ELECTRONIC RESEARCH ARCHIVE
Volume 28 Number 1, March 2020 Pages 347–367
eISSN: 2688-1594 AIMS (2020)
doi:10.3934/era.2020020

BLOW-UP IN DAMPED ABSTRACT NONLINEAR EQUATIONS

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Abstract. As a typical example of our analysis we consider a generalized Boussinesq equation, linearly damped and with a nonlinear source term. For any positive value of the initial energy, in particular for high energies, we give sufficient conditions on the initial data to conclude nonexistence of global solutions. We do our analysis in an abstract framework. We compare our results with those in the literature and we give more examples to illustrate the applicability of the abstract formulation.

1. Introduction. An example of study.

We first consider the following generalized Boussinesq equation with linear damping and a nonlinear source term

$$(\mathbf{GB})\begin{cases} u_{tt}-\alpha_1\Delta u-\alpha_2\Delta u_{tt}+\alpha_3\Delta^2 u+m^2u+\delta u_t-\delta\alpha_2\Delta u_t+\Delta f(u)=0,\\ u(0,x)=u_0(x),\ u_t(0,x)=u_1(x),\end{cases}$$

on $\mathbb{R}^+\times\mathbb{R}^N$, $\mathbb{R}^+\equiv[0,\infty)$, where $m^2>0,\delta\geq 0,\alpha_i>0,\ i=1,2,3,$ and f is a nonlinear function of the unknown u.

In [2], Boussinesq obtained an approximate equation from the Euler equation to describe bidirectional solitary waves propagating on the free surface of a constant depth in irrotational motion. This a particular case without dissipation of the one-dimensional version of $(\mathbf{GB})$ and it is called the classical Boussinesq equation, Since then, several modifications have been proposed. See [23, 3] for historical remarks and the physical foundations of several generalizations of the original Boussinesq equation to model nonlinear wave propagation in elasticity and fluid mechanics.

For an evolution equation, like the Cauchy problem $(\mathbf{GB})$, we have the following questions. With respect to a functional framework and in terms on the initial data, study:

- Local existence and uniqueness of solutions.
- Non-global existence: maximal time of existence $T_{MAX}<\infty$.
- Global existence: $T_{MAX}=\infty$.
- In the latter case, the behavior of the solution as time approaches infinity.

Here, we shall study the second point. In fact, we will give conditions on the initial data, u_0,u_1, for which the corresponding solution, in some sense given later, exists only up to a finite time. In order to do that, we are going to consider an abstract formulation of problem $(\mathbf{GB})$, so that can be applied to other equations.

Received by the editors December, 2019 and, in revised form, February, 2020.
2010 *Mathematics Subject Classification.* Primary: 35L70, 35L90, 35B44, 35L05; Secondary: 35B35, 35B40.
Key words and phrases. Hyperbolic equations, generalized Boussinesq equation, blow-up, high energies.
This author was supported by the Universidad Autónoma Metropolitana, Unidad Azcapotzalco.
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2. The abstract formulation, related results and preliminaries

We consider the following abstract differential equation.
For every initial data u_0, u_1, find $t \mapsto u(t),\ t \geq 0$, such that

$$(\mathbf{P}) \begin{cases} Pu_{tt} + Au + \delta P u_t = f(u), \\ u(0) = u_0,\ u_t(0) = u_1. \end{cases}$$

Here, $u_t \equiv \frac{d}{dt}u$, and we assume that the operators

$$P : W_P \to W_P^{'},\quad A : V_A \to V_A^{'},$$

are linear, continuous, positive and symmetric, and

$$V_A \subset W_P \subset H$$

are linear subspaces of the Hilbert space H with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Here, $H^{'}, W_P^{'}, V_A^{'}$, are the corresponding dual spaces and we identify $H = H^{'}$. Then,

$$H \subset W_P^{'} \subset V_A^{'}.$$

By means of the operators P and A, we define the following bilinear forms

$$\mathcal{P}(u,w) \equiv (Pu,w)_{W_P \times W_P^{'}},\ \ u,w \in W_P,\quad \mathcal{A}(u,w) \equiv (Au,w)_{V_A \times V_A^{'}},\ \ u,w \in V_A,$$

and corresponding norms

$$\|u\|_{W_P}^2 \equiv \mathcal{P}(u,u),\ \ u \in W_P,\quad \|u\|_{V_A}^2 \equiv \mathcal{A}(u,u),\ \ u \in V_A.$$

The following problem $(\mathbf{GB})^*$ has the abstract form $(\mathbf{P})$. Indeed, we remember that $(-\Delta)^{-s}u = \mathcal{F}^{-1}(|\omega|^{-2s}(\mathcal{F}u)(\omega))$, for $s > 0$, where $\mathcal{F}, \mathcal{F}^{-1}$ are the Fourier transform and the inverse Fourier transform, respectively. Applying $(-\Delta)^{-1}$, to $(\mathbf{GB})$, we get

$$(\mathbf{GB})^* \begin{cases} ((-\Delta)^{-1} + \alpha_2 I_d)u_{tt} + (-\alpha_3\Delta + m^2(-\Delta)^{-1} + \alpha_1 I_d)u \\ \qquad\qquad\qquad\qquad + \delta((-\Delta)^{-1} + \alpha_2 I_d)u_t = f(u), \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x), \end{cases}$$

on $\mathbb{R}^+ \times \mathbb{R}^N$.

Then, as usual $u(t) \equiv u(t,\cdot)$, and we define $H = L_2(\mathbb{R}^N)$, with inner product $(\cdot,\cdot)_2$ and norm $\|\cdot\|_2$. Also,

$$Pu_t = ((-\Delta)^{-1} + \alpha_2 I_d)u_t,\quad Au = (-\alpha_3\Delta + m^2(-\Delta)^{-1} + \alpha_1 I_d)u,$$

are defined, respectively, on the subspaces of H

$$W_P = \{u \in L_2(\mathbb{R}^N) : (-\Delta)^{-\frac{1}{2}}u \in L_2(\mathbb{R}^N)\},$$
$$V_A = \{u \in H^1(\mathbb{R}^N) : (-\Delta)^{-\frac{1}{2}}u \in L_2(\mathbb{R}^N)\}.$$

Moreover, if

$$\|u\|_*^2 = (u,u)_* \equiv ((-\Delta)^{-\frac{1}{2}}u, (-\Delta)^{-\frac{1}{2}}u)_2,$$

then

$$\|u\|_{W_P}^2 = \|u\|_*^2 + \alpha_2\|u\|_2^2,\ \ \|u\|_{V_A}^2 = \alpha_3\|\nabla u\|_2^2 + m^2\|u\|_*^2 + \alpha_1\|u\|_2^2.$$

The analysis of problem $(\mathbf{P})$ is in the framework of weak solutions in the sense of the following Definition 2.1. To this end, we consider the phase space

$$\mathcal{H} \equiv V_A \times W_P,$$

with norm

$$\|(u,v)\|_{\mathcal{H}}^2 \equiv \|v\|_{W_P}^2 + \|u\|_{V_A}^2,$$

and we assume that the following hypotheses hold.

There exists $c > 0$, such that

$$(H0) \qquad \|u\|^2_{V_A} \geq c\|u\|^2_{W_P}, \ \ \forall u \in V_A.$$

The nonlinear source term $f : V_A \to H$, is such that $f(0) = 0$, and it is a potential operator with potential $F : V_A \to \mathbb{R}$, that is, $f(u) = D_u F(u)$. Moreover, they satisfy

$$(H1) \qquad (f(u), u) - rF(u) \geq 0, \ \ \forall u \in V_A,$$

and some constant $r > 2 + \delta/\sqrt{c}$.

Definition 2.1. For every initial data $(u_0, u_1) \in \mathcal{H}$, the map $(u_0, u_1)) \mapsto (u(t), \dot{u}(t)) \in \mathcal{H}$, where $\dot{u}(t) \equiv \frac{d}{dt}u(t)$, is a weak local solution of problem (**P**), if there exists some $T > 0$, such that $(u, \dot{u}) \in C([0, T); \mathcal{H})$, with $u(0) = u_0, \ \dot{u}(0) = u_1$, and

$$\frac{d}{dt}\mathcal{P}(\dot{u}(t), w) + \mathcal{A}(u(t), w) + \delta\mathcal{P}(\dot{u}(t), w) = (f(u(t)), w),$$

a. e. in $(0, T)$ and for every $w \in V_A$. We shall consider that the solution in this sense is unique and satisfies the following energy equation for $T > t \geq t_0 \geq 0$,

$$\begin{aligned}
&E(u(t_0), \dot{u}(t_0)) = E(u(t), \dot{u}(t)) + \delta \int_{t_0}^{t} \|\dot{u}(\tau)\|^2_{W_P} \, d\tau, \\
&E(t) \equiv E(u(t), \dot{u}(t)) \equiv \frac{1}{2}\|\dot{u}(t)\|^2_{W_P} + J(u(t)), \\
&J(u(t)) \equiv \frac{1}{2}\|u(t)\|^2_{V_A} - F(u(t)) \\
&E(t) = \frac{1}{2}\|(u(t), \dot{u}(t))\|^2_{\mathcal{H}} - F(u(t)).
\end{aligned}$$

Furthermore, if the maximal time of existence $T_{MAX} < \infty$ then

$$\lim_{t \to T_{MAX}} \|(u(t), \dot{u}(t))\|_{\mathcal{H}} = \infty,$$

equivalently, by the energy equation,

$$\lim_{t \to T_{MAX}} F(u(t)) = \infty.$$

For the Boussinesq equation, hypothesis $(H0)$ holds with $c = \min\{m^2, \frac{\alpha_1}{\alpha_2}\}$. We assume that the source terms f and the corresponding potential operator F do not have any particular form but they satisfy $(H1)$. The existence and uniqueness of weak solutions of $(\mathbf{GB})^*$, in the sense of Definition 2.1 holds, see [29, 25, 33].

