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INITIAL-BOUNDARY VALUE PROBLEM FOR A FOURTH-ORDER PLATE EQUATION WITH HARDY-HÉNON POTENTIAL AND POLYNOMIAL NONLINEARITY

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Abstract. In this paper, the initial-boundary value problem for a fourth-order plate equation with Hardy-Hénon potential and polynomial nonlinearity is invsitgated. First, we establish the local well-posedness of solutions by means of the semigroup theory. Then by using ordinary differential inequalities, potential well theory and energy estimate, we study the conditions on global existence and finite time blow-up. Moreover, the lifespan (i.e., the upper bound of the blow-up time) of the finite time blow-up solution is estimated.

1. **Introduction and main results.** In this paper, we consider the following plate equation with Hardy-Hénon potential and polynomial nonlinearity:

$$\begin{cases} u_{tt} + \Delta^2 u + u = \sigma |x|^{-\alpha} * u + |u|^{p-2} u, & x \in \Omega, t > 0, \\ u = \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, ν is the unit outer normal to the boundary $\partial\Omega$ at x, $\alpha \in (-\infty, n)$ and $\sigma \in \mathbb{R}$ are constants, and

$$2 \le p \begin{cases} < \infty, & n = 1,2,3,4, \\ \le 2 + \dfrac{4}{n-4}, & n \ge 5. \end{cases} \tag{1.2}$$

The initial value $(u_0, u_1) \in H_0^2(\Omega) \times L^2(\Omega)$. Here,

$$|x|^{-\alpha} * u = \begin{cases} \displaystyle\int_\Omega |x-y|^{-\alpha} u(y)dy, & \text{if } \alpha \neq 0, \\ \displaystyle\int_\Omega u(y)dy, & \text{if } \alpha = 0. \end{cases}$$

Plate equations have been investigated for many years due to their importance in some physical areas such as vibration and elasticity theories of solid mechanics. For instance, in the case when σ is identically zero, equation (1.1) becomes an

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equation with polynomial nonlinearity which arises in aeroelasticity modeling (see, for example, [14, 15]), and the problem with (or without) damping, memory, time-delay etc. were studied extensively (see [17, 21, 25, 26, 27, 33, 34, 37, 40, 55, 44] and references therein for the topics on well-posedness, global existence, finite time blow-up, global attractor etc.).

The potential term $|x|^{-\alpha}*u$ is known in the literature as Hardy potential if $\alpha > 0$, while if $\alpha < 0$ it is known as Hénon potential. This type of potential is important in analyzing many aspects of physical phenomena with singular poles (at origin). For example, the Efimov states (the circumstances that the two-particle attraction is so weak that any two bosons can not form a pair, but the three bosons can be stable bound states): see e.g., [16]; effects on dipole-bound anions in polar molecules: see e.g., [4, 7, 28]; capture of matter by black holes (via near-horizon limits): see e.g., [9, 19]; the motions of cold neutral atoms interacting with thin charged wires (falling in the singularity or scattering): see e.g., [5, 12]; the renormalization group of limit cycle in nonrelativistic quantum mechanics: see e.g., [6, 8]; and so on. The are a lot of studies of evolution equations with this type of potentials, see, for example, [1, 2, 3, 45, 32, 50] for parabolic equation, [11, 49, 52, 39] for wave equations, and [24, 22, 23, 42, 47, 48, 53] for Schrödinger equation. However, as far as we know, there seems little studies of fourth-order plate equation with Hardy-Hénon potentials.

Motivated by the previous studies, in this paper, we will consider a fourth-order plate equation with Hardy-Hénon potential and polynomial nonlinearity, i.e, problem (1.1). We mainly concern with the well-posedness, and the conditions on global existence and finite time blow-up. To state the main results of this paper, we first introduce some notations used in this paper:

- Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two Banach space such that $X \hookrightarrow Y$ continuously. Then we denotes by $C_{X\to Y}$ the optimal constant of the embedding, i.e.,
$$C_{X\to Y} = \sup_{\phi\in X\setminus\{0\}} \frac{\|\phi\|_Y}{\|\phi\|_X}. \tag{1.3}$$
- The norm of the Lebesgue space $L^p(\Omega)$, $1 \le p \le \infty$ is denoted by $\|\cdot\|_p$. Especially, we denote $\|\cdot\|_2$ by $\|\cdot\|$ for simplicity.
- The inner product of the Hilbert space $L^2(\Omega)$ is denoted by $(\cdot,\cdot)$.
- The norm of the Sobolev space $H_0^2(\Omega)$ is denoted by $\|\cdot\|_{H^2}$ and
$$\|\phi\|_{H^2}^2 = \|\Delta\phi\|^2 + \|\phi\|^2.$$

Definition 1.1. Assume $\alpha \in (-\infty, n)$ and $\sigma \in \mathbb{R}$. Let $T > 0$, $u_0 \in H_0^2(\Omega)$, and $u_1 \in L^2(\Omega)$. By a weak solution to problem (1.1), we mean a function
$$u \in C\left([0,T]; H_0^2(\Omega)\right) \cap C^1\left([0,T]; L^2(\Omega)\right)$$
such that $u(0) = u_0$, $u_t(0) = u_1$, and
$$\begin{aligned}&\int_\Omega u_t\phi dx + \int_0^t\int_\Omega \Delta u\Delta\phi dx d\tau + \int_0^t\int_\Omega u\phi dx d\tau\\ =&\int_0^t\int_\Omega \left(\sigma|x|^{-\alpha} * u + |u|^{p-2}u\right)\phi dx d\tau + \int_\Omega u_1\phi dx\end{aligned} \tag{1.4}$$
holds for any $\phi \in H_0^2(\Omega)$ and $0 < t \le T$.

The local-well posedness of solutions to problem (1.1) is the following theorem:

Theorem 1.2. *Assume p satisfies (1.2), $\alpha \in (-\infty, n)$ and $\sigma \in \mathbb{R}$. Let $(u_0, u_1) \in H_0^2(\Omega) \times L^2(\Omega)$. Then there exists a positive constant T depending only on $\|u_0\|_{H^2} + \|u_1\|$ such that problem (1.1) admits a weak solution*

$$u \in C\left([0, T]; H_0^2(\Omega)\right) \cap C^1\left([0, T]; L^2(\Omega)\right).$$

The solution u can be extended to a maximal weak solution in $[0, T_{\max})$ such that either

1. *$T_{\max} = \infty$, i.e., the problem admits a global weak solution; or*
2. *$T_{\max} < \infty$, and*

$$\lim_{t \uparrow T_{\max}} \left(\|u(t)\|_{H^2} + \|u_t(t)\|\right) = \infty,$$

i.e., the solution blows up at a finite time $T_{\max}$.

Furthermore, then energy $E(t)$ is conservative, i.e.,

$$E(t) = E(0), \ 0 \le t < T_{\max}, \tag{1.5}$$

where

$$E(t) = \frac{1}{2}\|u_t\|^2 + J(u)(t), \ 0 \le t < T_{\max}, \tag{1.6}$$

and

$$E(0) = E(t)|_{t=0} = \frac{1}{2}\|u_1\|^2 + J(u_0). \tag{1.7}$$

Moreover, we have $(u_t, u) \in C^1[0, T_{\max})$, and

$$\frac{d}{dt}(u_t, u) = \|u_t\|^2 - I(u). \tag{1.8}$$

Here $J : H_0^2(\Omega) \to \mathbb{R}$ is a functional defined by

$$J(\phi) = \frac{1}{2}\|\Delta\phi\|^2 + \frac{1}{2}\|\phi\|^2 - \frac{1}{p}\|\phi\|_p^p - \frac{\sigma}{2}\int_\Omega\int_\Omega \frac{\phi(x)\phi(y)}{|x-y|^\alpha}dxdy, \tag{1.9}$$

and $I : H_0^2(\Omega) \to \mathbb{R}$ is a functional defined by

$$I(\phi) = \|\Delta\phi\|^2 + \|\phi\|^2 - \|\phi\|_p^p - \sigma\int_\Omega\int_\Omega \frac{\phi(x)\phi(y)}{|x-y|^\alpha}dxdy. \tag{1.10}$$

Based on Theorem 1.2, we study the conditions on global existence and finite time blow-up. The first result is about the case that the initial energy is non-positive.

Theorem 1.3. *Assume p satisfies (1.2), $\alpha \in (-\infty, n)$ and $\sigma \in \mathbb{R}$. Let $(u_0, u_1) \in H_0^2(\Omega) \times L^2(\Omega)$ satisfy*

1. *$E(0) < 0$; or*
2. *$E(0) = 0$ and $(u_0, u_1) > 0$.*

Then the weak solution got in Theorem 1.2 blows up in finite time, i.e., $T_{\max} < \infty$. Moreover, $T_{\max}$ satisfies the following estimates:

1. *If $|\sigma| \le \sigma^*$, then*

$$T_{\max} \le \begin{cases} \dfrac{2\|u_0\|^2}{(p-2)(u_0, u_1)}, & \text{if } E(0) \le 0 \text{ and } (u_0, u_1) > 0; \\ \dfrac{4\|u_0\|}{(p-2)\sqrt{-2E(0)}}, & \text{if } E(0) < 0 \text{ and } (u_0, u_1) = 0; \\ \overline{T}, & \text{if } E(0) < 0 \text{ and } (u_0, u_1) < 0, \end{cases}$$

where

$$\overline{T}=\frac{-4E(0)\left\|u_0\right\|^2+2\left(-(u_0,u_1)+\sqrt{(u_0,u_1)^2-2E(0)\|u_0\|^2}\right)^2}{-2E(0)(p-2)\sqrt{(u_0,u_1)^2-2E(0)\left\|u_0\right\|^2}}.$$

2. *If* $|\sigma|>\sigma^*$ *and* $(u_0,u_1)\geq 0$, *then*

$$T_{\max}\leq\begin{cases}\dfrac{(p\Lambda+2)\|u_0\|^2}{(p-2)(u_0,u_1)}, & \text{if } E(0)\leq 0 \text{ and } (u_0,u_1)>0;\\ \dfrac{2(p\Lambda+2)\|u_0\|}{(p-2)\sqrt{-2E(0)}}, & \text{if } E(0)<0 \text{ and } (u_0,u_1)=0.\end{cases}$$

Here,

$$\sigma^*=\inf_{\phi\in H_0^2(\Omega)\setminus\{0\}}\left|\frac{\|\Delta\phi\|^2+\|\phi\|^2}{\displaystyle\int_\Omega\int_\Omega|x-y|^{-\alpha}\phi(x)\phi(y)dxdy}\right| \tag{1.11}$$

and

$$\Lambda=\begin{cases}(|\sigma|-\sigma^*)R^{-\alpha}|\Omega|^{\frac{p}{2}}\|u_0\|^{2-p}, & \text{if } \alpha\leq 0;\\ \kappa(|\sigma|-\sigma^*)|\Omega|^{\frac{(p-2)^2}{2p}+\frac{p(2n-\alpha)-2n}{n}}\|u_0\|^{2-p}, & \text{if } 0<\alpha<n,\end{cases} \tag{1.12}$$

where

$$R=\sup_{x,y\in\Omega}|x-y|,\ \kappa=\pi^{\frac{\alpha}{2}}\frac{\Gamma(\frac{n-\alpha}{2})}{\Gamma(\frac{2n-\alpha}{2})}\left\{\frac{\Gamma(\frac{n}{2})}{\Gamma(n)}\right\}^{-\frac{n-\alpha}{n}}.$$

Remark 1. There are two remarks on the above theorem.

