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FINITE TIME BLOW-UP AND GLOBAL EXISTENCE OF SOLUTIONS FOR SEMILINEAR PARABOLIC EQUATIONS WITH NONLINEAR DYNAMICAL BOUNDARY CONDITION

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ABSTRACT. For a class of semilinear parabolic equations under nonlinear dynamical boundary conditions in a bounded domain, we obtain finite time blow-up solutions when the initial data varies in the phase space $H_0^1(\Omega)$ at positive initial energy level and get global solutions with the initial data at low and critical energy level. Our main tools are potential well method and concavity method.

1. INTRODUCTION

This paper deals with global well-posedness of semilinear parabolic equations with a nonlinear dynamical boundary condition

$$\begin{cases} u_t - \Delta u = \lambda |u|^{p-1}u, & x \in \Omega,\ t > 0, \\ u_\nu + u_t = \mu |u|^{q-1}u, & x \in \Gamma,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $\partial\Omega = \Gamma$, ν is the outer normal on Γ, $\lambda, \mu \in \{-1, 0, 1\}$, $\max\{\lambda, \mu\} = 1$, $\lambda p + \mu q > 0$, $1 < p, q < \infty$, $N = 1, 2$; $1 < p < \frac{N+2}{N-2}, 1 < q < \frac{N}{N-2}$, $N \geq 3$, and $u_0(x) \in H^1(\Omega)$.

The conditions of λ and μ imply that there are five cases for (λ, μ), i.e., $(\lambda, \mu) = (1, -1)$, $(\lambda, \mu) = (1, 0)$, $(\lambda, \mu) = (-1, 1)$, $(\lambda, \mu) = (0, 1)$, $(\lambda, \mu) = (1, 1)$. Obviously, when $\mu = 0$, the boundary condition becomes $u_\nu + u_t = 0$; when $\lambda = 0$, the equation in (1) becomes a homogeneous one.

The dynamical boundary conditions, although not too widely considered in the mathematical literature, are very natural in many mathematical models as heat transfer in a solid in contact with moving fluid, thermoelasticity, diffusion phenomena, problems in fluid dynamics, etc. (see [3, 5, 6, 7, 8, 9, 10, 11, 13, 19, 20] and the references therein). We note, however, that most of the literatures concern with the existence and uniqueness of local solutions, while few papers focus on global existence of solutions for parabolic equations with nonlinear dynamical boundary conditions.

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Hintermann [13] considered the local well-posedness of the following initial boundary value problem

$$\begin{aligned} &u_t - \Delta u = f, \quad \text{in} \ \ \Omega \times R_+, \\ &u_\nu + u_t = g, \quad \text{on} \ \ \partial\Omega \times R_+, \\ &u(0) = u_0, \qquad \text{in} \ \ \bar{\Omega}, \end{aligned}$$

where f, g may depend on space, time or the solutions. Under some assumptions, the author transformed the equations to abstract Cauchy problems, and applied the semigroup theory in Banach spaces to prove the local well-posedness results.

Local existence and uniqueness of solution to general quasilinear parabolic equation (systems) with dynamical boundary condition has been established in a series of papers by Escher [7, 8, 9]. For example, Escher in [7] studied the existence and uniqueness of local solutions for the following problem

$$\begin{aligned} &\partial_t u - \partial_j[a_{jk}(u,\cdot)\partial_k u] + a_j(u,\cdot)\partial_j u + a_0(u,\cdot)u = f(u), && \text{in } \Omega \times (0,\infty), \\ &\epsilon\partial_t u + a_{jk}(u,\cdot)\nu_j\partial_k u + b_0(u,\cdot)u = g_1(u), && \text{on } \Gamma_1 \times (0,\infty), \\ &a_{jk}(u,\cdot)(u,\cdot)\nu_j\partial_k u + b_0(u,\cdot)u = g_2(u), && \text{on } \Gamma_2 \times (0,\infty), \\ &u(\cdot,0) = u_0, && \text{in } \Omega. \end{aligned}$$

The author proved that if u_0 is in H_p^τ, the Besov potential space, then there exists a unique maximal weak solution in an open time interval. He also pointed out that if there is a uniform priori estimate of a solution on the space H_p^τ, then the solution can be extended to be a global one.

J. von Below and G. Pincet-Mailly [3] dealt with the blow-up phenomena of the nonlinear parabolic problem

$$\begin{aligned} &\partial_t u = \Delta u + f(u), \quad x \in \Omega, t > 0, \\ &\sigma\partial_t u + \partial_\nu u = 0, \qquad x \in \partial\Omega, t > 0, \\ &u(x,0) = \varphi \in C(\bar{\Omega}) \end{aligned}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$. Their main tools are comparison principle, energy methods and spectral comparison. The paper also contains some results concerning the upper bound of the blow-up time.

