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Research article

The impact of time-varying risk on stock returns: an experiment of cubic piecewise polynomial function model and the Fourier Flexible Form model

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Abstract: With fast evolving econometric techniques being adopted in asset pricing, traditional linear asset pricing models have been criticized by their limited function on capturing the time-varying nature of data and risk, especially the absence of data smoothing is of concern. In this paper, the impact of data smoothing is explored by applying two asset pricing models with non-linear feature: cubic piecewise polynomial function (CPPF) model and the Fourier Flexible Form (FFF) model are performed on US stock returns as an experiment. The traditional beta coefficient is treated asymmetrically as downside beta and upside beta in order to capture corresponding risk, and further, to explore the risk premia attached in a cross-sectional context. It is found that both models show better goodness of fit comparing to classic linear asset pricing model cross-sectionally. When appropriate knots and orders are determined by Akaike Information Criteria (AIC), the goodness of fit is further improved, and the model with both CPPF and FFF betas employed showed the best fit among other models. The findings fill the gap in literature, specifically on both investigating and pricing the time variation and asymmetric nature of systematic risk. The methods and models proposed in this paper embed advanced mathematical techniques of data smoothing and widen the options of asset pricing models. The application of proposed models is proven to superiorly provide high degree of explanatory power to capture and price time-varying risk in stock market.

Keywords: asset pricing; cubic piecewise polynomial function; Fourier Flexible Form; downside beta; upside beta

JEL Codes: G10, C22

1. Introduction

Since the Capital Asset Pricing Model and beta were introduced in modern finance, there has been arguments on whether beta is adequate to explain the complex nature of systematic risk in stock market. The complexity of systematic risk is primarily due to its asymmetry and time-varying nature. The asymmetry of systematic risk, particularly downside risk, begins drawing attention of researchers in recent two decades. Most studies of downside risk follow the classic approach, employing the linear market model to estimate beta. As a result, the time-varying nature of beta is either ignored or being weighted inappropriately. Therefore, adopting appropriate data smoothing technique is crucial to preserve the true time-varying nature of beta. In this paper, two models, the cubic piecewise polynomial function (CPPF) model and the Fourier Flexible Form (FFF) model are employed to model portfolio returns in order to examine the significance of beta, downside beta and upside beta estimates. Both models take flexible approaches, yet are parsimonious, allowing beta estimates to be time-varying with appropriate weight. Innovatively, various numbers of knots and orders are applied on the CPPF model and the FFF model, respectively, to smooth the sample. Also, the Akaike Information Criteria (AIC) (Akaike, 1974) is adopted to determine the most appropriate number of knots and order for the sample. With the AIC, the best fitted estimates of beta, upside beta and downside beta for both models are generated. These estimates are sorted into portfolios to examine the risk-return relationship. Moreover, Fama-Macbeth regressions are performed to discover the significance of the estimates in a cross-sectional context.

Taking the CPPF and the FFF approach is motivated by their flexibility, with both approaches allowing the beta estimates to vary over time with heavier weight on more recent data. The CPPF approach is analogous to cubic spline approach but with no constraints of intercept columns, and the estimates at each point in time are the product of a vector of initial estimates and a piecewise polynomial matrix. For the FFF approach, Sine and Cosine functions are adopted to construct a matrix which creates a non-linear pattern bounded between −1 and 1. The pattern is finally presented on the estimates at each point in time to allow time-variation. The importance of time-varying estimates is that estimates can present the true relationship between variables at each point of time, which allows us to discover the variation of co-movements among variables rather than a single estimate over the whole sample. Compared to the moving window approach, the advantage of the CPPF model and the FFF model is that the whole sample is considered, while the moving window approach is limited to past data and the length of the window used.

We find the estimates of beta, downside beta and upside beta estimates of both models to be highly significant to drive stock returns. The beta estimates positively drive stock returns. The downside and upside beta estimates demonstrate reversed impacts on stock returns, the downside beta has a negative impact on stock returns, while the upside beta, consistent with beta estimates, has a positive impact. This paper is arranged as follows: Section 2 provides literature reviews of both models, followed by Section 3 describes the data. Section 4 explains the econometric models and methods applied. Section 5 and Section 6 provide the empirical results and results of the Fama-Macbeth regressions. Section 7 concludes.