1. Firstly, by Lemma 2.2, if $\alpha\in(-\infty,n)$, σ^* is well-defined and
$$\sigma^*\geq\begin{cases}R^\alpha\left(C_{H_0^2\to L^1}\right)^{-2}, & \alpha\in(-\infty,0],\\ \frac{1}{\kappa}\left(C_{H_0^2\to L^{\frac{2n}{2n-\alpha}}}\right)^{-2}, & \alpha\in(0,n),\end{cases}$$
where $C_{H_0^2\to L^1}$ is the optimal constant of the embedding $H_0^2(\Omega)\hookrightarrow L^1(\Omega)$, $C_{H_0^2\to L^{\frac{2n}{2n-\alpha}}}$ is the optimal constant of the embedding $H_0^2(\Omega)\hookrightarrow L^{\frac{2n}{2n-\alpha}}(\Omega)$. So Theorem 1.3 makes sense.
2. Secondly, for $|\sigma|>\sigma^*$ and $(u_0,u_1)<0$. Due to technique reasons, we only show the solution will blow up in finite time, but the upper bound of the blow-up time $T_{\max}$ is not given. We left the study of this problem as an open question.

Theorem 1.3 is above the case $E(0)\leq 0$. In order to derive some results for the case $E(0)>0$, we use the potential well method (see, for example, [10, 20, 35, 36, 41, 43, 51]). Let

$$d=\inf\{J(\phi):\phi\in\mathcal{N}\}, \tag{1.13}$$

where J is the functional defined by (1.9), and $\mathcal{N}$ is the Nehari manifold defined as

$$\mathcal{N}=\{\phi\in H_0^2(\Omega)\setminus\{0\}:I(\phi)=0\}. \tag{1.14}$$

Here I is the functional defined by (1.10).

Remark 2. If we assume p satisfies (1.2), $\alpha \in (-\infty, n)$, and $\sigma \in (-\sigma^*, \sigma^*)$, where σ^* is the positive constant defined by (1.11), by Lemma 2.4, d is a positive constant and

$$d \geq \frac{p-2}{2p}\left(\left(1-\frac{|\sigma|}{\sigma^*}\right) C^2_{H_0^2 \to L^p}\right)^{\frac{p}{p-2}},$$

where $C_{H_0^2 \to L^p}$ is the optimal constant of the embedding $H_0^2(\Omega) \hookrightarrow L^p(\Omega)$.

Theorem 1.4. *(Global existence for $0 < E(0) \leq d$). Assume p satisfies (1.2), $\alpha \in (-\infty, n)$, and $\sigma \in (-\sigma^*, \sigma^*)$, where σ^* is the positive constant defined by (1.11). Let u be the weak solution got in Theorem 1.2 with $(u_0, u_1) \in H_0^2(\Omega) \times L^2(\Omega)$ satisfying $I(u_0) > 0$ and $0 < E(0) \leq d$, where $E(0), I, d$ are defined in (1.7),(1.10), (1.13) respectively. Then u exists globally, i.e., $T_{\max} = \infty$.*

Theorem 1.5. *(Blow-up for $0 < E(0) \leq d$) Assume p satisfies (1.2), $\alpha \in (-\infty, n)$, and $\sigma \in (-\sigma^*, \sigma^*)$, where σ^* is the positive constant defined by (1.11). Let u be the weak solution got in Theorem 1.2 with $(u_0, u_1) \in H_0^2(\Omega) \times L^2(\Omega)$ satisfying*

1. *$E(0) < d, I(u_0) < 0$; or*
2. *$E(0) = d, I(u_0) < 0, (u_0, u_1) > 0$,*

where $E(0), I, d$ are defined in (1.7),(1.10), (1.13) respectively. Then u blows up in finite time, i.e., $T_{\max} < \infty$. Moreover, $T_{\max}$ satisfies the following estimates:

$$T_{\max} \leq \begin{cases} \dfrac{2\|u_0\|^2}{(p-2)(u_0,u_1)}, & \text{if } E(0) \leq d \text{ and } (u_0,u_1) > 0; \\ \dfrac{4\|u_0\|}{(p-2)\sqrt{2(d-E(0))}}, & \text{if } E(0) < d \text{ and } (u_0,u_1) = 0; \\ \widehat{T}, & \text{if } E(0) < d \text{ and } (u_0,u_1) < 0, \end{cases}$$

where

$$\widehat{T} = \frac{4(d-E(0))\,\|u_0\|^2 + 2\left(-(u_0,u_1) + \sqrt{(u_0,u_1)^2 + (2(d-E(0)))\|u_0\|^2}\right)^2}{(2(d-E(0)))(p-2)\sqrt{(u_0,u_1)^2 + (2(d-E(0)))\,\|u_0\|^2}}.$$

The organization of the rest of this paper is as follows: In Section 2, we give some preliminaries, which will be used in this paper; In Section 3, we study the well-posed of solutions by semigroup theory and prove Theorem 1.2; In Section 4, we study the conditions on global existence and finite time blow-up and prove Theorems 1.3, 1.4 and 1.5.

2. Preliminaries.

The following well-known Hardy-Littlewood-Sobolev inequality can be found in [31]:

Lemma 2.1. *Let $q, r > 1$ and $0 < \theta < n$ with*

$$\frac{1}{q} + \frac{n-\theta}{n} + \frac{1}{r} = 2.$$

Let $u \in L^q(\mathbb{R}^n)$ and $v \in L^r(\mathbb{R}^n)$. Then there exists a sharp positive constant C depending only on n, α and q such that

$$\left| \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{u(x)v(y)}{|x-y|^{n-\theta}} dxdy \right| \leq \kappa \|u\|_q \|v\|_r.$$

If

$$q = r = \frac{2n}{n+\theta},$$

then

$$C = \kappa := \pi^{\frac{n-\theta}{2}} \frac{\Gamma(\frac{\theta}{2})}{\Gamma(\frac{n+\theta}{2})} \left\{ \frac{\Gamma(\frac{n}{2})}{\Gamma(n)} \right\}^{-\frac{\theta}{n}}. \tag{2.15}$$

If $u \in L^q(\Omega)$ *and* $v \in L^r(\Omega)$ *with* q *and* r *satisfying the assumptions, by valuing them* 0 *in* $\mathbb{R}^n \setminus \Omega$*, it also holds*

$$\left| \int_\Omega \int_\Omega \frac{u(x)v(y)}{|x-y|^{n-\theta}} dxdy \right| \leq C\|u\|_q \|v\|_r. \tag{2.16}$$

Lemma 2.2. *Let* $\alpha \in (-\infty, n)$ *and* σ^* *be the constant defined in* (1.11)*. Then* σ^* *is well-defined and*

$$\sigma^* \geq \begin{cases} R^\alpha \left(C_{H_0^2 \to L^1} \right)^{-2}, & \alpha \in (-\infty, 0], \\ \frac{1}{\kappa} \left(C_{H_0^2 \to L^{\frac{2n}{2n-\alpha}}} \right)^{-2}, & \alpha \in (0, n), \end{cases}$$

where $C_{H_0^2 \to L^1}$ *is the optimal constant of the embedding* $H_0^2(\Omega) \hookrightarrow L^1(\Omega)$, $C_{H_0^2 \to L^{\frac{2n}{2n-\alpha}}}$ *is the optimal constant of the embedding* $H_0^2(\Omega) \hookrightarrow L^{\frac{2n}{2n-\alpha}}(\Omega)$*,* κ *is the constant given in* (2.15) *with* $\theta = n - \alpha$*, and*

$$R = \sup_{x,y \in \Omega} |x-y| < \infty.$$

Proof. If $\alpha \in (-\infty, 0]$, we have,

$$\begin{aligned} \left| \int_\Omega \int_\Omega |x-y|^{-\alpha} \phi(x)\phi(y) dxdy \right| &\leq \left(\int_\Omega |\phi(x)| dx \right)^2 R^{-\alpha} \\ &\leq C_{H_0^2 \to L^1}^2 R^{-\alpha} \left(\|\Delta\phi\|^2 + \|\phi\|^2 \right). \end{aligned}$$

If $\alpha \in (0, n)$, by Hardy-Littlewood-Sobolev inequality (see (2.16) of Lemma 2.1) with $q = r = \frac{2n}{2n-\alpha}$ and $H_0^2(\Omega) \hookrightarrow L^{\frac{2n}{2n-\alpha}}(\Omega)$ with constant $C_{H_0^2 \to L^{\frac{2n}{2n-\alpha}}}$,

$$\begin{aligned} \left| \int_\Omega \int_\Omega |x-y|^{-\alpha} \phi(x)\phi(y) dxdy \right| &\leq \kappa \|\phi\|_{\frac{2n}{2n-\alpha}}^2 \\ &\leq \kappa C_{H_0^2 \to L^{\frac{2n}{2n-\alpha}}}^2 \left(\|\Delta\phi\|^2 + \|\phi\|^2 \right). \end{aligned}$$

□

Lemma 2.3. [29, 30] *Suppose* $F(t) \in C^2[0, T)$ *is a nonnegative function satisfying*

$$F''(t)F(t) - (1+r)\left(F'(t)\right)^2 \geq 0, \tag{2.17}$$

where $0 < T \leq +\infty$ *and* r *is a positive constant. If* $F(0) > 0$ *and* $F'(0) > 0$*, then*

$$T \leq \frac{F(0)}{rF'(0)} < +\infty \tag{2.18}$$

and $F(t) \to +\infty$ *as* $t \to T$.

Lemma 2.4. *Assume p satisfies (1.2), $\alpha \in (-\infty, n)$, and $\sigma \in (-\sigma^*, \sigma^*)$, where σ^* is the positive constant defined by (1.11). Let d be the constant defined in (1.13). Then we have*

$$d = \inf\left\{\sup_{\lambda\geq 0} J(\lambda\phi) : \phi \in H_0^2(\Omega)\setminus\{0\}\right\}, \tag{2.19}$$

and

$$d \geq \frac{p-2}{2p}\left(\left(1-\frac{|\sigma|}{\sigma^*}\right)C^2_{H_0^2\to L^p}\right)^{\frac{p}{p-2}}, \tag{2.20}$$

where $C_{H_0^2\to L^p}$ is the optimal constant of the embedding $H_0^2(\Omega)\hookrightarrow L^p(\Omega)$.

