

EXPONENTIAL STABILITY AND REGULARITY OF COMPRESSIBLE VISCOUS MICROPOLAR FLUID WITH CYLINDER SYMMETRY

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ABSTRACT. This paper is concerned with three-dimensional compressible viscous and heat-conducting micropolar fluid in the domain to the subset of R^3 bounded with two coaxial cylinders that present the solid thermoinsulated walls, being in a thermodynamical sense perfect and polytropic. We prove that the regularity and the exponential stability in H^2 .

1. Introduction. The model of micropolar fluids which respond to micro-rotational motions and spin inertia was first introduced by Eringen [16] in 1966. The mathematical theory of micropolar fluids has been developing in two directions. One explores incompressible and the other compressible flows. For more physical background, we can refer to [2], [3], [39]. In this paper we consider the compressible cylinder symmetric flow of the isotropic, viscous and heat-conducting micropolar fluid which is in the thermodynamical sense perfect and polytropic. The mathematical model of the described fluid is stated for example in the book of G. Lukaszewicz [32] and reads

$$\dot{\rho} = -\rho \nabla \cdot \mathbf{v}, \quad (1)$$

$$\rho \dot{\mathbf{v}} = \nabla \cdot \mathbf{T} + \rho \mathbf{f}, \quad (2)$$

$$\rho j_I \dot{\mathbf{w}} = \nabla \cdot \mathbf{C} + \mathbf{T}_x + \rho \mathbf{g}, \quad (3)$$

$$\rho \dot{\mathbf{E}} = -\nabla \dot{\mathbf{q}} + \mathbf{T} : \nabla \mathbf{v} + \mathbf{C} : \nabla \omega - \mathbf{T}_x \cdot \mathbf{w}, \quad (4)$$

$$\mathbf{T}_{ij} = (-p + \lambda v_{k,k}) \delta_{ij} + \mu (v_{j,i} - v_{i,j}) - 2\mu_r \varepsilon_{mij} w_m, \quad (5)$$

$$\mathbf{C} = c_0 w_{k,k} \delta_{ij} + c_d (w_{i,j} + w_{j,i}) + c_a (w_{j,i} - w_{i,j}), \quad (6)$$

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$$\mathbf{q} = -k_\theta \nabla \theta, \quad (7)$$

$$p = R\rho\theta, \quad (8)$$

$$\mathbf{E} = c_v \theta. \quad (9)$$

with notation:

- ρ – mass density
- $\mathbf{v}$ – velocity
- $\mathbf{w}$ – microrotation velocity
- E – internal energy density
- θ – absolute temperature
- $\mathbf{T}$ – stress tensor
- $\mathbf{C}$ – couple stress tensor
- $\mathbf{q}$ – heat flux density vector
- $\mathbf{f}$ – body force density
- $\mathbf{g}$ – body couple density
- p – pressure
- j_I – microinertia density (a positive constant)
- k_θ – heat conduction coefficient
- μ_r, c_0, c_a and c_d – coefficients of microviscosity
- R – specific gas constant
- c_v – specific heat (positive constant)
- $\mathbf{T}_x$ – axial vector ($(\mathbf{T}_x)_i = \varepsilon_{ijk} \cdot T_{jk}$)
- ε_{ijk} – Levi Civita symbol
- δ_{ij} – Kronecker delta.

Equations (1)-(4) are, respectively, local forms of the conservation laws for the mass, momentum, momentum moment and energy. Equations (5)-(6) are constitutive equations for the micropolar continuum. Equation (7) is the Fourier law and equations (8)-(9) present the assumptions that our fluid is perfect and polytropic. On account of the Clausius-Duhamel inequalities, they must have the following properties:

$$\mu \geq 0, \quad 3\lambda + 2\mu \geq 0, \quad \mu_r > 0, \quad (10)$$

$$c_d \geq 0, \quad 3c_0 + 2c_d \geq 0, \quad |c_d - c_a| \leq c_d + c_a. \quad (11)$$

For simplicity reasons, we assume that

$$f = g = 0.$$

In this paper, we consider the three dimensional case of (1)-(9) with the assumption of cylindrical symmetry, and we study the problem with homogeneous boundary conditions as in [13]:

$$t = 0 : (\rho, \mathbf{v}, \mathbf{w}, \theta)(r, 0) = (\rho_0(r), \mathbf{v}_0(r), \mathbf{w}_0(r), \theta_0(r)), \quad r \in G, \quad (12)$$

$$\mathbf{v}|_{\partial G} = \mathbf{0}, \quad \mathbf{w}|_{\partial G} = \mathbf{0}, \quad \frac{\partial \theta}{\partial r} \Big|_{\partial G} = 0, \quad t > 0, \quad (13)$$

where $G = \{(x_1, x_2, x_3) \in R^3, 0 < a < r < b < +\infty, x_3 \in R, r = \sqrt{x_1^2 + x_2^2}\}$ is the spatial domain of our problem and $\mathbf{v} = (v_1, v_2, v_3)$, $\mathbf{w} = (w_1, w_2, w_3)$ denote the velocity vector and microrotation velocity respectively. In the following work we give the mathematical model with cylindrical symmetry, first in the Eulerian description, which is then transformed to the Lagrangian description. The reduced

system of the three-dimensional equations in the Eulerian coordinate is now of the form [11] and [22]:

$$\frac{\partial \rho}{\partial t} + v_1 \frac{\partial \rho}{\partial r} + \rho \left(\frac{v_1}{r} + \frac{\partial v_1}{\partial r} \right) = 0, \quad (14)$$

$$\rho \left(\frac{\partial v_1}{\partial t} + v_1 \frac{\partial v_1}{\partial r} \right) = -R \frac{\partial}{\partial r} (\rho \theta) + (\lambda + 2\mu) \frac{\partial}{\partial r} \left(\frac{\partial v_1}{\partial r} + \frac{v_1}{r} \right) + \rho \frac{v_2^2}{r}, \quad (15)$$

$$\rho \left(\frac{\partial v_2}{\partial t} + v_1 \frac{\partial v_2}{\partial r} \right) = (\mu + \mu_r) \frac{\partial}{\partial r} \left(\frac{\partial v_2}{\partial r} + \frac{v_2}{r} \right) - \rho \frac{v_1 v_2}{r} - 2\mu_r \frac{\partial w_3}{\partial r}, \quad (16)$$

$$\rho \left(\frac{\partial v_3}{\partial t} + v_1 \frac{\partial v_3}{\partial r} \right) = (\mu + \mu_r) \left(\frac{\partial^2 v_3}{\partial r^2} + \frac{1}{r} \frac{\partial v_3}{\partial r} \right) + 2\mu_r \left(\frac{\partial w_2}{\partial r} + \frac{w_2}{r} \right), \quad (17)$$

$$\rho j_I \left(\frac{\partial w_1}{\partial t} + v_1 \frac{\partial w_1}{\partial r} \right) = (c_0 + 2c_d) \frac{\partial}{\partial r} \left(\frac{\partial w_1}{\partial r} + \frac{w_1}{r} \right) + \rho j_I \frac{w_2 v_2}{r} - 4\mu_r w_1, \quad (18)$$

$$\rho j_I \left(\frac{\partial w_2}{\partial t} + v_1 \frac{\partial w_2}{\partial r} \right) = (c_a + c_d) \frac{\partial}{\partial r} \left(\frac{\partial w_2}{\partial r} + \frac{w_2}{r} \right) - \rho j_I \frac{w_1 v_2}{r} - 2\mu_r \frac{\partial v_3}{\partial r} - 4\mu_r w_2, \quad (19)$$

$$\rho j_I \left(\frac{\partial w_3}{\partial t} + v_1 \frac{\partial w_3}{\partial r} \right) = (c_a + c_d) \left(\frac{\partial^2 w_3}{\partial r^2} + \frac{1}{r} \frac{\partial w_3}{\partial r} \right) + 2\mu_r \left(\frac{\partial v_2}{\partial r} + \frac{v_2}{r} \right) - 4\mu_r w_3, \quad (20)$$

$$\begin{aligned} c_V \rho \left(\frac{\partial \theta}{\partial t} + v_1 \frac{\partial \theta}{\partial r} \right) &= \kappa \left(\frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r} \frac{\partial \theta}{\partial r} \right) - R \rho \theta \left(\frac{\partial v_1}{\partial r} + \frac{v_1}{r} \right) + (\lambda + 2\mu) \left(\frac{\partial v_1}{\partial r} + \frac{v_1}{r} \right)^2 \\ &\quad - 4\mu_r \frac{v_1}{r} \frac{\partial v_1}{\partial r} + (\mu + \mu_r) \left(\frac{\partial v_2}{\partial r} + \frac{v_2}{r} \right)^2 - 4\mu_r \frac{v_2}{r} \frac{\partial v_2}{\partial r} + (\mu + \mu_r) \left(\frac{\partial v_3}{\partial r} \right)^2 \\ &\quad + (c_0 + 2c_d) \left(\frac{\partial w_1}{\partial r} + \frac{w_1}{r} \right)^2 - 4c_d \frac{w_1}{r} \frac{\partial w_1}{\partial r} + (c_d + c_a) \left(\frac{\partial w_2}{\partial r} + \frac{w_2}{r} \right)^2 \\ &\quad - 4c_d \frac{w_2}{r} \frac{\partial w_2}{\partial r} + (c_d + c_a) \left(\frac{\partial w_3}{\partial r} \right)^2 + 4\mu_r w_1^2 + 4\mu_r w_2^2 + 4\mu_r w_3^2 \\ &\quad + 4\mu_r w_2 \frac{\partial v_3}{\partial r} - 4\mu_r \left(\frac{\partial v_2}{\partial r} + \frac{v_2}{r} \right) w_3. \end{aligned} \quad (21)$$