2. Literature review

2.1. Literature of time -varying beta and downside risk

The existing literature of time-varying beta is extensive. There is a long history of literature that has argued that the Capital Asset Pricing Model (CAPM) proposed by Sharpe (1964) and Lintner (1965) is inadequate to explain the risk-return relationship due to the assumption of constant beta. To solve this drawback, relaxing the constant beta assumption and allowing time-varying beta is one possible method. There are a number of approaches to obtain time-varying betas, for instance, in Fama and Macbeth's (1973) study, moving window Ordinary Least Square (OLS) regression is applied to the market factor model to obtain time-varying beta. According to Härdle et al (1988), Härdle (1992), Wand and Jones (1995), Ang and Kristensen (2012), and Li and Yang (2011), various nonparametric approaches which are based on simulation are alternative methods to obtain the time-varying beta. Moreover, the time-varying beta also can be obtained by the multivariate GARCH based model proposed by Engle (2002), Andersen et al. (2002) and Nieto et al. (2011). More recently, Horvath et al. (2020) employ functional data regression to estimate time-varying beta in Chinese stock market, and Chakrbarti and Das (2021) adopt modified multivariate GARCH to capture time-varying risk in Indian and American stock market.

There is relatively fewer literature focusing on downside risk of stock market although research emerge in recent decade. Since Ang et al. (2006) point out that stocks with higher downside risk is compensated with additional risk premium, focus has primarily been lying on how to measure the downside beta. Giglio et al. (2016), Min and Kim (2016) and Li (2021) all adopt unique approaches to estimate downside risk in various financial markets.

Although existing literature are considerably well built on time-varying beta and downside risk separately, there is limited research focusing on time-varying feature of downside risk. Huang (2019) applies linear market model to estimate time-varying downside beta. More recently, Dobrynskaya (2021) investigates downside risk in cryptocurrency market by estimating downside beta at each point in time using high frequency data. The aforementioned studies use linear model to estimate time-varying downside beta, while fail to put appropriate timing weights (heavier weight on more recent data) when smoothing due to a lack of non-linear feature of the models. In this paper, to fill gap in the literature, specifically on both investigating and pricing the time variation and asymmetric nature of systematic risk. Instead of regressing the stock return upon the market portfolio return in a linear fashion, two alternative methods with non-linear feature are proposed to estimate the corresponding time-varying beta coefficients, namely the CPPF regression and the FFF regression, respectively. A brief literature of development of both methods is provided in following sections.

2.2. The foundation of CPPF approach: cubic spline method

The advantages of the CPPF approach are, firstly, data can be flexibly adjusted without considering the sample size. Secondly, for research with particular focus on data smoothing, it allows time weight to be considered when estimating among various selected knots. Thirdly, apart from the time weight, the nature of the original data is retained and there are no extra functions or patterns to be put into the model.

In order to introduce CPPF approach, a review of cubic spline method is essential. The cubic spline method was originally used in mathematics and engineering (Ferguson, 1963). Mathematically, as a third order piecewise polynomial function, the cubic spline is used to smooth discrete points into a continuous curve. According to Rorres and Anton (1984), a cubic spline can be expressed mathematically in the following form:

$$S(x) = \begin{cases} s_1(x)\ if\ x_1 \le x \le x_2 \\ s_2(x)\ if\ x_2 \le x \le x_3 \\ \vdots \\ s_{n-1}(x)\ if\ x_{n-1} \le x \le x_n \end{cases} \tag{1}$$

where it is assumed that s_i is the third order polynomial function defined by

$$s_i(x) = a_i(x - x_i)^3 + b_i(x - x_i)^2 + c_i(x - x_i) + d_i \tag{2}$$

for $i = 1, 2, 3 \ldots\ldots n - 1$.

The first and second order derivative of Equation (1) defines the fundamentals of the process. These derivatives are given by

$$s_i'(x) = 3a_i(x - x_i)^2 + 2b_i(x - x_i) + c_i \tag{3}$$

$$s_i''(x) = 6a_i(x - x_i) + 2b_i \tag{4}$$

for $i = 1, 2, 3 \ldots\ldots n - 1$

The piecewise polynomial function has the following properties:

1. The piecewise polynomial function interpolates all data points.
2. The *S(x)* function is continuous in the interval $[x_1, x_n]$.
3. The first derivative of the *S(x)* function is continuous in the interval $[x_1, x_n]$.
4. The second derivative of the *S(x)* function is continuous in the interval $[x_1, x_n]$.