An important set of solutions are the equilibria, that is solutions independent of time, that is, $\dot{u} = 0$. In this case, u satisfies

$$\mathcal{A}(u, w) = (f(u), w),$$

for every $w \in V_A$. In particular, for $w = u$,

$$\|u\|^2_{V_A} = (f(u), u),$$

and then

$$I(u) \equiv \|u\|^2_{V_A} - (f(u), u) = 0.$$

By $(H0)$, $(u,0) \equiv (0,0)$ is an equilibrium. The set of nonzero equilibria, denoted by $\mathcal{E}$, with minimal energy is called ground state, and the corresponding value of the energy is the mountain pass level denoted by d, see [32]. Indeed,

$$d \equiv \inf_{u\in\mathcal{N}} J(u) = \frac{1}{2}\inf_{u\in\mathcal{N}}((f(u),u) - 2F(u)), \tag{1}$$

where

$$\mathcal{N} \equiv \{u \neq 0 : I(u) = 0\},$$

is the Nehari manifold. By means of this number, the potential well method has been used to characterize the qualitative behavior for solutions of some equations of the type (**P**). For dissipative equations, $\delta > 0$, the behavior is completely different from the conservative case $\delta = 0$. Indeed, consider the initial and homogeneous Dirichlet boundary value problem of the nonlinear wave equation, which is a particular case of problem (**P**).

$$(\mathbf{NLW}) \begin{cases} u_{tt} - \alpha\Delta u + g(u_t) = f(u), & x \in \Omega \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x), & x \in \Omega \\ u(x,t) = 0, & x \in \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $t \in \mathbb{R}^+$, and $\alpha > 0$, $g(u_t) = \delta u_t$, $\delta > 0$, $f(u) = \mu|u|^{r-2}u$, $r > 2$. Here,

$$P = I_d,\quad H = W_P = L_2(\Omega), \qquad Au = -\alpha\Delta u,\quad V_A = H_0^1(\Omega).$$

Hypothesis $(H0)$ holds because of the Sobolev-Poincaré inequality,

$$\alpha\|\nabla u\|_2^2 \geq c\|u\|_2^2,$$

and $(H1)$ holds if $r > 2 + \delta/\sqrt{c}$, with $rF(u) = \mu\|u\|_r^r$, where $\|\cdot\|_r$ is the norm in $L_r(\Omega)$. The following result about the dynamics of (**NLW**) holds.

Theorem 2.2 ([7, 14]). *For every solution $(u,\dot{u})$ of* (**NLW**)*, in the sense of Definition 2.1, with initial data $(u_0,u_1) \in \mathcal{H} = V_A \times W_P \equiv H_0^1(\Omega) \times L_2(\Omega)$, only one of the following holds.*

- *There exists some $t_0 \geq 0$ such that $E(t_0) < d$, then either*

  $$u(t_0) \in V \equiv \{u : I(u) < 0, J(u) < d\}$$

  and the solution blows up in a finite time or

  $$u(t_0) \in W \equiv \{u : I(u) > 0 \text{ or } u = 0, J(u) < d\}$$

  and the solution is global, uniformly bounded in time in the norm of $\mathcal{H}$, and

  $$\lim_{t\to\infty} \|(u(t),\dot{u}(t))\|_{\mathcal{H}} = 0,$$

  exponentially in time.
- *$E(t) \geq d$, for all $t \geq 0$, then the solution is global, uniformly bounded in time in the norm of $\mathcal{H}$, and*

  $$(u(t),\dot{u}(t)) \to \mathcal{E}_\infty \text{ as } t \to \infty,$$

  strongly in $\mathcal{H}$, where

  $$\mathcal{E}_\infty \equiv \{(u,0) \in \mathcal{E} : J(u) = \lim_{t\to\infty} E(t)\}.$$

We notice that the only way to get blow up of in finite time is that, along the solution and since the energy is a non-increasing function, there exists some $t_0 \geq 0$ such that $E(t_0) < d$ and $u(t_0) \in V$. If the damping term is nonlinear of the form $g(u_t) \equiv \delta|u_t|^\lambda u_t,\ \lambda > 2$, although the dynamics is more complicated, a similar result holds, see [8]. Indeed, the solution blows up in a a finite time if and only if $r > \lambda$ and there exists some $t_0 \geq 0$ such that $E(t_0) < d$ and $u(t_0) \in V$. An open problem is to characterize the set of initial data such that $E(t_0) < d$ and $u(t_0) \in V$, for some $t_0 > 0$, especially when $E_0 \equiv E(0) \geq d$ and $u_0 \notin V$. Another open problem is to find conditions on the initial data in order to get $E(t) \geq d$, for all $t \geq 0$. Then, according with Theorem 2.2 those solutions are global and they approach to the set of nonzero equilibria. See also [9], where Theorem 2.2 is proved for the Timoshenko equation. In the conservative case, $\delta = 0$, we know the following result for $E_0 \leq d$.

Theorem 2.3 ([24, 10]). *Consider any solution $(u, \dot{u})$ of* (**NLW**)*, in the sense of Definition 2.1, with initial data $(u_0, u_1) \in \mathcal{H} = V_A \times W_P \equiv H_0^1(\Omega) \times L_2(\Omega)$ and $\delta = 0$.*

If $E_0 < d$, then

- *the solution blows up in a finite time if and only if $u_0 \in V$.*
- *the solution is global if and only if $u_0 \in W$, and in this case the solution is uniformly bounded in time in the norm of $\mathcal{H}$.*

If $E_0 = d$, and

- *there exists some $t_0 \geq 0$ such that $I(u(t_0)) > 0$, then the solution is global and uniformly bounded in time in the norm of $\mathcal{H}$.*
- *there exists some $t_0 \geq 0$ such that $I(u(t_0)) < 0$ and $(u(t_0), v(t_0))_2 \geq 0$, then the solution blows up in finite time.*

The asymptotic behavior of the global solutions with $E_0 \leq d$ is unknown, as well as the characterization of the qualitative behavior for $E_0 > d$ in terms of the initial data. We have to mention two remarkable papers, [18, 27] where the blow up of solutions for the Cauchy problem associated to (**NLW**) is studied, that is with $\Omega = \mathbb{R}^N$. In those works a nonlinear damping term of the form $g(u_t) \equiv \delta|u_t|^\lambda u_t$ is considered and it is proved that for any $(\eta, \xi) \in (0, \infty) \times [0, \infty)$, there are infinitely many compactly supported initial data (u_0, u_1) such that if $\|\nabla u_0\|_2 = \eta$ and $E_0 = \xi$, the corresponding solution blows up in finite time. We notice that this result is not true for the initial boundary value problem (**NLW**) with that nonlinearity $g(u_t)$. Indeed, for $\xi < d$ and $\eta^2 < \frac{2rd}{\alpha(r-2)}$ the solution is globally defined, see [8], because $u_0 \in W$ if and only if $\alpha\|\nabla u_0\|_2^2 < \frac{2r}{r-2}d$. However, the result in [18, 27] exhibits a complex dynamics because it is known that there are global solutions for the Cauchy problem associated to (**NLW**) with initial data such that η and ξ are small, see [27] and references therein.

For the (**GB**)* problem, characterizations of blow-up and global solutions with $E_0 < d$ by means of the potential well method were proved in [21, 22, 16] and in [34], for the one-dimensional and the multidimensional problems, respectively. For $E_0 \geq d$, there are only partial results on the dynamics of particular equations of the type (**GB**)*. Most of them are results for blow up of solutions, see [29, 25, 16]. Similar analysis, for $E_0 \geq d$, have been performed to prove the same qualitative properties for other equations, see [14, 28] to cite just some of the most influential papers on the subject, and see [4, 19, 20, 26, 30, 31, 35, 36, 37] and references therein for some recent works. In [6] a numerical study for the Cauchy problem of the focusing cubic nonlinear Klein-Gordon equation in three dimensions for radial initial data shows that for high energies, the qualitative behavior seems to be much more complicated than for $E < d$, and more research is required to find a threshold between globality and blow up of solutions. In [17] some numerical experiments were performed for the one-dimensional case of $(\mathbf{GB})^*$, with $\delta = 0$, to calculate the mountain pass level and confirm the range of validity of the potential well method. In particular, for $E_0 > d$, it was observed that the sign of $I(u_0)$ is irrelevant to get the blow up of solutions in finite time, but instead, the sign of the inner product $\mathcal{P}(u_0, u_1)$ is important to obtain nonexistence of global solution. In [11, 12] we have proved that this is true for the undamped nonlinear Klein-Gordon equation and for the problem $(\mathbf{P})$ with $\delta = 0$. Here, we are going to prove that, for any $\delta > 0$ and any positive E_0, the sign of $\mathcal{P}(u_0, u_1)$ is the main ingredient to prove the nonexistence of global solutions for problem $(\mathbf{P})$ and in particular for $(\mathbf{GB})^*$. However, we shall consider $\delta \geq 0$, in order to see the influence of the damping in the qualitative behavior.