To analyze the system and draw the desired results, it is convenient to transform the system (14)-(21) to Lagrangian coordinates. The Eulerian coordinates (r, t) are connected to the Lagrangian coordinates (ξ, t) by the relation

$$r(\xi, t) = r_0(\xi) + \int_0^t \tilde{v}_1(\xi, \tau) d\tau, \quad (22)$$

where $\tilde{v}_1(\xi, t) = v_1(r(\xi, t), t)$ and

$$r_0(\xi) = \eta^{-1}(\xi), \quad \eta(r) = \int_a^r s \rho_0(s) ds, \quad r \in G. \quad (23)$$

From (14) and it follows that

$$\frac{\partial}{\partial t} \left(\int_a^r s \rho(s, t) ds \right) = 0,$$

which implies

$$\int_a^r s \rho(s, t) ds = \int_a^{r_0} s \rho_0(s) ds = \xi. \quad (24)$$

Now, we have $\xi \in \Omega = [0, L]$, where

$$L = \int_a^b s \rho(s, t) ds = \int_a^b s \rho_0(s) ds, \quad \forall t \geq 0. \quad (25)$$

Moreover, differentiating (24) with respect to ξ yields

$$\frac{\partial r}{\partial \xi} = \frac{1}{r(\xi, t) \rho(r(\xi, t), t)}. \quad (26)$$

Let us introduce the temporary notation $\tilde{\phi}(\xi, t)$ for $\phi(r(\xi, t), t)$, we obtain

$$\frac{\partial \tilde{\phi}(\xi, t)}{\partial t} = \frac{\partial \phi(r, t)}{\partial t} + v_1 \frac{\partial \phi(r, t)}{\partial r}, \quad (27)$$

$$\frac{\partial \tilde{\phi}(\xi, t)}{\partial \xi} = \frac{\partial \phi(r, t)}{\partial r} \frac{\partial r(\xi, t)}{\partial \xi} = \frac{1}{r\rho(r, t)} \frac{\partial \phi(r, t)}{\partial r}, \quad (28)$$

$$\rho \frac{\partial}{\partial \xi} (r \tilde{\phi}(\xi, t)) = \frac{\partial \phi}{\partial r} + \frac{\phi}{r}. \quad (29)$$

Hereafter, without danger of confusion, we will write (x, t) instead of (ξ, t) and omit $\sim$. Subscripts t and x will denote the (partial) derivatives with respect to t and x , respectively, and we will use $u = \frac{1}{\rho}$ to denote the specific volume. Thus, by (27)-(29), system (14)-(21) can be rewritten using the new variables (x, t) , $x \in \Omega$, $t \geq 0$ as follows:

$$u_t = (rv_1)_x, \quad (30)$$

$$v_{1t} = r \left[(\lambda + 2\mu) \frac{(rv_1)_x}{u} - R \frac{\theta}{u} \right]_x + \frac{v_2^2}{r}, \quad (31)$$

$$v_{2t} = (\mu + \mu_r) r \left(\frac{(rv_2)_x}{u} \right)_x - \frac{v_1 v_2}{r} - 2\mu_r r w_{3x}, \quad (32)$$

$$v_{3t} = (\mu + \mu_r) r \left(\frac{(rv_3)_x}{u} \right)_x + (\mu + \mu_r) \frac{u v_3}{r^2} + 2\mu_r (r w_2)_x, \quad (33)$$

$$j_I w_{1t} = (c_0 + 2c_d) r \left(\frac{(r w_1)_x}{u} \right)_x + j_I \frac{w_2 v_2}{r} - 4\mu_r u w_1, \quad (34)$$

$$j_I w_{2t} = (c_d + c_a) r \left(\frac{(r w_2)_x}{u} \right)_x - j_I \frac{w_1 v_2}{r} - 2\mu_r r v_{3x} - 4\mu_r u w_2, \quad (35)$$

$$j_I w_{3t} = (c_d + c_a) r \left(\frac{(r w_3)_x}{u} \right)_x + (c_d + c_a) \frac{u w_3}{r^2} + 2\mu_r (r v_2)_x - 4\mu_r u w_3, \quad (36)$$

$$\begin{aligned} c_V \theta_t = & \kappa \left(r^2 \frac{\theta_x}{u} \right)_x + \frac{1}{u} [(\lambda + 2\mu)(rv_1)_x - R\theta] (rv_1)_x + (\mu + \mu_r) \frac{(rv_2)_x^2}{u} \\ & + (c_0 + 2c_d) \frac{(r w_1)_x^2}{u} + (c_d + c_a) \frac{(r w_2)_x^2}{u} + (\mu + \mu_r) \frac{r^2 v_{3x}^2}{u} \\ & + (c_d + c_a) \frac{r^2 w_{3x}^2}{u} - 2\mu(v_1^2 + v_2^2)_x - 2c_d(w_1^2 + w_2^2)_x \\ & + 4\mu_r(uw_1^2 + uw_2^2 + uw_3^2) + 4\mu_r r w_2 v_{3x} - 4\mu_r w_3 (r v_2)_x, \end{aligned} \quad (37)$$

together with the initial and boundary conditions

$$u(x, 0) = u_0(x), \quad \mathbf{v}(x, 0) = \mathbf{v}_0(x), \quad \mathbf{w}(x, 0) = \mathbf{w}_0(x), \quad \theta(x, 0) = \theta_0(x), \quad x \in \Omega, \quad (38)$$

$$\mathbf{v}(0, t) = \mathbf{v}(L, t) = \mathbf{w}(0, t) = \mathbf{w}(L, t) = \mathbf{0}, \quad \theta_x(0, t) = \theta_x(L, t) = 0, \quad t > 0. \quad (39)$$

A result of (22) and (26), we can conclude that $r(x, t)$ is determined by

$$r_t(x, t) = v_1(x, t), \quad r(x, t) r_x(x, t) = u(x, t), \quad (40)$$

$$r_0(x) = \left(a^2 + 2 \int_0^x u_0(y) dy \right)^{\frac{1}{2}}. \quad (41)$$

It is easy to see from (40) and (41) that the following is satisfied:

$$r_t(x, t) = v_1(x, t), \quad r(x, t) r_x(x, t) = u(x, t). \quad (42)$$

Let us mention some related results in this direction. When $w = 0$, it reduces to be classical Navier-Stokes equations, which provide a suitable model to motion of several important fluids, such as water, oil, air, etc., the existence and asymptotic behavior of Navier-Stokes equations has been regarded as an important problem in the fluid of dynamics, and has been receiving much attention for many researchers (see [1, 4, 19, 17, 18, 20, 28, 32, 30, 31] and references therein). Among them, Fujita and Kato [19] obtained the global well-posedness for small initial data and the local well-posedness for any initial data in $H^s(R^n)$ with $s \geq \frac{n}{2} - 1$. Kato [28] improved results have been established in $L^n(R^n)$. Recently, Lei and Lin [30] proved global well-posedness result in the space χ^{-1} . Li and Liang [29] proved large time behavior for one dimensional compressible Navier-Stokes equations in unbounded domains with large data.

For the micropolar fluids case (i.e., $w \neq 0$), compared with the classical Navier-Stokes equations, the angular velocity w in this model brings benefit and trouble. Benefit is the damping term $-vw$ can provides extra regularity of w , while the term vw^2 is bad, it increases the nonlinearity of the system. In the one dimensional case, Mujaković made a series of efforts in studying the local-in-time existence and uniqueness, the global existence and regularity of solutions to an initial-boundary value problem with both homogenous [33, 34, 35] and non-homogenous boundary conditions [36, 37, 38] respectively. Later, Huang and Nie [25] proved the exponential stability. Recently, the global attractor of this system has been established in [27]. Besides, we would also like to refer to the works in [5, 14, 15] for the 1D micropolar fluid model.