Wu [20] studied the asymptotic behavior of the solution to the parabolic problem with a linear dynamical boundary condition

$$\begin{aligned} &u_t - \Delta u + f(u) = 0, \quad (x,t) \in \Omega \times \mathbb{R}_+, \\ &\partial_\nu u + \mu u + u_t = 0, \qquad (x,t) \in \Gamma \times \mathbb{R}_+, \\ &u|_{t=0} = u_0(x), \end{aligned}$$

where $\Omega \subset \mathbb{R}^n$ $(n \in \mathbb{N}^+)$ is a bounded domain with smooth boundary Γ, ν is the outward normal direction to the boundary and $\mu \in \{0, 1\}$, f is analytic with respect to u. He proved that a global solution converges to an equilibrium, i.e., the solution of the concerned stationary problem, as time goes to infinity by means of a generalized Łojasiewicz-Simon-type inequality.

In [10], Fan and Zhong studied the global attractors parabolic equations with dynamic boundary conditions

$$u_t - \Delta u + f(u) = 0, \quad \text{in} \ \ \Omega,$$

$$u_t + \frac{\partial u}{\partial \nu} + f(u) = 0, \quad \text{on } \Gamma,$$
$$u(x,0) = u_0(x), \quad \text{in } \bar{\Omega}.$$

Under the assumption of $(u_0, \gamma(u_0)) \in L^2(\Omega) \times L^2(\Gamma)$, $f \in C^1$, $f'(s) \geq -l$, and

$$-C_0 + C_1|s|^p \leq f(s)s \leq C_0 + C_2|s|^p, \ p \geq 2,$$

they proved the local existence of solutions with Galerkin truncation and pointed out that the global existence of solutions can be studied by means of the comparison principle, as developed in the recent papers [5, 6] and reference therein. However, they did not give any global existence theorems. Later, they established the existence of a $(L^2(\Omega) \times L^2(\Gamma)), (W^{1,2}(\Omega) \cap L^p(\Omega) \times L^p(\Gamma))$-global attractor and proved the asymptotic compactness of the corresponding semigroup.

In [19], Vitillaro dealt with the local and global existence of solutions of the heat equation in bounded domains with nonlinear boundary conditions which involve damping and source terms (but the equation is homogeneous and linear)

$$\begin{aligned}
&u_t - \Delta u = 0, && x \in \Omega, \ t > 0,\\
&u = 0, && x \in \Gamma_0, \ t > 0,\\
&u_\nu = -|u_t|^{m-2}u_t + |u|^{p-2}u, && x \in \Gamma_1, \ t > 0,\\
&u(x,0) = u_0(x), && x \in \Omega,
\end{aligned} \tag{2}$$

where $\Omega \subset \mathbb{R}^n (n \geq 1)$ is a regular bounded domain, $\partial\Omega = \Gamma_0 \cup \Gamma_1$, $m > 1, 2 \leq p < r$, $r = \frac{2(n-1)}{n-2}$ when $n \geq 3$, $r = \infty$ when $n = 1, 2$. He proved the local existence of solutions in $H^1_{\Gamma_0}(\Omega)$ when $m > r/(r+1-p)$ or $n = 1, 2$, and global existence when $p \leq m$ or the initial datum is inside the potential well associated to the stationary problem. The method to prove global existence of (2) with small initial data can also be used in our work. However, the inhomogeneous equation in (1) makes the proof of global existence for problem (1) more difficult.

Based on the approaches in [7, 8, 9] etc., Fila and Quittner [11] considered several results concerning the asymptotic behavior of solutions to the parabolic problem (1). The authors obtained local existence and regularity (in Besov spaces) results by using the abstract approach by Amann and Escher in the framework of semigroup theory. They showed the boundedness of global solutions, i.e., there exists a constant $K = K(J(u_0)) > 0$ such that

$$\int_\Omega u^2 dx + \int_{\partial\Omega} u^2 dS \leq K, \ \liminf_{t\to\infty} \int_\Omega |\nabla u|^2 dx \leq K,$$

with the condition of $(N-2)p < N+2$, $(N-2)q < N$ and $u_0 \in H^1_0(\Omega)$. However, only based on the assumptions of the existence of the global solutions, they established the boundary estimates of the global solutions. That is to say, they did not give any global existence theories to assure the global existence. And of course, we do not know in what conditions the global solutions exist and also the boundedness results hold. So it is necessary to give a global existence theorem and point out in what conditions on both the nonlinear terms and the initial data, the local solutions exist for all the time. Later, they obtained the blow up results with negative initial energy, i.e. $J(u_0) < 0$. This is a very classical conclusion. And it's interesting to ask what will happen if $J(u_0) > 0$. If we denote the mountain pass level by d, the known works [4, 15, 16, 17, 21] tell us that the case $0 < J(u_0) \leq d$ can be considered to use the potential well method combined with concavity method to deal with the blow up of the solutions. But it is not trivial to study the high energy case, i.e. $J(u_0) > d$. So for this problem, the global existence and nonexistence for $J(u_0) < d$, $J(u_0) = d$ and $J(u_0) > d$ respectively is still open problem. In order to answer the above problems, we try to tackle (1) with potential well method and concavity method, as those methods can be well applied to the parabolic problems [22, 23, 24] and the hyperbolic problems [14, 25] to describe a whole picture of the energy levels.