Proof. Firstly, we prove (2.19). For any $\phi \in H_0^2(\Omega)\setminus\{0\}$, since $\sigma \in [0,\sigma^*)$, by means of a simple calculation, we find there exists a unique positive constant $\hat{\lambda}$ defined by

$$\hat{\lambda} = \left(\frac{\|\Delta\phi\|^2 + \|\phi\|^2 - \sigma\int_\Omega\int_\Omega |x-y|^{-\alpha}\phi(x)\phi(y)dxdy}{\|\phi\|_p^p}\right)^{\frac{1}{p-2}}, \tag{2.21}$$

such that

$$\sup_{\lambda\geq 0} J(\lambda\phi) = J(\hat{\lambda}\phi),\ \ \hat{\lambda}\phi \in \mathcal{N}. \tag{2.22}$$

Then,

$$\begin{aligned}\inf\left\{\sup_{\lambda\geq 0} J(\lambda\phi) : \phi \in H_0^2(\Omega)\setminus\{0\}\right\} &= \inf\left\{J(\hat{\lambda}\phi) : \phi \in H_0^2(\Omega)\setminus\{0\}\right\}\\ &\geq \inf\{J(\phi) : \phi\in\mathcal{N}\}.\end{aligned}$$

On the other hand, for any $\phi \in \mathcal{N}$, we have $\hat{\lambda} = 1$. Then

$$\begin{aligned}\inf\{J(\phi) : \phi\in\mathcal{N}\} &= \inf\left\{\sup_{\lambda\geq 0} J(\lambda\phi) : \phi\in\mathcal{N}\right\}\\ &\geq \inf\left\{\sup_{\lambda\geq 0} J(\lambda\phi) : \phi \in H_0^2(\Omega)\setminus\{0\}\right\}.\end{aligned}$$

Then (2.19) follows from the above two inequalities.

Secondly, we prove (2.20), by (1.11), we have

$$\begin{aligned} d =& \frac{p-2}{2p}\inf\left\{\hat{\lambda}^p\|\phi\|_p^p : \phi\in H_0^2(\Omega)\setminus\{0\}\right\}\\ =& \frac{p-2}{2p}\left(\inf_{\phi\in H_0^2(\Omega)\setminus\{0\}} \frac{\|\Delta\phi\|^2+\|\phi\|^2-\sigma\int_\Omega\int_\Omega |x-y|^{-\alpha}\phi(x)\phi(y)dxdy}{\|\phi\|_p^2}\right)^{\frac{p}{p-2}}\\ \geq& \frac{p-2}{2p}\left(\inf_{\phi\in H_0^2(\Omega)\setminus\{0\}} \frac{\|\Delta\phi\|^2+\|\phi\|^2-|\sigma|\left|\int_\Omega\int_\Omega |x-y|^{-\alpha}\phi(x)\phi(y)dxdy\right|}{\|\phi\|_p^2}\right)^{\frac{p}{p-2}}\\ \geq& \frac{p-2}{2p}\left(\inf_{\phi\in H_0^2(\Omega)\setminus\{0\}}\left(1-\frac{|\sigma|}{\sigma^*}\right)\frac{\|\Delta\phi\|^2+\|\phi\|^2}{\|\phi\|_p^2}\right)^{\frac{p}{p-2}}\\ =& \frac{p-2}{2p}\left(\left(1-\frac{\sigma}{\sigma^*}\right)C^2_{H_0^2\to L^p}\right)^{\frac{p}{p-2}}. \end{aligned} \tag{2.23}$$

□

Lemma 2.5. *Assume p satisfies (1.2), $\alpha \in (-\infty, n)$, and $\sigma \in (-\sigma^*, \sigma^*)$, where σ^* is the positive constant defined by (1.11). Let*

$$\mathcal{W} = \left\{\phi \in H_0^2(\Omega) : I(\phi) > 0, J(\phi) < d\right\} \cup \{0\}. \tag{2.24}$$

$$\mathcal{V} = \left\{u \in H_0^2(\Omega) : I(\phi) < 0, J(\phi) < d\right\}, \tag{2.25}$$

where d is the positive constant defined in (1.13). Then

$$\|\phi\|_p^p \le \frac{2pd}{p-2}, \ \forall \phi \in \mathcal{W} \tag{2.26}$$

and

$$\|\Delta\phi\|^2 + \|\phi\|^2 \ge \left(\left(1 - \frac{|\sigma|}{\sigma^*}\right)\left(C_{H_0^2 \to L^p}\right)^{-p}\right)^{\frac{2}{p-2}}, \ \forall \phi \in \mathcal{V}, \tag{2.27}$$

where $C_{H_0^2 \to L^p}$ is the optimal constant of the embedding $H_0^2(\Omega) \hookrightarrow L^p(\Omega)$. Moreover, $\mathcal{W} = \mathcal{W}_1$, and $\mathcal{V}_1 = \mathcal{V}$, where

$$\mathcal{W}_1 = \left\{\phi \in H_0^2(\Omega) : J(\phi) < d, \|\Delta\phi\|^2 + \|\phi\|^2 < \frac{2pd}{p-2} + \sigma C_\phi\right\}, \tag{2.28}$$

and

$$\mathcal{V}_1 = \left\{\phi \in H_0^2(\Omega) : J(\phi) < d, \|\Delta\phi\|^2 + \|\phi\|^2 > \frac{2pd}{p-2} + \sigma C_\phi\right\}. \tag{2.29}$$

Here,

$$C_\phi = \int_\Omega \int_\Omega \frac{\phi(x)\phi(y)}{|x-y|^\alpha} dxdy.$$

Proof. **Step 1.** We prove (2.26). By the definition of J and I (see (1.9) and (1.10)), we have

$$2J(\phi) - I(\phi) = \frac{p-2}{p}\|\phi\|_p^p, \ \forall \phi \in H_0^2(\Omega). \tag{2.30}$$

For any $\phi \in \mathcal{W}$, since $I(\phi) > 0$ and $J(\phi) < d$, (2.26) follows from the above inequality.

Step 2. We prove (2.27). For any $\phi \in V$, by the definition of I in (1.10), $|\sigma| < \sigma^*$, it follows from (1.11) and $I(\phi) < 0$ that

$$\begin{aligned}\left(C_{H_0^2 \to L^p}\right)^p \left(\|\Delta\phi\|^2 + \|\phi\|^2\right)^{\frac{p}{2}} &\ge \|\phi\|_p^p \\ &> \|\Delta\phi\|^2 + \|\phi\|^2 - \sigma \int_\Omega \int_\Omega \frac{\phi(x)\phi(y)}{|x-y|^\alpha} dxdy \\ &\ge \left(1 - \frac{|\sigma|}{\sigma^*}\right)\left(\|\Delta\phi\|^2 + \|\phi\|^2\right),\end{aligned} \tag{2.31}$$

which implies (2.27).

Step 3. We show that $\mathcal{V}_1 = \mathcal{V}$.

Firstly, for any $\phi \in \mathcal{V}_1$, by (1.9) and the definition of $\mathcal{V}_1$, we get

$$\|\Delta\phi\|^2 + \|\phi\|^2 > \frac{2pd}{p-2} + \sigma \int_\Omega \int_\Omega \frac{\phi(x)\phi(y)}{|x-y|^\alpha} dxdy,$$

$$\frac{1}{2}\left(\|\Delta\phi\|^2 + \|\phi\|^2\right) < d + \frac{1}{p}\|\phi\|_p^p + \frac{\sigma}{2} \int_\Omega \int_\Omega \frac{\phi(x)\phi(y)}{|x-y|^\alpha} dxdy,$$

which implies

$$\|\phi\|_p^p > \frac{2pd}{p-2}.$$

Thus, by the definition of I in (1.10) and the above two inequalities,

$$\begin{aligned} I(\phi) &= \|\Delta\phi\|^2 + \|\phi\|^2 - \|\phi\|_p^p - \sigma \int\limits_\Omega \int\limits_\Omega \frac{\phi(x)\phi(y)}{|x-y|^\alpha} dxdy \\ &\le 2d - \frac{p-2}{p}\|\phi\|_p^p \\ &< 0, \end{aligned}$$

which, together with $J(\phi) < d$, implies $\phi \subset \mathcal{V}$, then $\mathcal{V}_1 \subset \mathcal{V}$.

Secondly, we prove $\mathcal{V} \subset \mathcal{V}_1$. For any $\phi \in \mathcal{V}$, since $|\sigma| < \sigma^*$, it follow from (2.27) and (2.31) that

$$\begin{aligned} &\|\Delta\phi\|^2 + \|\phi\|^2 - \sigma \int\limits_\Omega \int\limits_\Omega \frac{\phi(x)\phi(y)}{|x-y|^\alpha} dxdy \\ &\ge \left(1 - \frac{|\sigma|}{\sigma^*}\right) \left(\left(1 - \frac{|\sigma|}{\sigma^*}\right) \left(C_{H_0^2 \to L^p}\right)^{-p} \right)^{\frac{2}{p-2}} > 0. \end{aligned}$$

Then, in view of $I(\phi) < 0$, i.e.,

$$\|\phi\|_p^p > \|\Delta\phi\|^2 + \|\phi\|^2 - \sigma \int\limits_\Omega \int\limits_\Omega \frac{\phi(x)\phi(y)}{|x-y|^\alpha} dxdy,$$

we get

$$\begin{aligned} &\|\phi\|_p^{\frac{2p}{p-2}} > \left(\|\Delta\phi\|^2 + \|\phi\|^2 - \sigma \int\limits_\Omega \int\limits_\Omega \frac{\phi(x)\phi(y)}{|x-y|^\alpha} dxdy \right)^{\frac{2}{p-2}} \\ \Leftrightarrow &\|\phi\|_p^{\frac{2p}{p-2}} > \left(\|\Delta\phi\|^2 + \|\phi\|^2 - \sigma \int\limits_\Omega \int\limits_\Omega \frac{\phi(x)\phi(y)}{|x-y|^\alpha} dxdy \right)^{\frac{p}{p-2}-1} \\ \Leftrightarrow &\|\Delta\phi\|^2 + \|\phi\|^2 - \sigma \int\limits_\Omega \int\limits_\Omega \frac{\phi(x)\phi(y)}{|x-y|^\alpha} dxdy \\ &\qquad > \left(\frac{\|\Delta\phi\|^2 + \|\phi\|^2 - \sigma \int_\Omega \int_\Omega \frac{\phi(x)\phi(y)}{|x-y|^\alpha} dxdy}{\|\phi\|_p^2} \right)^{\frac{p}{p-2}}, \end{aligned}$$

which, together with the second line of (2.23), implies

$$\|\Delta\phi\|^2 + \|\phi\|^2 - \sigma \int\limits_\Omega \int\limits_\Omega \frac{\phi(x)\phi(y)}{|x-y|^\alpha} dxdy > \frac{2pd}{p-2}.$$

Since $J(\phi) < d$, the above inequality infers that $\phi \in \mathcal{V}_1$, and then $\mathcal{V} \subset \mathcal{V}_1$.

Step 4. We show that $\mathcal{W} = \mathcal{W}_1$.