In the three dimensional case, for the spherical symmetric model of described micropolar fluid in a bounded annular domain, the local existence, uniqueness, global existence and the large time behavior and regularity of the solution has been proved in [6, 7, 8, 9, 10], and the exponential stability and regularity of the spherically symmetric solutions with large initial data has been established in [24, 23]. Recently, for the spherical symmetric model of described micropolar fluid in an exterior unbounded domain, we proved the large time behavior for spherically symmetric flow of viscous polytropic gas with large initial data in [26]. In the case of cylinder symmetry, which model described micropolar fluid in a bounded domain with two coaxial cylinders that present the solid thermoinsulated walls, Dražić and Mujaković [11] established the local existence of generalized solutions, then they proved global existence [12] and the uniqueness [13], Huang and Dražić [21, 22] studied the large time behavior of the cylindrically symmetric with small initial data, but the regularity is open. Besides, we would like to mention the work on the global wellposedness of the three-dimensional magnetohydrodynamic equations, Wang and Wang [41] obtained the global existence results for classical 3-D MHD ($\alpha = 1$). Wang and Qin [40] obtained global wellposedness and analyticity results to 3-D generalized magnetohydrodynamic equations. Later, Ye [42] obtained the global existence results for classical 3-D GMHD ($\frac{1}{2} \leq \alpha \leq 1$).

As mentioned above, the regularity and exponential stability of generalized (global) solutions in $H^2(\Omega)$ has never been studied for system (14)-(21) with boundary conditions (12) and initial conditions (13). Therefore, we shall continue the work by Huang and Dražić [22] and establish the regularity and exponential stability of solutions with small initial data.

Here we study the problem (30)-(37) on the spatial domain Ω . We introduce the space

$$H_+^1 = \{(u, \mathbf{v}, \mathbf{w}, \theta) \in H^1(0, L) \times H^1(0, L) \times H^1(0, L) \times H^1(0, L) : v(0, t) = v(L, T) = 0, w(0, t) = w(L, t) = 0\}, \quad (43)$$

$$H_+^2 = \{(u, \mathbf{v}, \mathbf{w}, \theta) \in H^2(0, L) \times H^2(0, L) \times H^2(0, L) \times H^2(0, L) : v(0, t) = v(L, T) = 0, w(0, t) = w(L, t) = 0, \theta_x(0, t) = \theta_x(L, t) = 0\}, \quad (44)$$

which becomes the metric space equipped with the metrics induced from the usual norms. In this paper we will denote by $L^{\bar{p}}$, $1 \leq \bar{p} \leq +\infty$, $W^{m, \bar{p}}$, $m \in \mathbb{N}$, $H^1 = W^{1, 2}$, $H_0^1 = W_0^{1, 2}$ denote the usual (Sobolev) spaces on $[0, 1]$. In addition, $\|\cdot\|_B$ denotes the norm in the space B , we also put $\|\cdot\| = \|\cdot\|_{L^2}$. Subscripts t and x denote the (partial) derivatives with respect to t and x , respectively. We use C_i ($i = 1, 2$) to denote the generic positive constant depending only on H^i norm of initial datum $(u_0, \mathbf{v}_0, \mathbf{w}_0, \theta_0)$, $\min_{x \in [0, L]} u_0(x)$ and $\min_{x \in [0, L]} \theta_0(x)$, but independent of variable t .

We assume that the initial data have the following properties

$$E_0 = \int_0^L \left(\frac{1}{2} |\mathbf{v}_0|^2 + \frac{j_I}{2} |\mathbf{w}_0|^2 + c_V \theta_0 \right) dx, \quad u_0, \theta_0 \geq m, \quad (45)$$

where m is a positive constant.

Now, we are in a position to state our main result.

Theorem 1.1. *Suppose that initial $(u_0, \mathbf{v}_0, \mathbf{w}_0, \theta_0) \in H_+^2$ and (45) hold, there exists a constant $\alpha_0 = \alpha_0(C_1) > 0$, such that if $E_0 \leq \alpha_0$, problem (30)-(39) has a unique generalized global solution $(u(t), \mathbf{v}(t), \mathbf{w}(t), \theta(t)) \in H_+^2$ verifying that for any $t > 0$,*

$$\begin{aligned} u - u^* &\in L^\infty([0, +\infty), H^2(\Omega)) \cap L^2([0, +\infty), H^2(\Omega)), \\ \mathbf{v} &\in L^\infty([0, +\infty), H^2(\Omega)) \cap L^2([0, +\infty), H^3(\Omega)), \\ \mathbf{w} &\in L^\infty([0, +\infty), H^2(\Omega)) \cap L^2([0, +\infty), H^3(\Omega)), \\ \theta - \theta^* &\in L^\infty([0, +\infty), H^2(\Omega)) \cap L^2([0, +\infty), H^3(\Omega)). \end{aligned} \quad (46)$$

Moreover, there exists a positive constant $\gamma_2 = \gamma_2(C_2)$ such that for any fixed $\gamma \in (0, \gamma_2]$ and for any $t > 0$, the following estimate holds

$$\|u - u^*, \mathbf{v}, \mathbf{w}, \theta - \theta^*\| \leq C_2 e^{-\gamma t} \quad (47)$$

where $u^* = \frac{1}{L} \int_0^L u_0(x) dx$, $\theta^* = \frac{1}{c_V L} \int_0^L \left(\frac{1}{2} |\mathbf{v}_0(x)|^2 + \frac{j_I}{2} |\mathbf{w}_0(x)|^2 + c_V \theta_0(x) \right) dx$, $r^* = (a^2 + 2u^*x)^{1/2}$.

2. Proof of Theorem 1.1. In this section, we shall complete the proof of Theorem 1.1. The global existence of cylindrically symmetric solutions for system (30)-(39) was proved in [22], we shall continue the work and prove the regularity and exponential of the solution. We begin with the following Lemma.

Lemma 2.1. (See [24] and [22]) *If $(u_0, \mathbf{v}_0, \mathbf{w}_0, \theta_0) \in H_+^1$ and $E_0 \leq \alpha_0$ are true, there exists a unique global weak solution $(u, \mathbf{v}, \mathbf{w}, \theta) \in H_+^1$ to the problem (30)-(39) satisfies the following estimates*

$$\begin{aligned} 0 &< a \leq r(x, t) \leq b, \quad (x, t) \in \Omega \times [0, +\infty), \\ 0 &< C_1^{-1} \leq u(x, t), \theta(x, t) \leq C_1 \quad (x, t) \in \Omega \times [0, \infty), \\ \|u - u^*\|_{H^1}^2 &+ \|v_1\|_{H^1}^2 + \|v_2\|_{H^1}^2 + \|v_3\|_{H^1}^2 + \|w_1\|_{H^1}^2 + \|w_2\|_{H^1}^2 \end{aligned}$$

$$\begin{aligned}
& + \|w_3\|_{H^1}^2 + \|\theta - \theta^*\|_{H^1}^2 + \int_0^t (\|u - u^*\|_{H^1}^2 + \|v_1\|_{H^2}^2 + \|v_2\|_{H^2}^2 + \|v_3\|_{H^2}^2 \\
& + \|w_1\|_{H^2}^2 + \|w_2\|_{H^2}^2 + \|w_3\|_{H^2}^2 + \|\theta - \theta^*\|_{H^2}^2 + \|v_{1t}\|^2 + \|v_{2t}\|^2 + \|v_{3t}\|^2 \\
& + \|w_{1t}\|^2 + \|w_{2t}\|^2 + \|w_{3t}\|^2 + \|\theta_t\|^2) ds \leq C_1.
\end{aligned} \tag{48}$$

Moreover, there exists constant $\gamma_1 = \gamma_1(C_1) > 0$, for any fixed $\gamma \in (0, \gamma_1]$ and $\forall t > 0$

$$\begin{aligned}
& e^{\gamma t} (\|u - u^*\|_{H^1}^2 + \|v_1\|_{H^1}^2 + \|v_2\|_{H^1}^2 + \|v_3\|_{H^1}^2 + \|w_1\|_{H^1}^2 + \|w_2\|_{H^1}^2 \\
& + \|w_3\|_{H^1}^2 + \|\theta - \theta^*\|_{H^1}^2) + \int_0^t e^{\gamma s} (\|u_x\|^2 + \|v_{1x}\|_{H^1}^2 + \|v_{2x}\|_{H^1}^2 + \|v_{3x}\|_{H^1}^2 \\
& + \|w_{1x}\|_{H^1}^2 + \|w_{2x}\|_{H^1}^2 + \|w_{3x}\|_{H^1}^2 + \|\theta_x\|_{H^1}^2 + \|v_{1t}\|^2 + \|v_{2t}\|^2 + \|v_{3t}\|^2 \\
& + \|w_{1t}\|^2 + \|w_{2t}\|^2 + \|w_{3t}\|^2 + \|\theta_t\|^2) ds \leq C_1,
\end{aligned} \tag{49}$$

where u^*, θ^* and r^* are the same as in Theorem 1.1.