There are a number of studies which employ the cubic spline approach in financial modeling, mainly focusing on estimation of the term structure, autoregressive conditional duration (ACD) models and volatility of high frequency data. Vasicek and Fong (1982) and Jarrow et al. (2004) employ the cubic spline approach in estimating interest rate term structure. Engle and Russell (1998) proposed an ACD model which treats the time between transactions as a stochastic process. Within the ACD model, a daily seasonal factor is modelled by a cubic spline series. Moreover, in Zhang et al. (2001), a threshold autoregressive conditional duration (TACD) model is proposed and shown to be superior to the classic ACD model.[1] More recently, Taylor (2004a, b) and adopted cubic splines in their studies in the context of modelling the volatility of high frequency data via the ACD model. Aside from the above studies, there are a few studies that have also adopted cubic splines as a modelling tool, such as Engle and Rivera (1991) who estimated the density factor by using cubic splines in an autoregressive conditional heterosecdasticity (ARCH) context. Yu and Ruppert (2002) introduced the cubic spline approach into the estimation of the single index model. Evans and Speight (2010) employed the cubic spline approach to model intraday exchange rate volatility.

[1]The cubic spline approach was particularly used to approximate seasonal factors within the model.

2.3. Competitive basis of cubic spline

According to Eilers and Marx (2004), there are mainly two approaches used in cubic spline regression: the B-spline basis and truncated power functions basis.

For the B-spline basis approach, Eilers and Marx (2004) use equally-spaced knots and spline function *B*. Mathematically, the B-spline model can be written as

$$E(y) = \mu = B\alpha \tag{5}$$

and the objective function to be minimized is

$$Q_B = |y - B\alpha|^2 + \lambda|D_d\alpha|^2 \tag{6}$$

which λ is a non-negative parameter, and D_d is the *d*-th difference of α, it can be written as

$$D_d = \Delta^d\alpha \tag{7}$$

and

$$\Delta\alpha_j = \alpha_j - \alpha_{j-1} \tag{8}$$

$$\Delta^2\alpha_j = \alpha_j - 2\alpha_{j-1} + \alpha_{j-2} \tag{9}$$

and so on for higher orders. So the objective function of Q_B leads the B-spline model to

$$(B'B + \lambda D_d^{'} D_d)\hat{\alpha} = B'y \tag{10}$$

It can be seen from Equation (10) that when λ=0, it becomes the classic equation of linear regression.

For the truncated power functions basis, according to Ruppert et al. (2003), for a given asset *i*, column *j* and degree *p*, the truncated power function of *F* is written as

$$F_{ij} = (x_i - t_j)^P I(x_i > t_j) \tag{11}$$

where *I(u)* is an indicator function, it is 0 when $u < 0$ and 1 otherwise. The vector *t* contains the knots, and the knots are placed as quantiles of *x*. Consequently, the model for *E(y)* can be written as

$$E(y_i) = \sum_{k=0}^{P} \beta_k x_i^k + \sum_{j=1}^{n-1} F_{ij} b_j \tag{12}$$

And the objective function to be minimized is given by

$$Q_F = |y - \beta x - Fb|^2 + \kappa|b|^2 \tag{13}$$

The increasing of κ will increase the smoothness.

Eliers and Marx (2004) point out that both bases allow a mixed model approach, and the B-spline basis can be derived from the truncated power basis. They also show that the truncated power basis has bad numerical properties, and could cause discontinuities in estimation, while the B-spline basis approach has no such issue. However, according to Taylor (2004), the truncated power basis is employed in the spline-based periodical GARCH model on high frequency commodity future return data, which produces excellent smooth estimates. Therefore, in light of Taylor's (2004) study, the truncated power basis is employed in the CPPF approach in this paper, and the detail of the piecewise polynomial matrix used will be introduced in Section 4.[2]

2.4. The FFF approach

The advantages of the FFF approach are: firstly, in the context of normal and high frequency data, the macroeconomic news announcement effect has been filtered by the periodic pattern of the FFF, so there is no need to model the macroeconomic news announcement effect; secondly, the FFF approach creates a smooth pattern for volatility dynamics and changes; thirdly, the FFF approach is based on sound mathematics and the fit of the periodicity of financial data is widely agreed.

