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DECOMPOSITION OF SPECTRAL FLOW AND BOTT-TYPE ITERATION FORMULA

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ABSTRACT. Let $A(t)$ be a continuous path of self-adjoint Fredholm operators, we derive a decomposition formula of spectral flow if the path is invariant under a matrix-like cogredient. As applications, we give the generalized Bott-type iteration formula for linear Hamiltonian systems.

1. Introduction

In this paper, we consider the decomposition of the spectral flow for a path of self-adjoint Fredholm operators. Let $\mathcal{H}$ be a separable Hilbert space, and we denote by $\mathcal{FS}(\mathcal{H})$ the set of all densely defined self-adjoint Fredholm operators on $\mathcal{H}$. We always equip $\mathcal{FS}(\mathcal{H})$ with the gap topology. For a continuous path $A(t) \in \mathcal{FS}(\mathcal{H})$, $t \in [a,b]$, the spectral flow $sf(A(t); t \in [a,b])$ is an integer that counts the net number of eigenvalues that change sign. This notation is first introduced by Atiyah-Patodi-Singer [2] in their study of index theory on manifolds with boundaries. Since then it had found many significant applications, see [27, 4] and references therein.

Some basic properties of spectral flow such as homotopy invariance, path additivity, direct sum e.t. are well known, please refer to the Appendix. We give the proof for another basic property which is called cogredient invariance property of spectral flow. For convenience, we first introduce some notations. Let $\mathcal{H}_1, \mathcal{H}_2$ be separable Hilbert space, we denote by $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ and $\mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$ the set of bounded and closed operators from $\mathcal{H}_1 \to \mathcal{H}_2$. Let $\mathcal{S}(\mathcal{H})$ be the set of self-adjoint operators on $\mathcal{H}$. For convenience, we denote by $\mathcal{L}^*(\mathcal{H}_1, \mathcal{H}_2), \mathcal{C}^*(\mathcal{H}_1, \mathcal{H}_2), \mathcal{S}^*(\mathcal{H}), \mathcal{FS}^*(\mathcal{H})$ the invertible subsets.

Lemma 1.1. *Let $M_s \in C([a,b], \mathcal{L}^*(\mathcal{H}_1, \mathcal{H}_2))$, $A_s \in C([a,b], \mathcal{FS}(\mathcal{H}_2))$, then*

$$M_s^* A_s M_s \in C([a,b], \mathcal{FS}(\mathcal{H}_1)), \tag{1.1}$$

and we have

$$sf(A_s; s \in [a,b]) = sf(M_s^* A_s M_s; [a,b]). \tag{1.2}$$

Remark 1.2. The cogredient invariance property is a nature property. In the case $\mathcal{H}_1 = \mathcal{H}_2$, we can get it by the homotopy invariant property, but the general case is not so easy. In a preprint paper [13], Fitzpatrick-Stuar-Pejsachowicz proved (1.2) in the case that M_s is constant, the domain of A_s is fixed and both A_a, A_b are

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invertible. Lemma 1.1 can be considered as a generalization of their result. For reader's convenience, we give the detail proof in Section 2.

Our main result is the decomposition formula based on the cogredient invariance property. Let $\mathcal{H}_i$ be closed subspaces of $\mathcal{H}$ for $i = 1, \cdots, m$, then we define

$$\sum_{1\le i\le m} \mathcal{H}_i = \mathcal{H}_1 + \cdots \mathcal{H}_m$$

which is the subspace spanned by $\mathcal{H}_i$, $i = 1, \cdots, m$. Suppose $g \in \mathcal{L}(\mathcal{H})$, we call g is a matrix-like operator if $\sigma(g) = \{\lambda_1, \cdots, \lambda_n\}$ is finite and there exist $m > 0$, such that

$$\mathcal{H} = \sum_{1\le i\le n} \ker(g - \lambda_i)^m. \tag{1.3}$$

We denote by $\mathcal{M}(\mathcal{H})$ the set of matrix-like operators. For $g \in \mathcal{M}(\mathcal{H})$, $\lambda \in \sigma(g)$, we set

$$\mathcal{H}_\lambda := \ker(g - \lambda)^m,$$

and denote

$$F_\lambda = \begin{cases} \mathcal{H}_\lambda, & if \quad \lambda \in \mathbb{U}; \\ \mathcal{H}_\lambda + \mathcal{H}_{\bar{\lambda}^{-1}}, & if \quad \lambda \notin \mathbb{U}. \end{cases} \tag{1.4}$$

Then we have

$$\mathcal{H} = \sum_{1\le i\le k} F_{\lambda_i}.$$

Moreover, let $\hat{F} = span\{F_\lambda, \lambda \in \sigma(g) \cap \mathbb{U}^c\}$, we have the next theorem.

Theorem 1.3. *Let $A_s \in C([0,1], \mathcal{FS}(\mathcal{H}))$. Suppose $g \in \mathcal{M}(\mathcal{H})$ is invertible and preserve the domain of A_s, $\sigma(g) \cap \mathbb{U} = \{\lambda_1, \cdots, \lambda_j\}$. Assume*

$$g^* A_s g = A_s, \quad for \quad s \in [0,1], \tag{1.5}$$

then we have

$$\begin{aligned} sf(A_s) &= sf(A_s|_{F_{\lambda_1}}) + \cdots + sf(A_s|_{F_{\lambda_j}}) \\ &\quad + \frac{1}{2}(dim \ker(A_1|_{\hat{F}}) - dim \ker(A_0|_{\hat{F}})). \end{aligned} \tag{1.6}$$

In [18], by assuming g is unitary and $\sigma(g)$ is finite, Hu-Sun proved the decomposition formula

$$sf(A_s) = sf(A_s|_{\ker(g-\lambda_1)}) + \cdots + sf(A_s|_{\ker(g-\lambda_j)}) \tag{1.7}$$

under the condition

$$A_s g = g A_s. \tag{1.8}$$

Obviously, we give a generalization of (1.7). In fact, a significant difference is that we do not assume g is unitary in Theorem 1.3, hence the subspaces $\mathcal{H}_\lambda$ are not orthogonal. To overcome this difficulty, we develop a new technique (Lemma 2.2) to prove the equality of spectral flow.

The second main result is a generalization for the Bott-type iteration formula which is a powerful tool to study the multiplicity and stability of periodic orbits in Hamiltonian systems. In 1956, Bott got his celebrated iteration formula for the Morse index of closed geodesics [5], and it was generalized by [3, 10, 9, 11]. The precise iteration formula of the general Hamiltonian system was established by

Long [22, 23]. In fact, the iteration could be regarded as a unitary group action. Motivated by the symmetry orbits in n-body problem [12], Hu-Sun [18] use this opinion to give generalization of Bott-type iteration formula to the system under a circle-type symmetry or reversible symmetry group action, and prove the stability of Figure Eight orbit [8]. The case of the reversible symmetry was deeply studied in [24, 20, 21, 16].

Based on Theorem 1.3, we prove the Bott-type iteration formula which cover all the previous cases and moreover give some new generalizations. Our generalized formula could be applied to the closed geodesics on Semi-Riemanian manifold and heteroclinic orbits with reversible symmetry.

Now we consider the linear Hamiltonian system

$$\dot{x}(t) = JB(t)x(t), \quad t \in \mathcal{I}, \tag{1.9}$$

where $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$, $\mathcal{I} \subset \mathbb{R}$ is a connected subinterval, $B(t) \in C(\mathcal{I}, \mathcal{S}(\mathbb{R}^{2n}))$. In the case $\mathcal{I}$ is finite, the boundary conditions are given by the Lagrangian subspaces. Let $(\mathbb{C}^{2n}, \omega)$ be the standard symplectic space with $\omega(x, y) = (Jx, y)$. A Lagrangian subspace V is a n-dimensional subspace with $\omega|_V = 0$. We denote the set of Lagrangian subspace by $Lag(2n)$. It is obvious $(\mathbb{C}^{2n} \oplus \mathbb{C}^{2n}, -\omega \oplus \omega)$ is a $4n$-dimensional symplectic space, then for $\mathcal{I} = [a, b]$, the boundary condition is given by

$$(x(a), x(b)) \in \Lambda \in Lag(4n). \tag{1.10}$$

In the case $\mathcal{I} = \mathbb{R}$, we always assume $B(\pm\infty) = \lim_{t\to\pm\infty} B(t)$ exist and $JB(\pm\infty)$ are both hyperbolic, that is

$$\sigma(JB(\pm\infty)) \cap i\mathbb{R} = \emptyset. \tag{1.11}$$

Let $\mathcal{H} = L^2(\mathcal{I}, \mathbb{C}^{2n})$ and E is $W^{1,2}(\mathcal{I}, \mathbb{C}^{2n})$ which satisfied some boundary conditions. Let

$$A := -J\frac{d}{dt} : E \subset \mathcal{H} \to \mathcal{H}.$$

and $B : \mathcal{H} \to \mathcal{H}$ be the multiplicity operator of $B(t)$. Let $A_s = A - sB$ for $s \in \mathbb{R}$, then $A_s \in \mathcal{FS}(\mathcal{H})$. For $g \in \mathcal{M}(\mathcal{H})$ which satisfies

$$g(E) = E, \quad g^*Ag = A, \quad g^*Bg = B, \tag{1.12}$$

we have $g^*A_sg = A_s$. By constructing g, we get the spectral flow decomposition of $sf(A_s)$. We list 6-cases which are common in applications of Hamiltonian systems. Our results generalize all the previous results, especially for the reversible symmetry of heteroclinic orbits (Case 5 and 6). Our result is new. Please see Section 4 for the detail.

It is well known that spectral flow is equal to Maslov index, and this is also true for the unbounded domain, see [6, 27, 26, 7, 15] and reference therein. The Maslov index is associated integer to a pair of continuous path $f(t) = (L_1(t), L_2(t))$, $t \in \mathcal{I}$, in $Lag(2n) \times Lag(2n)$ [6]. From the decomposition of spectral flow, we get the decomposition of Maslov index. Please refer to Section 5 for the detail. For reader's convenience, we give a brief review for the Maslov index and spectral flow in the Appendix.

This paper is organized as follows. We prove Lemma 1.1 in Section 2 and Theorem 1.3 in Section 3. In Section 4, we list 6-cases of decompositions in Hamiltonian systems. In Section 5, we give some cases of the Bott-type iteration formulas. At last, we briefly review the basic properties of spectral flow and Maslov index in the Section 6.