Lemma 2.2. Under the assumptions of Theorem 1.1, the following estimates hold for any $t > 0$:

$$\|v_{1t}(t)\|^2 + \|v_{1xx}(t)\|^2 + \|v_{1x}(t)\|_{L^\infty}^2 + \int_0^t \|v_{1tx}(s)\|^2 ds \leq C_2, \tag{50}$$

$$\|v_{2t}\|^2 + \|v_{2xx}\|^2 + \|v_{2x}\|_{L^\infty}^2 + \int_0^t \|v_{2tx}\|^2 ds \leq C_2, \tag{51}$$

$$\|v_{3t}\|^2 + \|v_{3xx}\|^2 + \|v_{3x}(t)\|_{L^\infty}^2 + \int_0^t \|v_{3tx}\|^2 ds \leq C_2. \tag{52}$$

Proof. Differentiating (31) with respect to t , multiplying the resulting equation by v_{1t} in $L^2(0, L)$, using an integration by parts, we have

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|v_{1t}\|^2 &= \int_0^L \left[r((\lambda + 2\mu) \frac{(rv_1)_x}{u} - R \frac{\theta}{u})_x \right] v_{1t} dx + \int_0^L \left(\frac{v_2^2}{r} \right)_t v_{1t} dx \\
&\leq -(\lambda + 2\mu) \int_0^L \frac{r^2 v_{1tx}^2}{u} dx + C_1 \|v_{1t}\| (\|v_{1x}\| + \|u_x\| + \|v_{1x} u_x\| \\
&\quad + \|\theta_x\| + \|v_{1xx}\| + \|v_{2t}\| + \|v_2^2\|) + C_1 \|v_{1tx}\| (\|v_{1t}\| + \|v_{1x}^2\| \\
&\quad + \|\theta_{1t}\| + \|v_{1x}\| + \|v_{1x} v_1\|) \\
&\leq -C_1^{-1} \|v_{1tx}\|^2 + C_1 (\|v_{1x}\|_{H^1}^2 + \|v_{1t}\|^2 + \|v_{2t}\|^2 + \|\theta_t\|^2 \\
&\quad + \|u_x\|^2 + \|\theta_x\|^2).
\end{aligned} \tag{53}$$

Integrating (53) with respect to t over $[0, t]$ ($t > 0$), and using Lemma 2.1, there holds

$$\|v_{1t}\|^2 + \int_0^t \|v_{1tx}\|^2 ds \leq C_2. \tag{54}$$

Moreover, integrating (31) with respect to x , and using Lemma 2.1 and Young's inequality, we obtain

$$\begin{aligned}
\|v_{1xx}\| &\leq C_1 (\|u_x\| + \|\theta_x\| + \|v_{1t}\| + \|u_x\| \|v_{1x}\|_{L^\infty} + \|v_2\|^2) \\
&\leq \frac{1}{2} \|v_{1xx}\| + C_1 (\|u_x\| + \|\theta_x\| + \|v_t\| + \|v_x\|).
\end{aligned} \tag{55}$$

Now the above facts along with the Gagliardo-Nirenberg interpolation inequality yields

$$\|v_{1x}\|_{L^\infty} \leq C_1 \|v_{1x}\|^{\frac{1}{2}} \|v_{1xx}\|^{\frac{1}{2}} + C_1 \|v_{1x}\|.$$

Combined (54) and (55) to arrive at

$$\|v_{1t}\|^2 + \|v_{1xx}\|^2 + \|v_{1x}\|_{L^\infty}^2 + \int_0^t \|v_{1tx}\|^2 ds \leq C_2.$$

Similarly, differentiating (32) with respect to t , multiplying the resulting equations by v_{2t} , and then integrating by parts, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|v_{2t}\|^2 &= (\mu + \mu_r) \int_0^L \left[r \left(\frac{(rv_2)_x}{u} \right) \right]_{x_t} v_{2t} dx - \int_0^L \left(\frac{v_1 v_2}{r} \right)_t v_{2t} dx \\ &\quad - 2\mu \int_0^L (rw_{3x})_t v_{2t} \\ &\leq -(\mu + \mu_r) \int_0^L \frac{r^2 v_{2xt}^2}{u} dx + C_1 \|v_{2t}\| (\|v_{2xx}\| + \|v_{2x}\| + \|u_x\| \\ &\quad + \|v_{2x} v_{1x}\| + \|v_{2t}\| + \|w_{3t}\|) + C_1 \|v_{2xt}\| (\|v_{2t}\| + \|v_{2x} v_{1x}\| \\ &\quad + \|v_{2x} v_1\| + \|v_{1x} v_2\| + \|w_{3t}\|) \\ &\leq -C_1^{-1} \|v_{2tx}\|^2 + C_1 (\|v_{2x}\|_{H^1}^2 + \|v_{2t}\|^2 + \|u_x\|^2 + \|w_{3t}\|^2). \end{aligned} \quad (56)$$

We integrate (56) with respect to t , and use Lemma 2.1 to have

$$\|v_{2t}\|^2 + \int_0^t \|v_{2tx}\|^2 ds \leq C_2. \quad (57)$$

Furthermore, integrating (32) with respect to x , and using the same way, we obtain

$$\begin{aligned} \|v_{2xx}\| &\leq C_1 (\|u_x\| + \|v_{2t}\| + \|w_{3x}\| + \|u_x\| \|v_{2x}\|_{L^\infty}) \\ &\leq \frac{1}{2} \|v_{2xx}\| + C_1 (\|u_x\| + \|v_{2t}\| + \|v_{2x}\| + \|w_{3x}\|). \end{aligned} \quad (58)$$

We use the Gagliardo-Nirenberg interpolation inequality to give

$$\|v_{2x}\|_{L^\infty} \leq C_1 \|v_{2x}\|^{\frac{1}{2}} \|v_{2xx}\|^{\frac{1}{2}} + C_1 \|v_{2x}\|.$$

Combining with (57)-(58), we arrive at

$$\|v_{2t}\|^2 + \|v_{2xx}\|^2 + \|v_{2x}\|_{L^\infty}^2 + \int_0^t \|v_{2tx}\|^2 ds \leq C_2.$$

Likewise, differentiating (33) with respect to t , multiplying by v_{3t} and integrating by parts, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|v_{3t}\|^2 &= (\mu + \mu_r) \int_0^L \left[r \left(\frac{(rv_3)_x}{u} \right) \right]_{x_t} v_{3t} dx + (\mu + \mu_r) \int_0^L \frac{uv_3}{r^2} v_{3t} dx \\ &\leq -(\mu + \mu_r) \int_0^L \frac{r^2 v_{3xt}^2}{u} dx + C_1 \|v_{3t}\| (\|v_{3x}\| + \|u_x\| + \|v_{3x} u_x\| \\ &\quad + \|v_{3xx}\|) + C_1 \|v_{3tx}\| (\|v_{3t}\| + \|v_{3x}\| + \|w_2\| + \|w_{2t}\|) \\ &\leq -C_1 \|v_{3tx}\|^2 + C_1 (\|v_{3t}\|_{H^1}^2 + \|v_{3t}\|^2 + \|u_x\|^2 + \|w_{2t}\|^2). \end{aligned} \quad (59)$$

Integrating (59) with respect to t over $[0, t]$ ($t > 0$), and using Lemma 2.1 to have

$$\|v_{3t}\|^2 + \int_0^t \|v_{3tx}\|^2 ds \leq C_2. \quad (60)$$

Similarly argument, we integrate (33) with respect to x , and apply Young's inequality and Lemma 2.1 to yield

$$\begin{aligned} \|v_{3xx}\| &\leq C_1(\|v_{3t}\| + \|u_x\| + \|u_x\|\|v_{3x}\|_{L^\infty}) \\ &\leq \frac{1}{2}\|v_{3xx}\| + C_1(\|u_x\| + \|v_{3t}\| + \|v_{3x}\|). \end{aligned} \quad (61)$$

Here, we have from the Gagliardo-Nirenberg interpolation inequality that

$$\|v_{3x}\|_{L^\infty} \leq C_1\|v_{3x}\|^{\frac{1}{2}}\|v_{3xx}\|^{\frac{1}{2}} + C_1\|v_{3x}\|,$$

which, combined with (60)-(61), gives

$$\|v_{3t}\|^2 + \|v_{3xx}\|^2 + \|v_{3x}\|_{L^\infty}^2 + \int_0^t \|v_{3tx}\|^2 ds \leq C_2.$$

Thus, we complete the proof. $\square$

Lemma 2.3. *The following estimates hold true for any $t > 0$*

$$\|w_{1t}\|^2 + \|w_{1xx}\|^2 + \|w_{1x}\|_{L^\infty}^2 + \int_0^t \|w_{1tx}\|^2 ds \leq C_2, \quad (62)$$

$$\|w_{2t}\|^2 + \|w_{2xx}\|^2 + \|w_{2x}\|_{L^\infty}^2 + \int_0^t \|w_{2tx}\|^2 ds \leq C_2, \quad (63)$$

$$\|w_{3t}\|^2 + \|w_{3xx}\|^2 + \|w_{3x}\|_{L^\infty}^2 + \int_0^t \|w_{3tx}\|^2 ds \leq C_2. \quad (64)$$