The FFF which was first proposed and refined by Gallant (1981, 1982 and 1984). This mathematical function, based on a Fourier series, was initially used to approximate the utility function and derive an appropriate expenditure system for the whole economy. Mathematically, it can be written as

$$\sum_{\alpha=1}^{A}\sum_{j=-J}^{J} a_{j\alpha} e^{ijk'_{\alpha}x} = \sum_{\alpha=1}^{A}\left\{u_{0\alpha} + 2\sum_{j=1}^{J}\left[u_{j\alpha}\, cos(jk'_{\alpha}x) - v_{j\alpha}\, sin(jk'_{\alpha}x)\right]\right\} \quad (14)$$

where

$$a_{j\alpha} = u_{j\alpha} + iv_{j\alpha}, \qquad \alpha = 1,2,3\ldots A, \qquad j = 0, \pm1, \pm2\ldots\pm$$

whereas i is defined as the imaginary unit, k is the order of the expansion, a_j is the coefficient given by

$$a_j = u_j + iv_j \quad (15)$$

Recently, the FFF was widely applied in two aspects of economics and finance: estimation of production and banking efficiency, and modeling high frequency volatility periodicity.

In the former aspect, Chung et al. (2001) and Huang and Wang (2001) both applied the FFF in estimating the scale and scope of the Asian banking industry. Huang and Wang (2004) expanded the FFF and applied it to panel data to estimate multiproduct banking efficiency. Featherstone and Cader (2005) employed the FFF in a Bayesian econometrics context to evaluate agricultural production. And Yu et al. (2007) adapted the FFF to estimate agricultural banking efficiency.

Within a volatility context, Andersen and Bollerslev (1998) introduced the FFF into high frequency data volatility modelling. In their study, under GARCH framework, the FFF was used to estimate an intraday periodicity component in order to capture volatility reactions to macroeconomic announcements. The FFF within their study has been simplified as

$$f(\theta,t,n) = \mu_0 + \sum_{k=1}^{D}\lambda_k \cdot I_k(t,n) + \sum_{p=1}^{P}\left(\delta_{c,p} \cdot cos\frac{p2\pi}{N}n + \delta_{s,p} \cdot sin\frac{p2\pi}{N}n\right) \quad (16)$$

where $I_k(t,n)$ is the indicator of event k during time interval n on day t, θ is the parameter vector to be estimated, and μ_0, λ_k, $\delta_{c,p}$ and $\delta_{s,p}$ are the fixed coefficients to be estimated (Andersen and Bollerslev, 1998). Moreover, Andersen et al. (2000) applied the FFF in the Japanese stock market, while Bollerslev

[2]The CPPF approach is derived from the cubic spline approach described above. However, we do not refer to CPPF as a spline because we allow for discontinuities at each knot.

et al. (2000) employed the FFF in analyzing the US bond market. More recently, Evans and Speight (2010) further adopted the FFF in the foreign exchange market.

Although the cubic spline approach and FFF approach are widely used in the financial literature, the majority of studies use high frequency data in a financial derivatives market, banking industry or foreign exchange market. There is very few study using both approaches to estimate the downside and upside components of risk in stock markets. This paper employs the CPPF and FFF as tools, with various numbers of knots and the AIC used to uncover the best fit of beta, downside beta and upside beta estimates of monthly data with a long span in the US stock market and to improve the goodness of fit of asset pricing models.

3. Data

As an experiment for CPPF and FFF models, data used in this paper are taken from Center for Research in Security Price (CRSP) database. This paper focuses on the ordinary common stocks listed on the New York Stock Exchange (NYSE), American Stock Exchange (AMEX) and NASDAQ measured on a monthly frequency from January 1960 to December 2010.[3] American depositary receipts (ADR), real estate investment trust (REIT), closed-end funds, foreign firms and other securities which do not have a CRSP share code of 10 or 11 are excluded from the sample. Each stock is required to have at least 5 years of consecutive monthly adjusted return observations with at most 5 missing observations. The return of each stock is adjusted for stock splits, mergers and acquisitions, and dividends (dividends are subtracted from stock prices for adjustment), giving 13,557 stocks. The value-weighted return of all listed stocks is taken as a measure of the market portfolio, and the one month Treasury bill rate represents the risk free rate.[4] A summary of stocks selected is shown in Table A.1 in Appendix.

4. Econometrics models and methods

All econometric models in this paper are based on the CAPM proposed by Sharpe (1964) shown as Equation (17)

$$R_{it} - R_{ft} = \alpha_{it} + \beta_{it} \cdot (R_{Mt} - R_{ft}) + \varepsilon_{it} \tag{17}$$

where R_{it} is the rate of return of stock i at time t, R_{ft} is the risk free rate at time t, α_{it} is the constant at time t, β_{it} is the coefficient to be estimated and represents the co-movement between stock i and the market at time t, R_{Mt} is the rate of return of market portfolio at time t, $(R_{Mt} - R_{ft})$ is the excess return of the market portfolio, and ε_{it} is the error term of stock i at time t. It is this equation that will be estimated using the CPPF and FFF models.