Motivated by the papers above, we focus on the global existence results and finite time blow up results for $J(u_0) \le d$ and the blow up results when $J(u_0) > 0$, which is an expansion of [11].

This paper is organized as follows. In Section 2, we recall some preliminary tools and definitions. In Section 3, we get global existence and finite time blow up of solutions when the initial data are at low and critical energy levels. In Section 4, based on the arbitrary high initial energy, we prove the finite time blowup of the solution as an interesting part in this paper.

2. Setup and notations

Let $T^*(u_0)$ denote the maximal existence time of the solution with initial condition $u_0 \in H^1(\Omega) = W^{1,2}(\Omega)$.

We denote by $\|\cdot\|_p$ the $L^p(\Omega)$ norm for $1 < p < \infty$, by $\|\cdot\|_{q,\Gamma}$ the $L^q(\Gamma)$ norm for $1 < q < \infty$.

For all $u, v \in H^1(\Omega)$, we put

$$(u, v)_H = \int_\Omega uv dx + \int_\Gamma uv dS = (u, v) + (u, v)_\Gamma,$$

and $\|u\|_H = (u, u)_H^{\frac{1}{2}}$.

We denote the Sobolev critical exponent of the imbedding $H^1(\Omega) \hookrightarrow L^{\bar{r}}(\Omega)$ by

$$\bar{r} = \begin{cases} \frac{2N}{N-2}, & \text{if } N \ge 3; \\ +\infty, & \text{if } N = 1, 2. \end{cases}$$

We also recall the trace theorem (Theorem 5.8 in [1]) the continuous imbedding $H^1(\Omega) \hookrightarrow L^{r'}(\Gamma_1)$, $2 \le r' \le \bar{r'}$, where

$$\bar{r'} = \begin{cases} \frac{2(N-1)}{N-2}, & \text{if } N \ge 3; \\ +\infty, & \text{if } N = 1, 2. \end{cases}$$

Hence we have the compact Sobolev constant for imbedding $H^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$ and the trace-Sobolev constant for imbedding $H^1(\Omega) \hookrightarrow L^{q+1}(\Gamma)$, which means that there exist S_{p+1} and S_{q+1} such that

$$0 < S_{p+1} = \min_{u \in H^1(\Omega)\setminus\{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_{p+1}^2},\ 0 < S_{q+1} = \min_{u \in H^1(\Omega)\setminus\{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_{q+1,\Gamma}^2}. \tag{3}$$

By the Sobolev imbedding theorem and the trace theorem, for the compact imbedding $L^{p+1}(\Omega) \hookrightarrow L^2(\Omega)$ and $L^{q+1}(\Gamma) \hookrightarrow L^2(\Gamma)$, we can get [1]

$$0 < C_1 = \min_{u \in H^1(\Omega)\setminus\{0\}} \frac{\|u\|_{p+1}^2}{\|u\|_2^2}, 0 < C_2 = \min_{u \in H^1(\Omega)\setminus\{0\}} \frac{\|u\|_{q+1,\Gamma}^2}{\|u\|_{2,\Gamma}^2}. \tag{4}$$

Now, we give the definition of the weak solution of problem (1) as follows.

Definition 2.1. (Weak solution) A function $u = u(x,t) \in L^\infty(0, T^*(u_0); H^1(\Omega)) \cap L^2(0, T^*(u_0); L^2(\Omega))$ with $u_t \in L^2(0, T^*(u_0); L^2(\Omega))$, is called a weak solution of (1) on $\Omega \times [0, T^*(u_0))$ if the following conditions are satisfied

(i) the trace of u on $[0, T^*(u_0)) \times \partial\Omega$ (which exists by the trace theorem) has a distributional time derivative on $[0, T^*(u_0)) \times \partial\Omega$, belonging to $L^2([0, T^*(u_0)) \times \partial\Omega)$;

(ii) for all $v \in H^1(\Omega) \cap L^2(\Gamma)$, and for almost all $t \in [0, T^*(u_0))$, we have

$$\int_\Omega u_t v dx + \int_\Omega \nabla u \nabla v dx + \int_\Gamma u_t v dS = \int_\Omega \lambda |u|^{p-1} u v dx + \int_\Gamma \mu |u|^{q-1} u v dS, \tag{5}$$

(iii) $u(x,0) = u_0(x)$ in $H^1(\Omega)$.

We define the energy functional $J(u)$ and the Nehari functional $I(u)$

$$J(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{\lambda}{p+1}\|u\|_{p+1}^{p+1} - \frac{\mu}{q+1}\|u\|_{q+1,\Gamma}^{q+1}, \tag{6}$$

$$I(u) = \|\nabla u\|_2^2 - \lambda\|u\|_{p+1}^{p+1} - \mu\|u\|_{q+1,\Gamma}^{q+1}. \tag{7}$$