3. Nonexistence of global solutions

We define

$$\begin{aligned}
&\Psi(u) \equiv \|u\|_{W_P}^2,\\
&\Phi(u, \dot{u}) \equiv \left(\frac{r - \delta/\sqrt{c}}{r}\right)\left(\frac{|\mathcal{P}(u, \dot{u})|^2}{\Psi(u)} + c\Psi(u)\right),
\end{aligned}$$

where $c > 0$ is the constant in $(H0)$, $\delta \geq 0$ is the damping coefficient, we have assumed that $r > 2 + \delta/\sqrt{c}$, see $(H1)$, and from the orthogonal decomposition

$$\dot{u} = \frac{\mathcal{P}(\dot{u}, u)}{\|u\|_{W_P}^2} u + h, \quad \mathcal{P}(u, h) = 0,$$

we have that

$$\|\dot{u}\|_{W_P}^2 = \|h\|_{W_P}^2 + \frac{|\mathcal{P}(\dot{u}, u)|^2}{\|u\|_{W_P}^2} \geq \frac{|\mathcal{P}(\dot{u}, u)|^2}{\|u\|_{W_P}^2} = \frac{|\mathcal{P}(\dot{u}, u)|^2}{\Psi(u)}.$$

We also define the following functions

$$\begin{aligned}
&\eta_{q,\delta}(u, \dot{u}) \equiv \frac{1}{2}\Phi(u, \dot{u}) - \frac{c}{r}\Psi(u)\left(\left(\frac{r - \delta/\sqrt{c}}{r}\right)\frac{c\Psi(u)}{\Phi(u, \dot{u})}\right)^q, \ q \geq 0,\\
&\mu_{\lambda,\delta}(u, \dot{u}) \equiv \frac{1}{2}\Phi(u, \dot{u}) - \frac{c}{r}\Psi(u)\left(\left(\frac{r - \delta/\sqrt{c}}{r}\right)\frac{\lambda c\Psi(u)}{\Phi(u, \dot{u})}\right)^{\frac{r-2-\delta/\sqrt{c}}{2}}, \ \lambda \in (0, 1),\\
&\sigma_\nu(u, \dot{u}) \equiv \frac{1}{2}\Phi(u, \dot{u}) - \frac{c\nu}{r}\Psi(u), \ \nu > 1.
\end{aligned}$$

If $\mathcal{P}(u, \dot{u}) > 0$ and for fixed δ, we notice that $q \mapsto \eta_{q,\delta}$ is strictly increasing, $\lambda \mapsto \mu_{\lambda,\delta}$ is strictly decreasing and $\nu \mapsto \sigma_\nu$ is strictly decreasing. They have the following relations

$$\begin{aligned}
&\lim_{\lambda \to 1} \mu_{\lambda,\delta}(u, \dot{u}) = \eta_{q,\delta}(u, \dot{u})\big|_{q = \frac{r-2-\delta/\sqrt{c}}{2}},\\
&\lim_{\nu \to 1} \sigma_\nu(u, \dot{u}) = \eta_{q,\delta}(u, \dot{u})|_{q=0},
\end{aligned}$$

and, $\sigma_\nu(u, \dot{u}) < \eta_{q,\delta}(u, \dot{u})|_{q=0} < \eta_{q,\delta}(u, \dot{u})\big|_{q = \frac{r-2-\delta/\sqrt{c}}{2}} < \mu_{\lambda,\delta}(u, \dot{u})$.

Now, we are able to present the main result of this work.

Theorem 3.1. *Consider any solution of problem* **(P)** *in the sense of Definition 2.1. Assume that hypotheses* $(H0)$ *and* $(H1)$ *hold, and that*

$$\|u_0\|^2_{W_P} > 0,\ \mathcal{P}(u_0,u_1) > 0, \tag{2}$$

are satisfied. Then, there exists a nonempty interval

$$\mathcal{I}_\delta \equiv (\alpha_\delta, \beta_\delta) \subset \left(0, \frac{1}{2}\Phi(u_0,u_1)\right),$$

where, by means of the functions $\sigma_{\nu,\delta}(u_0,u_1)$ *and* $\mu_{\lambda,\delta}(u_0,u_1)$*, we can obtain the size of such interval*

$$\alpha_\delta = \sigma_{\nu^*,\delta}(u_0,u_1) = \left(\frac{r-2-\delta/\sqrt{c}}{2r}\right)\left(\frac{c\Psi(u_0)}{\nu^{*\left(\frac{2}{r-2-\delta/\sqrt{c}}\right)}}\right),$$
$$\beta_\delta = \mu_{\lambda^*,\delta}(u_0,u_1) = \left(\frac{r-2-\delta/\sqrt{c}}{r-\delta/\sqrt{c}}\right)\left(\frac{\Phi(u_0,u_1)}{2\lambda^*}\right),$$

for some $\frac{r-2-\delta/\sqrt{c}}{r-\delta/\sqrt{c}} < \lambda^* < 1$ *and* $\nu^* > 1$*, with the following consequences:*

(i) If the initial energy is such that $E_0 \in \mathcal{I}_\delta$*, then the maximal time of existence of the solution is finite.*

(ii) For fixed $\Psi(u_0)$ *and* δ*,*

$$\mathcal{P}(u_0,u_1) \mapsto |\mathcal{I}_\delta| = \beta_\delta - \alpha_\delta,$$

is strictly increasing, and we have the limit values

$$\lim_{\mathcal{P}(u_0,u_1)\to\infty} \alpha_\delta = 0 = \lim_{\mathcal{P}(u_0,u_1)\to\infty} \left|\beta_\delta - \frac{1}{2}\Phi(u_0,u_1)\right|,$$
$$\lim_{\mathcal{P}(u_0,u_1)\to\infty} \nu^* = \infty, \qquad \lim_{\mathcal{P}(u_0,u_1)\to\infty} \lambda^* = \frac{r-2-\delta/\sqrt{c}}{r-\delta/\sqrt{c}}.$$

Corollary 1. *Assume that hypotheses of Theorem 3.1 are met. For every number* $\mathcal{K} > 0$*, we can choose initial data with* $\mathcal{P}(u_0,u_1)$ *large enough, so that* $\mathcal{K} \in \mathcal{I}_\delta$*, and then the corresponding solution with* $E_0 = \mathcal{K}$ *exists only up to a finite time.*

4. Proofs

Proof. (of Theorem 3.1.) We assume that $\Psi(u(t)) \equiv \|u\|^2_{W_P}$ exists for any $t \geq 0$. If such a thing does not happen, then the solution cannot be global. We observe that

$$\frac{d}{dt}\Psi(u(t)) = 2(\mathcal{P}(u(t), \dot{u}(t)),$$

and then we get the following estimate

$$\begin{aligned} 2(\mathcal{P}(u(t),\dot{u}(t)) &\leq \sqrt{c}\|u\|^2_{W_P} + \frac{1}{\sqrt{c}}\|\dot{u}(t)\|^2_{W_P} \\ &\leq \sqrt{c}\|u\|^2_{W_P} + \frac{1}{\sqrt{c}}\frac{|\mathcal{P}(\dot{u},u)|^2}{\|u\|^2_{W_P}}. \end{aligned}$$

Hence,

$$\frac{d}{dt}\Psi(u(t)) \leq \sqrt{c}\Psi(u(t)) + \frac{1}{4\sqrt{c}}\frac{\left(\frac{d}{dt}\Psi(u(t))\right)^2}{\Psi(u(t))} \tag{3}$$

By energy equation and hypotheses $(H0)$ and $(H1)$, we get the following chain of estimates

$$\begin{aligned}\frac{d^2}{dt^2}\Psi(u(t))+\delta\frac{d}{dt}\Psi(u(t)) &= 2(\|\dot u(t)\|_{W_P}^2 - I(u(t)))\\ &= 2(\|\dot u(t)\|_{W_P}^2 - I(u(t)))+2rE(t)-2rE(t)\\ &= (r+2)\|\dot u(t)\|_{W_P}^2+(r-2)\|u(t)\|_{V_A}^2-2rE(t)\\ &\geq (r+2)\frac{|\mathcal{P}(\dot u,u)|^2}{\|u\|_{W_P}^2}+c(r-2)\|u(t)\|_{W_P}^2-2rE_0\\ &= \frac{r+2}{4}\left(\frac{(\frac{d}{dt}\Psi(u(t)))^2}{\Psi(u(t))}\right)+c(r-2)\Psi(u(t))-2rE_0.\end{aligned}$$

Hence, by (3),

$$\begin{aligned}\frac{d^2}{dt^2}\Psi(u(t)) \geq \left(\frac{r+2-\delta/\sqrt{c}}{4}\right)\frac{\left(\frac{d}{dt}\Psi(u(t))\right)^2}{\Psi(u(t)}\\ +\left(c(r-2)-\delta\sqrt{c}\right)\Psi(u(t))-2rE_0.\end{aligned}\tag{4}$$

We define

$$\mathcal{F}(t)\equiv \Psi^{-\left(\frac{r-2-\delta/\sqrt{c}}{4}\right)}(u(t)).$$

Consequently, from (4),

$$\begin{aligned}&\frac{d^2}{dt^2}\mathcal{F}(t)\\ &= \left(\frac{r-2-\delta/\sqrt{c}}{4}\right)\Psi^{-\left(\frac{r+2-\delta/\sqrt{c}}{4}\right)}(u(t))\\ &\quad\times\left(\left(\frac{r+2-\delta/\sqrt{c}}{4}\right)\frac{(\frac{d}{dt}\Psi(u(t)))^2}{\Psi(u(t))}-\frac{d^2}{dt^2}\Psi(u(t))\right)\\ &\leq \left(\frac{r-2-\delta/\sqrt{c}}{4}\right)\Psi^{-\left(\frac{r+2-\delta/\sqrt{c}}{4}\right)}(u(t))\left\{-\left(c(r-2)-\delta\sqrt{c}\right)\Psi(u(t))+2rE_0\right\}.\end{aligned}$$

That is,

$$\frac{d^2}{dt^2}\mathcal{F}(t)\leq -\frac{c(r-2-\delta/\sqrt{c})^2}{4}\mathcal{F}(t)+E_0\frac{r(r-2-\delta/\sqrt{c})}{2}\mathcal{F}(t)^{\left(\frac{r+2-\delta/\sqrt{c}}{r-2-\delta/\sqrt{c}}\right)}.\tag{5}$$

