t)\|^2\bigg) + C_1, \end{aligned} \tag{3.8}$$

implies that

$$\begin{aligned} &\frac{d}{dt}E(\|u(t)\|^2) + \underline{\alpha} E(\|u(t)\|^2) + \frac{3}{2}\gamma_1 E(\|u(t)\|_p^p) \\ &\quad \le 2\sum_{j=1}^{\infty} \|\overline{h_j}\|^2 + \frac{1}{\underline{\lambda}}\|\overline{g}\|^2 + C_1, \end{aligned} \tag{3.9}$$

where $C_1 = \frac{p-q}{p}\big(\frac{p\gamma_1}{2q}\big)^{-\frac{q}{p-q}} (4c_\delta)^{\frac{p}{p-q}} \|\varphi_1\|_{\frac{2p}{p-q}}^{\frac{2p}{p-q}} + 4c_\delta\|\varphi_2\|^2 + 2\|\phi_1\|_1$. Multiplying (3.9) by $e^{\underline{\alpha} t}$ and integrating over $(0, t)$ to obtain

$$\begin{aligned} &E(\|u(t, 0, u_0)\|^2) + \frac{3}{2}\gamma_1 \int_0^t e^{\underline{\alpha}(r-t)} E(\|u(r, 0, u_0)\|_p^p) dr \\ &\quad \le e^{-\underline{\alpha} t} E(\|u_0\|^2) + C_2 \int_0^t e^{\underline{\alpha}(r-t)} dr, \end{aligned} \tag{3.10}$$

where $C_2 = 2\sum_{j=1}^{\infty} \|\overline{h_j}\|^2 + \frac{1}{\underline{\lambda}}\|\overline{g}\|^2 + C_1$. This completes the proof. □

Lemma 3.2. *Let* (2.1)–(2.7)*, and* (3.1) *be satisfied. Then for compact subset $\mathcal{K}$ of l^2, one can find a number $N_0 = N_0(\mathcal{K}) \in \mathbb{N}$ such that the solutions $u(t, 0, u_0)$ of* (2.15) *and* (2.16) *satisfy, for all $n \ge N_0$ and $t \ge 0$,*

$$E\bigg(\sum_{\|i\| \ge n} |u_i(t, 0, u_0)|^2\bigg) + \int_0^t e^{\underline{\alpha}(r-t)} E\bigg(\sum_{\|i\| \ge n} |u_i(r, 0, u_0)|^p\bigg) dr \le \varepsilon, \tag{3.11}$$

where $u_0 \in \mathcal{K}$ and $\|i\| := \max_{i \le j \le k} |i_j|$.

Proof. Define a smooth function $\xi : \mathbb{R} \to [0, 1]$ such that

$$\xi(s) = 0 \text{ for } |s| \le 1 \text{ and } \xi(s) = 1 \text{ for } |s| \ge 2. \tag{3.12}$$

Denote by

$$\xi_n = \bigg(\xi\Big(\frac{\|i\|}{n}\Big)\bigg)_{i \in \mathbb{Z}^k} \text{ and } \xi_n u = \bigg(\xi\Big(\frac{\|i\|}{n}\Big) u_i\bigg)_{i \in \mathbb{Z}^k}, \forall u = (u_i)_{i \in \mathbb{Z}^k}, n \in \mathbb{N}. \tag{3.13}$$

Similar notations will also be used for other terms. It follows from (2.15) that

$$\begin{aligned} & d(\xi_n u(t)) + \nu(t)\xi_n A_k u(t)dt + \lambda(t)\xi_n u(t)dt \\ & \qquad = \xi_n f(t,u(t))dt + \xi_n g(t)dt + \sum_{j=1}^{\infty}(\xi_n h_j(t) + \xi_n \sigma_j(t,u(t)))dW_j(t). \end{aligned} \tag{3.14}$$

By Ito's formula and (3.14) we have

$$\begin{aligned} & d\|\xi_n u(t)\|^2 + 2\nu(t)(A_k(u(t)), \xi_n^2 u(t))dt + 2\lambda(t)\|\xi_n u(t)\|^2 dt \\ & \quad = 2(f(t,u(t)), \xi_n^2 u(t))dt + 2(g(t), \xi_n^2 u(t))dt \\ & \qquad + \sum_{j=1}^{\infty} \|\xi_n h_j(t) + \xi_n \sigma_j(t,u(t))\|^2 dt + 2\sum_{j=1}^{\infty}(h_j(t) + \sigma_j(t,u(t)), \xi_n^2 u(t))dW_j. \end{aligned} \tag{3.15}$$