2. Spectral flow is preserved under cogredient

Let V be a closed subspace of $\mathcal{H}$, and P_V be the orthogonal projection from $\mathcal{H}$ to V. For $A \in \mathcal{C}(\mathcal{H})$, we denote the operator $P_V A P_V : V \to V$ by A_V. Obviously, if $A \in \mathcal{S}(\mathcal{H})$ then $A_V \in \mathcal{S}(V)$.

Definition 2.1. Let $A : [a,b] \to \mathcal{FS}(\mathcal{H})$ be a continuous curve. We call $A(t)$ is a positive curve if $\{t, \ker A(t) \neq 0\}$ is a distinct set and

$$sf(A(t);[0,1]) = \sum_{a<t\leq b} \dim \ker(A(t)). \tag{2.1}$$

Let $A \in \mathcal{FS}(\mathcal{H})$ and $B \in \mathcal{L}(\mathcal{H}) \cap \mathcal{S}(\mathcal{H})$, then $A + tB \in \mathcal{FS}(\mathcal{H})$ with $t \in \mathbb{R}$. Note that it is positive if $B|_{\ker(A+tB)>0}$ for any $t \in \{t|\ker(A+tB) \neq 0\}$. For example, $A + tI$ is a positive curve with $A \in \mathcal{FS}(H)$ for $t \in \mathbb{R}$.

Let $S \subset \mathcal{FS}(\mathcal{H})$ be a path connected subset. We assume there exists $K \in \mathcal{L}(\mathcal{H})$ such that $(Kx,x) > 0$, $\forall x \in \mathcal{H}$ and for any $A \in S$, there is a neighborhood U of A, and $\epsilon > 0$ such that $B + tK \in S$, $t \in [0,\epsilon]$ for each $B \in U$. Then $A + tK$, $t \in [0,\epsilon]$ is a positive curve in $\mathcal{FS}(\mathcal{H})$. Let $\{\mathcal{H}_k\}, 1 \leq k \leq n$ be a family of Hilbert spaces and $f_k : S \to \mathcal{FS}(\mathcal{H}_k)$ be a family of continuous maps.

We assume that

(a) For any $A \in S$, $f_k(A + tK), t \in [0,\epsilon]$ is a positive path in $\mathcal{FS}(\mathcal{H}_k)$.

(b) For any $A \in S$, $\sum_{1\leq k\leq n} \dim \ker f_k(A) = \dim \ker A$.

Then we have the following lemma.

Lemma 2.2. *Let $A \in C([0,1], \mathcal{FS}(\mathcal{H}))$ and satisfies condition (a) and (b), we have*

$$sf(A(t); t \in [0,1]) = \sum_{1\leq k\leq n} sf(f_k(A(t)); t \in [0,1]). \tag{2.2}$$

Proof. Since the spectral flow satisfies the path additivity property, we only need to prove (2.2) locally. Let $h_k(s,t) = f_k(A(t) + sK)$, $t \in [0,1], s \in [0,\epsilon]$, then for any $t \in [0,1]$, $h_k(s,t)$ is a positive curve with $1 \leq k \leq n$. Let $t_0 \in [0,1]$, since $(Kx,x) > 0$ for $x \in \ker A(t_0)$, there is $\delta > 0$ such that

$$\dim \ker(A(t_0) + \delta K) = 0.$$

It follows that $\dim \ker(h_k(\delta, t_0)) = 0$ for $1 \leq k \leq n$. Note that $A(t_0) + \delta K$ is a Fredholm operator, so there is $\delta_1 > 0$ such that

$$\dim \ker(A(t) + \delta K) = 0, \forall t \in [t_0 - \delta_1, t_0 + \delta_1].$$

It follows that $\dim \ker(h_k(\delta, t)) = 0$ for $t \in [t_0 - \delta_1, t_0 + \delta_1]$, $1 \leq k \leq n$. Then we have

$$\begin{cases} sf(A(t) + \delta K, t \in [t_0 - \delta_1, t_0 + \delta_1]) = 0 \\ sf(h_k(\delta, t), t \in [t_0 - \delta_1, t_0 + \delta_1]) = 0 \end{cases}.$$

By homotopy invariance of spectral flow, we have

$$\begin{aligned} sf(A(t); t \in [t_0 - \delta_1, t_0 + \delta_1]) &= sf(A(t_0 - \delta_1 + sK); s \in [0,\delta]) \\ &\quad - sf(A(t_0 + \delta_1 + sK); s \in [0,\delta]) \end{aligned} \tag{2.3}$$

and

$$sf(h_k(0,t); t \in [t_0-\delta_1, t_0+\delta_1]) = sf(h_k(s,t_0-\delta_1); s \in [0,\delta]) \\ -sf(h_k(s,t_0+\delta_1); s \in [0,\delta]). \tag{2.4}$$

Note that $A(t_0 \pm \delta_1) + sK$, $h_k(s,t_0 \pm \delta_1), 1 \le k \le n$ are positive paths. It follows that

$$\begin{aligned} sf(A(t_0 \pm \delta_1) + sK; s \in [0,\delta]) &= \sum_{0<s\le\delta} \dim \ker(A(t_0 \pm \delta_1) + sK) \\ &= \sum_{0<s\le\delta} \sum_{1\le k\le n} \dim \ker(h_k(s,t_0 \pm \delta_1)) \\ &= \sum_{1\le k\le n} sf(h_k(s,t_0 \pm \delta_1); s \in [0,\delta]). \end{aligned}$$

This completes the proof. □

Please note that Lemma 2.2 can be considered as a generalization of direct sum property of spectral flow.

In the next, we will prove that the spectral flow is invariant under the cogredient. The next Lemma is contained in [13], but for reader's convenience, we give details here.

Lemma 2.3. *Let E be the domain of $A \in \mathcal{FS}(\mathcal{H}_2)$. If $M \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ is invertible, then $M^*AM \in \mathcal{FS}(\mathcal{H}_1)$ with domain $M^{-1}(E)$.*

Proof. Since $A \in \mathcal{FS}(\mathcal{H}_2)$ with domain E, we have $\dim \ker A$, $\dim(\mathcal{H}_2/\operatorname{im} A) < +\infty$. Since M is invertible, we have $\ker(M^*AM) = \ker(AM) = M^{-1}\ker A$. Then M^{-1} induce an isomorphism from $\ker A$ to $\ker(M^*AM)$. Note that $\operatorname{im}(M^*AM) = M^*\operatorname{im}(A)$. Then M^* induce an isomorphism from $\mathcal{H}_2/\operatorname{im}(A)$ to $\mathcal{H}_1/\operatorname{im}(M^*AM)$. So M^*AM is a Fredholm operator.

Since $A \in \mathcal{FS}(\mathcal{H}_2)$ with domain E, we see that for each $x \in M^{-1}$, $(AMx, My) = (Mx, AMy)$ if and only if $y \in M^{-1}E$. It follows that $(M^*AM)^* = M^*AM$ with domain $M^{-1}E$. Then we can conclude that $M^*AM \in \mathcal{FS}(\mathcal{H}_1)$. □

Recall that the gap topology can be induced by the gap distance $\hat{\delta}$. Let X be a Banach space. Let M, N be two closed linear subspaces of X. Denote by S_M the unit sphere of M. Then gap distance is defined as

$$\hat{\delta}(M,N) = \max\{\delta(M,N), \delta(N,M)\}, \tag{2.5}$$

where

$$\delta\{M,N\} := \begin{cases} \sup_{u\in S_M} \operatorname{dist}(u,N), & \text{if } M \ne \{0\} \\ 0, & \text{if } M = \{0\} \end{cases}.$$

The gap distance has the following properties:

Lemma 2.4. *Let X, Y be two Hilbert spaces. Let M, N be two closed linear subspaces of X. Let $P, Q \in \mathcal{L}^*(X,Y)$. Then $\hat{\delta}(PM, QN) \le \hat{\delta}(M,N)\max\{\|P\|, \|Q\|\} + \|P-Q\|\max\{\|P^{-1}\|, \|Q^{-1}\|\}$.*

Proof. Without loss of generality, we assume that $M, N \ne \{0\}$, and let $d_1 = \hat{\delta}(M,N)$. Let $x \in PM$ with $\|x\| = 1$, we choose $y \in N$ such that $\|P^{-1}x - y\| =$

$\operatorname{dist}(P^{-1}x, N)$, then we have $\|y\| \le \|P^{-1}x\| \le \|P^{-1}\|$. Note that

$$\begin{aligned}\|x - Qy\| \le \|x - Py\| + \|Qy - Py\| &\le \|P\|\|P^{-1}x - y\| + \|Q - P\|\|y\| \\ &\le \|P\| \operatorname{dist}(P^{-1}x, N) + \|Q - P\|\|P^{-1}\| \\ &\le \|P\|\delta(M, N) + \|Q - P\|\|P^{-1}\|.\end{aligned}$$

It follows that $\delta(PM, QN) \le \|P\|\delta(M, N) + \|Q - P\|\|P^{-1}\|$. Similarly, we have $\delta(QN, PM) \le \|Q\|\delta(N, M) + \|Q - P\|\|Q^{-1}\|$. This conclude the proof. □

Lemma 2.5. *Suppose $M_s \in C([0,1], \mathcal{L}^*(\mathcal{H}_1, \mathcal{H}_2))$, $A_s \in C([0,1], \mathcal{FS}(\mathcal{H}_2))$, then $M_s^* A_s M_s \in C([0,1], \mathcal{FS}(\mathcal{H}_1))$.*

Proof. We only need to show that $M_s^* A_s M_s$ is a continuous curve with the gap topology. Let E_s be the domain of A_s. Note that

$$\operatorname{Gr}(M_s^* A_s M_s) = \{(M_s^* A_s x, M_s^{-1} x) | x \in E_s\}.$$