Firstly we prove $\mathcal{W} \subset \mathcal{W}_1$. In fact, for any $\phi \in \mathcal{W}$, if $\phi = 0$, it is obvious that $\phi \in \mathcal{W}_1$; if $\phi \neq 0$, in view of the definition of $\mathcal{W}$, we get

$$\|\Delta\phi\|^2 + \|\phi\|^2 - \sigma \int_\Omega \int_\Omega \frac{\phi(x)\phi(y)}{|x-y|^\alpha} dxdy > \|\phi\|_p^p,$$

$$\frac{1}{2}\left(\|\Delta\phi\|^2 + \|\phi\|^2\right) < d + \frac{1}{p}\|u\|_p^p + \frac{1}{2}\sigma \int_\Omega \int_\Omega \frac{\phi(x)\phi(y)}{|x-y|^\alpha} dxdy.$$

The above two inequalities imply

$$\Delta\phi\|^2 + \|\phi\|^2 < \frac{2pd}{p-2} + \sigma \int_\Omega \int_\Omega \frac{\phi(x)\phi(y)}{|x-y|^\alpha} dxdy,$$

which, together with $J(u) < d$, implies $\phi \in \mathcal{W}_1$, then $\mathcal{W} \subset \mathcal{W}_1$.

Secondly we prove $\mathcal{W}_1 \subset \mathcal{W}$ by contradiction argument. If there exists $\phi \in \mathcal{W}_1 \setminus \mathcal{W}$. Then we have

$$J(\phi) < d \text{ (by (2.28))}, \tag{2.32}$$

$$\|\Delta\phi\|^2 + \|\phi\|^2 < \frac{2pd}{p-2} + \sigma \int_\Omega \int_\Omega \frac{\phi(x)\phi(y)}{|x-y|^\alpha} dxdy \text{ (by (2.28))}, \tag{2.33}$$

$$I(\phi) \leq 0, \ \phi \neq 0 \text{ (by (2.24) and (2.32))}. \tag{2.34}$$

If $I(\phi) < 0$, the by (2.32), we get $\phi \in \mathcal{V}$ (see (2.25)), and then $\phi \in \mathcal{V}_1$ (since $\mathcal{V} = \mathcal{V}_1$ has been proved in Step 3), which, together with (2.29), contradicts (2.33); If $I(\phi) = 0$, then by $\phi \neq 0$, we get $\phi \in \mathcal{N}$ (see (1.14)), and then $J(\phi) \geq d$ (see (1.13)), which contradicts (2.32). So, $\phi \in \mathcal{W}$, and then $\mathcal{W}_1 \subset \mathcal{W}$. □

Lemma 2.6. *Assume p satisfies (1.2), $\alpha \in (-\infty, n)$, and $\sigma \in (-\sigma^*, \sigma^*)$, where σ^* is the positive constant defined by (1.11). Let*

$$u \in C\left([0, T_{\max}); H_0^2(\Omega)\right) \cap C^1\left([0, T_{\max}); L^2(\Omega)\right)$$

be the maximal weak solution to (1.1) with initial value $(u_0, u_1) \in H_0^2(\Omega) \times L^2(\Omega)$ got in Theorem 1.2.

1. *If there exists a $t_0 \in [0, T_{\max})$ such that $E(t_0) < d$, then $u(t) \in \mathcal{W}$ for $t \in [t_0, T_{\max})$ provided that $u(t_0) \in \mathcal{W}$;*
2. *If there exists a $t_0 \in [0, T_{\max})$ such that either $E(t_0) < d$ or $E(t_0) = d$ and $(u_t, u)_{t=t_0} \geq 0$, then $u(t) \in \mathcal{V}$ for $t \in [t_0, T_{\max})$ provided that $u(t_0) \in \mathcal{V}$,*

where $E(t)$ is the energy functional defined in (1.6), $\mathcal{W}$ and $\mathcal{V}$ is is the sets defined in (2.24) and (2.25) respectively, d constant defined in (1.13).

Proof. Firstly, we proof the first part by contradiction argument. Actually, if the conclusion is incorrect, by using $u \in C\left([0, T_{\max}); H_0^2(\Omega)\right) \cap C^1\left([0, T_{\max}); L^2(\Omega)\right)$ and $u(t_0) \in \mathcal{W}$, there must exist a $t_1 \in (t_0, T_{\max})$ such that

$$u(t) \in \mathcal{W}, \ t \in [t_0, t_1); \text{ and } u(t_1) \in \partial\mathcal{W}.$$

Since the energy is conservative (see (1.5)), $E(t) = \frac{1}{2}\|u_t\|^2 + J(u)(t)$ (see (1.6)), and $E(0) < d$, we get

$$J(u)(t_1) \leq E(t_1) = E(t_0) < d. \tag{2.35}$$

Then by $u(t_1) \in \partial\mathcal{W}$ and the definition of $\mathcal{W}$ (see (2.24)), it follows $I(u)(t_1) = 0$ and $u(t_1) \neq 0$. So $u(t_1) \in \mathcal{N}$ (see the definition of $\mathcal{N}$ in (1.14)), then by the definition of d (see (1.13)), it follows $J(u)(t_1) \geq d$, which contradicts (2.35).

Secondly, we proof the second part. In the case $E(t_0) < d$, in view of (2.27), the proof is similar to the first part. We only prove the case

$$E(t_0) = d \text{ and } (u_t, u)_{t=t_0} \geq 0$$

in detail. Arguing by contradiction, if the conclusion is incorrect, by $u(t_0) \in \mathcal{V}$ and $u \in C\left([0, T_{\max}); H_0^2\right)$, we obtain that there must exist a $t_1 \in (t_0, T_{\max})$ such that $u(\cdot, t) \in \mathcal{V}$, $t \in [t_0, t_1)$ and $u(t_1) \in \partial\mathcal{V}$, i.e. (see (2.25)),

(i)**:** $J(u)(t_1) < d, I(u)(t_1) = 0$; or

(ii)**:** $J(u)(t_1) = d, I(u)(t_1) \leq 0$.

Due to $u(t) \in \mathcal{V}$ for any $t \in [t_0, t_1)$ and $u \in C\left([0, T_{\max}); H_0^2\right)$, by (2.27), we get

$$\|\Delta u(t_1)\|^2 + \|u(t_1)\|^2 \geq \left(\left(1 - \frac{|\sigma|}{\sigma^*}\right)\left(C_{H_0^2 \to L^p}\right)^{-p}\right)^{\frac{2}{p-2}}. \tag{2.36}$$

If (i) is true, by using $I(u(t_1)) = 0$ and (2.36), we have $u(t_1) \in \mathcal{N}$ (see (1.14)), which implies $J(u)(t_1) \geq d$ (see (1.13)), a contradiction.

If (ii) is true, by (1.5) and $E(t) = \frac{1}{2}\|u_t\|^2 + J(u)(t)$ (see (1.6)), we get

$$d = E(t_0) = \frac{1}{2}\|u_t(t_1)\|^2 + J(u)(t_1). \tag{2.37}$$

Combining (2.37) and $J(u)(t_1) = d$, we have

$$\|u_t(t_1)\|^2 = 0. \tag{2.38}$$

Utilizing Cauchy-Schwartz's inequality, we obtain that

$$(u_t, u)|_{t=t_1} \leq \|u_t(t_1)\| \|u(t_1)\| = 0. \tag{2.39}$$

Integrating (1.8) over $[0, t]$, we obtain

$$(u_t, u) + \int_0^t I(u)(\tau) d\tau - \int_0^t \|u_\tau\|^2 d\tau = (u_1, u_0),\ 0 \leq t < T_{\max}. \tag{2.40}$$

By (2.40), we get

$$\begin{aligned}
&(u_t, u)|_{t=t_0} + \int_0^{t_0} I(u)(\tau) d\tau - \int_0^{t_0} \|u_\tau\|^2 d\tau \\
=&(u_t, u)|_{t=t_1} + \int_0^{t_1} I(u)(\tau) d\tau - \int_0^{t_1} \|u_\tau\|^2 d\tau.
\end{aligned}$$

Then,

$$(u_t, u)|_{t=t_0} - (u_t, u)|_{t=t_1} = \int_{t_0}^{t_1} I(u)(\tau) d\tau - \int_{t_0}^{t_1} \|u_\tau\|^2 d\tau.$$

Since $I(u)(t) < 0$ (by using $u(t) \in \mathcal{V}$ for $t \in [t_0, t_1)$) and $(u_t, u)|_{t=t_0} \geq 0$, we get from the above equality that

$$(u_t, u)|_{t=t_1} = (u_t, u)|_{t=t_0} - \int_{t_0}^{t_1} I(u)(\tau) d\tau + \int_{t_0}^{t_1} \|u_\tau\|^2 d\tau > 0,$$

which contradicts (2.39). □

3. **Local-well posedness.** In this section, we study local well-posedness of solutions to (1.1) by semigroup theory. To this end, first, we introduce some fundamental theory on semigroup theory.

Suppose that H is a Hilbert space with inner product $(\cdot,\cdot)_H$ and norm

$$\|\Phi\|_H = \sqrt{(\Phi,\Phi)_H},\ \Phi \in H.$$

Suppose F is a nonlinear operator from H into H. F is said to satisfy the local Lipschitz condition if for any positive constant $M > 0$, there is a positive constant L_M depending only on M such that when $U, V \in H$, $\|U\|_H \leq M$ and $\|V\|_H \leq M$,

$$\|F(U) - F(V)\|_H \leq M\|U - V\|_H. \tag{3.41}$$

Consider the following abstract semilinear evolution equation

$$\begin{cases} U_t + AU = F(U),\ t > 0, \\ U(0) = U_0, \end{cases} \tag{3.42}$$

where $A : D(A) \to H$ is a densely defined linear operator on H, i.e., A is linear and $D(A)$ is dense in H, where $D(A) = \{\Phi \in H : A\Phi \in H\}$.

First, we introduce the Lumer-Phillips theorem (see, for example, [38, Theorem 1.2.3] and [54, Lemma 2.2.3]):

Lemma 3.1. *The necessary and sufficient conditions for A generating a contraction C_0-semigroup $\{e^{tA}\}_{t\geq 0}$ on H are*

1. $(A\Phi,\Phi)_H \leq 0$ *for all* $\Phi \in D(A)$*, and*
2. $R(I - A) = H$.

Here $R(I + A) = \{\Phi + A\Phi : \Phi \in D(A)\}$ *is the range of the operator.*

Next, we introduce the local well-posedness results for (3.42), which can be found in [54, Theorems 2.5.4 and 2.5.5]:

Lemma 3.2. *Suppose that A generates a contraction C_0-semigroup $\{e^{tA}\}_{t\geq 0}$ on H, and F is a nonlinear operator from H into H satisfying the local Lipschitz condition. Then for any $U_0 \in H$, there is a positive constant T depending only on $\|U_0\|_H$ such that problem (3.42) admits a unique local mild solution $U(t)$, i.e., $U \in C([0,T],H)$ and satisfies*

$$U(t) = e^{tA}U_0 + \int_0^t e^{(t-\tau)A}F(U(\tau))d\tau, \quad t \in [0,T]. \tag{3.43}$$

The solution U can be extended to a maximal mild solution in $[0,T_{\max})$ such that either

1. $T_{\max} = \infty$*, i.e., the problem admits a global mild solution; or*
2. $T_{\max} < \infty$*, and*

$$\lim_{t\uparrow T_{\max}} \|U(t)\|_H = \infty,$$

i.e., the solution blows up at a finite time $T_{\max}$.

Furthermore, if $u_0 \in D(A)$, then $u \in C([0,T_{\max}); D(A)) \cap C^1([0,T_{\max}); H)$ is classical solution.