This yields

$$\begin{aligned} & \frac{d}{dt}E(\|\xi_n u(t)\|^2) + 2\nu(t)E(A_k(u(t)), \xi_n^2 u(t)) + 2\lambda(t)E(\|\xi_n u(t)\|^2) = 2E(f(t,u(t)), \xi_n^2 u(t)) \\ & \qquad + 2E(g(t), \xi_n^2 u(t)) + 2\sum_{j=1}^{\infty} E(\|\xi_n h_j(t)\|^2) + 2\sum_{j=1}^{\infty} E(\|\xi_n \sigma_j(t,u(t))\|^2)dt. \end{aligned} \tag{3.16}$$

For the second term on the left-hand side of (3.16), we have

$$\begin{aligned} & 2\nu(t)E(A_k(u(t)), \xi_n^2 u(t)) = 2\nu(t)\sum_{j=1}^{k} E(B_j u(t), B_j(\xi_n^2 u(t))) \\ & = 2\nu(t)E\Big(\sum_{j=1}^{k}\sum_{i\in\mathbb{Z}^k}(u_{i_1,\ldots,i_j+1,\ldots,i_k} - u_i) \\ & \qquad \times \Big(\xi^2\Big(\frac{\|(i_1,\ldots,i_j+1,\ldots,i_k)\|}{n}\Big)u_{(i_1,\ldots,i_j+1,\ldots,i_k)} - \xi^2\Big(\frac{\|i\|}{n}\Big)u_i\Big)\Big) \\ & = 2\nu(t)E\Big(\sum_{j=1}^{k}\sum_{i\in\mathbb{Z}^k}\xi^2\Big(\frac{\|i\|}{n}\Big)(u_{i_1,\ldots,i_j+1,\ldots,i_k} - u_i)^2\Big) \\ & \quad + 2\nu(t)E\Big(\sum_{j=1}^{k}\sum_{i\in\mathbb{Z}^k}\Big(\xi^2\Big(\frac{\|(i_1,\ldots,i_j+1,\ldots,i_k)\|}{n}\Big) - \xi^2\Big(\frac{\|i\|}{n}\Big)\Big) \\ & \qquad \times (u_{(i_1,\ldots,i_j+1,\ldots,i_k)} - u_i)u_{(i_1,\ldots,i_j+1,\ldots,i_k)}\Big). \end{aligned} \tag{3.17}$$

We first deal with the first term on the right-hand side of (3.17). Notice that

$$\begin{aligned}&2|\nu(t)|E\Big(\sum_{j=1}^{k}\sum_{i\in\mathbb{Z}^k}\xi^2\Big(\frac{\|i\|}{n}\Big)(u_{i_1,\dots,i_j+1,\dots,i_k}-u_i)^2\Big)\\&=2|\nu(t)|E\Big(\sum_{j=1}^{k}\sum_{i\in\mathbb{Z}^k}\Big|\xi\Big(\frac{\|i\|}{n}\Big)u_{(i_1,\dots,i_j+1,\dots,i_k)}-\xi\Big(\frac{\|i\|}{n}\Big)u_i\Big|^2\Big)\\&\le 4|\nu(t)|E\Big(\sum_{j=1}^{k}\sum_{i\in\mathbb{Z}^k}\Big|\Big(\xi\Big(\frac{\|i\|}{n}\Big)-\xi\Big(\frac{\|(i_1,\dots,i_j+1,\dots,i_k)\|}{n}\Big)\Big)u_{(i_1,\dots,i_j+1,\dots,i_k)}\Big|^2\Big)\\&\quad+4|\nu(t)|E\Big(\sum_{j=1}^{k}\sum_{i\in\mathbb{Z}^k}\Big|\xi\Big(\frac{\|(i_1,\dots,i_j+1,\dots,i_k)\|}{n}\Big)u_{(i_1,\dots,i_j+1,\dots,i_k)}-\xi\Big(\frac{\|i\|}{n}\Big)u_i\Big|^2\Big).\end{aligned}\tag{3.18}$$