Let $Q_s : \mathcal{H}_2 \oplus \mathcal{H}_2 \to \mathcal{H}_1 \oplus \mathcal{H}_1$ be $\begin{pmatrix} M_s^* & 0 \\ 0 & M_s^{-1} \end{pmatrix}$, then $Q_s \in C([0,1], \mathcal{L}^*(\mathcal{H}_2 \oplus \mathcal{H}_2, \mathcal{H}_1 \oplus \mathcal{H}_1))$, and we also have $\operatorname{Gr}(M_s^* A_s M_s) = Q_s \operatorname{Gr}(A_s)$. Since $\|Q_s\|$ and $\|Q_s^{-1}\|$ are continuous functions on $[0,1]$, we have $\|Q_s\| > 0$, $\|Q_s^{-1}\| > 0$ for $s \in [0,1]$. Let $C_1 = \sup(\|Q_s\|)$, $C_2 = \sup(\|Q_s^{-1}\|)$. For $s_0, s \in [0,1]$, by Lemma 2.4, we have

$$\begin{aligned}\hat{\delta}(\operatorname{Gr}(M_{s_0}^* A_{s_0} M_{s_0}, M_s^* A_s M_s) &= \hat{\delta}(Q_{s_0} \operatorname{Gr}(A_{s_0}), Q_s \operatorname{Gr}(A_s)) \\ &\le C_1 \hat{\delta}(A_{s_0}, A_s) + C_2 \|Q_s - Q_{s_0}\|.\end{aligned}$$

By the continuity of A_s and Q_s, we see that for any $\epsilon > 0$ there is $\delta_1 > 0$ such that for any $s \in (s_0 - \epsilon, s_0 - \epsilon)$, we have $\hat{\delta}(\operatorname{Gr}(A_{s_0}), \operatorname{Gr}(A_s)) < \epsilon/(2C_1)$ and $\|Q_s - Q_{s_0}\| < \epsilon/(2C_2)$. Then we have $\hat{\delta}(\operatorname{Gr}(M_{s_0}^* A_{s_0} M_{s_0}, M_s^* A_s M_s) < \epsilon$. This completes the proof. □

Now we give the proof of Lemma 1.1.

Proof of Lemma 1.1. Please note that (1.1) is from Lemma 2.5. We first prove the case $M_s \equiv M$. Let $S = \mathcal{FS}(\mathcal{H}_2)$, $K = I$, $f(A) = M^* A M$. Please note that

$$\dim \ker(M^* A M) = \dim (M^{-1} \ker A) = \dim \ker A$$

for each $A \in \mathcal{FS}(\mathcal{H}_2)$. Furthermore, we have $\frac{d}{dt} M^*(A + tI)M| = M^* M > 0$, so $M^*(A + tI)M$ is a positive curve. Then by Lemma 2.2, we have

$$sf(A_s; s \in [a,b]) = sf(M^* A_s M; s \in [a,b]) \quad for \quad M \in \mathcal{L}^*(\mathcal{H}_1, \mathcal{H}_2).$$

Now we consider the two family $M_{a+t(s-a)}^* A_s M_{a+t(s-a)}$, $(t,s) \in [0,1] \times [a,b]$. By the homotopy invariance property of spectral flow, we have

$$sf(M_a^* A_s M_a) = sf(M_s^* A_s M_s) - sf(M_{a+t(b-a)}^* A_b M_{a+t(b-a)}).$$

Note that $\dim \ker M_{a+t(b-a)}^* A_b M_{a+t(b-a)}$ is a constant which implies

$$sf(M_{a+t(b-a)}^* A_b M_{a+t(b-a)}) = 0.$$

It follows that

$$sf(M_a^* A_s M_a) = sf(M_s^* A_s M_s).$$

This completes the proof. □

As an example, we consider the one parameter family of linear Hamiltonian systems

$$\dot{z}(t) = JB_s(t)z(t), (s,t) \in [0,1] \times [0,T], \tag{2.6}$$

where $B(t) \in C([0,1] \times [0,T], \mathcal{S}(\mathbb{R}^{2n}))$. The boundary condition is given by

$$(x_s(0), x_s(T)) \in \Lambda_s \in Lag(4n),$$

where we assume Λ_s is continuous depend on s.

Let $A_s = -J\frac{d}{dt}|_{E(\Lambda_s)}$. It is a path of self adjoint operators on $\mathcal{H} := L^2([0,T], \mathbb{C}^{2n})$ with domain

$$E(\Lambda_s) = \{x \in W^{1,2}([0,T], \mathbb{C}^{2n}), (x(0), x(T)) \in \Lambda_s\}.$$

We define B_s by $(B_s x)(t) = B_s(t)x(t)$. It is well known that $A_s, A_s - B_s \in \mathcal{FS}(\mathcal{H})$ with domain E_s. Let $\gamma_s(t)$ be the fundamental solution of (2.6), i.e.

$$\dot{\gamma}_s(t) = JB_s(t)\gamma_s(t), \tag{2.7}$$

then

$$\gamma_s(t) \in \mathrm{Sp}(2n) := \{P \in \mathcal{L}^*(\mathbb{R}^{2n}), P^*JP = J\},$$

which implies $Gr(\gamma_s(T)) \in Lag(4n)$. The following formula which gives the relation of spectral flow and Maslov index (please refer to Theorem 6.1)

$$-sf(A_s - B_s) = \mu(\Lambda_s, Gr(\gamma_s(T))).$$

Let $P_s(t) \in C^1([0,1] \times [0,T], \mathrm{Sp}(2n))$, then $P_s \in C^1([0,1], \mathcal{L}^*(\mathcal{H}))$, hence

$$(P_s^*)^{-1}(A_s - B_s)P_s^{-1} \in \mathcal{FS}(\mathcal{H})$$

with domain

$$P_s E_s = \{x \in W^{1,2}([0,T], \mathbb{C}^{2n}), (x(0), x(T)) \in \hat{P}_s(T)\Lambda_s\},$$

where $\hat{P}_s(t) = diag(I_n, P_s(t))$. Direct calculation shows that

$$(P_s^*)^{-1}(-J\frac{d}{dt}|_{E(\Lambda_s)} - B_s)P_s^{-1} = A_s - \hat{B}_s,$$

where $\hat{B}_s(t) = -J\dot{P}_s(t)P_s^{-1}(t) + (P_s^*(t))^{-1}B(t)P_s^{-1}(t)$. From Lemma 1.1, we have

$$sf(-J\frac{d}{dt}|_{E(\hat{P}_s(T)\Lambda_s)} - \hat{B}_s) = sf((P_s^*)^{-1}(A_s - B_s)P_s^{-1}) = sf(A_s - B_s). \tag{2.8}$$

From (6.6), we can express the left of (2.8) as Maslov index. In fact, the fundamental solution is $P_s(t)\gamma_s(t)$, and the boundary conditions is given by $(\hat{P}_s(T)\Lambda_s$. Hence we have

$$sf(-J\frac{d}{dt}|_{E(\hat{P}_s(T)\Lambda_s)} - \hat{B}_s) = \mu(\hat{P}_s(T)\Lambda_s, \hat{P}_s(T)Gr(\gamma_s(T))).$$

Formula (2.8) implies that

$$\mu(\Lambda_s, Gr(\gamma_s(T))) = \mu(\hat{P}_s(T)\Lambda_s, \hat{P}_s(T)Gr(\gamma_s(T))),$$

which is just the symplectic invariance property (6.4) of Maslov index.

3. Decomposition of spectral flow under cogredient invariant

In this section, we will prove the decomposition formula for spectral flow. Suppose $g \in \mathcal{M}(\mathcal{H})$ with $\sigma(g) = \{\lambda_1, \cdots, \lambda_n\}$, then

$$\mathcal{H} = \sum_{1 \le i \le n} \mathcal{H}_{\lambda_i}, \tag{3.1}$$

where $\mathcal{H}_{\lambda_i} := \ker(g - \lambda_i)^m$ for large enough m. Note that $(\lambda - \lambda_1)^m$ and $(\lambda - \lambda_2)^m$ are coprime, then there are polynomials p_1, p_2 such that $p_1(\lambda)(\lambda - \lambda_1)^m + p_2(\lambda)(\lambda - \lambda_2)^m = 1$. For each $x \in \mathcal{H}_{\lambda_1} \cap \mathcal{H}_{\lambda_2}$, we have

$$x = p_1(g)(g - \lambda_1)^m x + p_2(g)(g - \lambda_2)^m x = 0.$$

Similarly we have $\mathcal{H}_{\lambda_i} \cap \mathcal{H}_{\lambda_j} = 0$ with $i \ne j$. So the decomposition (3.1) is a inner direct sum.

Lemma 3.1. *$g \in \mathcal{M}(\mathcal{H})$ if and only if there exist $\lambda_1, \cdots, \lambda_n \in \mathbb{C}$ such that $\Pi_{i=1}^n (g - \lambda_i)^m = 0$.*

Proof. We only need to show that $g \in \mathcal{M}(\mathcal{H})$ if $\Pi_{i=1}^n (g - \lambda_i)^m = 0$. Let $G_l(\lambda)$ be the polynomial $\Pi_{i=1}^{l-1} (\lambda - \lambda_i)^m \Pi_{i=l+1}^n (\lambda - \lambda_i)^m$. Then $G_1, G_2, \cdots, G_n$ are coprime polynomials. It follows that there are polynomials $a_i(\lambda), (1 \le i \le n)$, such that

$$\sum_{i=1}^n a_i(\lambda) G_i(\lambda) = 1.$$

It follows that $\sum_{i=1}^n a_i(g) G_i(g) = \mathrm{Id}$. Then we can conclude that

$$\mathcal{H} = \sum_{1 \le i \le n} G_i(g)\mathcal{H}.$$

We also have $(g - \lambda_i)^m G_i(g)\mathcal{H} = \Pi_{i=1}^n (g - \lambda_i)^m \mathcal{H} = 0$, which implies (3.1). □

We have the following lemmas.