Proof. Differentiating (34) with respect to t , multiplying the resulting identity by w_{1t} and integrating by parts, we obtain

$$\begin{aligned} &\frac{1}{2}j_I \frac{d}{dt} \|w_{1t}\|^2 \\ &= (c_0 + 2c_d) \int_0^L \left[r \left(\frac{(rw_1)_x}{u} \right) \right]_t w_{1t} dx + j_I \int_0^L \left(\frac{w_2 v_2}{r} \right)_t w_{1t} dx \\ &\quad - 4\mu_r \int_0^L (uw_1)_t w_{1t} dx \\ &\leq -(c_0 + 2c_d) \int_0^L \frac{r^2 w_{1tx}^2}{u} dx + C_1 \|w_{1t}\| (\|w_{1x}\| + \|w_{1xx}\| + \|w_{1x} u_x\| \\ &\quad + \|u_x\| + \|w_{1x} v_{1x}\| + \|w_{2t}\| + \|v_{2t}\|) + c_1 \|w_{1tx}\| (\|w_{1t}\| + \|w_{1x} v_{1x}\| \\ &\quad + \|w_{1x}\| + \|w_1 v_{1x}\|) \\ &\leq -C_1^{-1} \|w_{1tx}\|^2 + C_1 (\|w_{1x}\|_{H^1}^2 + \|w_{1t}\|^2 + \|u_x\|^2 + \|w_{2t}\|^2 + \|v_{2t}\|^2 \\ &\quad + \|v_{1x}\|^2). \end{aligned} \quad (65)$$

Integrating (65) with respect to t over $[0, t]$ ($t > 0$) and integrating (34) with respect to x , and using Lemma 2.1, respectively, we arrive at

$$\|w_{1t}\|^2 + \int_0^t \|w_{1tx}\|^2 ds \leq C_2, \quad (66)$$

$$\begin{aligned} \|w_{1xx}\| &\leq C_1(\|w_{1t}\| + \|u\| + \|w_{1x}\|_{L^\infty} \|u_x\| + \|w_2 v_2\| + \|w_1 u\|) \\ &\leq \frac{1}{2} \|w_{1xx}\| + C_1(\|w_{1t}\| + \|w_{1x}\| + \|u_x\|). \end{aligned} \quad (67)$$

We can use the Gagliardo-Nirenberg interpolation inequality to yield

$$\|w_{1x}\|_{L^\infty} \leq C_1 \|w_{1x}\|^{\frac{1}{2}} \|w_{1xx}\|^{\frac{1}{2}} + C_1 \|w_{1x}\|,$$

which together with (66)-(67) further implies that

$$\|w_{1t}\|^2 + \|w_{1xx}\|^2 + \|w_{1x}\|_{L^\infty}^2 + \int_0^t \|w_{1tx}\|^2 ds \leq C_2.$$

Similarly, differentiating (35) with respect to t , multiplying by w_{2t} and applying integration by parts to find

$$\begin{aligned} \frac{1}{2} j_I \frac{d}{dt} \|w_{2t}\|^2 &= (c_a + c_d) \int_0^L \left[r \left(\frac{(rw_2)_x}{u} \right) \right]_{x|_t} w_{2t} dx - j_I \int_0^L \left(\frac{w_1 v_2}{r} \right)_t w_{2t} dx \\ &\quad - 2\mu_r \int_0^L (rv_{3x})_t w_{2t} dx - 4\mu_r \int_0^L (uw_2)_t w_{2t} dx \\ &\leq -(c_a + c_d) \int_0^L \frac{r^2 w_{2xt}}{u} dx + C_1 \|w_{2t}\| (\|w_{2x}\| + \|w_{2xx}\| + \|w_{2u_x}\| \\ &\quad + \|w_{2x} u_x\| + \|w_{2v_{1x}}\| + \|w_{2x} v_{1x}\| + \|w_{2v_{1x}}\| + \|v_{3x}\| + \|v_{3xt}\|) \\ &\quad + C_1 \|w_{2tx}\| (\|w_{2t}\| + \|w_{2v_{1x}}\| + \|w_{2x} v_1\| + \|w_{1t} v_2\| + \|w_{1v_{2t}}\|) \\ &\leq -C_1^{-1} \|w_{2tx}\|^2 + C_1 (\|w_{2x}\|_{H^1}^2 + \|u_x\|^2 + \|v_{1x}\|^2 + \|w_{2x}\|^2 \\ &\quad + \|v_{3x}\|^2 + \|v_{3tx}\|^2 + \|w_{2t}\|^2 + \|w_{1t}\|^2 + \|v_{2t}\|^2). \end{aligned} \quad (68)$$

Integrating (68) with respect to t over $[0, t]$ ($t > 0$), and using Lemma 2.1, we arrive at

$$\|w_{2t}\|^2 + \int_0^t \|w_{2tx}\|^2 ds \leq C_2. \quad (69)$$

We integrate (35) with respect to x , and apply Lemma 2.1 and Young's inequality to obtain

$$\begin{aligned} \|w_{2xx}\| &\leq C_1 (\|w_{2t}\| + \|w_2\| + \|w_{2u_x}\| + \|w_{2x} u_x\| + \|w_1 v_2\| \\ &\quad + \|v_{3x}\| + \|w_2 u\|) \\ &\leq C_1 (\|w_{2t}\| + \|u_x\| + \|w_{2x}\|_{L^\infty} \|u_x\| + \|v_{3x}\| \\ &\quad + \|w_1\| + \|v_2\| + \|u\| + \|w_2\|) \\ &\leq \frac{1}{2} \|w_{2xx}\| + C_1 (\|u_x\| + \|w_{2t}\| + \|w_{2x}\| + \|v_{3x}\|). \end{aligned} \quad (70)$$

Similar argument, we deduce from the Gagliardo-Nirenberg interpolation inequality that

$$\|w_{2x}\|_{L^\infty} \leq C_1 \|w_{2x}\|^{\frac{1}{2}} \|w_{2xx}\|^{\frac{1}{2}} + C_1 \|w_{2x}\|.$$

Combining with (69)-(70), we have

$$\|w_{2t}\|^2 + \|w_{2xx}\|^2 + \|w_{2x}\|_{L^\infty}^2 + \int_0^t \|w_{2tx}\|^2 ds \leq C_2.$$

Differentiating (36) with respect to t , multiplying the resulting equation by w_{3t} and using integrating by parts, we obtain

$$\begin{aligned} &\frac{1}{2} j_I \frac{d}{dt} \|w_{3t}\|^2 \\ &= (c_a + c_d) \int_0^L \left[r \left(\frac{(rw_3)_x}{u} \right) \right]_{x|_t} w_{3t} dx + (c_a + c_d) \int_0^L \left(\frac{uw_3}{r^2} \right)_t w_{3t} dx \\ &\quad + 2\mu_r \int_0^L [(rv_2)_x]_t w_{3t} dx - 4\mu_r \int_0^L (uw_3)_t w_{3t} dx \end{aligned}$$

$$\begin{aligned}
&\leq -(c_a + c_d) \int_0^L \frac{r^2 w_{3tx}^2}{u} dx + C_1 \|w_{3t}\| (\|w_{3xx}\| + \|w_{3x} u_x\| + \|u_x\| + \|v_{1x}\| \\
&\quad + \|w_{3t}\| + \|w_{3x} v_{1x}\| + \|v_{2t}\| + \|v_{2x}\| + \|v_{2xt}\|) + C_1 \|w_{3tx}\| (\|w_{3v_{1x}}\| \\
&\quad + \|w_{3t}\| + \|w_{3x}\| + \|w_{3v_{1x}}\|) \\
&\leq -C_1^{-1} \|w_{3tx}\|^2 + C_1 (\|w_{3x}\|_{H^1}^2 + \|u_x\|^2 + \|w_{3t}\|^2 + \|v_{1x}\|^2 + \|v_{2t}\|^2 \\
&\quad + \|v_{2x}\|^2 + \|v_{2tx}\|^2). \tag{71}
\end{aligned}$$

Integrating (71) with respect to t over $[0, t]$ ($t > 0$), we can obtain

$$\|w_{3t}\|^2 + \int_0^t \|w_{3tx}\|^2 ds \leq C_2. \tag{72}$$

Next, integrating (36) with respect to x , and then applying the same way to yield

$$\begin{aligned}
\|w_{3xx}\| &\leq C_1 (\|w_{3t}\| + \|u_x\| + \|w_{3x}\|_{L^\infty} \|u_x(t)\|) \\
&\leq \frac{1}{2} \|w_{3xx}\| + C_1 (\|w_{3t}\| + \|u_x\| + \|w_{3x}\|), \tag{73}
\end{aligned}$$

where we have used the following simple Gagliardo-Nirenberg interpolation inequality

$$\|w_{3x}\|_{L^\infty} \leq C_1 \|w_{3x}\|^{\frac{1}{2}} \|w_{2xx}\|^{\frac{1}{2}} + C_1 \|w_{3x}\|.$$

By (72) and (73), we can get

$$\|w_{3t}\|^2 + \|w_{3xx}\|^2 + \|w_{3x}\|_{L^\infty}^2 + \int_0^t \|w_{3tx}\|^2 ds \leq C_2.$$

The proof is complete. $\square$

Lemma 2.4. *Under the assumptions of Theorem 1.1, the following estimate holds for any $t > 0$:*

$$\|\theta_t\|^2 + \|\theta_{xx}\|^2 + \|\theta_x\|_{L^\infty}^2 + \int_0^t \|\theta_{tx}\|^2 ds \leq C_2, \tag{74}$$