By multiplying (1) by $u_t(t)$ and integrating by parts, we get

$$\frac{d}{dt}\left(\frac{1}{2}\|\nabla u\|_2^2 - \frac{\lambda}{p+1}\|u\|_{p+1}^{p+1} - \frac{\mu}{q+1}\|u\|_{q+1,\Gamma}^{q+1}\right) = -\|u_t\|_2^2 - \|u_t\|_{2,\Gamma}^2,$$

i.e.,

$$\frac{d}{dt}J(u) = -\|u_t\|_2^2 - \|u_t\|_{2,\Gamma}^2 < 0. \tag{8}$$

All nontrivial stationary solutions belong to Nehari manifold $\mathscr{N}$ defined by

$$\mathscr{N} = \{u \in H^1(\Omega) \setminus \{0\} \mid I(u) = 0\},$$

and $\mathscr{N}$ separates two unbounded sets

$$\mathscr{N}_+ = \{u \in H^1(\Omega) \mid I(u) > 0\} \cup \{0\},$$

$$\mathscr{N}_- = \{u \in H^1(\Omega) \mid I(u) < 0\}.$$

The mountain-pass level d is characterized as

$$d = \min_{u \in H^1(\Omega)\setminus\{0\}} \max_{s \geq 0} J(su). \tag{9}$$

We also consider the sublevels of J

$$J^k := \{u \in H^1(\Omega) \mid J(u) \leq k\},$$

and we introduce the stable set W and the unstable set V defined by

$$W = J^d \cap \mathscr{N}_+, \; V = J^d \cap \mathscr{N}_-.$$

Then, the mountain-pass level d defined in (9) may also be characterized as

$$d = \min_{u \in \mathscr{N}} J(u).$$

3. Global existence and blow up of solutions for $J(u_0) \le d$

In this section, we show global existence and blow up of solutions when $J(u_0) \le d$ with potential well method and the classical concavity method.

We first give the following elementary statement.

Lemma 3.1. *There exists an $m > 1$ such that*

$$J(u) \ge \frac{m-1}{2(m+1)}\|\nabla u\|_2^2 + \frac{1}{m+1}I(u). \tag{10}$$

Proof. Since $(\lambda, \mu) \in \{-1, 0, 1\}$, $\lambda p + \mu q > 0$ and $p, q > 1$, we discuss J in details as follows.

(i) When $(\lambda, \mu) = (1, 1)$, then the functional J becomes

$$J(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1} - \frac{1}{q+1}\|u\|_{q+1,\Gamma}^{q+1}.$$

We take $m = \min\{p, q\}$, then

$$\begin{aligned} J(u) =& \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1} - \frac{1}{q+1}\|u\|_{q+1,\Gamma}^{q+1} \\ \ge& \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{m+1}\left(\|u\|_{p+1}^{p+1} + \|u\|_{q+1,\Gamma}^{q+1}\right) \\ =& \frac{m-1}{2(m+1)}\|\nabla u\|_2^2 + \frac{1}{m+1}I(u). \end{aligned}$$

(ii) When $(\lambda, \mu) = (1, 0)$, then the functional J becomes

$$J(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1},$$

then

$$J(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1} \ge \frac{p-1}{2(p+1)}\|\nabla u\|_2^2 + \frac{1}{p+1}I(u).$$

In this case, we just take $m = p$.

(iii) When $(\lambda, \mu) = (0, 1)$, then the functional J becomes

$$J(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{q+1}\|u\|_{q+1,\Gamma}^{q+1},$$

then

$$J(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{q+1}\|u\|_{q+1,\Gamma}^{q+1} \ge \frac{q-1}{2(q+1)}\|\nabla u\|_2^2 + \frac{1}{q+1}I(u).$$

In this case, we just take $m = q$.

(iv) When $(\lambda, \mu) = (1, -1)$, then the functional J becomes

$$J(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1} + \frac{1}{q+1}\|u\|_{q+1,\Gamma}^{q+1}.$$

Since $\lambda p + \mu q = p - q > 0$, we take $m = \min\{p, q\} = q$, then

$$\begin{aligned} J(u) =& \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1} + \frac{1}{q+1}\|u\|_{q+1,\Gamma}^{q+1} \\ \ge& \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{m+1}\left(\|u\|_{p+1}^{p+1} - \|u\|_{q+1,\Gamma}^{q+1}\right) \\ =& \frac{m-1}{2(m+1)}\|\nabla u\|_2^2 + \frac{1}{m+1}I(u). \end{aligned}$$

(v) When $(\lambda, \mu) = (-1, 1)$, then the functional J becomes

$$J(u) = \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{p+1}\|u\|_{p+1}^{p+1} - \frac{1}{q+1}\|u\|_{q+1,\Gamma}^{q+1}.$$

Since $\lambda p + \mu q = -p + q > 0$, we take $m = \min\{p, q\} = p$, then

$$\begin{aligned} J(u) =& \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{p+1}\|u\|_{p+1}^{p+1} - \frac{1}{q+1}\|u\|_{q+1,\Gamma}^{q+1} \\ \geq& \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{m+1}\left(\|u\|_{p+1}^{p+1} - \|u\|_{q+1,\Gamma}^{q+1}\right) \\ =& \frac{m-1}{2(m+1)}\|\nabla u\|_2^2 + \frac{1}{m+1}I(u). \end{aligned}$$

Then the proof is completed. □

Next, we give a global existence theorem for weak solutions of problem (1) in the sence of the Definition 2.1.