By the definition of function ξ, there exists a constant $C_3>0$ such that $|\xi'(s)|\le C_3$ for all $s\in\mathbb{R}$. Then the first term on the right-hand side of (3.18) is bounded by

$$\begin{aligned}&4|\nu(t)|E\Big(\sum_{j=1}^{k}\sum_{i\in\mathbb{Z}^k}\Big|\Big(\xi\Big(\frac{\|i\|}{n}\Big)-\xi\Big(\frac{\|(i_1,\dots,i_j+1,\dots,i_k)\|}{n}\Big)\Big)u_{(i_1,\dots,i_j+1,\dots,i_k)}\Big|^2\Big)\\&=4|\nu(t)|E\Big(\sum_{j=1}^{k}\sum_{i\in\mathbb{Z}^k}\Big|\xi\Big(\frac{\|i\|}{n}\Big)-\xi\Big(\frac{\|(i_1,\dots,i_j+1,\dots,i_k)\|}{n}\Big)\Big|^2\Big|u_{(i_1,\dots,i_j+1,\dots,i_k)}\Big|^2\Big)\\&\le\frac{4C_3^2}{n^2}|\nu(t)|E\Big(\sum_{j=1}^{k}\sum_{i\in\mathbb{Z}^k}\Big|u_{(i_1,\dots,i_j+1,\dots,i_k)}\Big|^2\Big)\le\frac{4C_3^2k}{n^2}|\nu(t)|E(\|u\|^2).\end{aligned}\tag{3.19}$$

By the definition of $|B_ju|_i$, the last term on the right-hand side of (3.18) is bounded by

$$\begin{aligned}&4|\nu(t)|E\Big(\sum_{j=1}^{k}\sum_{i\in\mathbb{Z}^k}\Big|\xi\Big(\frac{\|(i_1,\dots,i_j+1,\dots,i_k)\|}{n}\Big)u_{(i_1,\dots,i_j+1,\dots,i_k)}-\xi\Big(\frac{\|i\|}{n}\Big)u_i\Big|^2\Big)\\&\le 4|\nu(t)|E\Big(\sum_{j=1}^{k}\|B_j(\xi_nu(t))\|^2\Big)\le 16k|\nu(t)|E(\|\xi_nu(t)\|^2).\end{aligned}\tag{3.20}$$

Then we find from (3.18) to (3.20) that the first term on the right-hand side of (3.17) is bounded by

$$\begin{aligned}&2|\nu(t)|E\Big(\sum_{j=1}^{k}\sum_{i\in\mathbb{Z}^k}\xi^2\Big(\frac{\|i\|}{n}\Big)(u_{(i_1,\dots,i_j+1,\dots,i_k)}-u_i)^2\Big)\\&\le 16k|\nu(t)|E(\|\xi_nu(t)\|^2)+\frac{4C_3^2k}{n^2}|\nu(t)|E(\|u\|^2).\end{aligned}\tag{3.21}$$

In addition, we find that the last term on the right-hand side of (3.17) can be bounded by

$$
\begin{aligned}
&2\Big|\nu(t)E\Big(\sum_{j=1}^{k}\sum_{i\in\mathbb{Z}^k}\Big(\xi^2\Big(\frac{\|(i_1,\ldots,i_j+1,\ldots,i_k)\|}{n}\Big)-\xi^2\Big(\frac{\|i\|}{n}\Big)\Big)\\
&\qquad\qquad\qquad\qquad\times(u_{(i_1,\ldots,i_j+1,\ldots,i_k)}-u_i)u_{(i_1,\ldots,i_j+1,\ldots,i_k)}\Big)\Big|\\
&\le 2|\nu(t)|E\Big(\sum_{j=1}^{k}\sum_{i\in\mathbb{Z}^k}\Big|\xi^2\Big(\frac{\|(i_1,\ldots,i_j+1,\ldots,i_k)\|}{n}\Big)-\xi^2\Big(\frac{\|i\|}{n}\Big)\Big|\\
&\qquad\qquad\qquad\qquad\times|u_{(i_1,\ldots,i_j+1,\ldots,i_k)}-u_i||u_{(i_1,\ldots,i_j+1,\ldots,i_k)}|\Big)\\
&\le 4|\nu(t)|E\Big(\sum_{j=1}^{k}\sum_{i\in\mathbb{Z}^k}\Big|\xi\Big(\frac{\|(i_1,\ldots,i_j+1,\ldots,i_k)\|}{n}\Big)-\xi\Big(\frac{\|i\|}{n}\Big)\Big|\\
&\qquad\qquad\qquad\qquad\times|u_{(i_1,\ldots,i_j+1,\ldots,i_k)}-u_i||u_{(i_1,\ldots,i_j+1,\ldots,i_k)}|\Big)\\
&\le \frac{4C_3}{n}|\nu(t)|E\Big(\sum_{j=1}^{k}\sum_{i\in\mathbb{Z}^k}|u_{(i_1,\ldots,i_j+1,\ldots,i_k)}-u_i||u_{(i_1,\ldots,i_j+1,\ldots,i_k)}|\Big)\\
&\le \frac{8kC_3}{n}|\nu(t)|E(\|u\|^2).
\end{aligned}
\tag{3.22}
$$