Proof. Differentiating (37) with respect to t , multiplying by θ_t , and using integration by parts, we get

$$\begin{aligned}
\frac{1}{2} c_v \frac{d}{dt} \|\theta_t\|^2 &= \kappa \int_0^L \left(r^2 \frac{\theta_x}{u} \right)_{xt} \theta_t dx + \int_0^L \left[\frac{1}{u} [(\lambda + 2\mu)(rv_1)_x - R\theta](rv_1)_x \right]_t \theta_t dx \\
&\quad + (\mu + \mu_r) \int_0^L \left(\frac{(rv_2)_x^2}{u} \right)_t \theta_t dx + (c_0 + 2c_d) \int_0^L \left(\frac{(rw_1)_x^2}{u} \right)_t \theta_t dx \\
&\quad + (c_a + c_d) \int_0^L \left(\frac{(rw_2)_x^2}{u} \right)_t \theta_t dx + (\mu + \mu_r) \int_0^L \left(\frac{r^2 v_{3x}^2}{u} \right)_t \theta_t dx \\
&\quad + (c_a + c_d) \int_0^L \left(\frac{r^2 w_{3x}^2}{u} \right)_t \theta_t dx - 2\mu_r \int_0^L (v_1^2 + v_2^2)_{xt} \theta_t dx \\
&\quad - 2c_d \int_0^L (w_1^2 + w_2^2)_{xt} \theta_t dx + 4\mu_r \int_0^L (uw_1^2 + uw_2^2 + uw_3^2)_t \theta_t dx \\
&\quad + 4\mu_r \int_0^L (rw_2 v_{3x})_t \theta_t dx - 4\mu_r \int_0^L (w_3(rv_2)_x)_t \theta_t dx
\end{aligned}$$

$$\begin{aligned}
\leq & -C_1^{-1}\|\theta_{xt}\|^2 + C_1(\|\theta_x\|^2 + \|v_{1x}\|_{H^1}^2 + \|v_{1xt}\|^2 + \|v_{2x}\|_{H^1}^2 + \|v_{2xt}\|^2 \\
& + \|v_{3x}\|_{H^1}^2 + \|v_{3xt}\|^2 + \|w_{1x}\|_{H^1}^2 + \|w_{1xt}\|^2 + \|w_{2x}\|_{H^1}^2 + \|w_{2xt}\|^2 \\
& + \|w_{3x}\|_{H^1}^2 + \|w_{3xt}\|^2 + \|u_x\|^2 + \|v_{1t}\|^2 + \|v_{2t}\|^2 + \|v_{3t}\|^2 + \|w_{1t}\|^2 \\
& + \|w_{2t}\|^2 + \|w_{3t}\|^2 + \|\theta_t\|^2),
\end{aligned}$$

which, together with (50)-(52), (62)-(64) and Lemma 2.1, gives

$$\|\theta_t\|^2 + \|\theta_{xx}\|^2 + \int_0^t \|\theta_{tx}\|^2 ds \leq C_2. \quad (75)$$

By virtue of the Gagliardo-Nirenberg interpolation inequality and (75), we can obtain (74). The proof is complete. $\square$

Lemma 2.5. *Under the assumptions of Theorem 1.1, the following estimates hold for any $t > 0$:*

$$\|u_{xx}\|^2 + \int_0^t \|u_{xx}\|^2 ds \leq C_2, \quad (76)$$

$$\begin{aligned}
& \int_0^t (\|v_{1xxx}\|^2 + \|v_{2xxx}\|^2 + \|v_{3xxx}\|^2 + \|w_{1xxx}\|^2 + \|w_{2xxx}\|^2 \\
& + \|w_{3xxx}\|^2 + \|\theta_{xxx}\|^2) ds \leq C_2.
\end{aligned} \quad (77)$$

Proof. Differentiating (31) with respect to x , we have

$$\begin{aligned}
v_{1tx} &= \left[r \left((\lambda + 2\mu) \frac{(rv_1)_x}{u} - R \frac{\theta}{u} \right) \right]_x + \left(\frac{v_2^2}{r} \right)_x \\
&= \frac{uv_{1t}}{r^2} - \frac{uv_2^2}{r^2} + (\lambda + 2\mu) \frac{r(rv_1)_{xxx}}{u} - (\lambda + 2\mu) \frac{r(rv_1)_{xx}u_x}{u^2} \\
&\quad - (\lambda + 2\mu) \frac{r(rv_1)_{xx}u_x}{u^2} - (\lambda + 2\mu) \frac{r(rv_1)_x u_{xx}}{u^2} + 2(\lambda + 2\mu) \frac{r(rv_1)_x u_x^2}{u^3} \\
&\quad - rR(\lambda + 2\mu) \frac{\theta_{xx}}{u} + (\lambda + 2\mu)rR \frac{\theta_x u_x}{u^2} + (\lambda + 2\mu)rR \frac{\theta_x u_x}{u^2} \\
&\quad + (\lambda + 2\mu)rR \frac{\theta u_{xx}}{u^2} - 2(\lambda + 2\mu)rR \frac{\theta u_x^2}{u^3} + \frac{2v_2 v_{2x}}{r} - \frac{v_2^2 u}{r^3}.
\end{aligned}$$

By virtue of (30) ($u_{txx} = (rv_1)_{xxx}$), we can rewrite the above equation as

$$(\lambda + 2\mu) \frac{\partial}{\partial t} \left(\frac{u_{xx}}{u} \right) + (\lambda + 2\mu) R \frac{\theta u_{xx}}{u^2} = r^{-1} v_{1tx} + E(x, t), \quad (78)$$

where

$$\begin{aligned}
E(x, t) &= -r^{-3} uv_{1t} + r^{-3} uv_2^2 + 2(\lambda + 2\mu) \frac{(rv_1)_{xx}u_x}{u^2} \\
&\quad - 2(\lambda + 2\mu) \frac{(rv_1)_x u_x^2}{u^3} + R(\lambda + 2\mu) \frac{\theta_{xx}}{u} - 2(\lambda + 2\mu) R \frac{\theta_x u_x}{u^2} \\
&\quad + 2(\lambda + 2\mu) R \frac{\theta u_x^2}{u^3} - 2r^{-2} v_2 v_{2x} + r^{-4} v_2^2 u.
\end{aligned}$$

Multiplying (78) by $\frac{u_{xx}}{u}$, and using Young's inequality, by virtue of (50)-(52), (62)-(64), (74) and Lemma 2.1, we can obtain

$$\frac{d}{dt} \left\| \frac{u_{xx}}{u} \right\|^2 \leq -C_1^{-1} \left\| \frac{u_{xx}}{u} \right\|^2 + C_1 \|E(x, t)\|^2, \quad (79)$$

where

$$\begin{aligned} \|E(x, t)\|^2 &\leq \frac{1}{2C_1} \left\| \frac{u_{xx}}{u} \right\|^2 + C_1 (\|v_{1t}\|^2 + \|v_{1x}\|_{H^1}^2 + \|u_x\|^2 + \|\theta_x\|_{H^1}^2 \\ &\quad + \|v_{2x}\|^2). \end{aligned} \quad (80)$$

Integrating (79) with respect to t , together with (50)-(52), (62)-(64), (74) and Lemma 2.1 that

$$\|u_x\|^2 + \int_0^L \|u_{xx}\|^2 ds \leq C_2.$$

On the other hand, differentiating (31) with respect to x , we have from Lemma 2.1 and (50)-(52), (62)-(64), (74), (76) and the Gagliaro-Nirenberg inequality that

$$\begin{aligned} v_{1tx} &= \frac{uv_{1t}}{r^2} - \frac{uv_2^2}{r^2} + (\lambda + 2\mu)r \frac{(rv_1)_{xxx}}{u} - 2(\lambda + 2\mu)r \frac{(rv_1)_{xx}u_x}{u^2} \\ &\quad - (\lambda + 2\mu)r \frac{(rv_1)_x u_{xx}}{u^2} + 2(\lambda + 2\mu)r \frac{(rv_1)_x u_x^2}{u^3} - rR(\lambda + 2) \frac{\theta_{xx}}{u} \\ &\quad + 2(\lambda + 2\mu)rR \frac{\theta_x u_x}{u^2} + (\lambda + 2\mu)rR \frac{\theta u_{xx}}{u^2} - 2(\lambda + 2\mu)rR \frac{\theta u_x^2}{u^3} \\ &\quad + \frac{2v_2 v_{2x}}{r} - \frac{v_2^2 u}{r^3}, \end{aligned}$$

which implies

$$\|v_{1tx}\| \leq C_1 (\|v_{1x}\|_{H^2} + \|u_x\|_{H^1} + \|\theta_x\|_{H^1} + \|v_{2x}\|),$$

or

$$\|v_{1xxx}\| \leq C_1 (\|v_{1x}\|_{H^1} + \|v_{1tx}\| + \|u_x\|_{H^1} + \|\theta_x\|_{H^1} + \|v_{2x}\|).$$