Lemma 3.2. *Let $A \in \mathcal{FS}(\mathcal{H})$ with domain E. Suppose $g \in \mathcal{M}(\mathcal{H})$, which satisfied*

$$g^* A g = A, \quad gE = E,$$

then $\mathcal{H}_\lambda, \mathcal{H}_\mu$ are A-orthogonal if $\lambda\bar{\mu} \ne 1$, i.e.

$$(Ax, y) = 0, \quad if \quad x \in \mathcal{H}_\lambda \cap E, \quad y \in \mathcal{H}_\mu \cap E. \tag{3.2}$$

Proof. Let $x \in \ker(g - \lambda)^m \cap E$, $y \in \ker(g - \mu)^n \cap E$ with $m, n \ge 1$. We see that $(Ax, y) = 0$ if $m + n = 2$. In fact,

$$(Ax, y) = (Agx, gy) = \lambda\bar{\mu}(Ax, y)$$

implies $(Ax, y) = 0$ since $\lambda\bar{\mu} \ne 1$. Assume that $(Ax, y) = 0$ if $m + n \le k$. Note that $(g - \lambda)x \in \ker(g - \lambda)^{m-1} \cap E$, $(g - \mu)x \in \ker(g - \mu)^{n-1} \cap E$, $gx \in \ker(g - \lambda)^m \cap E$ and $gy \in \ker(g - \mu)^m \cap E$. If $m + n = k + 1$, We have

$$(Ax, y) = (Agx, gy) = (A(g - \lambda)x, gy) + (A\lambda x, (g - \mu)y) + \lambda\bar{\mu}(Ax, y) = \lambda\bar{\mu}(Ax, y).$$

Since $\lambda\bar{\mu} \ne 1$, we have $(Ax, y) = 0$. By induction, we have $(Ax, y) = 0$ with $x \in \mathcal{H}_\lambda \cap E$ and $y \in \mathcal{H}_\mu \cap E$. This complete the proof. □

Lemma 3.3. *Under the condition of Lemma 3.2, we have $\ker A = \sum_{1 \le i \le n} \ker A \cap \mathcal{H}_i$ and $E = \sum_{1 \le i \le n} E \cap \mathcal{H}_i$.*

Proof. Note that E is a invariant subspace of g. Then $\Pi_{1\le i\le n}(g-\lambda_i)^m = 0$ on E. It follows that $E = \sum_{1\le i\le n}\ker(g|_E - \lambda_i)^m = \sum_{1\le i\le n} E\cap\mathcal{H}_i$. We have

$$g^*Ag(\ker A) = A\ker A = 0.$$

It follows that $g(\ker A)\subset\ker A$. So $\ker A$ is a invariant subspace of g. Similarly, we have $\ker A = \sum_{1\le i\le n}\ker A\cap\mathcal{H}_i$. This complete the proof. □

For $A\in\mathcal{C}(\mathcal{H})$, assume that $\mathcal{H} = \sum_{1\le i\le k}\mathcal{H}_i$, where all of $\mathcal{H}_i$ are closed subspaces of $\mathcal{H}$. Let E be the domain of A and assume that $E = \sum_{1\le i\le k} E\cap\mathcal{H}_i$. $\mathcal{H}_i$, $\mathcal{H}_j$ are A-orthogonal if $i\ne j$. Recall that we set

$$F_\lambda = \begin{cases} \mathcal{H}_\lambda, \quad if \quad \lambda\in\mathbb{U}; \\ \mathcal{H}_\lambda + \mathcal{H}_{\bar\lambda^{-1}}, \quad if \quad \lambda\notin\mathbb{U}, \end{cases}$$

then we have $\mathcal{H} = \sum_{1\le i\le k} F_{\lambda_i}$ and $F_{\lambda_i}, F_{\lambda_j}$ are A-orthogonal if $i\ne j$.

Let $X = \bigoplus_{1\le i\le k} F_{\lambda_i}$, we define an inner product on X:

$$((x_1, x_2, \cdots, x_k), (y_1, y_2, \cdots, y_k)) = \sum_{1\le i\le k}(x_i, y_i),$$

where (x_i, y_i) is the inner product in $\mathcal{H}$. Then X is a Hilbert space and the map

$$M : (x_1, x_2, \cdots, x_k) \to \sum_{1\le i\le k} x_i$$

is a homeomorphism from X to $\mathcal{H}$.

Please note that $A|_{F_{\lambda_i}}$ is the map $M^*AM : M^{-1}F_{\lambda_i} \to M^{-1}(F_{\lambda_i})$. It is a self-adjoint Fredholm operator on $M^{-1}F_{\lambda_i}$ with domain $M^{-1}(E\cap F_{\lambda_i})$. It follows that

$$\ker(A|_{F_{\lambda_i}}) = \ker(AM)\cap M^{-1}(F_{\lambda_i}) = M^{-1}(\ker A\cap F_{\lambda_i}).$$

Proposition 3.4. *Suppose $g\in\mathcal{M}^*(\mathcal{H})$, $A_s\in C([0,1],\mathcal{FS}(\mathcal{H}))$ with fixed domain E and $gE = E$. We assume $g^*A_sg = A_s$ of $s\in[0,1]$, then we have*

$$sf(A_s) = sf(A_s|_{F_{\lambda_1}}) + \cdots + sf(A_s|_{F_{\lambda_k}}). \tag{3.3}$$

Proof. By Lemma 1.1, we have $M^*A_sM\in C([0,1],\mathcal{FS}(X))$, and

$$sf(A_s) = sf(M^*A_sM).$$

Note that $X = \bigoplus_{1\le i\le k} F_{\lambda_i}$ is an orthogonal decomposition. By the Direct sum property of spectral flow, we have

$$sf(A_s) = sf(M^*A_sM) = \sum_{1\le i\le k} sf(A_s|_{F_{\lambda_i}}).$$

This complete the proof. □

Lemma 3.5. *If $\lambda\notin\mathbb{U}$ then we have*

$$sf(A_s|_{F_\lambda}) = \frac{1}{2}(dim\ker(A_1|_{F_\lambda}) - dim\ker(A_0|_{F_\lambda})). \tag{3.4}$$

Proof. Recall that $A_s|_{F_\lambda}$ is the operator $M^*A_sM : M^{-1}(F_\lambda)\to M^{-1}(F_\lambda)$ and $M^{-1}(F_\lambda) = M^{-1}\mathcal{H}_\lambda + M^{-1}\mathcal{H}_{\bar\lambda^{-1}}$. We also have $M^{-1}\mathcal{H}_\lambda \perp M^{-1}\mathcal{H}_{\bar\lambda^{-1}}$. Let Q be the map $x + y \to -x + y$ with $x\in M^{-1}\mathcal{H}_\lambda, y\in M^{-1}\mathcal{H}_{\bar\lambda^{-1}}$. Then Q is invertible and $Q^* = Q$. Let $x_1, x_2 \in M^{-1}(F_\lambda\cap E)$, $y_1, y_2\in M^{-1}(F_{\bar\lambda^{-1}}\cap E)$. We have

$$(QM^*A_sMQ(x_1+y_1), (x_2+y_2)) = -(M^*A_sM(x_1+y_1), (x_2+y_2)).$$

It follows that $-A_s|_{F_\lambda} = Q(A_s|_{F_\lambda})Q$. Then by Lemma 1.1, we have

$$\begin{aligned}2sf(A_s|_{F_\lambda}) &= sf(A_s|_{F_\lambda}) + sf(QA_s|_{F_\lambda}Q) = sf(A_s) + sf(-A_s)\\ &= dim\ker(A_1|_{F_\lambda}) - dim\ker(A_0|_{F_\lambda}).\end{aligned}$$

The lemma then follows. □

Proof of Theorem 1.3. By Proposition 3.4 and Lemma 3.5 , we only need to show that

$$\frac{1}{2}(dim\ker(A_1|_{\hat F}) - dim\ker(A_0|_{\hat F})) = \sum_{\lambda\notin\mathbb{U}}\frac{1}{2}(dim\ker(A_1|_{F_\lambda}) - dim\ker(A_0|_{F_\lambda})).$$

In fact $\ker A_1|_{\hat F} = \ker A_1\cap\hat F$. By Lemma 3.3, we see that

$$\ker A_1\cap\hat F = \sum_{\lambda\notin\mathbb{U}}\ker(A_1)\cap F_\lambda.$$

It follows that $dim\ker(A_1|_{\hat F}) = \sum_{\lambda\notin\mathbb{U}} dim\ker(A_1|_{F_\lambda})$. It is also true for A_0. The theorem then follows. □

Corollary 3.6. *Under the condition of Theorem 1.3, if $\sigma(M)\cap\mathbb{U} = \emptyset$, then*

$$sf(A_s) = \frac{1}{2}(dim\ker(A_1) - dim\ker(A_0)). \tag{3.5}$$

If the path is closed, then

$$sf(A_s) = 0.$$

Remark 3.7. In the case B is compact with respect to A, the spectral flow $A - sB$ is only depend on the end points, thus we define the relative Morse index by (follows [27])

$$I(A, A-B) = -sf(A - sB; s\in[0,1]). \tag{3.6}$$

Especially, when A is positive, then $I(A, A-B) = m^-(A-B)$ is just the Morse index of $A-B$, i.e. the total number of negative eigenvalues. It is obvious that Theorem 1.3 and Corollary 3.6 give the decomposition formula of relative Morse index and Morse index.

4. Applications to Hamiltonian systems

In this section, we will give the applications for Hamiltonian systems. We list 6 cases which are common in applications.

For $\Lambda\in Lag(4n)$, we consider the solution of the flowing linear Hamiltonian systems

$$\dot z(t) = JB(t), \quad (z(0), z(T))\in\Lambda, \tag{4.1}$$

where $B(t)\in C([0,T], \mathcal{S}(\mathbb{R}^{2n}))$. Recall that $A = -J\dfrac{d}{dt}$ is self adjoint operator on $\mathcal{H} := L^2([0,T],\mathbb{C}^{2n})$ with domain

$$E_\Lambda = \{x\in W^{1,2}([0,T],\mathbb{C}^{2n}), (x(0), x(T))\in\Lambda\},$$

then $A, A-B\in\mathcal{FS}(\mathcal{H})$. We will construct $g\in\mathcal{M}(\mathcal{H})$ such that

$$g^*Ag = A, \quad g^*Bg = B, \quad gE_\Lambda = E_\Lambda. \tag{4.2}$$

In order to make $gE_\Lambda = E_\Lambda$, g is always assumed to preserve the boundary condition, that is $g\Lambda = \Lambda$ which means

$$((gx)(0), (gx)(T))\in\Lambda \quad if \quad (x(0), x(T))\in\Lambda.$$

Hence we have

$$g^*(A - sB)g = A - sB, \quad s \in \mathbb{R}.$$

and get the decomposition formula (1.7).