Theorem 3.2. *(Global existence of solutions for $J(u_0) \leq d$) Assume that $1 < p, q < \infty$, $N = 1, 2$; $1 < p < \frac{N+2}{N-2}, 1 < q < \frac{N}{N-2}$, $N \geq 3$ and $u_0 \in H^1(\Omega)$. Then if $J(u_0) \leq d$ and $I(u_0) > 0$ or $\|u_0\|_{H^1} = 0$, then problem (1) admits a global weak solution $u(t) \in L^\infty(0, \infty; H^1(\Omega)) \cap L^2(0, \infty; L^2(\Omega) \cap L^2(\Gamma))$ with $u_t \in L^2(0, \infty; L^2(\Omega) \cap L^2(\Gamma))$ and $u(t) \in W$ for $0 \leq t < \infty$.*

Proof. By (8), it is derived that the map J is decreasing respect to t. We can conclude $u(t) \in W$ from $I(u_0) > 0$ for any $t \in [0, T^*(u_0))$. By contradictory arguments, we can suppose that there exists a first time $\bar{t} \in (0, T^*(u_0))$ making $u(\bar{t}) \in \mathscr{N}$. From (9), we see that

$$d \leq J(u(\bar{t})) < J(u_0) \leq d,$$

which is self-contradictory. Thus there holds $u(t) \in W$ for any $t \in [0, T^*(u_0))$.

Lemma 3.1 together with the inequality (8) implies that there exists an $m > 1$ satisfies

$$\begin{aligned} d \geq J(u_0) \geq& J(u(t)) + \int_0^t \|u_\tau\|_2^2 + \|u_\tau\|_{2,\Gamma}^2 d\tau \\ \geq& \frac{m-1}{2(m+1)}\|\nabla u(t)\|_2^2 + \frac{1}{m+1}I(u(t)) + \int_0^t \|u_\tau\|_2^2 + \|u_\tau\|_{2,\Gamma}^2 d\tau \\ \geq& \frac{m-1}{2(m+1)}\|\nabla u(t)\|_2^2 + \int_0^t \|u_\tau\|_2^2 + \|u_\tau\|_{2,\Gamma}^2 d\tau, \end{aligned}$$

then we obtain

$$\|\nabla u(t)\|_2^2 \leq \frac{2(m+1)}{m-1}d, \quad \int_0^t \|u_\tau\|_2^2 + \|u_\tau\|_{2,\Gamma}^2 d\tau \leq d.$$

By (3) and (4), we obtain

$$\begin{aligned} \|u(t)\|_{p+1}^2 \leq& \frac{1}{S_{p+1}}\|\nabla u(t)\|_2^2 \leq \frac{2(m+1)}{S_{p+1}(m-1)}d, \\ \|u(t)\|_{q+1,\Gamma}^2 \leq& \frac{1}{S_{q+1}}\|\nabla u(t)\|_2^2 \leq \frac{2(m+1)}{S_{q+1}(m-1)}d, \end{aligned}$$

$$\|u(t)\|_2^2 \leq \frac{1}{C_1}\|u(t)\|_{p+1}^2 \leq \frac{2(m+1)}{C_1 S_{p+1}(m-1)}d,$$
$$\|u(t)\|_{2,\Gamma}^2 \leq \frac{1}{C_2}\|u(t)\|_{q+1,\Gamma}^2 \leq \frac{2(m+1)}{C_2 S_{q+1}(m-1)}d,$$

and

$$\|u(t)\|_2^2 + \|u(t)\|_{2,\Gamma}^2 \leq \left(\frac{1}{C_1 S_{p+1}} + \frac{1}{C_2 S_{q+1}}\right)\frac{2(m+1)}{m-1}d \quad \text{for any} \quad t \in [0, T^*(u_0)).$$

Hence, we conclude the corresponding results. □

Now, we show blow-up of solutions by potential well method and the classical concavity method similarly as in the proofs of corresponding results in [2, 18]. Before the blow up theorem, we give a basic lemma which implies that the initial datum in $\mathscr{N}_-$ can generate blow up solutions.

Lemma 3.3. *Let $u_0 \in H^1(\Omega)$. For all $t \in [0, T^*(u_0))$, then*

$$\frac{d}{dt}\left(\|u\|_2^2 + \|u\|_{2,\Gamma}^2\right) = -2I(u). \tag{11}$$

Proof. Multiplying (1) by $u(t)$ and integrate the resultant equation with respect to x over Ω we obtain

$$\int_\Omega u_t u(t) - \int_\Omega \Delta u u(t) = \lambda \int_\Omega |u|^{p-1} u u(t),$$
$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 - \left(-\frac{1}{2}\frac{d}{dt}\|u\|_{2,\Gamma}^2 - \mu\|u\|_{q+1,\Gamma}^{q+1}\right) + \|\nabla u\|_2^2 = \lambda\|u\|_{p+1}^{p+1},$$
$$\frac{1}{2}\frac{d}{dt}\left(\|u\|_2^2 + \|u\|_{2,\Gamma}^2\right) = -\left(\|\nabla u\|_2^2 - \lambda\|u\|_{p+1}^{p+1} - \mu\|u\|_{q+1,\Gamma}^{q+1}\right),$$
$$\frac{d}{dt}\left(\|u\|_2^2 + \|u\|_{2,\Gamma}^2\right) = -2I(u),$$

for all $t \in [0, T^*(u_0))$. This completes the proof. □

Now, we arrive at a blow up result for solutions of (1). In our proof, we first prove the invariance of the unstable set V, which improves the processing of blow up results in [2, 18].