Differentiating (32) with respect to x , using Lemma 2.1, (50)-(52), (62)-(64), (74), (76) and the Gagliaro-Nirenberg inequality to find

$$\begin{aligned} v_{2tx} &= (\mu + \mu_r) \frac{r_x (rv_2)_{xx}}{u} - (\mu + \mu_r) \frac{r_x (rv_2)_x u_x}{u^2} + (\mu + \mu_r) \frac{r (rv_2)_{xxx}}{u^3} \\ &\quad - (\mu + \mu_r) \frac{r (rv_2)_{xx} u_x}{u^2} - (\mu + \mu_r) \frac{r (rv_2)_x u_x^2}{u^2} - (\mu + \mu_r) \frac{r (rv_2)_x u_x u_x}{u^2} \\ &\quad + 2(\mu + \mu_r) \frac{r (rv_2)_x u_x^2}{u^3} - \frac{(v_{2x} v_1 + v_2 v_{1x}) r - v_2 v_1 r_x}{r^2} \\ &\quad - 2\mu_r (r_x w_{3x} + r w_{3xx}), \end{aligned}$$

thus

$$\|v_{2tx}\| \leq C_1 (\|v_{2x}\|_{H^2} + \|u_x\|_{H^1} + \|v_{1x}\| + \|w_{3x}\|_{H^1}),$$

or

$$\|v_{2xxx}\| \leq C_1 (\|v_{2x}\|_{H^1} + \|v_{2tx}\| + \|u_x\|_{H^1} + \|w_{3x}\|_{H^1} + \|v_{1x}\|).$$

Similarly, differentiating (33) with respect to x , and applying Lemma 2.1, (50)-(52), (62)-(64), (74), (76) and the Gagliaro-Nirenberg inequality to deduce

$$\begin{aligned} v_{3tx} &= (\mu + \mu_r) \frac{r_x (rv_3)_{xx}}{u} - (\mu + \mu_r) \frac{r_x (rv_3)_x u_x}{u^2} + (\mu + \mu_r) \frac{r (rv_3)_{xxx} u - (rv_3)_{xx} u_x}{u^2} \\ &\quad - (\mu + \mu_r) r \frac{[(rv_3)_{xx} u_x + (rv_3)_x u_{xx}] u^2 - 2(rv_3)_x u_x^2 u}{u^4} + (\mu + \mu_r) \frac{u_x v_3 + u v_{3x}}{r^2} \\ &\quad - (\mu + \mu_r) \frac{2u v_3 r_x}{r^3} + 2\mu_r (r_{xx} w_2 + r_x w_{2x} + r_x w_{2x} + r w_{2xx}), \end{aligned}$$

thus

$$\|v_{3tx}\| \leq C_1(\|v_{3x}\|_{H^2} + \|u_x\|_{H^1} + \|w_{2x}\|_{H^1}),$$

or

$$\|v_{3xxx}\| \leq C_1(\|v_{3x}\|_{H^1} + \|v_{3tx}\| + \|u_x\|_{H^1} + \|w_{2x}\|_{H^1}).$$

We can differentiate (34) with respect to x , combine Lemma 2.1 and the Gagliaro-Nirenberg inequality to get

$$\begin{aligned} j_I w_{1tx} &= (c_0 + 2c_d)r_x \frac{(rw_1)_{xx}}{u} - (c_0 + 2c_d)r_x \frac{(rw_1)_x u_x}{u^2} \\ &\quad + (c_0 + 2c_d)r \frac{(rw_1)_{xxx}u - (rw_1)_{xx}u_x}{u^2} \\ &\quad - (c_0 + 2c_d)r \frac{[(rw_1)_{xx}u_x + (rw_1)_x u_{xx}]u^2 - 2(rw_1)_x u_x^2 u}{u^4} \\ &\quad + j_I \frac{w_{2x}v_2 + w_2v_{2x}}{r} - j_I \frac{w_2v_2r_x}{r^2} - 4\mu_r(u_x w_1 + u w_{1x}). \end{aligned}$$

Then

$$\|w_{1tx}\| \leq C_1(\|w_{1x}\|_{H^2} + \|u_x\|_{H^1} + \|v_{2x}\| + \|w_{2x}\|),$$

or

$$\|w_{1xxx}\| \leq C_1(\|w_{1x}\|_{H^1} + \|w_{1tx}\| + \|u_x\|_{H^1} \|w_{2x}\| + \|v_{2x}\|).$$

Differentiating (35) with respect to x , using Lemma 2.1, (50)-(52), (62)-(64), (74), (76) and the Gagliaro-Nirenberg inequality to yield

$$\begin{aligned} j_I w_{2tx} &= (c_a + c_d)r_x \frac{(rw_2)_{xx}}{u} - (c_a + c_d)r_x \frac{(rw_2)_x u_x}{u^2} \\ &\quad + (c_a + c_d)r \frac{(rw_2)_{xxx}u - (rw_2)_{xx}u_x}{u^2} - j_I \frac{w_{1x}v_2 + w_1v_{2x}}{r} \\ &\quad - (c_a + c_d)r \frac{[(rw_2)_{xx}u_x + (rw_2)_x u_{xx}]u^2 - 2(rw_2)_x u_x^2 u}{u^4} \\ &\quad + j_I \frac{w_1v_2r_x}{r^2} - 2\mu_r(r_x w_{3xx} + r v_{3xx}) - 4\mu_r(u_x w_2 + u w_{2x}), \end{aligned}$$

thus

$$\begin{aligned} \|w_{2tx}\| &\leq C_1(\|w_{2x}\|_{H^2} + \|u_x\|_{H^1} + \|v_{2x}\| + \|w_{1x}\| \\ &\quad + \|w_{3x}\| + \|v_{3x}\|_{H^1}), \end{aligned}$$

or

$$\begin{aligned} \|w_{2xxx}\| &\leq C_1(\|w_{2x}\|_{H^1} + \|w_{2tx}\| + \|u_x\|_{H^1} + \|w_{1x}\| + \|v_{2x}\| \\ &\quad + \|w_{3x}\| + \|v_{3x}\|_{H^1}). \end{aligned}$$

Likewise, we differentiate (36) with respect to x , apply Lemma 2.1 and the Gagliaro-Nirenberg inequality to arrive at

$$\begin{aligned} j_I w_{3xt} &= (c_a + c_d)r_x \frac{(rw_3)_{xx}}{u} - (c_a + c_d)r_x \frac{(rw_3)_x u_x}{u^2} \\ &\quad + (c_a + c_d)r \frac{(rw_3)_{xxx}u - (rw_3)_{xx}u_x}{u^2} - (c_a + c_d)r \frac{(rw_3)_{xx}u_x + (rw_3)_x u_{xx}}{u^2} \\ &\quad + (c_a + c_d)r \frac{2(rw_3)_{xx}u_x^2}{u^3} + (c_a + c_d) \frac{u_x w_3 + u w_{3x}}{r^2} - 2(c_a + c_d) \frac{u^2 w_3}{r^4} \\ &\quad + 2\mu_r(r_{xx}v_2 + r_x v_{2x} + r_x v_{2x} + r v_{2xx}) - 4\mu_r(u_x w_3 + u w_{3x}). \end{aligned}$$

Then we have

$$\|w_{3tx}\| \leq C_1(\|w_{3x}\|_{H^2} + \|u_x\|_{H^1} + \|v_{2x}\|_{H^1}),$$

or

$$\|w_{3xxx}\| \leq C_1(\|w_{3x}\|_{H^1} + \|w_{3tx}\| + \|u_x(t)\|_{H^1} + \|v_{2x}\|_{H^1}).$$

We can differentiate (37) with respect to x , and easily deduce from Lemma 2.1, (50)-(52), (62)-(64), (74), (76) and the Gagliaro-Nirenberg inequality that

$$\begin{aligned} \|\theta_{tx}\| &\leq C_2(\|\theta_x\|_{H^2} + \|u_x\|_{H^1} + \|v_{1x}\|_{H^1} + \|v_{2x}\|_{H^1} + \|v_{3x}\|_{H^1} \\ &\quad + \|w_{1x}\|_{H^1} + \|w_{2x}\|_{H^1} + \|w_{3x}\|_{H^1}), \end{aligned}$$

or

$$\begin{aligned} \|\theta_{xxx}\| &\leq C_2(\|\theta_x\|_{H^1} + \|\theta_{xt}\| + \|u_x\|_{H^1} + \|v_{1x}\|_{H^1} + \|v_{2x}\|_{H^1} \\ &\quad + \|v_{3x}\|_{H^1} + \|w_{1x}\|_{H^1} + \|w_{2x}\|_{H^1} + \|w_{3x}\|_{H^1}). \end{aligned}$$