By introduction $U=(u,v):=(u,u_t)$, $U_0=(u_0,u_1)$, and

$$\begin{aligned} A&=\begin{pmatrix} 0 & I \\ -\Delta^2-I & 0 \end{pmatrix}, \\ F(U)&=\begin{pmatrix} 0 \\ \sigma\,|x|^{-\alpha}*u+|u|^{p-2}\,u \end{pmatrix}, \end{aligned} \tag{3.44}$$

where I is the identity operator, (1.1) can be equivalently written as the following system

$$\begin{cases} U_t=AU+F(U) & x\in\Omega, t>0, \\ U=\dfrac{\partial U}{\partial\nu}=0, & x\in\partial\Omega, t>0 \\ U(x,0)=U_0(x), & x\in\Omega. \end{cases} \tag{3.45}$$

In the next lemma, we show A generates a contraction semigroup $\{e^{tA}\}_{t\geq 0}$ on $H_0^2(\Omega)\times L^2(\Omega)$.

Lemma 3.3. *Let A be the operator defined in (3.44). Then A generates a contraction semigroup $\{e^{tA}\}_{t\geq 0}$ on $H_0^2(\Omega)\times L^2(\Omega)$.*

Proof. Let $H:=H_0^2(\Omega)\times L^2(\Omega)$, then H is a Hilbert space with inner produce $(\cdot,\cdot)_H$ defined as

$$(\Phi,\Psi)_H:=\int\limits_\Omega(\Delta\varphi_1\Delta\psi_1+\varphi_1\psi_1+\varphi_2\psi_2)\,dx, \tag{3.46}$$

where $\Psi=(\varphi_1,\varphi_2),\ \Psi=(\psi_1,\psi_2)\in H$. Then

$$\|\Phi\|_H=(\Phi,\Phi)_H=\|\phi_1\|_{H^2}+\|\varphi_2\|.$$

Let A be the linear operator defined in (3.44), then

$$A:D(A)=(H^4(\Omega)\cap H_0^2(\Omega))\times H_0^2(\Omega)\subset H\to H.$$

Next we show A generates a C_0-semigroup on H by using Lemma 3.1. It is obvious $D(A)$ is dense in H, and for any $\Phi=(\varphi_1,\varphi_2)\in D(A)$, we have

$$\begin{aligned} (A\Phi,\Phi)_H&=\left(\left(\varphi_2,-\Delta^2\varphi_1-\varphi_1\right),(\varphi_1,\varphi_2)\right)_H \\ &=\int\limits_\Omega\left(\Delta\varphi_2\Delta\varphi_1+\varphi_2\varphi_1+\left(-\Delta^2\varphi_1-\varphi_1\right)\varphi_2\right)dx \\ &=0. \end{aligned} \tag{3.47}$$

Next we show $R(I-A)=H$. Fixed any $f=(f_1,f_2)\in H$, since $f_1+f_2\in L^2(\Omega)$, by standard theory of elliptic equation, the following problem

$$\begin{cases} \Delta^2u+2u=f_1+f_2, & x\in\Omega, \\ u=\dfrac{\partial u}{\partial\nu}=0, & x\in\partial\Omega \end{cases} \tag{3.48}$$

admits a unique solution $u \in H^4(\Omega) \cap H_0^2(\Omega)$. Let $v = u - f_1 \in H_0^2(\Omega)$. Then $U = (u, v)$ satisfies

$$\begin{aligned}(I-A)U &= \begin{pmatrix} I & -I \\ \Delta^2 + I & I \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \\ &= \begin{pmatrix} u - v \\ \Delta^2 u + u + v \end{pmatrix} \\ &= \begin{pmatrix} f_1 \\ \Delta^2 u + 2u - f_1 \end{pmatrix} \\ &= \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = f,\end{aligned}$$

which implies $R(I - A) = H$. Then by Lemma 3.1, A generates a contraction C_0-semigroup on H. □

Next, we show (3.45) admits a mild solution.

Lemma 3.4. *Assume $\alpha \in (-\infty, n)$ and $\sigma \geq 0$. Let H be the Hilbert space defined in Lemma 3.3, and $U_0 = (u_0, u_1) \in H = H_0^2(\Omega) \times L^2(\Omega)$. Then there exists a positive constant T depending only on $\|U_0\|_H = \|u_0\|_{H^2} + \|u_1\|$ such that problem (3.45) admits a unique mild solution $U(t)$, i.e., $U = (u, u_t) \in C([0, T]; H)$ and satisfies*

$$U(t) = e^{tA} U_0 + \int_0^t e^{(t-\tau)A} F(U(\tau)) d\tau, \ 0 \leq t \leq T. \tag{3.49}$$

The solution U can be extended to a maximal weak solution in $[0, T_{\max})$ such that either

1. *$T_{\max} = \infty$, i.e., the problem admits a global mild solution; or*
2. *$T_{\max} < \infty$, and*

$$\lim_{t \uparrow T_{\max}} \|U(t)\|_H = \lim_{t \uparrow T_{\max}} (\|u(t)\|_{H^2} + \|u_t(t)\|) = \infty,$$

i.e., the solution blows up at a finite time $T_{\max}$.

Furthermore, it holds

$$\|U(t)\|_H = \|U_0\|_H + 2 \int_0^t (F(U(\tau)), U(\tau))_H d\tau. \tag{3.50}$$

Proof. Let F be the nonlinear function defined in (3.44). In view of Lemmas 3.2 and 3.3, to prove local existence, uniqueness, and extension of mild solutions, we only need to show $F : H \to H$ satisfying the local Lipschitz condition.

First, we show $F(H) \subset H$. For any $U = (u, v) \in H$, by (3.44), to prove $F(U) \in H$, we only need to prove,

$$\sigma |x|^{-\alpha} * u + |u|^{p-2} u \in L^2(\Omega), \ \forall u \in H_0^2(\Omega). \tag{3.51}$$

Since $H_0^2(\Omega) \hookrightarrow L^{2(p-1)}(\Omega)$ (see (1.2)), it is obvious $|u|^{p-2}u \in L^2(\Omega)$.

Next we show $|x|^{-\alpha} * u \in L^2(\Omega)$. According the range of α, we divide the proof into two cases: $\alpha \leq 0$ and $\alpha \in (0, n)$.

Case 1. $\alpha \leq 0$. Since Ω is bound, we have

$$R = \sup_{x,y \in \Omega} |x - y| < \infty.$$

Then, by Hölder's inequality,

$$\begin{aligned}\left\||x|^{-\alpha} * u\right\|^2 &= \int\limits_\Omega \left(\int\limits_\Omega |x-y|^{-\alpha} u(y)dy \right)^2 dx \\ &\le R^{-2\alpha}|\Omega| \left(\int\limits_\Omega u(y)dy \right)^2 \\ &\le R^{-2\alpha}|\Omega|^2 \|u\|^2 < \infty.\end{aligned} \tag{3.52}$$

Case 2. $0 < \alpha < n$. Since $\alpha < n$, it follows $\frac{2n}{2n-\alpha} < 2$. For any $\phi \in L^2(\Omega)$, there exists a positive C_Ω depending only on Ω such that $\|\phi\|_{\frac{2n}{2n-\alpha}} \le C_\Omega \|\phi\|$. Since

$$\frac{1}{\frac{2n}{2n-\alpha}} + \frac{\alpha}{n} + \frac{1}{\frac{2n}{2n-\alpha}} = 2,$$

by using (2.16) with $q = r = \frac{2n}{2n-\alpha}$, $\theta = n - \alpha$, we get

$$\begin{aligned}\int\limits_\Omega \left(|x|^{-\alpha} * u\right)(x)\phi(x)dx &= \int\limits_\Omega \int\limits_\Omega \frac{u(y)\phi(x)}{|x-y|^\alpha} dxdy \\ &\le \kappa \|u\|_{\frac{2n}{2n-\alpha}} \|\phi\|_{\frac{2n}{2n-\alpha}} \\ &\le \kappa C_\Omega^2 \|u\| \|\phi\|.\end{aligned}$$

Then we get

$$\begin{aligned}\left\||x|^{-\alpha} * u\right\| &= \sup_{\phi \in L^2(\Omega), \|\phi\|=1} \int\limits_\Omega \left(|x|^{-\alpha} * u\right)(x)\phi(x)dx \\ &\le \kappa C_\Omega^2 \|u\| < \infty.\end{aligned} \tag{3.53}$$

So (3.51) is true.

Next we show F is locally Lipschitz continuous. Let $U_1 = (u_1, v_1) \in H$ and $U_2 = (u_2, v_2) \in H$ be such that

$$\|U_1\|_H = \|u_1\|_{H^2} + \|v_1\| \le M, \ \|U_2\|_H = \|u_2\|_{H^2} + \|v_2\| \le M, \tag{3.54}$$

where M is a positive constant. Let

$$\chi = \begin{cases} R^{-\alpha}|\Omega|, & \alpha \le 0 \\ \kappa C_\Omega^2, & 0 < \alpha < n. \end{cases}$$

Then, by (3.52) and (3.53),

$$\begin{aligned}\|F(U_1) - F(U_2)\|_H \le& |\sigma| \left\||x|^{-\alpha} * (u_1 - u_2)\right\| + \left\||u_1|^{p-2}u_1 - |u_2|^{p-2}u_2\right\| \\ \le& |\sigma|\chi\|u_1 - u_2\| \\ &+ (p-2)\left\| \int\limits_0^1 |\theta u_1 + (1-\theta)u_2|^{p-2} d\theta (u_1 - u_2) \right\|.\end{aligned}$$

Case 1. $n = 1, 2, 3, 4$. Since $H_0^2(\Omega) \hookrightarrow L^\infty(\Omega)$ with optimal constant $C_{H_0^2 \to L^\infty}$, in view of (3.54), we have

$$\begin{aligned}\left\| \int_0^1 |\theta u_1 + (1-\theta)u_2|^{p-2} d\theta (u_1 - u_2) \right\| &\le \left\| \int_0^1 |\theta u_1 + (1-\theta)u_2|^{p-2} d\theta \right\|_\infty \|u_1 - u_2\| \\ &\le (\|u_1\|_\infty + \|u_2\|_\infty)^{p-2} \|u_1 - u_2\| \\ &\le \left(2C_{H_0^2 \to L^\infty} M\right)^{p-2} \|u_1 - u_2\|.\end{aligned}$$

Case 2. $n \ge 5$. Since $H_0^2(\Omega) \hookrightarrow L^{\frac{2n}{n-4}}(\Omega)$ with optimal constant $C_{H_0^2 \to L^{\frac{2n}{n-4}}}$ and $H_0^2(\Omega) \hookrightarrow L^{\frac{n(p-2)}{2}}(\Omega)$ with optimal constant $C_{H_0^2 \to L^{\frac{n(p-2)}{2}}}$, in view of (3.54), we have

$$\begin{aligned}&\left\| \int_0^1 |\theta u_1 + (1-\theta)u_2|^{p-2} d\theta (u_1 - u_2) \right\|^2 \\ \le& \left(\int_\Omega \left(\int_0^1 |\theta u_1 + (1-\theta)u_2|^{p-2} d\theta \right)^{\frac{n}{2}} dx \right)^{\frac{n}{4}} \|u_1 - u_2\|^2_{\frac{2n}{n-4}} \\ \le& 2^{\frac{np}{2}} \left(C_{H_0^2 \to L^{\frac{2n}{n-4}}} \right)^2 \left(\int_\Omega \left(|u_1|^{\frac{n(p-2)}{2}} + |u_2|^{\frac{n(p-2)}{2}} \right) dx \right)^{\frac{n}{4}} \|u_1 - u_2\|^2_{H^2} \\ \le& 2^{\frac{np}{2} + \frac{n}{4}} \left(C_{H_0^2 \to L^{\frac{2n}{n-4}}} \right)^2 \left(C_{H_0^2 \to L^{\frac{n(p-2)}{2}}} M \right)^{\frac{n^2(p-2)}{8}} \|u_1 - u_2\|^2_{H^2}\end{aligned}$$

In view of the above three inequalities, we get F is locally Lipschitz continuous. Then the local existence and extension of mild solutions follows.