Due to (2) and since $r>2+\delta/\sqrt{c}$, see $(H1)$, we have that the following inequality is true

$$\begin{aligned}\frac{d}{dt}\mathcal{F}(t) &= -\left(\frac{r-2-\delta/\sqrt{c}}{4}\right)\Psi^{-\left(\frac{r+2-\delta/\sqrt{c}}{4}\right)}(u(t))\frac{d}{dt}\Psi(u(t))\\ &= -\left(\frac{r-2-\delta/\sqrt{c}}{2}\right)\Psi^{-\left(\frac{r+2-\delta/\sqrt{c}}{4}\right)}(u(t))\,\mathcal{P}(u(t),\dot u(t))<0,\end{aligned}$$

for any $t\geq 0$ sufficiently close to zero. That is,

$$\frac{d}{dt}\mathcal{F}(t)=-\left(\frac{r-2-\delta/\sqrt{c}}{4}\right)\mathcal{F}^{-\left(\frac{r+2-\delta/\sqrt{c}}{r+2-\delta/\sqrt{c}}\right)}(u(t))\,\frac{d}{dt}\Psi(u(t))<0.$$

Now, we multiply (5) by $\frac{d}{dt}\mathcal{F}(t) < 0$. Then, after an easy integration, we get

$$\left(\frac{d}{dt}\mathcal{F}(t)\right)^2 \geq \left(\frac{r-2-\delta/\sqrt{c}}{2}\right)^2 \times \left(\left(\frac{2r}{r-\delta/\sqrt{c}}\right) E_0 \mathcal{F}^{\left(\frac{2(r-\delta/\sqrt{c})}{r-2-\delta/\sqrt{c}}\right)}(t) - c\mathcal{F}^2(t)\right) + \mathcal{C}_0, \tag{6}$$

where

$$\mathcal{C}_0 \equiv \left(\frac{d}{dt}\mathcal{F}(0)\right)^2 - \left(\frac{r-2-\delta/\sqrt{c}}{2}\right)^2 \left(\left(\frac{2r}{r-\delta/\sqrt{c}}\right) E_0 \mathcal{F}^{\left(\frac{2(r-\delta/\sqrt{c})}{r-2-\delta/\sqrt{c}}\right)}(0) - c\mathcal{F}^2(0)\right).$$

We shall prove that there exists a constant $\kappa_0 > 0$ such that

$$\left(\frac{d}{dt}\mathcal{F}(t)\right)^2 \geq \kappa_0^2 > 0, \tag{7}$$

and then

$$\frac{d}{dt}\mathcal{F}(t) \leq -\kappa_0 < 0.$$

Hence,

$$0 \leq \mathcal{F}(t) \leq -\kappa_0 t + \mathcal{F}(0).$$

Which is impossible for any $t > \frac{\mathcal{F}(0)}{\kappa_0}$. Then, the solution only exits up to a finite time.

Next, we prove that (7) holds. To this end, we consider the right hand side of (6) and define, for $s \geq 0$,

$$\mathcal{G}(s) \equiv \left(\frac{r-2-\delta/\sqrt{c}}{2}\right)^2 \left(\left(\frac{2r}{r-\delta/\sqrt{c}}\right) E_0 s^{\left(\frac{r-\delta/\sqrt{c}}{r-2-\delta/\sqrt{c}}\right)} - cs\right) + \mathcal{C}_0,$$

and we notice that $\mathcal{G}$ attains an absolute minimum, that is

$$\mathcal{G}(s) \geq \mathcal{G}(s_0), \ \forall s \geq 0,$$

with $s_0 \equiv \left(\frac{c(r-2)-\delta\sqrt{c}}{2rE_0}\right)^{\left(\frac{r-2-\delta/\sqrt{c}}{2}\right)} > 0$, since $c(r-2) - \delta\sqrt{c} > 0$, and

$$\begin{aligned}
\mathcal{G}(s_0) &= \left(\frac{r-2-\delta/\sqrt{c}}{2}\right)^2 \left(\left(\frac{2r}{r-\delta/\sqrt{c}}\right) E_0 s_0^{\left(\frac{r-\delta/\sqrt{c}}{r-2-\delta/\sqrt{c}}\right)} - cs_0\right) + \mathcal{C}_0, \\
&= -rE_0 \left(\frac{r-2-\delta/\sqrt{c}}{r-\delta/\sqrt{c}}\right) \left(\frac{c(r-2)-\delta\sqrt{c}}{2rE_0}\right)^{\left(\frac{r-\delta/\sqrt{c}}{2}\right)} + \mathcal{C}_0,
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\mathcal{C}_0 = &\left(\frac{r-2-\delta/\sqrt{c}}{2}\right)^2 \Psi(u_0)^{-\left(\frac{r+2-\delta/\sqrt{c}}{2}\right)} \left(\mathcal{P}(u_0,u_1)\right)^2 \\
&- \left(\frac{r-2-\delta/\sqrt{c}}{2}\right)^2 \Psi(u_0)^{-\left(\frac{r-2-\delta/\sqrt{c}}{2}\right)} \left(\frac{2rE_0}{r-\delta/\sqrt{c}}\Psi(u_0)^{-1} - c\right).
\end{aligned}$$

Notice that (7) is satisfied for $\kappa_0^2 = \mathcal{G}(s_0)$, and $\mathcal{G}(s_0) > 0$ holds if and only if

$$c\left(\frac{c(r-2)-\delta\sqrt{c}}{2rE_0}\right)^{\left(\frac{r-2-\delta/\sqrt{c}}{2}\right)} + E_0\Psi(u_0)^{-\left(\frac{r-\delta/\sqrt{c}}{2}\right)}$$
$$< \left(\frac{r-\delta/\sqrt{c}}{2r}\right)\left(\Psi(u_0)^{-\left(\frac{r+2-\delta/\sqrt{c}}{2}\right)}\left(\mathcal{P}(u_0,u_1)\right)^2 + c\Psi(u_0)^{-\left(\frac{r-2-\delta/\sqrt{c}}{2}\right)}\right),$$

which is equivalent to

$$E_0 + \left(\left(\frac{c(r-2)-\delta\sqrt{c}}{2rE_0}\right)\Psi(u_0)\right)^{\left(\frac{r-2-\delta/\sqrt{c}}{2}\right)}\frac{c}{r}\Psi(u_0) < \frac{1}{2}\Phi(u_0,u_1), \tag{8}$$

where, we remember that

$$\Phi(u_0,u_1) \equiv \left(\frac{r-\delta/\sqrt{c}}{r}\right)\left(\frac{\left(\mathcal{P}(u_0,u_1)\right)^2}{\Psi(u_0)} + c\Psi(u_0)\right).$$

We consider the left hand side of (8) to define, for $s \geq 0$,

$$\mathcal{J}(s) \equiv s + \left(\left(\frac{c(r-2)-\delta\sqrt{c}}{2rs}\right)\Psi(u_0)\right)^{\left(\frac{r-2-\delta/\sqrt{c}}{2}\right)}\frac{c}{r}\Psi(u_0).$$

We observe that (8) is satisfied if and only if

$$\mathcal{J}(E_0) < \frac{1}{2}\Phi(u_0,u_1), \tag{9}$$

We notice that $\mathcal{J}(s) \to \infty$ as, either $s \to 0$ or $s \to \infty$, and $\mathcal{J}$ attains an absolute minimum, that is

$$\mathcal{J}(s) \geq \mathcal{J}(s_1) = c\left(\frac{r-\delta/\sqrt{c}}{2r}\right)\Psi(u_0),\ \forall s \geq 0,$$

for $s_1 \equiv \left(\frac{c(r-2)-\delta\sqrt{c}}{2r}\right)\Psi(u_0)$. Moreover, by (2), there exist exactly two different roots of $\mathcal{J}(s) = \frac{1}{2}\Phi(u_0,u_1)$, denoted by α_δ and β_δ, such that

$$0 < \alpha_\delta < s_1 < \beta_\delta < \frac{1}{2}\Phi(u_0,u_1),$$

and

$$c\left(\frac{r-\delta/\sqrt{c}}{2r}\right)\Psi(u_0) < \mathcal{J}(s) < \frac{1}{2}\Phi(u_0,u_1),\ \forall s \in \mathcal{I}_\delta \equiv (\alpha_\delta,\beta_\delta), s \neq s_1.$$

And since $\mathcal{J}$ is strictly monotone for $s < s_1$ and $s > s_1$, it follows that, for fixed $\Psi(u_0)$, the interval $\mathcal{I}_\delta$ grows as $\mathcal{P}(u_0,u_1)$ grows. Precisely,

$$\lim_{\mathcal{P}(u_0,u_1)\to\infty}\left|\frac{1}{2}\Phi(u_0,u_1) - \beta_\delta\right| = 0 = \lim_{\mathcal{P}(u_0,u_1)\to\infty}\alpha_\delta.$$

Then, (8) holds if and only if the initial energy satisfies (9). That is, if and only if $E_0 \in \mathcal{I}_\delta$. This proves that the maximum time of existence must be finite if the initial energy is within this interval.