For convenience, we define

$$xR_{it} = R_{it} - R_{ft} \tag{18}$$

and

[3]The NASDAQ data are only available from January 1972.

[4]Using the same criteria Ang et al. (2007) adopted.

$$xR_{Mt} = R_{Mt} - R_{ft} \tag{19}$$

where xR_{it} is the excess rate of return of stock i at time t, and xR_{Mt} is the excess rate of return of the market portfolio at time t.

4.1. The CPPF model

In this section, the CPPF (with knots) model is described. By using the CPPF model, the excess rate of return on the market portfolio xR_M will be divided into different numbers of series depending on the number of knots selected, thus allowing the betas to vary over time.

Deciding the number of knots to use is an interesting tradeoff (Stone, 1986). If a small number of knots are chosen, the estimates will be over-smooth with less variability, and could also be biased. By contrast, if a high number of knots is selected, the bias can be avoided, however, it will also lead to a high variability of estimates in the fit and could result in overfitting. Eilers and Marx (1996) discovered that up to 4 to 5 knots is most appropriate for most applications, therefore, the number of knots selected for the CPPF model will vary from 0 to 5.

Placement of knots follows the quintile method proposed by Stone (1986). In his study, he found that placing knots according to the quintile point with respect to the total number of observations results in less bias than placing knots according to a fixed number of observations. Therefore, the knots are placed at the quintile points as follows:

Table 1. Placement points of knots.

Number of knots	0	1	2	3	4	5
Placement points		50%	33.3%	25%	20%	16.6%
			66.6%	50%	40%	33.30%
				75%	60%	50%
					80%	66.6%
						83.3%

The econometric models used here take advantage of the CPPF approach, and apply it to the classic market model. To estimate the beta coefficient of each stock, the model, in matrix terms, can be written as

$$\mathbf{XR_i} = \boldsymbol{\alpha}_\mathbf{i} + (\mathbf{XR_M} \odot \mathbf{S_N})\cdot\mathbf{B_i} + \boldsymbol{\varepsilon}_\mathbf{i} \qquad N = 0, 1, 2, 3, 4, 5 \tag{20}$$

where $\mathbf{XR_i}$ is a (t × 1) column vector of excess returns of stock i, $\boldsymbol{\alpha}_\mathbf{i}$ is a t × 1 column vector, $\mathbf{XR_M}$ is a (t × 1) column vector of excess returns of the market portfolio, $\mathbf{S_N}$ is a (t × n) piecewise polynomial matrix with N representing the number of knots, ʘ is the element to element multiplication sign which results in ($\mathbf{XR_M} \odot \mathbf{S_N}$) becoming a (t × n) matrix,[5] $\mathbf{B_i}$ is the (n × 1) estimated beta column vector, and $\boldsymbol{\varepsilon}_\mathbf{i}$ is the (t × 1) column vector error term.

Specifically, the piecewise polynomial matrix, $\mathbf{S_N}$, varies along with the number of knots selected. When the CPPF has no knots, $\mathbf{S_0}$ can be expressed as:

[5]ʘ is conventionally used as an element to element multiplication sign when two matrices are in the same rank, we borrow it here for different rank matrices for the sake of simplicity.

$$S_0 = \begin{bmatrix} 1^0 & 1 & 1^2 & 1^3 \\ 2^0 & 2 & 2^2 & 2^3 \\ 3^0 & 3 & 3^2 & 3^3 \\ 4^0 & 4 & 4^2 & 4^3 \\ \vdots & \vdots & \vdots & \vdots \\ t^0 & t & t^2 & t^3 \end{bmatrix}. \tag{21}$$

Moreover, when one knot is selected, the knot will be placed at the 50% point of observations, with the $\mathbf{S_0}$ elements remaining in $\mathbf{S_1}$, plus new elements added in with elements valued 0 above the knot, therefore $\mathbf{S_1}$ can be written as