It is well known that for $P \in \mathrm{Sp}(2n)$, if $\lambda \in \sigma(P)$, then $\bar{\lambda}, \lambda^{-1}, \bar{\lambda}^{-1} \in \sigma(P)$ and possess the same geometric and algebraic multiplicities [23]. Case 1 is given by symplectic matrix.

Case 1. For $P \in \mathrm{Sp}(2n)$, and satisfied $P\Lambda = \Lambda$ which means if $(x(0), x(T)) \in \Lambda$, then $(Px(0), Px(T)) \in \Lambda$. Let

$$(gx)(t) = Px(t), \tag{4.3}$$

then it is obvious that $(g^*x)(t) = P^*x(t)$, $g^*Ag = A$, $g\Lambda = \Lambda$. Moreover, we assume $P^*B(t)P = B(t)$, then $g^*Bg = g$, hence we have (4.2). It is obvious that $g \in \mathcal{M}(\mathcal{H})$ and

$$\sigma(g) = \sigma(P).$$

Let $V_\lambda = \ker(P - \lambda)^{2n}$, then $\mathcal{H}_\lambda = L^2([0,T], V_\lambda)$.

Case 2. For $S \in \mathrm{Sp}(2n)$, we consider the S-periodic solution of (4.1), that is

$$z(0) = Sz(T), \tag{4.4}$$

and moreover we assume

$$S^*B(0)S = B(T). \tag{4.5}$$

We assume (4.1) with S-periodic boundary conditions admits a $\mathbb{Z}_k$ symmetry. More precisely, let $P \in \mathrm{Sp}(2n)$ and $PS = SP$, the group generator g is defined by

$$(gx)(t) = \Big\{ \; Px(t + \tfrac{T}{k}), \quad t \in [0, \tfrac{k-1}{k}T]; S^{-1}Px(t + \tfrac{T}{k} - T), \quad t \in [\tfrac{k-1}{n}T, T]. \tag{4.6}$$

Easy computation show that $g \in \mathcal{L}(\mathcal{H})$ and $gE = E$. By direct computation, we get the adjoint operator g^*.

Lemma 4.1. *The adjoint operator g^* is given by*

$$(g^*x)(t) = \begin{cases} P^*(S^*)^{-1}x(t + T - \frac{T}{k}), & t \in [0, \frac{T}{k}]; \\ P^*x(t - \frac{T}{k}), & t \in [\frac{T}{k}, T]. \end{cases} \tag{4.7}$$

Proof. Let $y \in L^2([0,T], \mathbb{C}^{2n})$. We see that

$$\int_0^{\frac{k-1}{k}T} (Px(t + T/k), y(t))dt = \int_{T/k}^T (x(t), P^*y(t - T/k))dt,$$

and

$$\int_{\frac{k-1}{k}T}^T (S^{-1}Px(t + T/k - T), y(t))dt = \int_0^{T/k} (x(t), P^*(S^*)^{-1}y(t + T - T/k)).$$

Then we have checked $\langle gx, y\rangle_{L^2} = \langle x, g^*y\rangle_{L^2}$ for each $x, y \in L^2([0,T], \mathbb{C}^{2n})$. □

We assume $B(t)$ satisfied

$$B(t) = \begin{cases} P^*(S^*)^{-1}B(t + T - \frac{T}{k})S^{-1}P, & t \in [0, \frac{T}{n}]; \\ P^*B(t - \frac{T}{k})P, & t \in [\frac{T}{n}, T]. \end{cases} \tag{4.8}$$

Please note that (4.8) implies (4.5), and (4.2) is satisfied. Since

$$(g^kx)(t) = S^{-1}P^kx(t),$$

which is a multiplicity operator on $\mathcal{H}$. Then

$$\sigma(g^k) = \sigma(S^{-1}P^k).$$

To simplify the notation, for $\Omega \in \mathbb{C}$, we define

$$\Omega^{\frac{1}{k}} = \{z \in \mathbb{C}, z^k \in \Omega\}.$$

By this notation, we have $\sigma(g) \in (\sigma(S^{-1}P^k))^{\frac{1}{k}}$. For $\lambda \in \sigma(g)$, $\mathcal{H}_\lambda = \ker(g-\lambda)^{2n}$.

Case 3. We consider the generalized reversible symmetry. We call a matrix M anti-symplectic if it satisfied

$$M^*JM = -J. \tag{4.9}$$

We denote by $\mathrm{Sp}_a(2n)$ the set of anti-symplectic matrices. For $M_1, M_2 \in \mathrm{Sp}_a(2n)$ and $M_3 \in \mathrm{Sp}(2n)$, then it is obvious that

$$M_1M_2 \in \mathrm{Sp}(2n), \quad M_1M_3 \in \mathrm{Sp}_a(2n).$$

We list some basic property of $\mathrm{Sp}_a(2n)$ follows.

Lemma 4.2. *If $M \in \mathrm{Sp}_a(2n)$, $\lambda \in \sigma(M)$, then $\bar{\lambda}, -\lambda^{-1}, -\bar{\lambda}^{-1} \in \sigma(M)$ and possess the same geometric and algebraic multiplicities.*

Proof. Note that $M^* = -JM^{-1}J^{-1}$. Let $\lambda \in \mathbb{C}\backslash\{0\}$. It follows that $(M^*-\lambda) = -J(M^{-1}+\lambda)J^{-1}$. Then we have

$$\dim\ker(M-\bar{\lambda}) = \dim\ker(M^*-\lambda) = \ker(M^{-1}+\lambda) = \dim\ker(M+\lambda^{-1}).$$

And we also have

$$\overline{\det(M-\bar{\lambda})} = \det(-M^{-1}-\lambda) = \det(-M^{-1}\lambda)\det(M+\lambda^{-1}).$$

It follows that $\dim\ker(M-\bar{\lambda})^{2n} = \dim\ker(M+\lambda^{-1})^{2n}$. So $\lambda, -\lambda^{-1} \in \sigma(M)$ and posses the same geometric and algebraic multiplicities. Specially, if M is a real matrix, $\lambda, \bar{\lambda}, -\lambda^{-1}, -\bar{\lambda}^{-1} \in \sigma(M)$ and posses the same geometric and algebraic multiplicities. □

Similar with the symplectic matrix, we have the following results.

Lemma 4.3. *Let $M \in \mathrm{Sp}_a(2n)$. Let $\lambda, \mu \in \sigma(M)$. Let $V_\lambda = \ker(M-\lambda)^{2n}$, $V_\mu = \ker(M-\mu)^{2n}$. Then we have $(Jx,y) = 0$ if $\lambda\bar{\mu} \neq -1$.*

Proof. Let $x \in \ker(M-\lambda)^p$,$y \in \ker(g-\mu)^q$ with $p,q \geq 0$. We see that $(Jx,y) = 0$ if $p+q = 0$. Assume that $(Jx,y) = 0$ if $p+q \leq k$. Note that $(M-\lambda)x \in \ker(M-\lambda)^{p-1}$,$(M-\mu)x \in \ker(M-\mu)^{q-1}$. If $p+q = k+1$, We have

$$\begin{aligned}(Jx,y) = -(JMx,My) = -(J(M-\lambda)x,My) - (J\lambda x,(M-\mu)y) - \lambda\bar{\mu}(Jx,y)\\ = -\lambda\bar{\mu}(Jx,y).\end{aligned}$$

Since $\lambda\bar{\mu} \neq -1$, we have $(Ax,y) = 0$. By induction, we have $(Jx,y) = 0$ with $x \in \ker(M-\lambda)^{2n}$ and $y \in \ker(M-\mu)^{2n}$. This completes the result. □

We assume (4.1) admits a N-reversible symmetry. More exactly, for $N \in \mathrm{Sp}_a(2n)$, let

$$(gx)(t) = Nx(T-t). \tag{4.10}$$

We assume $g\Lambda = \Lambda$, that is

$$(Nx(T), Nx(0)) \in \Lambda,\ if \quad (x(0), x(T)) \in \Lambda, \tag{4.11}$$

then $gE = E$. Obviously, $(g^*x)(t) = N^*x(T-t)$. We assume

$$N^*B(T-t)N = B(t), \tag{4.12}$$

then (4.2) is satisfied.

Please note that for the S-periodic boundary conditions, $NS^{-1} = SN$ implies $g\Lambda = \Lambda$. Separated boundary conditions is another kind of important boundary conditions. More precisely, we consider solution of (4.1) under the boundary conditions

$$x(0) \in V_0, \quad x(T) \in V_1,$$

where $V_0, V_1 \in Lag(2n)$. In this case g is defined by (4.10), for $N \in \mathrm{Sp}_a(2n)$ which satisfied

$$NV_0 = V_1, \quad NV_1 = V_0,$$

then $g\Lambda = \Lambda$.

Obviously, we have

$$(g^2x)(t) = N^2x(t),$$

hence $g \in \mathcal{M}(\mathcal{H})$ and

$$\sigma(g) = (\sigma(N^2))^{\frac{1}{2}}.$$

For $\lambda \in \sigma(g)$, $\mathcal{H}_\lambda = \ker(g-\lambda)^{2n}$.

From theorem 1.3, we get the decomposition of spectral flow. Since on the finite interval B is relative compact with respect to A, then from Remark 3.7, we have

$$I(A, A-B) = \sum_{i=1}^{m} I(A|_{F_{\lambda_i}}, A|_{F_{\lambda_i}} - B|_{F_{\lambda_i}}) + \frac{1}{2}(dim\ker((A-B)|_{\hat{F}}) - dim\ker(A|_{\hat{F}})). \tag{4.13}$$

All the above discussions can be applied to Sturm-Liouville systems, so we don't give the details in all cases, instead we only consider the following two cases which have clear background.