Next we prove (3.50). Suppose firstly $U_0 \in D(A) = (H^4(\Omega) \cap H_0^2(\Omega)) \times H_0^2(\Omega)$, then by Lemma 3.2, $U \in C([0, T_{\max}); D(A)) \cap C^1([0, T_{\max}); H)$ is a classical solution. Then it follows from (3.45) and (3.47) that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|U(t)\|_H^2 &= (U, U_t)_H \\ &= (U, A(U)) + (U, F(U)) = (U, F(U)), \ 0 \le t < T_{\max}.\end{aligned}$$

For fixed $t \in [0, T_{\max})$, integrating the above equality over $[0, t]$, we get (3.50).

In general case $U_0 \in H$, since $D(A)$ is dense in H, we approximate U_0 by a sequence $\{U_{n0}\}_{n=1}^\infty$, and then we pass to the limit to obtain (3.50). □

Proof of Theorem 1.2. **Step 1.** Existence of maximal weak solution. By Lemma 3.4 and Definition 1.1, to show the existence of maximal weak solution, we only need to prove the mild solution $U = (u, u_t)$ got in Lemma 3.4 satisfies (1.4).

We denote the inner produce of the Hilbert space $L^2(\Omega) \times L^2(\Omega)$ by $((\cdot, \cdot))$, i.e.,

$$((U, V)) = \int_\Omega (u_1 v_1 + u_2 v_2) dx, \ \forall\ U = (u_1, u_2), \ V = (v_1, v_2) \in L^2(\Omega) \times L^2(\Omega).$$

Since $C_0^\infty(\Omega)$ is dense in $H_0^2(\Omega)$, by density arguments, we only need to prove (1.4) with $\phi \in C_0^\infty(\Omega)$. Let $U(t) = (u, u_t) \in C\left([0, T_{\max}); H_0^2(\Omega) \times L^2(\Omega)\right)$ be the mild solution of (3.45) got in Lemma 3.4 and $\Phi = (0, \phi)$. For fixed $t \in [0, T_{\max})$, by using (3.49), we get

$$((U, \Phi)) = \left(\left(e^{tA}U_0, \Phi\right)\right) + \left(\left(\int_0^t e^{(t-\tau)A} F(U(\tau)), \Phi\right)\right).$$

We differentiate to obtain

$$\frac{d}{dt}((U, \Phi)) = \frac{d}{dt}\left(\left(e^{tA}U_0, \Phi\right)\right) + \frac{d}{dt}\left(\left(\int_0^t e^{(t-\tau)A} F(U(\tau)), \Phi\right)\right). \tag{3.55}$$

Now, using the standard properties of the semigroup (see for example, [54, Chapter 2]), we obtain

$$\frac{d}{dt}\left(\left(e^{tA}U_0, \Phi\right)\right) = \left(\left(e^{tA}U_0, A^*\Phi\right)\right) + \left(\left(e^{tA}U_0, \Phi_t\right)\right) \tag{3.56}$$

where

$$A^* = \begin{pmatrix} 0 & -\Delta^2 - I \\ I & 0 \end{pmatrix}$$

is the adjoint operator of A; and

$$\begin{aligned} \frac{d}{dt}&\left(\left(\int_0^t e^{(t-\tau)A} F(U(\tau)), \Phi\right)\right) \\ &= ((F(U(t)), \Phi)) + \left(\left(\int_0^t e^{(t-\tau)A} F(U(\tau)), A^*\Phi\right)\right) \\ &\quad + \left(\left(\int_0^t e^{(t-\tau)A} F(U(\tau)), \Phi_t\right)\right). \end{aligned} \tag{3.57}$$

Then it follows from (3.55)-(3.57) and (3.49) that

$$\frac{d}{dt}((U, \Phi)) = ((F(U(t)), \Phi)) + ((U, A^*\Phi)) + ((U, \Phi_t)). \tag{3.58}$$

Since $U = (u, u_t)$ and $\Phi = (0, \phi)$, we have

$$((U, \Phi)) = \int_\Omega u_t \phi dx,$$

$$((F(U(t)), \Phi)) = \int_\Omega \left(\sigma|x|^{-\alpha} * u + |u|^{p-2}u\right) \phi dx,$$

$$((U, A^*\Phi)) = \left(\left((u, u_t), (-\Delta^2\phi - \phi, 0)\right)\right) = -\int_\Omega \left(u\Delta^2\phi + u\phi\right) dx,$$

$$((U, \Phi_t)) = \int_\Omega u_t \phi_t dx.$$

Then it follows from (3.58) that

$$\frac{d}{dt}\int_\Omega u_t\phi dx+\int_\Omega \left(u\Delta^2\phi+u\phi\right)dx=\int_\Omega \left(\sigma|x|^{-\alpha}*u+|u|^{p-2}u\right)\phi dx+\int_\Omega u_t\phi_t dx.$$

Since $u\in C\left([0,T_{\max});H_0^2(\Omega)\right)$, integrating by parts, we get

$$\begin{aligned}&\frac{d}{dt}\int_\Omega u_t\phi dx+\int_\Omega \left(\Delta u\Delta\phi+u\phi\right)dx\\&=\int_\Omega \left(\sigma|x|^{-\alpha}*u+|u|^{p-2}u\right)\phi dx+\int_\Omega u_t\phi_t dx.\end{aligned}\tag{3.59}$$

Note $\phi_t=0$, integrating in time over $[0,t]$ for any $t\in(0,T_{\max})$, we obtain (1.4).

Step 2. Proof of (u_t,u) in $C^1[0,T_{\max})$ and the equality (1.8). Since

$$u(t)\in C\left([0,T_{\max});H_0^2(\Omega)\right)\ \text{ and }\ u_t\in C\left([0,T_{\max});L^2(\Omega)\right),$$

by taking $\phi=u(t)$ in (3.59), we get

$$\begin{aligned}\frac{d}{dt}(u_t,u)=\|u_t\|^2-&\left(\overbrace{\|\Delta u\|^2+\|u\|^2-\|u\|_p^p-\sigma\int_\Omega\int_\Omega\frac{u(x,t)u(y,t)}{|x-y|^\alpha}dxdy}^{=I(u)}\right)\\&\in C[0,T_{\max}),\end{aligned}$$

i.e, (u_t,u) in $C^1[0,T_{\max})$ and (1.8) holds.

Step 3. Proof of the equality (1.5). The energy identity (1.5) follows from (3.50) directly. In fact by using $U=(u,u_t)$, (3.44), and (3.46), we have

$$\begin{aligned}\|U\|_H&=\|\Delta u\|^2+\|u\|^2+\|u_t\|^2,\\(F(U),U)_H&=\frac{d}{dt}\left(\frac{\sigma}{2}\int_\Omega\int_\Omega\frac{u(x,t)u(y,t)}{|x-y|^\alpha}dxdy+\frac{1}{p}\|u\|_p^p\right).\end{aligned}$$

Then by (3.50), we get (1.5). □

4. Global existence and finite time blow-up.

Proof of Theorem 1.3. Let $u\in C\left([0,T_{\max});H_0^2(\Omega)\right)\cap C^1\left([0,T_{\max});L^2(\Omega)\right)$ be the maximal weak solution got in Theorem 1.2. By $E(0)\le 0$ and (1.5), it holds,

$$E(t)=E(0)\le 0,\ \ 0\le t<T_{\max}.$$

By the definitions of J and I (see (1.9) and (1.10)), we get

$$\begin{aligned}I(u)&=2J(u)-\frac{p-2}{p}\|u\|_p^p,\\I(u)&=pJ(u)-\frac{p-2}{2}\left(\|\Delta u\|^2+\|u\|^2-\sigma\int_\Omega\int_\Omega\frac{u(x,t)u(y,t)}{|x-y|^\alpha}dxdy\right).\end{aligned}$$

By (1.5) and (1.6), it follows $J(u)=E(0)-\frac{1}{2}\|u_t\|^2$. Then, by the above two inequalities, we get

$$I(u)=2E(0)-\|u_t\|^2-\frac{p-2}{p}\|u\|_p^p\tag{4.60}$$

and

$$
\begin{aligned}
I(u) =& pE(0) - \frac{p}{2}\|u_t\|^2 \\
&- \frac{p-2}{2}\left(\|\Delta u\|^2 + \|u\|^2 - \sigma \int_\Omega \int_\Omega \frac{u(x,t)u(y,t)}{|x-y|^\alpha} dxdy\right).
\end{aligned} \tag{4.61}
$$

By (1.11), we get

$$
\left|\sigma \int_\Omega \int_\Omega \frac{u(x,t)u(y,t)}{|x-y|^\alpha} dxdy\right| \leq \|\Delta u\|^2 + \|u\|^2 \text{ if } |\sigma| \leq \sigma^*. \tag{4.62}
$$

In the following we divide the proof into two cases: $|\sigma| \leq \sigma^*$ and $|\sigma| > \sigma^*$.