$$S_1 = \begin{bmatrix} 1^0 & 1 & 1^2 & 1^3 & 0 & 0 & 0 & 0 \\ 2^0 & 2 & 2^2 & 2^3 & 0 & 0 & 0 & 0 \\ 3^0 & 3 & 3^2 & 3^3 & 0 & 0 & 0 & 0 \\ 4^0 & 4 & 4^2 & 4^3 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \left(\frac{t}{2}\right)^0 & \left(\frac{t}{2}\right) & \left(\frac{t}{2}\right)^2 & \left(\frac{t}{2}\right)^3 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & 1^0 & 1^1 & 1^2 & 1^3 \\ \vdots & \vdots & \vdots & \vdots & 2^0 & 2^1 & 2^2 & 2^3 \\ \vdots & \vdots & \vdots & \vdots & 3^0 & 3^1 & 3^2 & 3^3 \\ \vdots & \vdots & \vdots & \vdots & 4^0 & 4^1 & 4^2 & 4^4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ t^0 & t & t^2 & t^3 & \left(\frac{t}{2}\right)^0 & \left(\frac{t}{2}\right) & \left(\frac{t}{2}\right)^2 & \left(\frac{t}{2}\right)^3 \end{bmatrix} \tag{22}$$

The expression for $\mathbf{S_N}$ with two knots (N = 2), three knots (N = 3), four knots (N = 4) and five knots (N = 5) can be found in Illustration A.1–A.4 in Appendix. It can be seen from the expression for $\mathbf{S_0}$ and $\mathbf{S_1}$ that as the number of knots increases, the number of columns in $\mathbf{S_N}$ will increase. More precisely, for every one extra knot placed, the number of columns in $\mathbf{S_N}$ will increase by 4, so the dimensions of $\mathbf{S_0}$, $\mathbf{S_1}$, $\mathbf{S_2}$, $\mathbf{S_3}$, $\mathbf{S_4}$ and $\mathbf{S_5}$ will be (t × 4), (t × 8), (t × 12), (t × 16), (t × 20) and (t × 24) respectively.

The OLS regression is applied to each stock to get the vector of beta estimates. Then, beta estimates for each stock at each point in time ($\mathbf{B_S}$ in vector form) can be calculated as follows:

$$\mathbf{B_S} = \mathbf{S_N} \cdot \mathbf{B_i} \tag{23}$$

It can be seen from Equation (14) that $\mathbf{B_S}$ is a product of a (t × n) matrix $\mathbf{S_N}$ and a (n × 1) estimated beta vector $\mathbf{B_i}$, therefore regardless of the number of knots placed in the function, the rank of $\mathbf{B_S}$ will always be t × 1. Since the number of knots varies from 0 to 5, there will be 6 possible $\mathbf{B_i}$ vectors for each stock corresponding to the number of knots used. In order to find the best fit for each stock, we follow Eilers and Marx's (1996) study, and use the AIC.[6] The AIC can be expressed as

$$AIC = -2\,ln(L) + 2k \tag{24}$$

where L is the maximum value of the likelihood function, and k is the number of parameters within the model. There are discontinuities at knots points, while the fitted values are smooth between each knot.

[6]The Schwarz Information Criteria (SIC) can also be used to determine the appropriate number of knots, in this paper, AIC is chosen instead of SIC since the SIC shows less tolerance when the number of parameters in the model is high, according to Eilers and Marx's study (1996).

These discontinuities are due to the column of ones in the piecewise polynomial matrix, and in these cases, we let the data to decide the appropriate value of the estimates.

In order to calculate the downside and upside beta estimates by using the CPPF model, the same logic is used with Equation (11) modified. Referring to Ang et al. (2006), the downside beta and upside beta are calculated as

$$\beta^{-} = \frac{cov(xR_i, xR_M | xR_M < \overline{xR_M})}{var(xR_M | xR_M < \overline{xR_M})} \tag{25}$$

and

$$\beta^{+} = \frac{cov(xR_i, xR_M | xR_M \geq \overline{xR_M})}{var(xR_M | xR_M \geq \overline{xR_M})} \tag{26}$$

where $\overline{xR_M}$ is the average market excess return over the sample period of the stock, and previous notations hold. In light of Ang et al. (2006), dummy variables (vectors) D_{1i} and D_{2i} are created and employed for each stock. D_1 and D_2 (with time subscript t) can be expressed as

$$D_{1i} = 1 \text{ and } D_{2i} = 0 \text{ if } xR_{M,t} < \overline{xR_M} \tag{27}$$

and

$$D_{1i} = 0 \text{ and } D_{2i} = 1 \text{ if } xR_{M,t} \geq \overline{xR_M} \tag{28}$$