We shall use the functions σ_ν and $\mu_{\lambda,\delta}$ to find the values of α_δ and β_δ, respectively. Remember that these are the roots of $\mathcal{J}(s) = \frac{1}{2}\Phi(u_0,u_1)$. To find α_δ, we consider the equation

$$\mathcal{J}(\sigma_\nu(u_0,u_1)) = \frac{1}{2}\Phi(u_0,u_1), \tag{10}$$

where

$$\sigma_\nu(u_0,u_1) \equiv \frac{1}{2}\Phi(u_0,u_1) - \frac{c\nu}{r}\Psi(u_0),$$

is defined for $\nu > 1$. Notice that (10) holds if and only if

$$\frac{1}{\nu^{\left(\frac{2}{r-2-\delta/\sqrt{c}}\right)}} = \frac{2r}{c(r-2)-\delta\sqrt{c}}\left(\frac{\sigma_\nu(u_0,u_1)}{\Psi(u_0)}\right).$$

Which is equivalent to

$$\frac{2}{r}\nu + \left(\frac{r-2-\delta/\sqrt{c}}{r}\right)\frac{1}{\nu^{\left(\frac{2}{r-2-\delta/\sqrt{c}}\right)}} = \frac{\Phi(u_0,u_1)}{c\Psi(u_0)}. \tag{11}$$

We consider the function, defined for $s > 0$,

$$f(s) \equiv \frac{2}{r}s + \left(\frac{r-2-\delta/\sqrt{c}}{r}\right)\frac{1}{s^{\left(\frac{2}{r-2-\delta/\sqrt{c}}\right)}},$$

and notice that

$$f(\nu) \to \infty,$$

as $\nu \to 0$ and $\nu \to \infty$. Furthermore, f has an absolute minimum, that is

$$f(s) \geq f(1) = \frac{r-\delta/\sqrt{c}}{r},\ \forall s > 0.$$

Moreover, from (2) and the definition of $\Phi(u_0,u_1)$,

$$\frac{\Phi(u_0,u_1)}{c\Psi(u_0)} > \frac{r-\delta/\sqrt{c}}{r}.$$

Then, equation (11) equivalently (10), has two roots and only one bigger than one, that is, $\nu^* > 1$. Furthermore, at this root,

$$\alpha_\delta = \sigma_{\nu^*}(u_0,u_1) = \left(\frac{r-2-\delta/\sqrt{c}}{2r}\right)\left(\frac{c\Psi(u_0)}{\nu^{*\left(\frac{2}{r-2-\delta\sqrt{c}}\right)}}\right),$$

and

$$\lim_{\mathcal{P}(u_0,u_1)\to\infty} \nu^* = \infty.$$

Next, we consider the equation

$$\mathcal{J}(\mu_{\lambda,\delta}(u_0,u_1)) = \frac{1}{2}\Phi(u_0,u_1), \tag{12}$$

where

$$\mu_{\lambda,\delta}(u_0,u_1) \equiv \frac{1}{2}\Phi(u_0,u_1) - \frac{c}{r}\Psi(u_0)\left(\frac{(r-\delta/\sqrt{c})}{r}\frac{\lambda c\Psi(u_0)}{\Phi(u_0,u_1)}\right)^{\left(\frac{r-2-\delta/\sqrt{c}}{2}\right)},$$

is defined for $0 < \lambda < 1$. We observe that (12) holds if and only if

$$\left(\frac{c(r-2-\delta/\sqrt{c})}{2r}\frac{\Psi(u_0)}{\mu_{\lambda,\delta}(u_0,u_1)}\right)^{\left(\frac{r-2-\delta/\sqrt{c}}{2}\right)} = \left(\left(\frac{r-\delta/\sqrt{c}}{r}\right)\frac{\lambda c\Psi(u_0)}{\Phi(u_0,u_1)}\right)^{\left(\frac{r-2-\delta/\sqrt{c}}{2}\right)}$$

And this is characterized by

$$\frac{r-2-\delta/\sqrt{c}}{2(r-\delta/\sqrt{c})} = \lambda\frac{\mu_{\lambda,\delta}(u_0,u_1)}{\Phi(u_0,u_1)}.$$

Which is equivalent to

$$\frac{2}{r-\delta/\sqrt{c}}\left(\lambda\left(\frac{r-\delta/\sqrt{c}}{r}\right)\frac{c\Psi(u_0)}{\Phi(u_0,u_1)}\right)^{\left(\frac{r-\delta/\sqrt{c}}{2}\right)} = \lambda - \frac{r-2-\delta/\sqrt{c}}{r-\delta/\sqrt{c}}. \tag{13}$$

We consider the functions, defined for $s \in [s_2, 1]$, with $s_2 = \frac{r-2-\delta/\sqrt{c}}{r-\delta/\sqrt{c}}$,

$$g(s) \equiv \frac{2}{r-\delta/\sqrt{c}}\left(s\left(\frac{r-\delta/\sqrt{c}}{r}\right)\frac{c\Psi(u_0)}{\Phi(u_0,u_1)}\right)^{\left(\frac{r-\delta/\sqrt{c}}{2}\right)},$$

and

$$h(s) \equiv s - \frac{r-2-\delta/\sqrt{c}}{r-\delta/\sqrt{c}},$$

are strictly monotone increasing, and

$$g(s_2) > h(s_2) = 0, \;\; g(1) < h(1) = \frac{2}{r-\delta/\sqrt{c}},$$

since, from (2) and definition of $\Phi(u_0, u_1)$,

$$\left(\frac{r-\delta/\sqrt{c}}{r}\right)\frac{c\Psi(u_0)}{\Phi(u_0,u_1)} < 1.$$

Then, there exists one and only one $\lambda^* \in \left(\frac{r-2-\delta/\sqrt{c}}{r-\delta/\sqrt{c}}, 1\right)$ where $g(\lambda^*) = h(\lambda^*)$. That is, only one root λ^* of equation (13), equivalently (12). Moreover,

$$\beta_\delta = \mu_{\lambda^*,\delta}(u_0,u_1) = \left(\frac{r-2-\delta/\sqrt{c}}{r-\delta/\sqrt{c}}\right)\left(\frac{\Phi(u_0,u_1)}{2\lambda^*}\right),$$

and

$$\lim_{\mathcal{P}(u_0,u_1)\to\infty} \lambda^* = \frac{r-2-\delta/\sqrt{c}}{r-\delta/\sqrt{c}}.$$

□

Proof. (of Corollary 1.) Since

$$\mathcal{P}(u_0,u_1) \to \infty \;\; \Rightarrow \;\; \alpha_\delta \to 0 \;\text{ and }\; \beta_\delta \to \infty$$

then, for every $\mathcal{K} > 0$ there exists $\mathcal{L} > 0$, such that

$$\mathcal{P}(u_0,u_1) > \mathcal{L} \;\; \Rightarrow \;\; \mathcal{K} \in \mathcal{I}_\delta = (\alpha_\delta, \beta_\delta).$$

Hence, the corresponding solution with initial energy $E_0 = \mathcal{K}$ satisfying (2) exists only up to a finite time. □

5. Discussion and some remarks

Remark 1. It is well known that when the potential well method is applied, for $E_0 < d$, the qualitative behavior is determinated by the sign of $I(u_0)$. In particular, the blow-up of a solution is characterized if the initial data are such that $I(u_0) < 0$. For any positive value of the initial energy, this is not the case. Indeed, under the assumptions of Theorem 3.1, we get from energy equation, if $E_0 < \beta_\delta$, and since $\frac{r-2-\delta/\sqrt{c}}{r-\delta/\sqrt{c}} < \lambda_\delta^* < 1$, the following consequences

$$\begin{aligned} I(u_0) &= 2E_0 - \|u_1\|^2_{W_P} + 2F(u_0) - (f(u_0), u_0) \\ &\le 2E_0 - \frac{|\mathcal{P}(u_0,u_1)|^2}{\|u_0\|^2_{W_P}} + 2F(u_0) - (f(u_0), u_0) \\ &< 2\beta_\delta - \frac{|\mathcal{P}(u_0,u_1)|^2}{\Psi(u_0)} + 2F(u_0) - (f(u_0), u_0) \\ &= \left(\frac{r-2-\delta/\sqrt{c}}{r-\delta/\sqrt{c}}\right)\frac{\Phi(u_0,u_1)}{\lambda_\delta^*} - \frac{|\mathcal{P}(u_0,u_1)|^2}{\Psi(u_0,v_0)} + 2F(u_0) - (f(u_0), u_0) \\ &= \left(\left(\frac{r-2-\delta/\sqrt{c}}{r-\delta/\sqrt{c}}\right)\frac{1}{\lambda_\delta^*} - \frac{r}{r-\delta/\sqrt{c}}\right)\Phi(u_0,u_1) - ((f(u_0),u_0) - 2F(u_0) - c\Psi(u_0)) \\ &< \left(1 - \frac{r}{r-\delta/\sqrt{c}}\right)\Phi(u_0,u_1) - ((f(u_0),u_0) - 2F(u_0) - c\Psi(u_0)). \end{aligned}$$

Hence,

$$I(u_0) < -\left(\frac{\delta/\sqrt{c}}{r-\delta/\sqrt{c}}\right)\Phi(u_0,u_1) - ((f(u_0),u_0) - 2F(u_0) - c\Psi(u_0)) \tag{14}$$

Let us assume that the source term is large enough, that is,

$$(f(u_0),u_0) - 2F(u_0) \ge c\Psi(u_0), \tag{15}$$

then, by (14),

$$I(u_0) < -\left(\frac{\delta/\sqrt{c}}{r-\delta/\sqrt{c}}\right)\Phi(u_0,u_1) \le 0.$$

From $(H1)$, (15) holds if the source term is such that

$$F(u_0) \ge \frac{1}{r-2}c\Psi(u_0).$$

Then, in this case the inequality $I(u_0) < 0$ is a necessary condition for nonexistence of global solutions. However, it seems that the condition $I(u_0) < 0$, alone, does not imply nonexistence of global solutions for high energies, see [14, 28]. Moreover, the sign of $I(u_0)$ is not required in the proof of Theorem 3.1.

By $(H1)$ and the energy equation

$$\{2E_0 - I(u_0) - (f(u_0),u_0) + 2F(u_0)\}\Psi(u_0)\{2(E_0 + F(u_0)) - \|u_0\|^2_{V_A}\}\Psi(u_0) \ge |\mathcal{P}(u_0,u_1)|^2.$$

If $E_0 < d$, the mountain pass level defined in (1), and $f(u) \equiv |u|^{r-2}u$, then $rF(u) = \|u\|_r^r$, and

$$\left\{2d - I(u_0) - \frac{r-2}{r}\|u_0\|_r^r\right\}\Psi(u_0) > |\mathcal{P}(u_0,u_1)|^2.$$

From Corollary 1, global nonexistence is obtained if $\mathcal{P}(u_0,u_1) > 0$. Hence, in order to conclude global nonexistence for $E_0 < d$, it is necessary that

$$I(u_0) < 2d - \frac{r-2}{r}\|u_0\|_r^r.$$

We remember that for $E_0 < d$, see [7], we have that $I(u_0) < 0$ if and only if $2rd < (r-2)\|u_0\|_r^r$. Apparently, only for energies $E_0 < d$, the condition $I(u_0) < 0$ characterizes the nonexistence of global solutions of problem $(\mathbf{P})$.