Case 1. $|\sigma| \leq \sigma^*$. It follows from (4.61) and (4.62) that

$$
I(u) \leq pE(0) - \frac{p}{2}\|u_t\|^2. \tag{4.63}
$$

Let

$$
h(t) = \|u(t)\|^2 + \beta(t+\gamma)^2, \; 0 \leq t < T_{\max}, \tag{4.64}
$$

where $\beta \geq 0$ and $\gamma \geq 0$ are two constants to be determined later. Then by using (1.8) and (4.63), we have

$$
h'(t) = 2(u_t, u) + 2\beta(t+\gamma), \tag{4.65}
$$

$$
h''(t) = 2\|u_t\|^2 - 2I(u)(t) + 2\beta \geq -2pE(0) + (p+2)\|u_t\|^2 + 2\beta. \tag{4.66}
$$

By Cauchy-Schwartz's inequality,

$$
\begin{aligned}
(h'(t))^2 &\leq 4\left(\|u_t\|\|u\| + \beta(t+\gamma)\right)^2 \\
&= 4\left(\|u\|^2 \|u_t\|^2 + \beta^2(t+\gamma)^2 + 2\beta(t+\gamma)\|u\|\|u_t\|\right) \\
&\leq 4\left(\|u\|^2 \|u_t\|^2 + \beta^2(t+\gamma)^2 + \beta(t+\gamma)^2\|u_t\|^2 + \beta\|u\|^2\right) \\
&\leq 4\left(\left(\|u\|^2 + \beta(t+\gamma)^2\right)\left(\|u_t\|^2 + \beta\right)\right) \\
&= 4h(t)\left(\|u_t\|^2 + \beta\right).
\end{aligned} \tag{4.67}
$$

Then by (4.66) and (4.67), it follows

$$
\begin{aligned}
h''(t)h(t) - \left(1 + \frac{p-2}{4}\right)(h'(t))^2 &\geq p(-2E(0) - \beta)h(t) \\
&\geq 0 \text{ for } 0 \leq \beta \leq -2E(0).
\end{aligned} \tag{4.68}
$$

Subcase 1. $E(0) \leq 0$ and $(u_0, u_1) > 0$. We take $\beta = \gamma = 0$, then $h(0) = \|u_0\|^2 > 0$ and $h'(0) = 2(u_0, u_1) > 0$. Then, it follows from Lemma 2.3 that h blows up at a finite time $\hat{T}$, $\hat{T} \geq T_{\max}$ (by Theorem 1.2), and

$$
\hat{T} \leq \frac{h(0)}{\frac{p-2}{4}h'(0)} = \frac{2\|u_0\|^2}{(p-2)(u_0, u_1)}.
$$

Subcase 2. $E(0) < 0$ and $(u_0, u_1) = 0$. We take

$$
\beta = -2E(0) \text{ and } \gamma = \frac{\|u_0\|}{\sqrt{\beta}},
$$

then $h(0)=\|u_0\|^2+\beta\gamma^2=2\|u_0\|^2>0$, $h'(0)=2\beta\gamma=2\sqrt{-2E(0)}\|u_0\|>0$. Then, it follows from Lemma 2.3 that h blows up at a finite time $\hat{T}$, $\hat{T}\geq T_{\max}$ (by Theorem 1.2), and

$$\hat{T}\leq\frac{h(0)}{\frac{p-2}{4}h'(0)}=\frac{4\|u_0\|}{(p-2)\sqrt{-2E(0)}}.$$

Subcase 3. $E(0)<0$ and $(u_0,u_1)<0$. We take

$$\beta=-2E(0),\ \gamma=\frac{-(u_0,u_1)+\sqrt{(u_0,u_1)^2+\beta\|u_0\|^2}}{\beta},$$

then,

$$\begin{aligned}h(0)&=\|u_0\|^2+\beta\gamma^2=\|u_0\|^2+\left(-(u_0,u_1)+\sqrt{(u_0,u_1)^2+\beta\|u_0\|^2}\right)^2/\beta>0,\\ h'(0)&=2(u_0,u_1)+2\beta\gamma=2\sqrt{(u_0,u_1)^2+\beta\|u_0\|^2)}>0.\end{aligned}$$

Then, it follows from Lemma 2.3 that h blows up at a finite time $\hat{T}$, $\hat{T}\geq T_{\max}$ (by Theorem 1.2), and

$$\begin{aligned}\hat{T}\leq\frac{h(0)}{\frac{p-2}{4}h'(0)}&=\frac{2\left(\|u_0\|^2+\frac{\left(-(u_0,u_1)+\sqrt{(u_0,u_1)^2+\beta\|u_0\|^2)}\right)^2}{\beta}\right)}{(p-2)\sqrt{(u_0,u_1)^2+\beta\|u_0\|^2)}}\\ &=\frac{2\beta\left\|u_0\right\|^2+2\left(-(u_0,u_1)+\sqrt{(u_0,u_1)^2+\beta\|u_0\|^2}\right)^2}{\beta(p-2)\sqrt{(u_0,u_1)^2+\beta\left\|u_0\right\|^2}}\\ &=\frac{-4E(0)\left\|u_0\right\|^2+2\left(-(u_0,u_1)+\sqrt{(u_0,u_1)^2-2E(0)\|u_0\|^2}\right)^2}{-2E(0)(p-2)\sqrt{(u_0,u_1)^2-2E(0)\left\|u_0\right\|^2}}\end{aligned}$$

Case 2. $|\sigma|>\sigma^*$. Firstly we estimate $\int_\Omega\int_\Omega|x-y|^{-\alpha}u(x,t)u(y,t)dxdy$.

If $\alpha\leq 0$, let $R=\sup_{x,y\in\Omega}|x-y|$, then by Hölder's inequality, we get

$$\begin{aligned}\left|\int_\Omega\int_\Omega\frac{u(x,t)u(y,t)}{|x-y|^\alpha}dxdy\right|&\leq R^{-\alpha}\left(\int_\Omega|u|dx\right)^2\\ &\leq R^{-\alpha}|\Omega|^{\frac{2(p-1)}{p}}\|u\|_p^2.\end{aligned}\tag{4.69}$$

If $0<\alpha<n$, since

$$\frac{1}{\frac{2n}{2n-\alpha}}+\frac{\alpha}{n}+\frac{1}{\frac{2n}{2n-\alpha}}=2$$

and $\frac{2n}{2n-\alpha}<p$, by using (2.16) with $q=r=\frac{2n}{2n-\alpha}$, $\theta=n-\alpha$ and Hölder's inequality, we get

$$\begin{aligned}\left|\int_\Omega\int_\Omega\frac{u(x,t)u(y,t)}{|x-y|^\alpha}dxdy\right|&\leq\kappa\|u\|^2_{\frac{2n}{2n-\alpha}}\\ &\leq\kappa|\Omega|^{\frac{p(2n-\alpha)-2n}{n}}\|u\|_p^2,\end{aligned}\tag{4.70}$$

where κ is the positive constant defined in (2.15).

Let

$$\Theta = \begin{cases} R^{-\alpha}|\Omega|^{\frac{2(p-1)}{p}}, & \text{if } \alpha \le 0, \\ \kappa|\Omega|^{\frac{p(2n-\alpha)-2n}{n}}, & \text{if } 0 < \alpha < n. \end{cases}$$

By (4.69) and (4.70), we get

$$\left| \int_\Omega \int_\Omega \frac{u(x,t)u(y,t)}{|x-y|^\alpha} dxdy \right| \le \Theta \|u\|_p^2. \tag{4.71}$$

Subcase 1. $E(0) < 0$ and $(u_0, u_1) \ge 0$; or $E(0) = 0$ and $(u_0, u_1) > 0$. By (1.8), (4.60), and $E(0) \le 0$, we get

$$\frac{d}{dt}(u_t, u) \ge 0.$$

Then

$$\frac{d}{dt}\|u(t)\|^2 = (u_t, u) \ge (u_1, u_0) \ge 0,$$

and then $\|u\| \ge \|u_0\|$. Then it follows from the Hölder's inequality that

$$\|u_0\|^2 \le \|u\|^2 \le |\Omega|^{\frac{p-2}{p}} \|u\|_p^2,$$

which, together with (4.71) implies

$$\left| \int_\Omega \int_\Omega \frac{u(x,t)u(y,t)}{|x-y|^\alpha} dxdy \right| \le \Theta \|u\|_p^{2-p}\|u\|_p^p \le \Theta |\Omega|^{\frac{(p-2)^2}{2p}} \|u_0\|^{2-p}\|u\|_p^p. \tag{4.72}$$

In view of (1.11), (4.61) and (4.72), we obtain

$$\begin{aligned} I(u) \le & pE(0) - \frac{p}{2}\|u_t\|^2 \\ & - \frac{p-2}{2}\left(\|\Delta u\|^2 + \|u\|^2 - |\sigma| \left| \int_\Omega \int_\Omega \frac{u(x,t)u(y,t)}{|x-y|^\alpha} dxdy \right| \right) \\ = & pE(0) - \frac{p}{2}\|u_t\|^2 \\ & - \frac{p-2}{2}\left(\|\Delta u\|^2 + \|u\|^2 - \sigma^* \left| \int_\Omega \int_\Omega \frac{u(x,t)u(y,t)}{|x-y|^\alpha} dxdy \right| \right) \\ & + \frac{p-2}{2}(|\sigma| - \sigma^*) \left| \int_\Omega \int_\Omega \frac{u(x,t)u(y,t)}{|x-y|^\alpha} dxdy \right| \\ \le & pE(0) - \frac{p}{2}\|u_t\|^2 + \frac{p-2}{2}\Lambda \|u\|_p^p, \end{aligned} \tag{4.73}$$

where

$$\Lambda = (|\sigma| - \sigma^*)\Theta|\Omega|^{\frac{(p-2)^2}{2p}} \|u_0\|^{2-p}. \tag{4.74}$$

It follows $(\frac{\Lambda}{2} \times (4.60) + \frac{1}{p} \times (4.73)) \times \frac{2p}{p\Lambda+2}$ that

$$I(u) \le \frac{2p(\Lambda+1)}{p\Lambda+2}E(0) - \frac{p(\Lambda+1)}{p\Lambda+2}\|u_t\|^2 \tag{4.75}$$

Let h be the function defined in (4.64). Then by using (1.8) and (4.75), we have

$$h'(t) = 2(u_t, u) + 2\beta(t+\gamma) \tag{4.76}$$

and

$$\begin{aligned} h''(t) &= 2\|u_t\|^2 - 2I(u)(t) + 2\beta \\ &\geq -\frac{4p(\Lambda+1)}{p\Lambda+2}E(0) + \left(\frac{2p(\Lambda+1)}{p\Lambda+2} + 2\right)\|u_t\|^2 + 2\beta. \end{aligned} \tag{4.77}$$

Then it follows from (4.67) that

$$\begin{aligned} &h''(t)h(t) - \left(1 + \frac{p-2}{2(p\Lambda+2)}\right)(h'(t))^2 \\ \geq &\frac{2p(\Lambda+1)}{p\Lambda+2}\left(-2E(0) - \beta\right)h(t) \geq 0 \text{ for } 0 \leq \beta \leq -2E(0). \end{aligned} \tag{4.78}$$

If $E(0) \leq 0$ and $(u_0, u_1) > 0$, we take $\beta = \gamma = 0$, then $h(0) = \|u_0\|^2 > 0$ and $h'(0) = 2(u_0, u_1) > 0$. Then, it follows from Lemma 2.3 that h blows up at a finite time $\hat{T}$, $\hat{T} \geq T_{\max}$ (by Theorem 1.2), and

$$\hat{T} \leq \frac{h(0)}{\frac{p-2}{2(p\Lambda+2)}h'(0)} = \frac{(p\Lambda+2)\|u_0\|^2}{(p-2)(u_0,u_1)}.$$

If $E(0) < 0$ and $(u_0, u_1) = 0$, we take

$$\beta = -2E(0) \text{ and } \gamma = \frac{\|u_0\|}{\sqrt{\beta}},$$

then $h(0) = \|u_0\|^2 + \beta\gamma^2 = 2\|u_0\|^2 > 0$, $h'(0) = 2\beta\gamma = 2\sqrt{-2E(0)}\|u_0\| > 0$. Then, it follows from Lemma 2.3 that h blows up at a finite time $\hat{T}$, $\hat{T} \geq T_{\max}$ (by Theorem 1.2), and

$$\hat{T} \leq \frac{h(0)}{\frac{p-2}{2(p\Lambda+2)}h'(0)} = \frac{2(p\Lambda+2)\|u_0\|}{(p-2)\sqrt{-2E(0)}}.$$

Subcase 2. $E(0) < 0$ and $(u_0, u_1) < 0$. We prove u blows up by contradiction argument. Assume u exists globally, i.e., $T_{\max} = \infty$.