Theorem 3.4. *(Blow up of solutions for $J(u_0) \leq d$) Assume that $1 < p, q < \infty$, $N = 1, 2$; $1 < p < \frac{N+2}{N-2}, 1 < q < \frac{N}{N-2}$, $N \geq 3$ and $u_0 \in H^1(\Omega)$. Then if $J(u_0) \leq d$ and $I(u_0) < 0$, solutions of problem (1) blow up in finite time.*

Proof. The proof is based on the argument by contradiction. We assume that $T^*(u_0) = \infty$. By (8), we know that the map J is decreasing respect to t. Then if $I(u_0) < 0$ holds, we have $u(t) \in V$ for all $t \in [0, T^*(u_0))$. Indeed, if it was not the case, there would exist a first time $\bar{t} \in (0, T^*(u_0))$ such that $u(\bar{t}) \in \mathscr{N}$, i.e., $I(u(\bar{t})) = 0$. Then we know that $I(u) < 0$ for $0 \leq t < \bar{t}$ and $I(u) > 0$ for $\bar{t} < t < T^*(u_0)$. By the variational characterization (9) and (8),

$$d \leq J(u(\bar{t})) < J(u_0) \leq d,$$

whose contradiction directly gives $u(t) \in V$ for all $t \in [0, T^*(u_0))$.

Now, we consider $M : [0, T^*(u_0)) \to \mathbb{R}_+$ defined by

$$M(t) = \int_0^t \|u\|_2^2 d\tau + \int_0^t \|u\|_{2,\Gamma}^2 d\tau,$$

then $M'(t) = \|u(t)\|_2^2 + \|u(t)\|_{2,\Gamma}^2$, and

$$\begin{aligned} M''(t) =& 2\int_\Omega uu_t dx + 2\int_\Gamma uu_t dS \\ =& -2\left(\|\nabla u\|_2^2 - \lambda\|u\|_{p+1}^{p+1} - \mu\|u\|_{q+1,\Gamma}^{q+1}\right) \\ =& -2I(u). \end{aligned}$$

Since $I(u) < 0$, we have $M''(t) > 0$.

By the proof of Lemma 3.1 in [11], we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}(\|u\|_2^2 + \|u\|_{2,\Gamma}^2) \geq& -(2+\delta)J(u_0) + \frac{\delta}{2}\|\nabla u\|_2^2 \\ &+ \left(1 - \frac{2+\delta}{p+1}\right)\lambda\|u\|_{p+1}^{p+1} + \left(1 - \frac{2+\delta}{q+1}\right)\lambda\|u\|_{q+1,\Gamma}^{q+1}, \end{aligned}$$

and

$$\frac{1}{2}M'' \geq (2+\delta)\left(\int_0^t \left(\|u_\tau\|_2^2 + \|u_\tau\|_{2,\Gamma}^2\right) d\tau - J(u_0) + \varepsilon M' - c\right), \tag{12}$$

where ε, c are positive constants and δ is defined as

$$\delta = \begin{cases} q-1, & \text{if } (\lambda,\mu) = (1,-1) \\ \frac{p-1}{2}, & \text{if } (\lambda,\mu) = (1,0) \\ p-1, & \text{if } (\lambda,\mu) = (-1,1) \\ \frac{q-1}{2}, & \text{if } (\lambda,\mu) = (0,1) \\ \frac{1}{2}(\min\{p,q\}-1), & \text{if } (\lambda,\mu) = (1,1). \end{cases}$$

Observe that $M'(0) = \|u_0\|_2^2 + \|u_0\|_{2,\Gamma}^2$, we know that $M' = \|u\|_2^2 + \|u\|_{2,\Gamma}^2$ is increasing respect to t. Then there exists a $t_0 > 0$ such that $\varepsilon M' - J(u_0) - c > 0$ and when $t \geq t_0$, (12) becomes

$$\frac{1}{2}M''(t) \geq (2+\delta)\int_0^t \left(\|u_\tau\|_2^2 + \|u_\tau\|_{2,\Gamma}^2\right) d\tau.$$

Then we have

$$\begin{aligned} M(t)M''(t) \geq& 2(2+\delta)\int_0^t \left(\|u\|_2^2 + \|u\|_{2,\Gamma}^2\right) d\tau \int_0^t \left(\|u_\tau\|_2^2 + \|u_\tau\|_{2,\Gamma}^2\right) d\tau \\ =& 2(2+\delta)\bigg(\int_0^t \|u\|_2^2 d\tau \int_0^t \|u_\tau\|_2^2 d\tau + \int_0^t \|u\|_{2,\Gamma}^2 d\tau \int_0^t \|u_\tau\|_{2,\Gamma}^2 d\tau \\ &+ \int_0^t \|u_\tau\|_2^2 d\tau \int_0^t \|u\|_{2,\Gamma}^2 d\tau + \int_0^t \|u_\tau\|_{2,\Gamma}^2 d\tau \int_0^t \|u\|_2^2 d\tau\bigg). \end{aligned}$$