It can be seen from Equations (18) and (19) that $D_{1i} = 1$ and $D_{2i} = 0$ if the market excess return at time t is below the average market excess return, while $D_{1i} = 0$ and $D_{2i} = 1$ if the market excess return at time t is above the average market excess return. Then two more variables are created as follows:

$$D_{1i}xR_M = D_{1i} \odot xR_M \tag{29}$$

$$D_{2i}xR_M = D_{2i} \odot xR_M \tag{30}$$

It can be seen from Equations (20) and (21) that two new variables $D_{1i}xR_M$ and $D_{2i}xR_M$ are the element to element products of dummy variables of stock i and the corresponding excess market return over the sample period of the stock. For the former, observations are excess market returns if they are below the average excess market return over the sample period, and 0 otherwise. For the latter, observations are excess market returns if they are above the average excess market return over the sample period, and 0 otherwise.

The econometric model used to estimate downside and upside betas for each stock, in matrix form, can be written as

$$\mathbf{XR_i} = \mathbf{D_{1i}} + \mathbf{D_{2i}} + (\mathbf{D_{1i}XR_M} \odot \mathbf{S_N}) \cdot \mathbf{B_i^-} + (\mathbf{D_{2i}XR_M} \odot \mathbf{S_N}) \cdot \mathbf{B_i^+} + \boldsymbol{\varepsilon_i} \tag{31}$$

where $N = \{0, 1, 2, 3, 4, 5\}$, $\mathbf{B_i^-}$ and $\mathbf{B_i^+}$ are the (n × 1) estimated downside and upside beta estimate column vectors. Since the number of knots varies from 0 to 5, there will be 6 pairs of $\mathbf{B_i^-}$ and $\mathbf{B_i^+}$ vectors. The best downside and upside beta estimates as determined by the AIC for a stock at each point of time, ($\mathbf{B_S^{-*}}$ and $\mathbf{B_S^{+*}}$ in vector form) can be conducted as follows

$$\mathbf{B_S^{-*}} = \mathbf{S_N} \cdot \mathbf{B_i^-} \tag{32}$$

and

$$\mathbf{B}_\mathbf{S}^{+*} = \mathrm{S_N}\cdot\mathbf{B}_\mathbf{i}^{+} \quad (33)$$

As mentioned in the previous paragraph, regardless of the number of knots placed in the function, the dimensions of $\mathbf{B}_\mathbf{S}^{-*}$ and $\mathbf{B}_\mathbf{S}^{+*}$ will always be t × 1. Both downside and upside betas can be interpreted in an analogous manner to classic beta regarding to downside and upside market.

4.2. The FFF model

In this section, the FFF model is described in detail. In light of Andersen and Bollerslev (1998), Andersen et al. (2000), Bollerslev et al. (2000), and Evans and Speight (2010), the econometric model with the FFF specification employed in this paper is defined as

$$xR_{it} = \alpha_{it} + \sum_{p=1}^{P}[\beta_{cos,p} \cdot (cos\frac{p2\pi}{N}n \cdot xR_{Mt}) + \beta_{sin,p} \cdot (sin\frac{p2\pi}{N}n \cdot xR_{Mt})] + \varepsilon_{it} \quad (34)$$

where α_{it} is the constant, $\beta_{cos,p}$ and $\beta_{sin,p}$ are the coefficients to be estimated for stock i, N is the total number of observations of stock i, n is the order of observations with $n = \{1, 2, 3\ldots t\}$, ε_{it} is the error term of stock i at time t, and p is the order of the FFF. The order of the FFF can vary from 1 to infinity. However, in order to provide efficient and unbiased estimates, according to previous studies, we chose up to 4.[7] In this paper, the order from 1 to 4 is selected to examine and discover the best fit of the estimates.

The OLS regression is applied to each stock to get the $\beta_{cos,p}$ and $\beta_{sin,p}$ estimates. The AIC will then be used for each regression. Since an order of 1 to 4 is examined, there are 4 AICs for each stock. Taking advantage of the nature of the AIC, the regression that produces the least AIC will indicate the optimal fit. To calculate the best estimate for a stock at each point of time, the AIC supported estimates of $\beta_{cos,p}$ and $\beta_{sin,p}$ for each stock are used to get the best estimates at each point of time, specifically,

$$\beta_F^* = \sum_{p=1}^{P}(\beta_{cos,p} \cdot cos\frac{p2\pi}{N}n + \beta_{sin,p} \cdot sin\frac{p2\pi}{N}n) \quad (35)$$