Case 4. We consider the one parameter family Sturm-Liouville system

$$-(G_s(t)\dot{x})' + R_s(t)x(t) = 0, \quad x(0) = Sx(T),\ \dot{x}(0) = S\dot{x}(T), s \in [0,1] \tag{4.14}$$

where $S \in \mathcal{L}^*(\mathbb{R}^n)$. We suppose $G_s(t), R_s(t) \in \mathcal{S}(n)$, instead of the Legendre convex condition we only assume $G_s(t)$ is invertible. Let $P \in \mathcal{L}^*(\mathbb{R}^n)$ and $PS = SP$, the group generator g is defined as same form of (4.6). We assume

$$G_s(t) = \begin{cases} P^*(S^*)^{-1}G_s(t+T-\frac{T}{n})S^{-1}P, & t \in [0, \frac{T}{n}]; \\ P^*G_s(t-\frac{T}{n})P, & t \in [\frac{T}{n}, T]. \end{cases} \tag{4.15}$$

$$R_s(t) = \begin{cases} P^*(S^*)^{-1}R_s(t+T-\frac{T}{n})S^{-1}P, & t \in [0, \frac{T}{n}]; \\ P^*R_s(t-\frac{T}{n})P, & t \in [\frac{T}{n}, T]. \end{cases} \tag{4.16}$$

Then

$$g^*(-(G_s(t)\frac{d}{dt})' + R_s)g = -(G_s(t)\frac{d}{dt})' + R_s,$$

and we could give the decomposition of spectral flow from Theorem 1.3.

This case includes the Bott-type formula of Semi-Riemann manifold [17]. Let c be a space-like or time-like closed geodesic on $n+1$ dimension Semi-Riemann manifold $(M, \mathfrak{g})$ with period T. We choose a parallel $\mathfrak{g}$-orthonormal frame $e_i(t)$ alone c, and satisfied $\mathfrak{g}(e_i(t), \dot{c}(t)) = 0$. Assume

$$\mathfrak{g}(e_i, e_j) = \begin{cases} 0, & i \neq j; \\ 1, & 1 \leq i = j \leq n - \nu; \\ -1, & n - \nu \leq i = j \leq n \end{cases}$$

and

$$(e_1(0), \cdots, e_n(0)) = (e_1(T), \cdots, e_n(T))P,$$

then $P^T GP = G$ with $G = diag(I_{n-\nu}, -I_\nu)$.

Writing the $\dot{c}$ $\mathfrak{g}$-orthogonal Jacobi vector field alone c as $J(t) = \sum_{i=1}^n u_i(t)e_i(t)$, then we get the linear second order system of ordinary differential equations

$$-G\ddot{u} + R(t)u(t) = 0, \quad t \in [0, T], \tag{4.17}$$

where R is symmetry matrices which is get by the curvature. A period solution is satisfied

$$u(0) = Pu(T).$$

For $\omega \in \mathbb{U}$, let

$$E^2_{\omega,T} := \{u \in W^{2,2}([0, T], \mathbb{C}^n) | u(0) = \omega Pu(T), \dot{u}(0) = \omega P\dot{u}(T)\},$$

then

$$A^\omega_{s,T} = -G\frac{d^2}{dt^2} + R(t) + sG$$

are self-adjoint Fredholm operators on $L^2([0, T], \mathbb{C}^n)$ with domain $E^2_{\omega,T}$.

It has proved in [17] that there exist s_0 sufficiently large such that for $s \geq s_0$, A^ω_s is non-degenerate. The ω spectral index of c is defined by

$$i^\omega_{spec}(c) := sf(A^\omega_{s,T}; s \in [0, +\infty)).$$

Let $c^{(m)}$ be the m-th iteration of c, then

$$i^\omega_{spec}(c^{(m)}) := sf(A^\omega_{s,mT}; s \in [0, +\infty)).$$

Let $S = P^m$, $G_s = G$, $R_s = R(t) + sG$, $(gu)(t) = Pu(t + T)$, then from Case 4. we get the decomposition of spectral flow. Since $g^m = \omega$, then

$$\sigma(g) = \{\omega\}^{\frac{1}{m}}.$$

Let ω_j be the m-th root of ω, then

$$\mathcal{H}_{\omega_j} = \ker(g - \omega_j) = \{u(t) = \omega_j Pu(t + T)\}.$$

We have

$$sf(A^\omega_{s,mT}; s \in [0, +\infty)) = \sum_{\omega_j^m=\omega} sf(A^{\omega_j}_{s,T}; s \in [0, +\infty)).$$

Hence we get the Bott-type iteration formula [17]

$$i^\omega_{spec}(c^{(m)}) = \sum_{\omega_j^m=\omega} i^{\omega_j}_{spec}(c). \tag{4.18}$$

Obviously, we can consider the case of reversible symmetry, since it is similar, we omit the detail.

Case 5. Now we consider the case of heteroclinic orbits, for the one parameter family linear Hamiltonian system

$$\dot{x} = JB_s(t)x(t), \quad t \in \mathbb{R}, \quad s \in [0,1]. \tag{4.19}$$

Let $B_s(\pm\infty) = \lim_{t\to\pm\infty} B_s(t)$ exist and satisfied the hyperbolic condition, i.e.

$$\sigma(JB_s(\pm\infty)) \cap i\mathbb{R} = \emptyset, \quad s \in [0,1].$$

Let $\mathcal{H} = L^2(\mathbb{R}, \mathbb{C}^{2n})$, it is well known that $A - B_s \in \mathcal{FS}(\mathcal{H})$ with domain $E = W^{1,2}(\mathbb{R}, \mathbb{C}^{2n})$.

We assume

$$N^* B_s(-t)N = B_s(t),$$

then $g^* B_s g = g$. Easy computation show that $g^* A g = A$, then we have

$$g^*(A - B_s)g = A - B_s, \quad s \in [0,1].$$

Obviously, we have

$$(g^2 x)(t) = N^2 x(t),$$

hence $g \in \mathcal{M}(\mathcal{H})$ and then

$$\sigma(g) = (\sigma(N^2))^{\frac{1}{2}}.$$

For $\lambda \in \sigma(g)$, $\mathcal{H}_\lambda = \ker(g-\lambda)^{2n}$, then we get the decomposition formula from Theorem 1.3.

In the case $N^2 = I$, let

$$\mathcal{H}_\pm = \ker g \mp I = \{x \in \mathcal{H}, Nx(-t) = \pm x(t)\}, \tag{4.20}$$

we have

$$sf(A - B_s) = sf(A|_{\mathcal{H}_+} - B_s|_{\mathcal{H}_+}) + sf(A|_{\mathcal{H}_-} - B_s|_{\mathcal{H}_-}). \tag{4.21}$$

Now we consider the case of homoclinic orbits. For the linear Hamiltonian system

$$\dot{x}(t) = JB(t)x(t), \quad t \in \mathbb{R}, \tag{4.22}$$

assume $\lim_{t\to\pm\infty} B(t) = B_*$ and JB_* is hyperbolic. In this case, $B - B_*$ is relative compact with respect to $A - B_*$, where $A = -J\frac{d}{dt}$. The relative index is defined by

$$I(A - B_*, A - B) = -sf(A - B_* + s(B - B_*)).$$

In the case that (4.22) is a linear system of homoclinic orbits z, the index of z is defined by [7]

$$i(z) = I(A - B_*, A - B).$$

Assume $N^* B(-t)N = B(t)$ and $N^2 = I$, from (4.21), we have

$$\begin{aligned} I(A - B_*, A - B) = {} & I(A|_{\mathcal{H}_+} - B_*|_{\mathcal{H}_+}, A|_{\mathcal{H}_+} - B|_{\mathcal{H}_+}) \\ & + I(A|_{\mathcal{H}_-} - B_*|_{\mathcal{H}_-}, A|_{\mathcal{H}_-} - B|_{\mathcal{H}_-}). \end{aligned} \tag{4.23}$$

Case 6. We consider the one parameter Sturm-Liouville system on $\mathbb{R}$

$$-(G(t)\dot{x})' + R(t)x(t) = 0, \quad t \in \mathbb{R} \tag{4.24}$$

where $G(t), R(t) \in \mathcal{S}(n)$. We assume there exist $\delta > 0$, such that $G(t) > \delta$ for $t \in \mathbb{R}$, and there exist $T, \delta_1, \delta_2 > 0$ such that

$$\delta_1 < R(t) < \delta_2, \quad for \quad t \geq |T|. \tag{4.25}$$

The Morse index of $\mathcal{A} := -(G(t)\frac{d}{dt})' + R(t)$ is defined by the maximum dimension of the subspace such that $\mathcal{A}$ restricted on it is negative definite. It is well known that $m^-(\mathcal{A})$ is finite under the condition (4.25). Obviously

$$m^-(\mathcal{A}) = sf(\mathcal{A} + sG(0); s \in [0, +\infty)).$$

We assume there exist $N \in \mathrm{Sp}_a(n)$, such that

$$N^* R(-t) N = R(t), \quad N^* G(-t) N = G(t).$$

Let

$$gx(t) = Nx(-t),$$

then $g^*(\mathcal{A} + sG(0))g = \mathcal{A} + sG(0)$. Since $g^2 = N$, then $\sigma(g) = (\sigma(N))^{\frac{1}{2}}$, and we get the decomposition formula of Morse index.

$$m^-(\mathcal{A}) = \sum_{i=1}^{j} m^-(\mathcal{A}|_{F_{\lambda_i}}) - dim \ker \mathcal{A}|_{\hat{F}}.$$

In the case $N^2 = I$, we have

$$m^-(\mathcal{A}) = m^-(\mathcal{A}|_{\mathcal{H}_+}) + m^-(\mathcal{A}|_{\mathcal{H}_-}). \tag{4.26}$$

5. Relation with the Maslov index

In this section, we will give some Bott-type iteration formulas of Maslov-index. In what follows g is pointed as the Matrix-like operator which appears in cases 2, 3, 5. To avoid discussing too many technique details, we only consider the case

$$g^m = \omega I$$

for some $\omega \in \mathbb{U}$. Let $\omega_1, \cdots, \omega_m$ be the m-th roots of ω, and let $\mathcal{H}_i = \ker(g - \omega_i)$.