Using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} &\int_0^t \|u\|_2^2 d\tau \int_0^t \|u_\tau\|_2^2 d\tau \geq \left(\int_0^t (u,u_\tau) d\tau\right)^2, \\ &\int_0^t \|u\|_{2,\Gamma}^2 d\tau \int_0^t \|u_\tau\|_{2,\Gamma}^2 d\tau \geq \left(\int_0^t (u,u_\tau)_\Gamma d\tau\right)^2, \end{aligned}$$

and

$$\int_0^t (u,u_\tau) d\tau \int_0^t (u,u_\tau)_\Gamma d\tau$$

$$
\begin{aligned}
&\leq \left(\int_0^t \|u\|_2^2 d\tau\right)^{\frac{1}{2}} \left(\int_0^t \|u_\tau\|_2^2\right)^{\frac{1}{2}} \left(\int_0^t \|u\|_{2,\Gamma}^2 d\tau\right)^{\frac{1}{2}} \left(\int_0^t \|u_\tau\|_{2,\Gamma}^2 d\tau\right)^{\frac{1}{2}} \\
&\leq \left(\int_0^t \|u\|_2^2 d\tau \int_0^t \|u_\tau\|_{2,\Gamma}^2 d\tau\right)^{\frac{1}{2}} \left(\int_0^t \|u_\tau\|_2^2 \int_0^t \|u\|_{2,\Gamma}^2 d\tau d\tau\right)^{\frac{1}{2}} \\
&\leq \frac{1}{2}\left(\int_0^t \|u\|_2^2 d\tau \int_0^t \|u_\tau\|_{2,\Gamma}^2 d\tau + \int_0^t \|u_\tau\|_2^2 \int_0^t \|u\|_{2,\Gamma}^2 d\tau d\tau\right).
\end{aligned}
$$

Therefore, we obtain

$$M(t)M''(t) \geq \frac{2+\delta}{2}\left(2\int_0^t (u,u_\tau)d\tau + 2\int_0^t (u,u_\tau)_\Gamma d\tau\right)^2.$$

Observe that

$$
\begin{aligned}
&2\int_0^t (u,u_\tau)d\tau = \|u\|_2^2 - \|u_0\|_2^2, \\
&2\int_0^t (u,u_\tau)_\Gamma d\tau = \|u\|_{2,\Gamma}^2 - \|u_0\|_{2,\Gamma}^2,
\end{aligned}
$$

we have

$$M(t)M''(t) \geq \frac{2+\delta}{2}\left(M'(t) - M'(0)\right)^2 = \left(1+\frac{\delta}{2}\right)\left(M'(t) - M'(0)\right)^2,$$

and there exists a $t_1 > t_0$ such that when $t > t_1$,

$$M(t)M''(t) \geq \left(1+\frac{\delta}{2}\right)\left(M'(t) - M'(0)\right)^2 > \left(1+\frac{\delta}{4}\right)\left(M'(t)\right)^2.$$

Thus the function $M(t)^{-\frac{\delta}{4}}$ is concave for $t > t_1$, and

$$\left(M(t)^{-\theta}\right)'' = \frac{-\theta}{M^{\theta+2}(t)}\left(M(t)M''(t) - (1+\theta)\left(M'(t)\right)^2\right) < 0,$$

where $\theta = \frac{\delta}{4} > 0$, which indicates that $M(t)^{-\theta}$ hits zero in finite time, then there exists a $T_1 \in [0, T^*(u_0))$ such that

$$\lim_{t\to T_1} M(t) = \infty,$$

which contradicts $T^*(u_0) = \infty$.

This completes the proof. □

From Theorem 3.2 and Theorem 3.4, we can obtain a sharp condition for global existence of solutions for problem (1) with $J(u_0) \leq d$ as follows.

Remark 1. Assume that $1 < p, q < \infty$, $N = 1, 2$; $1 < p < \frac{N+2}{N-2}, 1 < q < \frac{N}{N-2}$, $N \geq 3$, $u_0 \in H^1(\Omega)$ and $J(u_0) \leq d$. Then if $I(u_0) > 0$ or $\|u_0\|_{H^1} = 0$, problem (1) admits a global weak solution; if $I(u_0) < 0$, solutions of problem (1) blow up in finite time.