By (3.21), (3.22) and (3.17), we infer that the second term on the left-hand side of (3.16) satisfied

$$
2|\nu(t)E(A_k(u(t)),\xi_n^2u(t))|\le C_4|\nu(t)|\Big(\frac{1}{n}+\frac{1}{n^2}\Big)E(\|u\|^2)+16k|\nu(t)|E(\|\xi_nu(t)\|^2),
\tag{3.23}
$$

where $C_4=4kC_3(2+C_3)$. For the first term on the right-hand side of (3.16), we find from (2.1) that

$$
\begin{aligned}
2E(f(t,u(t)),\xi_n^2u(t))&\le -2\gamma_1E\Big(\sum_{i\in\mathbb{Z}^k}\xi^2\Big(\frac{\|i\|}{n}\Big)|u_i(t)|^p\Big)+2E\Big(\sum_{i\in\mathbb{Z}^k}\xi^2\Big(\frac{\|i\|}{n}\Big)|\phi_{1,i}|\Big)\\
&\le -2\gamma_1E\Big(\sum_{i\in\mathbb{Z}^k}\xi^2\Big(\frac{\|i\|}{n}\Big)|u_i(t)|^p\Big)+2\sum_{\|i\|\ge n}|\phi_{1,i}|.
\end{aligned}
\tag{3.24}
$$

For the second term on the right-hand side of (3.16), we infer from Young's inequality that

$$
\begin{aligned}
2E(g,\xi_n^2u(t))&\le \underline{\lambda}E(\|\xi_nu(t)\|^2)+\frac{1}{\underline{\lambda}}E\Big(\sum_{i\in\mathbb{Z}^k}\xi^2\Big(\frac{\|i\|}{n}\Big)|g_i|^2\Big)\\
&\le \underline{\lambda}E(\|\xi_nu(t)\|^2)+\frac{1}{\underline{\lambda}}\sum_{\|i\|\ge n}|g_i|^2.
\end{aligned}
\tag{3.25}
$$

For the last term on the right-hand side (3.16), we infer from (2.4) and Young's inequality that

$$\begin{aligned}
2\sum_{j=1}^{\infty} E\Big(\|\xi_n\sigma_j(t,u(t))\|^2\Big) &= 2\sum_{j=1}^{\infty} E\Big(\sum_{i\in\mathbb{Z}^k}\Big|\xi\Big(\frac{\|i\|}{n}\Big)\delta_{i,j}\hat{\sigma}_{i,j}(t,u_i(t))\Big|^2\Big)\\
&\le 4\sum_{j=1}^{\infty} E\Big(\sum_{i\in\mathbb{Z}^k}\xi^2\Big(\frac{\|i\|}{n}\Big)|\delta_{i,j}|^2\Big(|\varphi_{1,i}|^2|u_i(t)|^q+|\varphi_{2,i}|^2\Big)\Big)\\
&\le 4c_\delta E\Big(\sum_{i\in\mathbb{Z}^k}\xi^2\Big(\frac{\|i\|}{n}\Big)\Big(|\varphi_{1,i}|^2|u_i(t)|^q+|\varphi_{2,i}|^2\Big)\Big)\\
&\le \gamma_1 E\Big(\sum_{i\in\mathbb{Z}^k}\xi^2\Big(\frac{\|i\|}{n}\Big)|u_i(t)|^p\Big)+\frac{p-q}{p}\Big(\frac{p\gamma_1}{q}\Big)^{-\frac{q}{p-q}}(4c_\delta)^{\frac{p}{p-q}}\sum_{i\in\mathbb{Z}^k}\xi^2\Big(\frac{\|i\|}{n}\Big)|\varphi_{1,i}|^{\frac{2p}{p-q}}\\
&\quad+4c_\delta\sum_{i\in\mathbb{Z}^k}\xi^2\Big(\frac{\|i\|}{n}\Big)|\varphi_{2,i}|^2\\
&\le \gamma_1 E\Big(\sum_{i\in\mathbb{Z}^k}\xi^2\Big(\frac{\|i\|}{n}\Big)|u_i(t)|^p\Big)+\frac{p-q}{p}\Big(\frac{p\gamma_1}{q}\Big)^{-\frac{q}{p-q}}(4c_\delta)^{\frac{p}{p-q}}\sum_{\|i\|\ge n}|\varphi_{1,i}|^{\frac{2p}{p-q}}\\
&\quad+4c_\delta\sum_{\|i\|\ge n}|\varphi_{2,i}|^2.
\end{aligned}\tag{3.26}$$