Remark 2. We shall prove the following lower bound for β_δ

$$\beta_\delta > \frac{1}{2}\Phi(u_0, u_1) - \frac{1}{r}c\Psi(u_0),$$

which is equivalent to

$$\beta_\delta > \left(\frac{r-\delta/\sqrt{c}}{2r}\right)\frac{|\mathcal{P}(u_0,u_1)|^2}{\Psi(u_0)} + \left(\frac{r-2-\delta/\sqrt{c}}{2r}\right)c\Psi(u_0).$$

To this end, consider first the following inequality

$$\beta_\delta > \left(\frac{r-\delta/\sqrt{c}}{2r}\right)\frac{|\mathcal{P}(u_0,u_1)|^2}{\Psi(u_0)}.$$

Notice that it is equivalent to

$$\left(\lambda^*\left(\frac{r-\delta/\sqrt{c}}{r-2-\delta/\sqrt{c}}\right)-1\right)|\mathcal{P}(u_0,u_1)|^2 < c\Psi^2(u_0),$$

where $\frac{r-2-c/\sqrt{c}}{r-c/\sqrt{c}} < \lambda^* < 1$.

In order to prove last inequality, let us define for any $s>0$, the positive function

$$l(s) \equiv \left(\lambda^*\left(\frac{r-\delta/\sqrt{c}}{r-2-\delta/\sqrt{c}}\right)-1\right)s > 0,$$

and, from the proof of Theorem 3.1, we remember that λ^* is a function of $s \equiv |\mathcal{P}(u_0,u_1)|^2$, defined implicitly by

$$\frac{2}{r-\delta/\sqrt{c}}\left(\lambda^*\left(\frac{r-\delta/\sqrt{c}}{r}\right)\frac{c\Psi_0}{\Phi_0}\right)^{\left(\frac{r-\delta/\sqrt{c}}{2}\right)} = \lambda^* - \frac{r-2-\delta/\sqrt{c}}{r-\delta/\sqrt{c}},$$

where

$$\Psi_0 \equiv \Psi(u_0),\ \Phi_0 \equiv \Phi(u_0,u_1) = \left(\frac{r-\delta/\sqrt{c}}{r}\right)\left(c\Psi_0 + \frac{s}{\Psi_0}\right).$$

Also, from the proof of Theorem 3.1 we know that

$$\lim_{s\to\infty}\lambda^* = \frac{r-2-\delta/\sqrt{c}}{r-\delta/\sqrt{c}},\quad \lim_{s\to 0}\lambda^* = 1.$$

Then, from the definition of λ^* and Φ_0,

$$\begin{aligned}\lim_{s\to\infty} l(s) &= \left(\frac{2}{r-2-\delta/\sqrt{c}}\right)\lim_{s\to\infty} s\left(\lambda^*\left(\frac{r-\delta/\sqrt{c}}{r}\right)\frac{c\Psi_0}{\Phi_0}\right)^{\frac{r-\delta/\sqrt{c}}{2}} \\ &= \frac{2}{r-2}\lim_{s\to\infty} s\left(\lambda^*\frac{c\Psi_0^2}{c\Psi_0^2+s}\right)^{\left(\frac{r-\delta/\sqrt{c}}{2}\right)} = 0.\end{aligned}$$

Also,

$$\lim_{s\to 0} l(s) = 0.$$

Consequently, there is some $s^* \in (0,\infty)$, such that $l(s^*) = \max_{s\in(0,\infty)} l(s)$. After some calculations, we find that

$$s^* = c\Psi_0^2\left(\frac{(r-2-\delta/\sqrt{c})(1-\lambda^*)}{(r-\delta/\sqrt{c})\lambda^* - (r-2-\delta/\sqrt{c})(1-\lambda^*)}\right),$$

$$l(s^*) = c\Psi_0^2\left(\frac{((r-\delta/\sqrt{c})\lambda^* - (r-2-\delta/\sqrt{c}))(1-\lambda^*)}{(r-\delta/\sqrt{c})\lambda^* - (r-2-\delta/\sqrt{c})(1-\lambda^*)}\right),$$

and consequently

$$l(s) \le l(s^*) \text{ for any } s > 0.$$

Hence,

$$\left(\lambda^*\left(\frac{r-\delta/\sqrt{c}}{r-2-\delta/\sqrt{c}}\right)-1\right)|\mathcal{P}(u_0,u_1)|^2<c\Psi_0^2\eta(\lambda^*), \tag{16}$$

where

$$\eta(\lambda^*)\equiv\left(\frac{((r-\delta/\sqrt{c})\lambda^*-(r-2-\delta/\sqrt{c}))(1-\lambda^*)}{(r-\delta/\sqrt{c})\lambda^*-(r-2-\delta/\sqrt{c})(1-\lambda^*)}\right)<1.$$

Notice that (16) is equivalent to

$$\beta_\delta>\left(\frac{r-\delta/\sqrt{c}}{2r}\right)\frac{|\mathcal{P}(u_0,u_1)|^2}{\Psi(u_0)}+\zeta(\lambda^*)c\Psi(u_0),$$

where

$$\zeta(\lambda^*)\equiv\frac{1}{2r}\left(\frac{(r-\delta/\sqrt{c})(r-2-\delta/\sqrt{c})\lambda^*}{(r-\delta/\sqrt{c})\lambda^*-(r-2-\delta/\sqrt{c})(1-\lambda^*)}\right)>\frac{r-2-\delta/\sqrt{c}}{2r},$$

since $\lambda^*<1$.

Consequently,

$$\beta_\delta>\left(\frac{r-\delta/\sqrt{c}}{2r}\right)\frac{|\mathcal{P}(u_0,u_1)|^2}{\Psi(u_0)}+\left(\frac{r-2-\delta/\sqrt{c}}{2r}\right)c\Psi(u_0)=\frac{1}{2}\Phi(u_0,u_1)-\frac{1}{r}c\Psi(u_0),$$

Remark 3. We shall prove the following upper bound for α_δ

$$\alpha_\delta<\left(\frac{r-2-\delta/\sqrt{c}}{2r}\right)c\Psi(u_0).$$

To this end, notice that this inequality is equivalent to

$$\nu^*-\left(\frac{r-\delta/\sqrt{c}}{2c}\right)\frac{|\mathcal{P}(u_0,u_1)|^2}{\Psi^2(u_0)}>1,$$

where $\nu^*>1$.

In order to prove last inequality, we define for any $s>0$, the function

$$l(s)\equiv\nu^*-\left(\frac{r-\delta/\sqrt{c}}{2c}\right)\frac{s}{\Psi^2(u_0)},$$

where we remember from the proof of Theorem 3.1 that ν^* depends of $s\equiv|\mathcal{P}(u_0,u_1)|^2$, defined implicitly by

$$\frac{2}{r}\nu^*+\left(\frac{r-2-\delta/\sqrt{c}}{r}\right)\frac{1}{\nu^{*\left(\frac{2}{r-2-\delta/\sqrt{c}}\right)}}=\frac{\Phi_0}{c\Psi_0},$$

where like in Remark 2

$$\Psi_0\equiv\Psi(u_0),\ \Phi_0\equiv\Phi(u_0,u_1)=\left(\frac{r-\delta/\sqrt{c}}{r}\right)\left(c\Psi_0+\frac{s}{\Psi_0}\right).$$

Furthermore, from the proof of Theorem 3.1 we know that

$$\lim_{s\to\infty}\nu^*=\infty,\ \ \lim_{s\to0}\nu^*=1$$

Now, from the definition of ν^* and Φ_0,

$$\lim_{s\to\infty}l(s)=\frac{r-\delta/\sqrt{c}}{2}-\lim_{s\to\infty}\left(\frac{r-2-\delta/\sqrt{c}}{2}\right)\frac{1}{\nu^{*\left(\frac{2}{r-2-\delta/\sqrt{c}}\right)}}=\frac{r-\delta/\sqrt{c}}{2}.$$

Also,

$$\lim_{s\to0}l(s)=1.$$

After some calculations, it follows that,

$$\frac{d}{ds}l(s) = \left(\frac{r-\delta/\sqrt{c}}{2c\Psi_0^2}\right)\left(\frac{1}{1-\frac{1}{\nu^*\left(\frac{r-\delta/\sqrt{c}}{r-2-\delta/\sqrt{c}}\right)}}-1\right) > 0.$$

Moreover,

$$\lim_{s\to\infty}\frac{d}{ds}l(s) = 0, \ \ \lim_{s\to 0}\frac{d}{ds}l(s) = \infty.$$

Then, $l(s)$ is strictly increasing and bounded by

$$1 < l(s) < \frac{r-\delta/\sqrt{c}}{2}.$$

And this is equivalent to

$$0 < \alpha_\delta < \left(\frac{r-2-\delta/\sqrt{c}}{2r}\right)c\Psi(u_0).$$

Remark 4. The length of $\mathcal{I}_\delta$ depends on the size of the damping coefficient $\delta \geq 0$ and decreases as δ grows. Therefore, as the damping coefficient increases, the set of initial energies where we can have global non existence becomes smaller. In fact, from Remarks 2 and 3, we have that

$$\left(\frac{r-\delta/\sqrt{c}}{2r}\right)\frac{|\mathcal{P}(u_0,u_1)|^2}{\Psi(u_0)} < |\mathcal{I}_\delta| < \left(\frac{r-\delta/\sqrt{c}}{2r}\right)\left(\frac{|\mathcal{P}(u_0,u_1)|^2}{\Psi(u_0)}+c\Psi(u_0)\right).$$

In [1], the abstract problem (**P**) is studied with a more general linear damping term. In that paper, the nonexistence of global solutions is analyzed for any positive value of the initial energy. In fact, it is proved that the following functional, written in terms of our problem (**P**),

$$\mathcal{P}(u(t),u(t)) + \delta\int_0^t \mathcal{P}(u(s),u(s))\,ds,$$

blows up in finite time if $(H0), (H1)$ and

$$0 \leq E_0 < \frac{\mathcal{P}(u_0,u_1)}{\|u_0\|_{W_P}} - \frac{\delta}{4}\left(\sqrt{\frac{r+2}{r}}-1\right)\|u_0\|_{W_P},$$

hold. We notice that the last condition implies $\|u_0\|_{W_P} > 0$, $\mathcal{P}(u_0,u_1) > 0$, and that for large values of E_0, necessarily correspond large values of $\mathcal{P}(u_0,u_1)$, conclusion that agrees with our results.