In cases 2, 3, 5, $\mathcal{H}(\mathcal{I}) = L^2(\mathcal{I}, \mathbb{C}^{2n})$ where $\mathcal{I}$ is some finite interval or $\mathbb{R}$, and E is $W^{1,2}(\mathcal{I}, \mathbb{C}^{2n})$ which satisfies some boundary conditions. We choose a subinterval $\hat{\mathcal{I}} \subset \mathcal{I}$, and let $\mathcal{T}$ be the restriction map from $\mathcal{H}$ to $\mathcal{H}(\hat{\mathcal{I}}) := L^2(\hat{\mathcal{I}}, \mathbb{C}^{2n})$, that is

$$(\mathcal{T} f)(x) = f(x), \quad x \in \hat{\mathcal{I}}. \tag{5.1}$$

$\hat{\mathcal{I}}$ is called a fundamental domain if for any $i = 1, \cdots, m$, $\mathcal{T}$ is a bijection from $\mathcal{H}_i$ to $L^2(\hat{\mathcal{I}}, \mathbb{C}^{2n})$. Recall that $E_i = E \cap \mathcal{H}_i$ is domain of $A_s|_{\mathcal{H}_i}$, then $\mathcal{T} E_i$ is closed in the $W^{1,2}$ norm. Let $\hat{A}^i_s = -J\frac{d}{dt} - B_s$ be the operator on $\mathcal{H}(\hat{\mathcal{I}})$ with domain $\mathcal{T} E_i$.

Lemma 5.1. *Suppose for $s \in [a, b]$, $\hat{A}^i_s$ is self-adjoint and $dim \ker(A_s|_{\mathcal{H}_i}) = dim \ker(\hat{A}^i_s)$, then*

$$sf(A_s|_{\mathcal{H}_i}; s \in [a, b]) = sf(\hat{A}^i_s; s \in [a, b]). \tag{5.2}$$

Proof. Note that $(A_s + t\,\mathrm{Id})|_{\mathcal{H}_i} = P_{\mathcal{H}_i}(A_s + t\,\mathrm{Id})P_{\mathcal{H}_i} = P_{\mathcal{H}_i} A_s P_{\mathcal{H}_i} + tP_{\mathcal{H}_i}$. Since $A_s \in \mathcal{FS}(\mathcal{H})$, there is $\epsilon > 0$, such that for each $t \in [0, \epsilon]$, $(A_s + t\,\mathrm{Id}) \in \mathcal{FS}(\mathcal{H})$. Then $(A_s + t\,\mathrm{Id})|_{\mathcal{H}_i}$ is a positive curve on $\mathcal{FS}(H_i)$ with $t \in [0, \epsilon]$. Note that $-J\frac{d}{dt} - B_s + t\,\mathrm{Id}$ is also a positive curve on $\mathcal{FS}(\mathcal{H}(\hat{I}))$, then (5.2) is from Lemma 2.2. This completes the proof. □

Now we consider Case 2. We assume $\omega S = P^m$, then

$$\mathcal{H}_i = \ker(g-\omega_i) = \{x \in \mathcal{H},\, \omega_i x(t) = Ps(t+\frac{T}{m})\}.$$

We choose $\hat{\mathcal{I}} = [0, \frac{T}{m}]$ be the fundamental domain, then

$$\mathcal{T}E_i = \{x \in W^{1,2}([0,\frac{T}{m}], \mathbb{C}^{2n}), \quad \omega_i x(0) = Ps(\frac{T}{m})\}.$$

From Corollary 6.2, we have

$$-sf(A_s) = \mu(Gr(\omega S), Gr(\gamma(t)); t \in [0,T]),$$
$$-sf(\hat{A}_s^{(i)}) = \mu(Gr(\omega_i), Gr(\gamma(t)); t \in [0, T/m]).$$

Then we have

$$\mu(Gr(S^{-1}), Gr(\gamma(t)); t \in [0,T]) = \sum_{i=1}^{m} \mu(Gr(\omega_i P^{-1}), Gr(\gamma(t)); t \in [0,T/m]). \tag{5.3}$$

Remark 5.2. In the case $P = I_{2n}$, (5.3) is the standard Bott-type iteration formula for Hamiltonian systems, please refer [22], [23] for the detail. In the case $P \in \mathrm{Sp}(2n) \cap \mathbb{O}(2n)$, (5.3) is established by Hu and Sun [18], the general case is proved by Liu and Tang [19].

For Case 3. We assume $N \in \mathrm{Sp}_a(2n)$ and $N^2 = I$. Since N is anti-symplectic, we have $(Jx, x) = -(JNx, Nx) = -(Jx, x) = 0$ with $x \in \ker(N-I)$. So $\ker(N-I)$ is a Lagrange subspace of the symplectic space $(\mathbb{R}^{2n}, J)$. Recall that $(gx)(t) = Nx(T-t)$ and $g\Lambda = \Lambda$. Let

$$\mathcal{H}_\pm = \ker g \mp I = \{x \in \mathcal{H}, Nx(T-t) = \pm x(t)\},$$

then $\hat{\mathcal{I}} = [0, T/2]$ is a fundamental domain.

For the S-periodic boundary conditions, that is $x(0) = Sx(T)$, then

$$\mathcal{T}E_\pm = \{x \in W^{1,2}([0,T/2], \mathbb{C}^{2n}), x(0) \in V^\pm(SN), x(T/2) \in V^\pm(N)\}.$$

We have

$$\begin{aligned}&\mu(Gr(S), Gr(\gamma(t)); t \in [0,T])\\ &= \mu(V^+(N), \gamma(t)V^+(SN); t \in [0,T/2]) + \mu(V^-(N), \gamma(t)V^-(SN); t \in [0,T/2]).\end{aligned}$$

Similarly, if the boundary condition is given by $x(0) \in V_0$, $x(T) \in V_1$ for $V_0, V_1 \in Lag(2n)$ and $NV_0 = V_1$, $NV_1 = V_0$. Similar discussion with above, we have

$$\begin{aligned}\mu(V_1, \gamma(t)V_0; t \in [0,T]) &= \mu(V^+(N), \gamma(t)V_0; t \in [0,T/2])\\ &\quad + \mu(V^-(N), \gamma(t)V_0; t \in [0,T/2]).\end{aligned} \tag{5.4}$$

Remark 5.3. To our knowledge, in the case $S = I_{2n}$, $N^2 = I$, (5.4) is first established by Long Zhang and Zhu [24]. A deep study is given by Liu and Zhang [20] [21]. Hu and Sun had established the case of $S, N \in \mathbb{O}(2n)$, for the case of dihedral group please refer [16].

Now we consider Case 5. For $\lambda \in [0,1]$, let $\gamma_\lambda(\tau, t)$ be the fundamental solution of (4.19), that is

$$\dot{\gamma}_\lambda(\tau, t) = JB_\lambda(t)\gamma_\lambda(\tau, t), \quad \gamma_\lambda(\tau, \tau) = I_{2n}. \tag{5.5}$$

Let

$$V^s_\lambda(\tau) := \{v \in \mathbb{R}^{2n} | \lim_{\tau\to\infty} \gamma_\lambda(\tau,t)v = 0\}, \quad V^u_\lambda(\tau) := \{v \in \mathbb{R}^{2n} | \lim_{\tau\to-\infty} \gamma_\lambda(\tau,t)v = 0\},$$

be the stable and unstable paths, then $V^s_\lambda(\tau), V^u_\lambda(\tau) \in Lag(2n)$.

Recall that in this case, $\mathcal{H} = L^2(\mathbb{R}, \mathbb{R}^{2n})$ and

$$A_\lambda := -J\frac{d}{dt} - B_s(t) : E = W^{1,2}(\mathbb{R}, \mathbb{R}^{2n}) \subset \mathcal{H} \to \mathcal{H}.$$

Let $\mathbb{R}^-$ be the fundamental domain, then

$$\mathcal{T}E_\pm = \{x \in W^{1,2}(\mathbb{R}^-, \mathbb{R}^{2n}), x(0) \in V^\pm(N)\}.$$

Let $A^\pm_\lambda$ be the restricted operators on $\mathcal{H}(\mathbb{R}^-)$ with domain $\mathcal{T}E_\pm$.

From Prop 3.7 of [15], we have

$$-sf(A_\lambda; \lambda \in [0,1]) = \mu(V^s_\lambda(0), V^u_\lambda(0); \lambda \in [0,1]).$$

Similarly

$$-sf(A^\pm_\lambda; \lambda \in [0,1]) = \mu(V^\pm(N), V^u_\lambda(0); \lambda \in [0,1]).$$

Then we have

$$\begin{aligned}(5.6)\quad \mu(V^s_\lambda(0), V^u_\lambda(0); \lambda \in [0,1]) = {}& \mu(V^+(N), V^u_\lambda(0); \lambda \in [0,1]) \\ &+ \mu(V^-(N), V^u_\lambda(0); \lambda \in [0,1]).\end{aligned}$$

In the case of homoclinic orbits, let $A_\lambda = A - B_* - \lambda(B - B_*)$, then from [7] or [15] the index satisfied

$$\begin{aligned}(5.7)\quad -sf(A_\lambda; \lambda \in [0,1]) &= \mu(V^s(+\infty), V^u(t); t \in \mathbb{R}) \\ &= -\mu(V^s(t), V^u(-t); t \in \mathbb{R}^+).\end{aligned}$$

We have

$$\begin{aligned}(5.8)\quad -sf(A^\pm_\lambda; \lambda \in [0,1]) &= \mu(V^\pm(N), V^u(t); t \in \mathbb{R}^-) \\ &= -\mu(V^\pm(N), V^u(-t); t \in \mathbb{R}^+).\end{aligned}$$

Compare (5.7) and (5.8), we have

$$\begin{aligned}(5.9)\quad \mu(V^s(+\infty), V^u(t); t \in \mathbb{R}) = {}& \mu(V^+(N), V^u(t); t \in \mathbb{R}^-) \\ &+ \mu(V^-(N), V^u(t); t \in \mathbb{R}^-),\end{aligned}$$

or equivalently

$$\begin{aligned}(5.10)\quad \mu(V^s(t), V^u(-t); t \in \mathbb{R}^+) = {}& \mu(V^+(N), V^u(-t); t \in \mathbb{R}^+) \\ &+ \mu(V^-(N), V^u(-t); t \in \mathbb{R}^+).\end{aligned}$$

Obviously, we can use $\mathbb{R}^+$ as the fundamental domain, for reader's convenience, we list the formulas below. Here we let $A^\pm_\lambda$ be the restricted operators on $\mathcal{H}(\mathbb{R}^+)$ with domain $\mathcal{T}(E_\pm)$.

$$(5.11)\quad -sf(A^\pm_\lambda; \lambda \in [0,1]) = \mu(V^s_\lambda(0), V^\pm(N); \lambda \in [0,1]).$$

$$\begin{aligned}(5.12)\quad \mu(V^s_\lambda(0), V^u_\lambda(0); \lambda \in [0,1]) = {}& \mu(V^s_\lambda(0), V^+(N); \lambda \in [0,1]) \\ &+ \mu(V^s_\lambda(0), V^-(N); \lambda \in [0,1]).\end{aligned}$$

In the case of homoclinic orbits,

$$\mu(V^s(t), V^u(-\infty); t \in \mathbb{R}) = \mu(V^s(t), V^u(-t); t \in \mathbb{R}^+)$$
$$= \mu(V^s(t), V^+(N); t \in \mathbb{R}^+) + \mu(V^s(t), V^-(N); t \in \mathbb{R}^+).$$

In study the stability problem of homographic solution in planar n-body problem, Hu and Ou [14] use the McGehee blow up method to get linear heteroclinic system, this system with reversible symmetry if the corresponding central configurations with a symmetry property. Please refer [14] for the detail.