4. Blow up of solutions for $J(u_0) > 0$

This section is devoted to finite time blowup with the arbitrarily high positive initial energy, which will be proved by the spirits of [12]

Theorem 4.1. *(Blow up of solutions for $J(u_0) > 0$) Assume that $1 < p, q < \infty$, $N = 1, 2$; $1 < p < \frac{N+2}{N-2}$, $1 < q < \frac{N}{N-2}$, $N \geq 3$ and $u_0 \in H^1(\Omega)$. In addition, assume that $I(u_0) < 0$ such that*

$$J(u_0) > 0, \|u_0\|_2^2 + \|u_0\|_{2,\Gamma}^2 > C_0 = \frac{2(m+1)}{m-1} J(u_0) \left(\frac{1}{S_{p+1}C_1} + \frac{1}{S_{q+1}C_2} \right),$$

where $S_{p+1}, S_{q+1}, C_1, C_2$ are defined in (3) (4), and m is defined as in Lemma 3.1. Then the solutions of problem (1) blow up in finite time.

Proof. We will prove this theorem by two steps.

Step I. We firstly show that

$$I(u(t)) < 0 \text{ for all } t \in [0, T^*(u_0)).$$

By contradictory arguments, we assume that there exists a finite $t_0 \in (0, T^*(u_0))$ such that $I(u(t_0)) = 0$. Then consider $L : [0, T^*(u_0)) \to \mathbb{R}_+$ defined by

$$L(t) = \|u\|_2^2 + \|u\|_{2,\Gamma}^2.$$

Then we obtain

$$L'(t) = 2 \int_\Omega u u_t dx + 2 \int_\Gamma u u_t dS = -2I(u).$$

Since $I(u) < 0$ for $0 \leq t < t_0$, then $L'(t) > 0$ for $0 \leq t < t_0$. Observe that $L(0) = \|u_0\|_2^2 + \|u_0\|_{2,\Gamma}^2 > 0$, we know that $L(t)$ is strictly increasing on $(0, t_0)$. Thus,

$$L(t) > \|u_0\|_2^2 + \|u_0\|_{2,\Gamma}^2 > C_0 \quad \text{for all } t \in (0, t_0).$$

Hence, we get

$$L(t_0) > C_0.$$

On the other hand, by (8), we have

$$J(u(t_0)) = \frac{1}{2}\|\nabla u(t_0)\|_2^2 - \frac{\lambda}{p+1}\|u(t_0)\|_{p+1}^{p+1} - \frac{\mu}{q+1}\|u(t_0)\|_{q+1,\Gamma}^{q+1} < J(u_0).$$

By the result of Lemma 3.1 and $I(u(t_0)) = 0$, we know

$$\frac{m-1}{2(m+1)}\|\nabla u(t_0)\|_2^2 < J(u(t_0)) < J(u_0),$$

where m is defined as in Lemma 3.1.

Then from (3) and (4), we get

$$\begin{aligned} L(t_0) =& \|u(t_0)\|_2^2 + \|u(t_0)\|_{2,\Gamma}^2 \leq \frac{1}{C_1}\|u(t_0)\|_{p+1}^2 + \frac{1}{C_2}\|u(t_0)\|_{q+1,\Gamma}^2 \\ \leq& \frac{1}{S_{p+1}C_1}\|\nabla u(t_0)\|_2^2 + \frac{1}{S_{q+1}C_2}\|\nabla u(t_0)\|_2^2 \\ \leq& \frac{2(m+1)}{m-1} J(u_0) \left(\frac{1}{S_{p+1}C_1} + \frac{1}{S_{q+1}C_2} \right) = C_0, \end{aligned}$$

which is a contradiction. Thus we have proved that

$$I(u(t)) < 0 \text{ for all } t \in [0, T^*(u_0)).$$

Step II. We prove the blow up results now. As we discussed in Theorem 3.4, we assume $T^*(u_0)=\infty$ and consider $M:[0,T^*(u_0))\to\mathbb{R}_+$ defined by

$$M(t)=\int_0^t\|u\|_2^2d\tau+\int_0^t\|u\|_{2,\Gamma}^2d\tau.$$

By the same spirit of Theorem 3.4, we obtain the inequality (12). And since $I(u)<0$ and $M'(0)>0$, we get $M'(t)$ is increasing respect to t. Thus, there exists a $t_0>0$ such that $\varepsilon M'(t_0)-J(u_0)-c>0$ and (12) becomes

$$\frac{1}{2}M''(t)\geq(2+\delta)\int_0^t\left(\|u_\tau\|_2^2+\|u_\tau\|_{2,\Gamma}^2\right)d\tau.$$

Then we obtain there exists a $t_1>0$ satisfies

$$M(t)M''(t)\geq\left(1+\frac{\delta}{2}\right)\left(M'(t)-M'(0)\right)^2>\left(1+\frac{\delta}{4}\right)\left(M'(t)\right)^2,$$

thus the function $M(t)^{-\frac{\delta}{4}}$ is concave for $t>t_1$, and

$$\left(M(t)^{-\theta}\right)''=\frac{-\theta}{M^{\theta+2}(t)}\left(M(t)M''(t)-(1+\theta)\left(M'(t)\right)^2\right)<0,$$

where $\theta=\frac{\delta}{4}>0$, which indicates that $M(t)^{-\theta}$ hits zero in finite time, then there exists a $T_1\in[0,T^*(u_0))$ such that

$$\lim_{t\to T_1}M(t)=\infty,$$

which contradicts $T^*(u_0)=\infty$.

□

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