Substituting (3.23)–(3.26) into (3.16) we get

$$\begin{aligned}
&\frac{d}{dt}E(\|\xi_n u(t)\|^2)+\underline{\alpha}E(\|\xi_n u(t)\|^2)+\gamma_1 E\Big(\sum_{i\in\mathbb{Z}^k}\xi^2\Big(\frac{\|i\|}{n}\Big)|u_i(t)|^p\Big)\\
&\le C_4|\nu|\Big(\frac{1}{n}+\frac{1}{n^2}\Big)E(\|u\|^2)+C_5\Big(\sum_{\|i\|\ge n}\Big(\overline{|g_i|}^2+|\varphi_{1,i}|^{\frac{2p}{p-q}}+|\varphi_{2,i}|^2+|\phi_{1,i}|\Big)+\sum_{\|i\|\ge n}\sum_{j=1}^{\infty}\overline{|h_{i,j}|}^2\Big),
\end{aligned}\tag{3.27}$$

where $C_5=2+\frac{1}{\underline{\lambda}}+\frac{p-q}{p}(\frac{p\gamma_1}{q})^{-\frac{q}{p-q}}(4c_\delta)^{\frac{p}{p-q}}+4c_\delta$. One can multiply (3.27) by $e^{\underline{\alpha}t}$ and integrate over $(0,t)$ in order to obtain

$$\begin{aligned}
&E(\|\xi_n u(t,0,u_0)\|^2)+\gamma_1\int_0^t e^{\underline{\alpha}(r-t)}E\Big(\sum_{i\in\mathbb{Z}^k}\xi^2\Big(\frac{\|i\|}{n}\Big)|u_i(r,0,u_0)|^p\Big)dr\\
&\le e^{-\underline{\alpha}t}E(\|\xi_n u_0\|^2)+C_4|\nu|\Big(\frac{1}{n}+\frac{1}{n^2}\Big)\int_0^t e^{\underline{\alpha}(r-t)}E(\|u(r,0,u_0)\|^2)dr\\
&\quad+\frac{C_5}{\underline{\alpha}}\Big(\sum_{\|i\|\ge n}\Big(\overline{|g_i|}^2+|\varphi_{1,i}|^{\frac{2p}{p-q}}+|\varphi_{2,i}|^2+|\phi_{1,i}|\Big)+\sum_{\|i\|\ge n}\sum_{j=1}^{\infty}\overline{|h_{i,j}|}^2\Big).
\end{aligned}\tag{3.28}$$

Since $\mathcal{K}$ is a compact subset of l^2 we infer from (3.1) that

$$\lim_{n\to\infty}\sup_{u_0\in\mathcal{K}}\sup_{t\ge 0}e^{-\underline{\alpha}t}E(\|\xi_n u_0\|^2)\le\lim_{n\to\infty}\sup_{u_0\in\mathcal{K}}E(\sum_{\|i\|\ge n}|u_{0,i}|^2)=0.\tag{3.29}$$