By (4.60), $-I(u)(t) \geq -2E(0) > 0$, $t \in [0, \infty)$. Then it follows (1.8),

$$\frac{d}{dt}(u_t, u) \geq -I(u)(t) \geq -2E(0) > 0.$$

Integrating this inequality from 0 to t we get

$$(u_t, u) \geq (u_0, u_1) - 2E(0)t.$$

Let $t_0 = \frac{(u_0,u_1)}{2E(0)}$, then $(u(t_0), u_t(t_0)) \geq 0$. Moreover, $E(t_0) = E(0) < 0$. We come back to subcase 1. Then $\|u\|$ will become infinite in finite time, which contradicts $T_{\max} = \infty$. □

Proof of Theorem 1.4. Let $u \in C\left([0, T_{\max}); H_0^2(\Omega)\right) \cap C^1\left([0, T_{\max}); L^2(\Omega)\right)$ be the weak solution got in Theorem 1.2 with $(u_0, u_1) \in H_0^2(\Omega) \times L^2(\Omega)$ satisfying $I(u_0) > 0$ and $0 < E(0) \leq d$.

Step 1. $E(0) < d$; or $E(0) = d$ and $\|u_1\| > 0$. By (1.7), $J(u_0) < d$. Since, we also have $I(u_0) > 0$, we get $u_0 \in \mathcal{W}$ (see (2.24) for the definition of $\mathcal{W}$). Then, by Lemma 2.6, $u(t) \in \mathcal{W}$ for all $t \in [0, T_{\max})$. Since $\mathcal{W} = \mathcal{W}_1$ (see Lemma 2.5), it follows from the definition of $\mathcal{W}_1$ (see (2.28)) that

$$\|\Delta u\|^2 + \|u\|^2 < \frac{2pd}{p-2} + \sigma \int\limits_\Omega \int\limits_\Omega \frac{u(x,t)u(y,t)}{|x-y|^\alpha} dxdy,\ 0 \le t < T_{\max}. \tag{4.79}$$

By (1.11), it follows

$$\left| \int\limits_\Omega \int\limits_\Omega \frac{u(x,t)u(y,t)}{|x-y|^\alpha} dxdy \right| \le \frac{1}{\sigma^*} \left(\|\Delta u\|^2 + \|u\|^2 \right),$$

which, together with (4.79) and $|\sigma| < \sigma^*$, implies

$$\|\Delta u\|^2 + \|u\|^2 \le \left(1 - \frac{|\sigma|}{\sigma^*}\right)^{-1} \frac{2pd}{p-2},\ 0 \le t < T_{\max}. \tag{4.80}$$

Since $E(t) = E(0)$ (see (1.5)), $E(t) = \frac{1}{2}\|u_t\|^2 + J(u)(t)$ (see (1.6)), $2J(u) - I(u) = \frac{p-2}{p}\|u\|_p^p$ (see (2.30)), $I(u)(t) > 0$ (since $u(t) \in \mathcal{W}$) for all $t \in [0, T_{\max})$, and $E(0) \le d$, it holds

$$\|u_t\|^2 = 2E(0) - 2J(u)(t) \le 2d - I(u) < 2d,\ 0 \le t < T_{\max}. \tag{4.81}$$

By (4.80), (4.81), and Theorem 1.2, we get $T_{\max} = \infty$.

Step 2. $E(0) = d$ and $\|u_1\| = 0$. Since $E(t) = E(0)$ (see (1.5)), $E(t) = \frac{1}{2}\|u_t\|^2 + J(u)(t)$ (see (1.6)) and $E(0) = \frac{1}{2}\|u_1\|^2 + J(u_0)$ (see (1.7)), we get

$$J(u)(t) \le d,\ t \in [0, T_{\max}) \text{ and } J(u_0) = d.$$

If $J(u(t)) \equiv d$ for all $t \in [0, T_{\max})$, then by (1.6), $\|u_t\| \equiv 0$ for $t \in [0, T_{\max})$, then $u(t) \equiv u_0$, so by Theorem 1.2, $T_{\max} = \infty$; If there exists $t_1 \in [0, T_{\max})$ such that $J(u)(t_1) < d$, we claim

- there exists a constant $\sigma > 0$ sufficient small, and a sequence $\{t_n\}_{n=1}^\infty$ such that $\sigma \ge t_n \downarrow 0$ as $n \uparrow \infty$ and $J(u)(t_n) < d$ for $n = 1, 2, \cdots$.

In fact, if the claim is not true, there must exists a constant $\delta > 0$ such that $J(u)(t) = d$ for $t \in [0, \delta]$. Then by analysis as the first case, $u(t) \equiv u_0$ for $t \in [0, \delta]$; and then $u(t) \equiv u_0$ for $t \in [\delta, 2\delta]$, $[2\delta, 3\delta]$, etc. So $u(t) \equiv u_0$ for all $t \in [0, T_{\max})$, then $J(u)(t_1) = J(u_0) = d$, a contradiction. So the claim is true, since $I(u_0) > 0$ and $I(u)(t) \in C[0, T_{\max})$, $\lim_{n\uparrow\infty} I(u)(t_n) = I(u_0) > 0$. So for n large enough, we have $J(u)(t_n) < d$ and $I(u)(t_n) > 0$, we come back to step 1. □

Proof of Theorem 1.5. Let $u \in C\left([0, T_{\max}); H_0^2(\Omega)\right) \cap C^1\left([0, T_{\max}); L^2(\Omega)\right)$ be the weak solution got in Theorem 1.2 with $(u_0, u_1) \in H_0^2(\Omega) \times L^2(\Omega)$.

If $E(0) < d$, by (1.7), we get $J(u_0) < d$; If $E(0) = d$ and $(u_0, u_1) > 0$, we must have $\|u_1\| > 0$, and by (1.7) again, we get $J(u_0) < d$. Now we prove $\|u_1\| > 0$ by contradiction argument. In fact if $\|u_1\| = 0$, by Cauchy-Schwartz's inequality, we have $0 < (u_0, u_1) \le \|u_0\|\|u_1\| = 0$, a contradiction.

Since $J(u_0) < d$ and $I(u_0) < 0$, we get $u_0 \in \mathcal{V}$ (see (2.25)). Then by Lemma 2.6, $u(t) \in \mathcal{V}$ for $t \in [0, T_{\max})$, where $\mathcal{V}$ is the set defined in (2.25). Since $\mathcal{V} = \mathcal{V}_1$ (see Lemma 2.5), by (2.29) and (4.61),

$$I(u) \le pE(0) - \frac{p}{2}\|u_t\|^2 - pd,\ 0 \le t < T_{\max}. \tag{4.82}$$

Let $h(t)$ be the function defined in (4.64), with $\beta \geq 0$ and $\gamma \geq 0$ to be determined later. Then by using (1.8) and (4.82), we have

$$h'(t) = 2(u_t, u) + 2\beta(t+\gamma), \tag{4.83}$$

$$h''(t) = 2\|u_t\|^2 - 2I(u)(t) + 2\beta \geq -2pE(0) + (p+2)\|u_t\|^2 + 2pd + 2\beta. \tag{4.84}$$

Then it follows from (4.67) that

$$\begin{aligned} h''(t)h(t) - \left(1 + \frac{p-2}{4}\right)(h'(t))^2 &\geq p[2(d-E(0)) - \beta] \\ &\geq 0 \text{ for } 0 \leq \beta \leq 2(d-E(0)). \end{aligned} \tag{4.85}$$

Case 1. $E(0) \leq d$ and $(u_0, u_1) > 0$. We take $\beta = \gamma = 0$, then $h(0) = \|u_0\|^2 > 0$ and $h'(0) = 2(u_0, u_1) > 0$. Then, it follows from Lemma 2.3 that h blows up at a finite time $\hat{T}$, $\hat{T} \geq T_{\max}$ (by Theorem 1.2), and

$$\hat{T} \leq \frac{h(0)}{\frac{p-2}{4}h'(0)} = \frac{2\|u_0\|^2}{(p-2)(u_0, u_1)}.$$

Case 2. $E(0) < d$ and $(u_0, u_1) = 0$. We take

$$\beta = 2(d - E(0)), \ \gamma = \frac{\|u_0\|}{\sqrt{\beta}},$$

then $h(0) = \|u_0\|^2 + \beta\gamma^2 = 2\|u_0\|^2 > 0$, $h'(0) = 2\beta\gamma = 2\sqrt{2(d-E(0))}\|u_0\| > 0$. Then, it follows from Lemma 2.3 that h blows up at a finite time $\hat{T}$, $\hat{T} \geq T_{\max}$ (by Theorem 1.2), and

$$\hat{T} \leq \frac{h(0)}{\frac{p-2}{4}h'(0)} = \frac{4\|u_0\|}{(p-2)\sqrt{2(d-E(0))}}.$$

Case 3. $E(0) < d$ and $(u_0, u_1) < 0$. We take

$$\beta = 2(d - E(0)), \ \gamma = \frac{-(u_0, u_1) + \sqrt{(u_0, u_1)^2 + \beta\|u_0\|^2)}}{\beta},$$

then,

$$h(0) = \|u_0\|^2 + \beta\gamma^2 = \|u_0\|^2 + \left(-(u_0, u_1) + \sqrt{(u_0, u_1)^2 + \beta\|u_0\|^2)}\right)^2 / \beta > 0,$$

$$h'(0) = 2(u_0, u_1) + 2\beta\gamma = 2\sqrt{(u_0, u_1)^2 + \beta\|u_0\|^2)} > 0.$$

Then, it follows from Lemma 2.3 that h blows up at a finite time $\hat{T}$, $\hat{T} \geq T_{\max}$ (by Theorem 1.2), and

$$\begin{aligned} \hat{T} \leq \frac{h(0)}{\frac{p-2}{4}h'(0)} &= \frac{2\left(\|u_0\|^2 + \frac{\left(-(u_0,u_1)+\sqrt{(u_0,u_1)^2+\beta\|u_0\|^2)}\right)^2}{\beta}\right)}{(p-2)\sqrt{(u_0, u_1)^2 + \beta\|u_0\|^2)}} \\ &= \frac{2\beta\left\|u_0\right\|^2 + 2\left(-(u_0, u_1) + \sqrt{(u_0, u_1)^2 + \beta\|u_0\|^2}\right)^2}{\beta(p-2)\sqrt{(u_0, u_1)^2 + \beta\left\|u_0\right\|^2}} \end{aligned}$$

$$= \frac{4(d - E(0)) \|u_0\|^2 + 2\left(-(u_0, u_1) + \sqrt{(u_0, u_1)^2 + (2(d - E(0)))\|u_0\|^2}\right)^2}{(2(d - E(0)))(p - 2)\sqrt{(u_0, u_1)^2 + (2(d - E(0)))\|u_0\|^2}}.$$

□

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