6. Appendix: Spectral flow and Maslov index

Spectral flow was introduced by Atiyah, Patodi and Singer in their study of index theory on manifold with boundary [2]. Let $\{A_t, t \in [0,1]\}$ be a continuous path of self-adjoint Fredholm operators on a Hilbert space $\mathcal{H}$. The spectral flow $sf\{A_t\}$ of A_t counts the algebraic multiplicities of the spectral points of A_t cross the line $\lambda = -\epsilon$ with some small positive number ϵ. For reader's convenience, we list some basic properties of spectral flow.

(Stratum homotopy relative to the ends) If $A_{s,\lambda} \in C([a,b] \times [0,1], \mathcal{FS}(\mathcal{H}))$, such that dim ker $A_{a,\lambda}$ and dim ker $A_{b,\lambda}$ is constant, then

$$sf(A_{s,0}; s \in [a,b]) = sf(A_{s,1}; s \in [a,b]).$$

(Path additivity) If $A^1, A^2 \in C\big([a,b]; \mathcal{FS}(\mathcal{H})\big)$ such that $A^1(b) = A^2(a)$, then

$$sf(A^1_t * A^2_t; t \in [a,b]) = sf(A^1_t; t \in [a,b]) + sf(A^2_t; t \in [a,b])$$

where $*$ denotes the usual catenation between the two paths.

(Direct sum) If $\mathcal{H}_i$ are Hilbert space for $i = 1, 2$, and $A^i \in C\big([a,b]; \mathcal{FS}(\mathcal{H}_i)\big)$, then

$$sf(A^1_t \oplus A^2_t; t \in [a,b]) = sf(A^1_t; t \in [a,b]) + sf(A^2_t; t \in [a,b]).$$

(Nullity) If $A \in C\big([a,b]; \mathcal{FS}(\mathcal{H})\big)$, then $sf(A_t; t \in [a,b]) = 0$;

(Reversal) Denote the same path travelled in the reverse direction in $\mathcal{FS}(\mathcal{H})$ by $\widehat{A}(t) = A(-t)$. Then

$$sf(A_t; t \in [a,b]) = -sf(\widehat{A}_t; t \in [-b,-a]).$$

The spectral flow is related to Maslov index in Hamiltonian systems. We now briefly reviewing the Maslov index theory [1, 6, 25]. Let $(\mathbb{R}^{2n}, \omega)$ be the standard symplectic space and $Lag(2n)$ the Lagrangian Grassmanian. For two continuous paths $L_1(t), L_2(t)$, $t \in [a,b]$ in $Lag(2n)$, the Maslov index $\mu(L_1, L_2)$ is an integer invariant. Here we use the definition from [6]. We list several properties of the Maslov index. The details could be found in [6].

(Reparametrization invariance) Let $\phi : [c,d] \to [a,b]$ be a continuous and piecewise smooth function with $\phi(c) = a$, $\phi(d) = b$, then

$$\mu(L_1(t), L_2(t)) = \mu(L_1(\phi(\tau)), L_2(\phi(\tau))). \tag{6.1}$$

(Homotopy invariant with end points) For two continuous family of Lagrangian path $L_1(s,t)$, $L_2(s,t)$, $0 \le s \le 1$, $a \le t \le b$, and satisfies $dimL_1(s,a) \cap L_2(s,a)$ and $dimL_1(s,b) \cap L_2(s,b)$ is constant, then

$$\mu(L_1(0,t), L_2(0,t)) = \mu(L_1(1,t), L_2(1,t)). \tag{6.2}$$

(Path additivity) If $a < c < b$, then then

$$\mu(L_1(t), L_2(t)) = \mu(L_1(t), L_2(t)|_{[a,c]}) + \mu(L_1(t), L_2(t)|_{[c,b]}). \tag{6.3}$$

(Symplectic invariance) Let $\gamma(t)$, $t\in[a,b]$ is a continuous path in $\mathrm{Sp}(2n)$, then

$$\mu(L_1(t),L_2(t))=\mu(\gamma(t)L_1(t),\gamma(t)L_2(t)). \tag{6.4}$$

(Symplectic additivity) Let W_i, $i=1,2$ be symplectic space with

$$L_1,L_2\in C([a,b],Lag(W_1)) \quad \text{and} \quad \hat{L}_1,\hat{L}_2\in C([a,b],Lag(W_2)),$$

then

$$\mu(L_1(t)\oplus\hat{L}_1(t),L_2(t)\oplus\hat{L}_2(t))=\mu(L_1(t),L_2(t))+\mu(\hat{L}_1(t),\hat{L}_2(t)). \tag{6.5}$$

The next Theorem give the relation of spectral flow and Maslov index.

Theorem 6.1.

$$-sf(A_s-B_s)=\mu(\Lambda_s,Gr(\gamma_s(T))) \tag{6.6}$$

The above Theorem is well known [15], we like to give a direct proof here.

Proof. We use the idea of Lemma 2.2. We only need to prove the theorem locally. Let $s_0\in[0,1]$. $A_{s_0}-B_{s_0}+rI, r\in[0,1]$ is a positive path in $\mathcal{FS}(\mathcal{H})$. There is $\epsilon>0$ such that $\ker(A_{s_0}-B_{s_0}+\epsilon I)=0$. Then there is $\delta>0$ such that $\ker(A_s-B_s+\epsilon I)=0$, $\forall s\in[s_0-\delta,s_0+\delta]$. Without loss of generality, we can assume that

$$\ker(A_s-B_s+I)=0,\ \forall s\in[0,1].$$

Let $\gamma_{s,r}(t)$ be the fundamental solution of the equation

$$\dot{z}(t)=J(B_s(t)-rI)z(t),(s,t)\in[0,1]\times[0,T],\ r\in[0,1]. \tag{6.7}$$

Recall that $\dim(\mathrm{Gr}(\gamma_{s,r}(T))\cap\Lambda_s)=\dim\ker(A_s-B_s+rI)$, then we have

$$\mu(\Lambda_s,Gr(\gamma_{s,1}(T)))=0.$$

Then use the homotopy invariant property of spectral flow and Maslov index, we have

$$\begin{cases} sf(A_s-B_s,s\in[0,1])=sf(A_0-B_0+rI)-sf(A_1-B_1+rI)\\ \mu(\Lambda_s,\mathrm{Gr}(\gamma_s(T)))=\mu(\Lambda_0,\mathrm{Gr}(\gamma_{0,r}(T)))-\mu(\Lambda_1,\mathrm{Gr}(\gamma_{0,r}(T)))\end{cases}. \tag{6.8}$$

Note that A_0-B_0+rI is a positive path in $\mathcal{FS}(\mathcal{H})$, then we have

$$sf(A_0-B_0+rI)=\sum_{0<r\le 1}\dim\ker(A_0-B_0+rI).$$

Let $Q_t((x,\gamma(t)x),(y,\gamma(t)y))=<-J\gamma(t)^{-1}\dot{\gamma}(t)x,y>$ which is a quadratic form on $\mathrm{Gr}(\gamma(t))$. Recall that the crossing form of the Lagrangian pair $(\Lambda,\mathrm{Gr}(\gamma(t)))$ is given by $Q_t|_{Gr(\gamma(t))\cap\Lambda}$.

Note that

$$\begin{aligned}\frac{\partial}{\partial t}(\gamma_s(t)^{-1}\frac{\partial}{\partial s}\gamma_s(t))&=-\gamma_s(t)^{-1}\frac{\partial}{\partial t}\gamma_s(t)\gamma_s(t)^{-1}\frac{\partial}{\partial s}\gamma_s(t)+\gamma_s(t)^{-1}\frac{\partial}{\partial s}(\frac{\partial}{\partial t}\gamma_s(t))\\ &=-\gamma_s(t)^{-1}\frac{\partial}{\partial t}\gamma_s(t)\gamma_s(t)^{-1}\frac{\partial}{\partial s}\gamma_s(t)+\gamma_s(t)^{-1}\frac{\partial}{\partial s}(JB_s)\gamma_s(t)+\gamma_s(t)^{-1}JB_s\frac{\partial}{\partial s}\gamma_s(t)\\ &=\gamma_s(t)^{-1}\frac{\partial}{\partial s}(JB_s)\gamma_s(t).\end{aligned}$$

Then we have $\frac{\partial}{\partial t}(-J\gamma_{0,r}(t)^{-1}\frac{\partial}{\partial r}\gamma_{0,r}(t))=-I$, and it follows that

$$-J\gamma_{0,r}(t)^{-1}\frac{\partial}{\partial r}\gamma_{0,r}(t)=-TI.$$

Thus we have

$$\begin{aligned}\mu(\Lambda_0, \mathrm{Gr}(\gamma_{0,r}(T))) &= -\sum_{0<r\leq 1} \dim\left(\Lambda_0 \cap \mathrm{Gr}(\gamma_{0,r}(T))\right) \\ &= -\sum_{0<r\leq 1} \dim\left(\ker(A_0 - B_0 + rI)\right) = -sf(A_0 - B_0 + rI).\end{aligned}$$

Similarly

$$\mu(\Lambda_1, \mathrm{Gr}(\gamma_{1,r}(T))) = -sf(A_1 - B_1 + rI).$$

Then by (6.8), we get (6.6). □

From the homotopy invariance of Maslov index, we have

Corollary 6.2.

$$-sf(A - sB) = \mu(\Lambda, Gr(\gamma(t)), t \in [0, T]). \tag{6.9}$$

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