By Lemma 3.1, we find that for all $u_0 \in \mathcal{K}$ and $t \geq 0$, as $n \to \infty$,

$$
\begin{aligned}
&\Big(\frac{1}{n} + \frac{1}{n^2}\Big) \int_0^t e^{\underline{\alpha}(r-t)} E(\|u(r, 0, u_0)\|^2) dr \\
&\leq \frac{L_1}{\underline{\alpha}} \Big(\frac{1}{n} + \frac{1}{n^2}\Big)\Big(E(\|u_0\|^2) + \sum_{j=1}^{\infty} \overline{\|h_j\|}^2 + \overline{\|g\|}^2 + \|\varphi_1\|_{\frac{2p}{p-q}}^{\frac{2p}{p-q}} + \|\varphi_2\|^2 + \|\phi_1\|_1\Big) \\
&\leq \frac{L_1}{\underline{\alpha}} \Big(\frac{1}{n} + \frac{1}{n^2}\Big)\Big(C_6 + \sum_{j=1}^{\infty} \overline{\|h_j\|}^2 + \overline{\|g\|}^2 + \|\varphi_1\|_{\frac{2p}{p-q}}^{\frac{2p}{p-q}} + \|\varphi_2\|^2 + \|\phi_1\|_1\Big) \to 0,
\end{aligned} \tag{3.30}
$$

where L_1 is the same number of (3.1) and $C_6 > 0$ is a constant depending only on u_0. By $\varphi_1 \in l^{\frac{2p}{p-q}}, \varphi_2 \in l^2, \phi_1 \in l^1$, (2.6) and (3.1), we infer that

$$
\sum_{\|i\| \geq n} \Big(\overline{|g_i|}^2 + |\varphi_{1,i}|^{\frac{2p}{p-q}} + |\varphi_{2,i}|^2 + |\phi_{1,i}|\Big) + \sum_{\|i\| \geq n} \sum_{j=1}^{\infty} \overline{|h_{i,j}|}^2 \to 0 \text{ as } n \to \infty. \tag{3.31}
$$

It follows from (3.28) to (3.31) that as $n \to \infty$,

$$
\sup_{u_0 \in \mathcal{K}} \sup_{t \geq 0} \Big(E(\|\xi_n u(t, 0, u_0)\|^2) + \int_0^t e^{\underline{\alpha}(r-t)} E\Big(\sum_{i \in \mathbb{Z}^k} \xi^2\Big(\frac{\|i\|}{n}\Big)|u_i(r, 0, u_0)|^p\Big) dr\Big) \to 0. \tag{3.32}
$$

Then for every $\varepsilon > 0$ we can find a number $N_0 = N_0(\mathcal{K}) \in \mathbb{N}$ satisfying, for all $n \geq N_0$ and $t \geq 0$,

$$
\begin{aligned}
&\Big(E\Big(\sum_{\|i\| \geq 2n} |u_i(t, 0, u_0)|^2\Big) + \int_0^t e^{\underline{\alpha}(r-t)} E\Big(\sum_{\|i\| \geq 2n} |u_i(t, 0, u_0)|^p\Big) dr\Big) \\
&\leq \Big(E\Big(\|\xi_n u(t, 0, u_0)\|^2\Big) + \int_0^t e^{\underline{\alpha}(r-t)} E\Big(\sum_{i \in \mathbb{Z}^k} \xi^2\Big(\frac{\|i\|}{n}\Big)|u_i(t, 0, u_0)|^p\Big) dr\Big) \leq \varepsilon,
\end{aligned} \tag{3.33}
$$

uniformly for $u_0 \in \mathcal{K}$ and $t \geq 0$. This concludes the proof. □

4. Existence of periodic measures

In the sequel, we use $\mathcal{L}(u(t, 0, u_0))$ to denote the probability distribution of the solution $u(t, 0, u_0)$ of (2.15)–(2.16) which has initial condition u_0 at initial time 0. Then we have the following tightness of a family of distributions of solutions.

Lemma 4.1. *Suppose* (2.1)–(2.7) *and* (3.1) *hold. Then the family* $\{\mathcal{L}(u(t, 0, u_0)) : t \geq 0\}$ *of the distributions of the solutions of* (2.15)–(2.16) *is tight on* l^2.