6. Examples

6.1. **Generalized Boussinesq equation.** We consider the equation $(\mathbf{GB})^*$. We remember that hypothesis $(H0)$ holds with $c = \min\{m, \frac{\alpha_1}{\alpha_2}\}$. Also, we assumed that the source terms f and the corresponding potential operator F do not have any particular form but they satisfy $(H1)$. The solution in the sense of Definition 2.1 holds, see [29, 25, 33], then nonexistence of global solutions is due to blow-up. By Theorem 3.1, if the initial data are such that

$$\|u_0\|_*^2 + \alpha_2\|u_0\|_2^2 > 0, \ \ (u_0,u_1)_* + \alpha_2(u_0,u_1)_2 > 0,$$

and the initial energy

$$E_0 = \frac{1}{2}\left(\|u_1\|_*^2 + \alpha_2\|u_1\|_2^2 + \alpha_3\|\nabla u_0\|_2^2 + m^2\|u_0\|_*^2 + \alpha_1\|u_0\|_2^2\right) - F(u_0),$$

is such that $E_0 \in \mathcal{I}_\delta = (\alpha_\delta, \beta_\delta)$, given in Theorem 3.1, then the corresponding solution is not global and blows up in finite time. Moreover, by Corollary 1, for every positive initial energy E_0, there exists initial data such that imply the nonexistence of global solutions in the norm of $\mathcal{H}$.

There are several results in the literature showing blow up for large positive values of the initial energy for equations of the type $(\mathbf{GB})^*$. They consider either $\delta = 0$, see [16, 4, 36], or a linear damping term, which can be either weak or strong, see [25, 26, 29, 30, 31]. In most of them, the blow up is proved under the assumption (2) and if the initial energy satisfies an inequality of the type

$$E_0 < C_1 \Psi(u_0) + C_2 \frac{|\mathcal{P}(u_0, u_1)|^2}{\Psi(u_0)}, \tag{17}$$

with $C_j \geq 0,\ j = 1, 2,\quad C_1 + C_2 > 0$, are numbers depending on the damping constant δ and on the numbers c, r in $(H0), (H1)$. See Remark 2 to compare this condition with a lower bound of β_δ.

In [16], the one dimensional equation $(\mathbf{GB})^*$ with a source term of polynomial type, $\delta = 0$ and $m^2 = 1 = \alpha_j,\ j = 1, 2, 3$, and consequently with $c = 1$, is considered. There, blow up is proved if E_0 satisfies, implicitly, the condition given in (8).

Very recently, characterizations for blow up and globality where given in [5] for a one dimensional sixth order nonlinear double dispersive equation with a linear restoring force and with $\delta = 0$. This equation is of the type $(\mathbf{P})$, with

$$Pu_t = (-\Delta + I_d + (-\Delta)^{-1})u_t, \quad Au = Pu,$$

defined on the subspace of $H = L_2(\mathbb{R}^N)$,

$$W_P = V_A = \{u \in H^1(\mathbb{R}^N) : (-\Delta)^{-\frac{1}{2}} u \in L_2(\mathbb{R}^N)\},$$

with norms

$$\|u\|^2_{W_P} = \|u\|^2_{V_A} = \|\nabla u\|^2_2 + \|u\|^2_2 + \|u\|^2_*,$$

where

$$\|u\|^2_* = (u, u)_* \equiv ((-\Delta)^{-\frac{1}{2}} u, (-\Delta)^{-\frac{1}{2}} u)_2.$$

Hence $c = 1$ in $(H0)$ and the source term is of polynomial type and $(H1)$ is satisfied. The characterizations are as follows.

Theorem 6.1 ([5])**.** *For this particular example, the following characterizations hold.*

- *The solution blows up in a finite time, $T_{MAX} < \infty$, if and only if*
$$\limsup_{t \to T_{MAX}} I(u(t)) < 0.$$
- *The solution is global, $T_{MAX} = \infty$, if and only if*
$$\liminf_{t \to T_{MAX}} I(u(t)) \geq 0.$$

Those conditions are not easy to verify. Hence, sufficient conditions in terms of the initial data are needed. In [5], blow up in finite time is guaranteed if

$$0 < E_0 < \frac{\sqrt{r-2}}{r} \mathcal{P}(u_0, u_1) + \frac{r-2}{2r} \Psi(u_0). \tag{18}$$

We observe that this inequality is not of the type (17), like the upper bound of β_δ in Remark 2. However for $\mathcal{P}(u_0, u_1)$ large enough, β_δ is larger than (18).

6.2. **Nonlinear Klein-Gordon and wave equations.** Now, we consider the following problem with a linear damping term

$$(\mathbf{KG}) \begin{cases} u_{tt} - \Delta u + m^2 u + \delta u_t = f(u), \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x), \end{cases}$$

on $\mathbb{R}^+ \times \mathbb{R}^N$, where $\delta \geq 0,\ m^2 > 0$.

This equation was studied in [28] with $\delta = 0$. Here,

$$P = I_d,\quad H = W_P = L_2(\mathbb{R}^N),\qquad Au = -\Delta u + m^2 u,\quad V_A = H^1(\mathbb{R}^N).$$

Hypothesis $(H0)$ holds with $c = \min\{1, m^2\}$, and $(H1)$ holds since f is assumed of a polynomial type.

The solution in the sense of Definition 2.1 holds and nonexistence of global solutions is due to blow-up, see [28] for the details in the undamped case. Consequently, by Theorem 3.1, if the initial data satisfy

$$\|u_0\|_2^2 > 0,\ \ (u_0, u_1)_2 > 0,$$

and the initial energy is such that $E_0 \in \mathcal{I}_\delta$, the corresponding solution blows up in finite time in the norm of $\mathcal{H}$. And by Corollary 1 for every positive initial energy E_0, there exist initial data satisfying last inequality such that the solution blows up.

In [28], blow up was proved under the assumption (2), and if the following inequalities hold

$$0 < E_0 < \frac{r-2}{2r}\|u_0\|_2^2,\ \ I(u_0) < 0.$$

We observe that in Theorem 3.1 we did not assume any sign of $I(u_0)$ and β_δ is larger than the upper bound on E_0 given in [28], see Remark 2.

The one dimensional case of $(\mathbf{KG})$, with $\delta = 0$ and $m^2 = 1$, was studied in [4] and blow up is showed under the assumption (2), and if the initial energy satisfies the implicit inequality given in (8).

In [37] the following initial and homogeneous Dirichlet boundary value problem of a nonlinear wave equation of the type $(\mathbf{KG})$, with a strong linear damping term, a source f of a polynomial type and $m^2 = 0$, is studied.

$$(\mathbf{KG})^* \begin{cases} u_{tt} - \Delta u - \omega \Delta u_t + \delta u_t = f(u),\ \ x \in \Omega,\ t > 0, \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x),\ \ x \in \Omega, \\ u(x,t) = 0,\ \ x \in \partial\Omega,\ t > 0, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, is a bounded domain with smooth boundary

The blow up is proved under the assumption (2), and if the initial energy satisfies the inequality

$$0 < E_0 < C\ (u_0, u_1)_2, \tag{19}$$

where $C > 0$, depends on ω, δ and c, r in $(H0), (H1)$, and in the case that $\omega = 0$, the constant is equal to $C = \frac{c(r-2)}{r(1+\delta)}$. This inequality is not of the type of the upper bound of β_δ in Remark 2, but for $(u_0, u_1)_2$ large enough, β_δ is larger than the bound in (19).

6.3. **Kirchhoff equation.** Many works have been published about the Kirchhoff equation. In particular, in [15] blow up of solutions of the following problem was studied.

$$(\mathbf{KE})\begin{cases} u_{tt}-\phi(\|\nabla u\|_2^2)\Delta u+\delta u_t=g(u), & x\in\Omega,\ t>0,\\ u(0,x)=u_0(x),\ u_t(0,x)=u_1(x), & x\in\Omega,\\ u(x,t)=0, & x\in\partial\Omega,\ t>0,\end{cases}$$

where $\Omega\subset\mathbb{R}^N$, is a bounded domain with smooth boundary

In [15], existence and uniqueness is proved in the phase space

$$\mathcal{H}=(H_0^1(\Omega)\cap H^2(\Omega))\times H(\Omega)$$

for a set of functions g and ϕ, satisfying several conditions. In particular, g satisfies $(H1)$, that is,

$$(g(u),u)-\theta G(u)\geq 0,\ \ u\in H_0^1(\Omega)\cap H^2(\Omega),\ \ \theta>2,$$

where G is the potential of g, and a typical form of ϕ is

$$\phi(\|\nabla u\|_2^2)=c_1\|\nabla u\|_2^2+c_2\|\nabla u\|_2^{2q}$$

with $c_j>0,\ j=1,2$ and $q\geq 1$. In this case, if $c_1=c_2=1$ for simplicity,

$$f(u)=g(u)+\|\nabla u\|_{2q}^2\Delta u,$$

has the potential

$$F(u)=G(u)-\frac{1}{2(q+1)}\|\nabla u\|_2^{2q},$$

and satisfies $(H1)$ with

$$\theta\geq r\geq 2(q+1)\geq 4,$$

that is, the nonlinearity of the source term g is stronger than the one of ϕ. In contrast, for a Timoshenko equation where these two nonlinearities appear, if $\theta<2(q+1)$ holds then all the solutions are global and uniformly bounded in the phase space $\mathcal{H}$ defined above. Furthermore, all the solutions are attracted by the set of equilibria as times goes to infinity, see [13].