According to the estimate above, we have

$$\|v_{1tx}\| \leq C_2(\|v_{1x}\|_{H^2} + \|u_x\|_{H^1} + \|\theta_x\|_{H^1} + \|v_{2x}\|), \quad (81)$$

$$\|v_{2tx}\| \leq C_2(\|v_{2x}\|_{H^2} + \|u_x\|_{H^1} + \|v_{1x}\| + \|w_{3x}\|_{H^1}), \quad (82)$$

$$\|v_{3tx}\| \leq C_2(\|v_{3x}\|_{H^2} + \|u_x\|_{H^1} + \|w_{2x}\|_{H^1}), \quad (83)$$

$$\|w_{1tx}\| \leq C_2(\|w_{1x}\|_{H^2} + \|u_x\|_{H^1} + \|v_{2x}\| + \|w_{2x}\|), \quad (84)$$

$$\begin{aligned} \|w_{2tx}\| &\leq C_2(\|w_{2x}\|_{H^2} + \|u_x\|_{H^1} + \|v_{2x}\| + \|w_{1x}\| \\ &\quad + \|w_{3x}\| + \|v_{3x}\|_{H^1}), \end{aligned} \quad (85)$$

$$\|w_{3tx}\| \leq C_2(\|w_{3x}\|_{H^2} + \|u_x\|_{H^1} + \|v_{2x}\|_{H^1}), \quad (86)$$

$$\begin{aligned} \|\theta_{tx}\| &\leq C_2(\|\theta_x\|_{H^2} + \|u_x\|_{H^1} + \|v_{1x}\|_{H^1} + \|v_{2x}\|_{H^1} \\ &\quad + \|v_{3x}\|_{H^1} + \|w_{1x}\|_{H^1} + \|w_{2x}\|_{H^1} + \|w_{3x}\|_{H^1}), \end{aligned} \quad (87)$$

or

$$\|v_{1xxx}\| \leq C_2(\|v_{1x}\|_{H^1} + \|v_{1tx}\| + \|u_x\|_{H^1} + \|\theta_x\|_{H^1} + \|v_{2x}\|), \quad (88)$$

$$\|v_{2xxx}\| \leq C_2(\|v_{2x}\|_{H^1} + \|v_{2tx}\| + \|u_x\|_{H^1} + \|w_{3x}\|_{H^1} + \|v_{1x}\|), \quad (89)$$

$$\|v_{3xxx}\| \leq C_2(\|v_{3x}\|_{H^1} + \|v_{3tx}\| + \|u_x\|_{H^1} + \|w_{2x}\|_{H^1}), \quad (90)$$

$$\|w_{1xxx}\| \leq C_2(\|w_{1x}\|_{H^1} + \|w_{1tx}\| + \|u_x\|_{H^1} + \|w_{2x}\| + \|v_{2x}\|), \quad (91)$$

$$\begin{aligned} \|w_{2xxx}\| &\leq C_2(\|w_{2x}\|_{H^1} + \|w_{2tx}\| + \|u_x\|_{H^1} + \|w_{1x}\| + \|v_{2x}\| \\ &\quad + \|w_{3x}\| + \|v_{3x}\|_{H^1}), \end{aligned} \quad (92)$$

$$\|w_{3xxx}\| \leq C_2(\|w_{3x}\|_{H^1} + \|w_{3tx}\| + \|u_x\|_{H^1} + \|v_{2x}\|_{H^1}), \quad (93)$$

$$\begin{aligned} \|\theta_{xxx}\| &\leq C_2(\|\theta_x\|_{H^1} + \|\theta_{xt}\| + \|u_x\|_{H^1} + \|v_{1x}\|_{H^1} + \|v_{2x}\|_{H^1} \\ &\quad + \|v_{3x}\|_{H^1} + \|w_{1x}\|_{H^1} + \|w_{2x}\|_{H^1} + \|w_{3x}\|_{H^1}). \end{aligned} \quad (94)$$

By (50)-(52), (62)-(64), (74), (76), (81)-(87) and Lemma 2.1, we get (77). The proof is complete. $\square$

Lemma 2.6. *Under assumptions of Theorem 1.1, there exists a positive constant $\gamma_2 = \gamma_2(C_2)$ such that for any fixed $\gamma \in (0, \gamma_2]$, the following estimate holds for any*

$t > 0$

$$\begin{aligned}
& e^{\gamma t} (\|v_{1xx}\|^2 + \|v_{2xx}\|^2 + \|v_{3xx}\|^2 + \|w_{1xx}\|^2 + \|w_{2xx}\|^2 \\
& \quad + \|w_{3xx}\|^2 + \|u_{xx}\|^2 + \|\theta_{xx}\|^2 + \|v_{1t}\|^2 + \|v_{2t}\|^2 \\
& \quad + \|v_{3t}\|^2 + \|w_{1t}\|^2 + \|w_{2t}\|^2 + \|w_{3t}\|^2 + \|\theta_t\|^2) \\
& \quad + \int_0^t e^{\gamma s} (\|v_{1x}\|_{H^2}^2 + \|v_{2x}\|_{H^2}^2 + \|v_{3x}\|_{H^2}^2 + \|w_{1x}\|_{H^2}^2 \\
& \quad + \|w_{2x}\|_{H^2}^2 + \|w_{3x}\|_{H^2}^2 + \|\theta_x\|_{H^2}^2 + \|u_x\|_{H^1}^2 + \|v_{1tx}\|^2 \\
& \quad + \|v_{2tx}\|^2 + \|v_{3tx}\|^2 + \|w_{1tx}\|^2 + \|w_{2tx}\|^2 + \|w_{3tx}\|^2 \\
& \quad + \|\theta_{tx}\|^2) ds \leq C_1. \tag{95}
\end{aligned}$$

Proof. We multiply (53) by $e^{\gamma t}$, integrate the resulting over $[0, t]$, and then apply Lemma 2.1 and Lemma 2.2 to deduce

$$\begin{aligned}
& e^{\gamma t} \|v_{1t}\|^2 + \int_0^t e^{\gamma s} \|v_{1tx}\|^2 ds \\
& \leq C_2 + C_1 \int_0^t e^{\gamma s} (\|v_{1x}\|_{H^1}^2 + \|v_{1t}\|^2 + \|v_{2t}\|^2 + \|\theta_t\|^2 + \|u_x\|^2 + \|\theta_x\|^2) ds \\
& \leq C_2. \tag{96}
\end{aligned}$$

Multiplying (56), (59), (65), (68), (71) and (75) by $e^{\gamma t}$, and adding them together, by virtue of Lemmas 2.1-2.2, we can conclude

$$\begin{aligned}
& e^{\gamma t} (\|v_{2t}\|^2 + \|v_{3t}\|^2 + \|w_{1t}\|^2 + \|w_{2t}\|^2 + \|w_{3t}\|^2 + \|\theta_t\|^2) \\
& \quad + \int_0^t e^{\gamma s} (\|v_{1tx}\|^2 + \|v_{2tx}\|^2 + \|v_{3tx}\|^2 + \|w_{1tx}\|^2 + \|w_{2tx}\|^2 \\
& \quad + \|w_{3tx}\|^2 + \|\theta_{tx}\|^2) ds \leq C_2. \tag{97}
\end{aligned}$$

On the other hand, multiplying (79) by $e^{\gamma t}$, integrating over $[0, t]$ to have

$$\begin{aligned}
& e^{\gamma t} \left\| \frac{u_{xx}}{u} \right\|^2 + \frac{1}{2C_1} \int_0^t e^{\gamma s} \left\| \frac{u_{xx}}{u} \right\|^2 ds \\
& \leq C_2 + \gamma \int_0^t e^{\gamma s} \left\| \frac{u_{xx}}{u} \right\|^2 ds + C_1 \int_0^t e^{\gamma s} (\|v_{1x}\|_{H^1}^2 + \|u_x\|^2 + \|v_{1t}\|^2 \\
& \quad + \|\theta_x\|_{H^1}^2 + \|v_{2x}\|^2) ds. \tag{98}
\end{aligned}$$

Picking $\gamma_2 = \min\{\frac{1}{4C_1}, \gamma_1\}$ such that for any fixed $\gamma \in (0, \gamma_2]$, using Lemmas 2.1 - 2.2 and (96)-(97), we obtain

$$e^{\gamma t} \|u_{xx}\|^2 + \int_0^t e^{\gamma s} \|u_{xx}\|^2 ds \leq C_2. \tag{99}$$

By (31)-(37), (81)-(87), (96)-(97) and (99), we have

$$\begin{aligned}
& e^{\gamma t} (\|v_{1xx}\|^2 + \|v_{2xx}\|^2 + \|v_{3xx}\|^2 + \|v_{1xx}\|^2 + \|w_{1xx}\|^2 \\
& \quad + \|w_{2xx}\|^2 + \|w_{3xx}\|^2 + \|\theta_{xx}\|^2) + \int_0^t e^{\gamma s} (\|v_{1xx}\|^2 + \|v_{2xx}(t)\|^2 \\
& \quad + \|v_{3xx}\|^2 + \|w_{1xx}\|^2 + \|w_{2xx}\|^2 + \|w_{3xx}\|^2 + \|\theta_{xx}\|^2) \leq C_2,
\end{aligned}$$

which, combined with (96)-(97) and (99), yields (95). The proof is complete. $\square$

Proof of Theorem 1.1. According to Lemmas 2.2-2.6, Theorem 1.1 is complete. $\square$

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