Proof. For simplicity, we will write the solution $u(t, 0, u_0)$ as $u(t)$ from now on. It follows from Lemma 3.1 that there exists a constant $c_1 > 0$ such that

$$
E\Big(\|u(t)\|^2\Big) \leq c_1, \quad \text{for all} \quad t \geq 0. \tag{4.1}
$$

By Chebyshev's inequality, we get from (4.1) that for all $t \geq 0$,

$$
P\Big(\|u(t)\|^2 \geq R\Big) \leq \frac{c_1}{R^2} \to 0 \quad \text{as} \quad R \to \infty.
$$

Hence for every $\epsilon > 0$, there exists $R_1 = R_1(\epsilon) > 0$ such that for all $t \geq 0$,

$$P\left\{\|u(t)\|^2 \geq R_1\right\} \leq \frac{1}{2}\epsilon. \tag{4.2}$$

By Lemma 3.2, we infer that for each $\epsilon > 0$ and $m \in \mathbb{N}$, there exists an integer $n_m = n_m(\epsilon, m)$ such that for all $t \geq 0$,

$$E\left(\sum_{|i|>n_m} |u_i(t)|^2\right) < \frac{\epsilon}{2^{2m+2}},$$

and hence for all $t \geq 0$ and $m \in \mathbb{N}$,

$$P\left(\left\{\sum_{|i|>n_m} |u_i(r)|^2 \geq \frac{1}{2^m}\right\}\right) \leq 2^m E\left(\sum_{|i|>n_m} |u_i(r)|^2\right) < \frac{\epsilon}{2^{m+2}}. \tag{4.3}$$

It follows from (4.3) for all $t \geq 0$,

$$P\left(\bigcup_{m=1}^{\infty}\left\{\sum_{|i|>n_m} |u_i(t)|^2 \geq \frac{1}{2^m}\right\}\right) \leq \sum_{m=1}^{\infty} \frac{\epsilon}{2^{m+2}} \leq \frac{1}{4}\epsilon,$$

which shows that for all $t \geq 0$,

$$P\left(\left\{\sum_{|i|>n_m} |u_i(t)|^2 \leq \frac{1}{2^m} \text{ for all } m \in \mathbb{N}\right\}\right) > 1 - \frac{\epsilon}{2}. \tag{4.4}$$

Given $\epsilon > 0$, set

$$Y_{1,\epsilon} = \left\{v \in l^2 : \|v\| \leq R_1(\epsilon)\right\}, \tag{4.5}$$

$$Y_{2,\epsilon} = \left\{v \in l^2 : \sum_{|i|>n_m} |v_i(r)|^2 \leq \frac{1}{2^m} \text{ for all } m \in \mathbb{N}\right\}, \tag{4.6}$$

and

$$Y_\epsilon = Y_{1,\epsilon} \cap Y_{2,\epsilon}. \tag{4.7}$$

By (4.2) and (4.4) we get, for all $t \geq 0$,

$$P(\{u(t) \in Y_\epsilon\}) > 1 - \epsilon. \tag{4.8}$$

Now, we show the precompactness of $\{v : v \in Y_\epsilon\}$ in l^2. Given $\kappa > 0$, choose an integer $m_0 = m_0(\kappa) \in \mathbb{N}$ such that $2^{m_0} > \frac{8}{\kappa^2}$. Then by (4.6) we obtain

$$\sum_{|i|>n_{m_0}} |v_i|^2 \leq \frac{1}{2^{m_0}} < \frac{\kappa^2}{8}, \quad \forall v \in Y_\epsilon. \tag{4.9}$$

On the other hand, by (4.5) we see that the set $\{(v_i)_{|i|\leq m_0} : v \in Y_\epsilon\}$ is bounded in the finite-dimensional space R^{2m_0+1} and hence precompact. Consequently, $\{v : v \in Y_\epsilon\}$ has a finite open cover of balls with radius $\frac{\kappa}{2}$, which along with (4.9) implies that the set $\{v : v \in Y_\epsilon\}$ has a finite open cover of balls with radius κ in l^2 . Since $\kappa > 0$ is arbitrary, we find that the set $\{v : v \in Y_\epsilon\}$ is precompact in l^2. This completes the proof. □