The blow up result showed in [15] holds if the initial data are such that

$$0<E_0<\frac{1}{2\|u_0\|_2^2}\left((u_0,u_1)_2-\frac{2\delta}{r-2}\|u_0\|_2^2\right)^2,\ \ (u_0,u_1)_2>0.$$

Note that this upper bound of E_0 is smaller than

$$\frac{1}{2}\frac{|(u_0,u_1)_2|^2}{\|u_0\|_2^2}+2\left(\frac{\delta}{r-2}\right)^2\|u_0\|_2^2,$$

which is of the type (17). And also, it is smaller than β_δ in Theorem 3.1.

Acknowledgments

I thank Professor Howard Levine for calling our attention to his work with Professor Todorova, [18]. We also thank to Professor Varga Kalantarov for sharing his articles, especially [1]. Finally, we thank to the anonymous referees for their valuable comments that improved the final form of this article. This work was supported by the Universidad Autónoma Metropolitana, Unidad Azcapotzalco.

References

[1] B. A. Bilgin and V. K. Kalantarov, Non-existence of global solutions to nonlinear wave equations with positive initial energy, *Commun. Pure Appl. Anal.*, **17** (2018), no. 3, 987–999. MR 3809110

[2] J. V. Boussinesq, Théorie des ondes et des remous qui se propagent le long d'un canal rectangulaire en communiquant au liquide contenu dans ce canal des vitesses sensiblement pareilles de la surface au fond, *J. Math. Pures Appl.*, **17** (1872), 55–108. MR 3363411

[3] I. C. Christo, G. A. Maugin and A. V. Porubov, On Boussinesq's paradigm in nonlinear wave propagation, *Comptes Rendus Mécanique*, **335** (2007), 521–535.

[4] M. Dimova, N. Kolkovska and N. Kutev, Blow up of solutions to ordinary differential equations arising in nonlinear dispersive problems, *Electron. J. Differential Equations*, (2018) Paper No. 68, 16 pp. MR 3781183

[5] M. Dimova, N. Kolkovska and N. Kutev, On the solvability of sixth order nonlinear double dispersive equations, *AIP Conf. Proc.*, **2159** (2019), 030008, 10 pp.

[6] R. Donninger and W. Schlag, Numerical study of the blowup/global existence dichotomy for the focusing cubic nonlinear Klein-Gordon equation, *Nonlinearity*, **24** (2011), no. 9, 2547–2562. MR 2824020

[7] J. A. Esquivel-Avila, The dynamics of a nonlinear wave equation, *J. Math. Anal. Appl.*, **279** (2003), no. 1, 135–150. MR 1970496

[8] J. A. Esquivel-Avila, Qualitative analysis of a nonlinear wave equation, *Discrete Contin. Dyn. Syst.*, **10** (2004), no. 3, 787–804. MR 2018880

[9] J. A. Esquivel-Avila, Dynamic analysis of a nonlinear Timoshenko equation, *Abstr. Appl. Anal.*, **2011** (2011), Art. ID 724815, 36 pp. MR 2802846

[10] J. A. Esquivel-Avila, Blow up and asymptotic behavior in a nondissipative nonlinear wave equation, *Appl. Anal.*, **93** (2014), no. 9, 1963–1978. MR 3227792

[11] J. A. Esquivel-Avila, Remarks on the qualitative behavior of the undamped Klein-Gordon equation, *Math. Methods Appl. Sci.*, **41** (2018), no. 1, 103–111. MR 3745359

[12] J. A. Esquivel-Avila, Nonexistence of global solutions of abstract wave equations with high energies, *J. Inequal. Appl.*, **2017** (2017) Paper No. 268, 14 pp. MR 3716687

[13] J. A. Esquivel-Avila, Global attractor for a nonlinear Timoshenko equation with source terms, *Math. Sci. (Springer)*, **7** (2013), Art. 32, 8 pp. MR 3091563

[14] F. Gazzola and M. Squassina, Global solutions and finite time blow up for damped semilinear wave equations, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **23** (2006), 185–207. MR 2201151

[15] M. O. Korpusov, On blow-up of solutions to a Kirchhoff type dissipative wave equation with a source and positive energy, *Sib. Math. J.*, **53** (2012), no. 4, 702–717. MR 3013533

[16] N. Kutev, N. Kolkovska and M. Dimova, Global behavior of the solutions to Boussinesq type equation with linear restoring force, *AIP Conf. Proc.*, **1629** (2014), no. 1, 172–185.

[17] N. Kutev, N. Kolkovska, M. Dimova and C. I. Christov, Theoretical and numerical aspects for global existence and blow up for the solutions to Boussinesq Paradigm Equation, *AIP Conf. Proc.*, **1404** (2011), no. 1, 68–76.

[18] H. A. Levine and G. Todorova, Blow up of solutions of the Cauchy problem for a wave equation with nonlinear damping and source terms and positive initial energy, *Proc. Amer. Math. Soc.*, **129** (2001), no. 3, 793–805. MR 1792187

[19] W. Lian and R. Xu, Global well-posedness of nonlinear wave equation with weak and strong damping terms and logarithmic source term, *Adv. Nonlinear Anal.*, **9** (2020), no. 1, 613–632. MR 3980417

[20] W. Lian, R. Xu, V. D. Radulescu, Y. Yang and N. Zhao, Global well-posedness for a class of fourth order nonlinear strongly damped wave equations, *Advances in Calculus of Variations*, published online, (2019).

[21] Y. Liu and R. Xu, Potential well method for Cauchy problem of generalized double dispersion equations, *J. Math. Anal. Appl.*, **338** (2008), no. 2, 1169–1187. MR 2386490

[22] Y. Liu and R. Xu, Potential well method for initial boundary value problem of the generalized double dispersion equations, *Commun. Pure Appl. Anal.*, **7** (2008), no. 1, 63–81. MR 2358354

[23] G. A. Maugin, *Nonlinear Waves in Elastic Crystals*, Oxford University Press, Oxford, UK, 1999. MR 1772390

[24] L. E. Payne and D. H. Sattinger, Saddle points and instability of nonlinear hyperbolic equations, *Israel J. Math.*, **22** (1975), no. 3-4, 273–303. MR 0402291

[25] N. Polat and A. Ertaş, Existence and blow-up of solution of Cauchy problem for the generalized damped multidimensional Boussinesq equation, *J. Math. Anal. Appl.*, **349** (2009), 10–20. MR 2455726
[26] X. Su and S. Wang, The initial-boundary value problem for the generalized double dispersion equation, *Z. Angew. Math. Phys.*, **68** (2017), Art. 53, 21 pp. MR 3632916
[27] G. Todorova and E. Vitillaro, Blow-up for nonlinear dissipative wave equations in $\mathbb{R}^n$, *J. Math. Anal. Appl.*, **303** (2005), no. 1, 242–257. MR 2113879
[28] Y. Wang, A sufficient condition for finite time blow up of the nonlinear Klein-Gordon equations with arbitrarily positive initial energy, *Proc. Amer. Math. Soc.*, **136** (2008), no. 10, 3477–3482. MR 2415031
[29] S. Wang and G. Chen, Cauchy problem of the generalized double dispersion equation, *Nonlinear Anal.*, **64** (2006), no. 1, 159–173. MR 2183835
[30] S. Wang and X. Su, The Cauchy problem for the dissipative Boussinesq equation, *Nonlinear Anal. Real World Appl.*, **45** (2019), 116–141. MR 3854297
[31] S. Wang and X. Su, Global existence and nonexistence of the initial-boundary value problem for the dissipative Boussinesq equation, *Nonlinear Anal.*, **134** (2016), 164–188. MR 3462623
[32] M. Willem, *Minimax Theorems*, Progress in Nonlinear Differential Equations and Applications, Vol. 24, Birkhäuser Boston, Inc., Boston, MA, 1996. MR 1400007
[33] R. Xu, Y. Liu and T. Yu, Global existence of solution for Cauchy problem of multidimensional generalized double dispersion equations, *Nonlinear Anal.*, **71** (2009), no. 10, 4977–4983. MR 2548729
[34] R. Xu and Y. Liu, Global existence and nonexistence of solution for Cauchy problem of multidimensional double dispersion equations, *J. Math. Anal. Appl.*, **359** (2009), no. 2, 739–751. MR 2546791
[35] R. Xu, Y. Yang, B. Liu, J. Shen and S. Huang, Global existence and blowup of solutions for the multidimensional sixth-order "good" Boussinesq equation, *Z. Angew. Math. Phys.*, **66** (2015), no. 3, 955–976. MR 3347419
[36] R. Xu, M. Zhang, S. Chen, Y. Yang and J. Shen, The initial-boundary value problems for a class of sixth order nonlinear wave equation, *Discrete Contin. Dyn. Syst.*, **37** (2017), no. 11, 5631–5649. MR 3681953
[37] Y. Yang and R. Xu, Nonlinear wave equation with both strongly and weakly damped terms: Supercritical initial energy finite time blow up, *Commun. Pure Appl. Anal.*, **18** (2019), no. 3, 1351–1358. MR 3917710

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