If $\phi : l^2 \to \mathbb{R}$ is a bounded Borel function, then for $0 \le r \le t$ and $u_0 \in l^2$, we set

$$(p_{r,t}\phi)(u_0) = E(\phi(u(t, r, u_0)))$$

and

$$p(r, u_0; t, \Gamma) = (p_{r,t}1_\Gamma)(u_0),$$

where $\Gamma \in \mathcal{B}(l^2)$ and 1_Γ is the characteristic function of Γ. The operators $p_{s,t}$ with $0 \le s \le t$ are called the transition operators for the solutions of (2.15)–(2.16). Recall that a probability measure ν on l^2 is periodic for (2.15)–(2.16) if

$$\int_{l^2} (p_{0,t+T}\phi)(u_0)d\nu(u_0) = \int_{l^2} (p_{0,t}\phi)(u_0)d\nu(u_0), \qquad \forall t \ge 0.$$

Lemma 4.2. *[21] Let $\varrho(\psi, \omega)$ be a scalar bounded measurable random function of ψ, independent of $\mathcal{F}_s$. Let ς be an $\mathcal{F}_s$-measurable random variable. Then*

$$E\left(\varrho\left(\varsigma, \omega\right) | \mathcal{F}_s\right) = E\left(\varrho\left(\varsigma, \omega\right)\right).$$

The transition operators $\{p_{r,t}\}_{0 \le r \le t}$ have the following properties.

Lemma 4.3. *Assume that* (2.1)–(2.7) *and* (3.1) *hold. Then:*

(i) $\{p_{r,t}\}_{0 \le r \le t}$ is Feller; that is, for every bounded and continuous $\phi : l^2 \to \mathbb{R}$, the function $p_{r,t}\phi : l^2 \to \mathbb{R}$ is also bounded and continuous for all $0 \le r \le t$.

(ii) The family $\{p_{r,t}\}_{0 \le r \le t}$ is T-periodic; that is, for all $0 \le r \le t$,

$$p(r, u_0; t, \cdot) = p(r + T, u_0; t + T, \cdot), \qquad \forall u_0 \in l^2.$$

(iii) $\{u(t, 0, u_0)\}_{t \ge 0}$ is a l^2-valued Markov process.

Finally, we present our main result on the existence of periodic measures for problem (2.15)–(2.16).

Theorem 4.4. *Assume that* (2.1)–(2.7) *and* (3.1) *hold. Then problem* (2.15)–(2.16) *has a periodic measure on l^2.*

Proof. We apply Krylov-Bogolyubov's method to prove the existence of periodic measures of (2.15)–(2.16), define a probability measure μ_n by

$$\mu_n = \frac{1}{n} \sum_{l=1}^{n} p(0, 0; lT, \cdot). \tag{4.10}$$

By Lemma 4.1 we see the sequence $\{\mu_n\}_{n=1}^{\infty}$ is tight on l^2, and hence there exists a probability measure μ on l^2 such that, up to a subsequence,

$$\mu_n \to \mu, \qquad \text{as } n \to \infty. \tag{4.11}$$

By (4.10)–(4.11) and Lemma 4.3, we infer that for every $t \geq 0$ and every bounded and continuous function $\phi : l^2 \to \mathbb{R}$,

$$\begin{aligned}
&\int_{l^2} (p_{0,t}\phi)(u_0)\, d\mu(u_0) = \int_{l^2} \left(\int_{l^2} \phi(y)\, p(0, u_0; t, dy) \right) d\mu(u_0) \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{l=1}^{n} \int_{l^2} \left(\int_{l^2} \phi(y)\, p(0, u_0; t, dy) \right) p(0, 0; lT, du_0) \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{l=1}^{n} \int_{l^2} \left(\int_{l^2} \phi(y)\, p(kT, u_0; t + lT, dy) \right) p(0, 0; kT, du_0) \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{l=1}^{n} \int_{l^2} \phi(y)\, p(0, 0; t + lT, dy) \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{l=1}^{n} \int_{l^2} \phi(y)\, p(0, 0; t + lT + T, dy) \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \int_{l^2} \left(\int_{l^2} \phi(y)\, p(0, u_0; t + T, dy) \right) p(0, 0; lT, du_0) \\
&= \int_{l^2} \left(\int_{l^2} \phi(y)\, p(0, u_0; t + T, dy) \right) d\mu(u_0) \\
&= \int_{l^2} (p_{0,t+T}\phi)(u_0)\, d\mu(u_0),
\end{aligned} \tag{4.12}$$

which shows that μ is a periodic measure of (2.15)–(2.16), as desired. □

Conflict of interest

The author declares there is no conflict of interest.

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