In order to calculate the downside and upside beta estimates using the above FFF model, the same logic is followed as in the CPPF case. The same variables created in Equations (20) and (21) are created and employed for each stock in the new FFF model and the new market model is given by

$$xR_{it} = D_{1i} + D_{2i} + \sum_{p=1}^{P}[\beta_{cos,p}^{-} \cdot (cos\frac{p2\pi}{N}n \cdot D_{1i}xR_{Mt}) + \beta_{sin,p}^{-} \cdot (sin\frac{p2\pi}{N}n \cdot D_{1i}xR_{Mt})]$$

$$+ \sum_{p=1}^{P}[\beta_{cos,p}^{+} \cdot (cos\frac{p2\pi}{N}n \cdot D_{2i}xR_{Mt}) + \beta_{sin,p}^{+} \cdot (sin\frac{p2\pi}{N}n \cdot D_{2i}xR_{Mt})] + \varepsilon_{it} \quad (36)$$

where $\beta_{cos,p}^{-}$ and $\beta_{sin,p}^{-}$ are the downside market coefficients to be estimated for stock i, and $\beta_{cos,p}^{+}$ and $\beta_{sin,p}^{+}$ are the upside market coefficients to be estimated for stock i, and previous notations hold. As in Equation (22), there is no conventional constant term in the model, rather, full set of dummy variables are instead used. Since the order of the FFF examined varies from 1 to 4, there will be 4 groups of $\beta_{cos,p}^{-}$, $\beta_{sin,p}^{-}$, $\beta_{cos,p}^{+}$ and $\beta_{sin,p}^{+}$ for each stock. For each group an AIC value is calculated and the lowest value indicates the best fit group of $\beta_{cos,p}^{-}$, $\beta_{sin,p}^{-}$, $\beta_{cos,p}^{+}$ and $\beta_{sin,p}^{+}$.

[7]See Andersen and Bollerslev (1998), and Evans and Speight (2010) for similar assumptions.

Furthermore, the best downside and upside beta estimates for each point in time are given by

$$\beta_F^{-*} = \sum_{p=1}^{P}(\beta_{cos,p}^{-} \cdot cos\frac{p2\pi}{N}n + \beta_{cos,p}^{-} \cdot sin\frac{p2\pi}{N}n) \quad (37)$$

and

$$\beta_F^{+*} = \sum_{p=1}^{P}(\beta_{cos,p}^{+} \cdot cos\frac{p2\pi}{N}n + \beta_{cos,p}^{+} \cdot sin\frac{p2\pi}{N}n) \quad (38)$$

and all previous notations hold. In the next section, the empirical results will be demonstrated and analyzed in detail.

5. Empirical results

As the method explained in the previous section, the best fits for both the CPPF and FFF models are obtained. In order to illustrate the results in a clearer way, having the best fitted estimates β_S^*, β_S^{-*} and β_S^{+*} for the CPPF model and β_F^*, β_F^{-*} and β_F^{+*} for the FFF model, the number of stocks with corresponding numbers of knots or orders and the percentage of the whole sample are shown in Table 2 and Table 3, respectively.

For the CPPF model, it can be seen from Table 2 that for 9409 stocks, best estimates are obtained when no knots are used. For other knot values, the number of stocks decreases. Typically, when 5 knots are used, just 563 stocks produced the best estimates. Similar results are obtained when the downside and upside beta estimates are constructed.

Table 2. Stocks with corresponding knots to construct β_S^*, β_S^{-*} and β_S^{+*}.

Knots		0	1	2	3	4	5
β_S^*	Number of Stocks	9409	1455	861	655	614	563
	Percentage to Whole sample	69.40%	10.73%	6.35%	4.83%	4.53%	4.15%
$\beta_S^{-*}$and β_S^{+*}	Number of Stocks	9399	903	483	416	729	1627
	Percentage to Whole sample	69.33%	6.66%	3.56%	3.07%	5.38%	12.00%

Note: This table reports the number and percentage of stocks with different knots to construct the best fit estimates of CPPF model.

Table 3. Stocks with corresponding orders to construct β_F^*, β_F^{-*} and β_F^{+*}.

Order		1	2	3	4
β_F^*	Number of Stocks	6204	2746	2099	2508
	Percentage to Whole sample	45.76%	20.26%	15.48%	18.50%
$\beta_F^{-*}$and β_F^{+*}	Number of Stocks	8377	2293	1429	1458
	Percentage to Whole sample	61.79%	16.91%	10.54%	10.75%

Note: This table reports the number and percentage of stocks in different orders to construct the best fit estimates of the FFF model.

For the FFF model, it is clear from Table 3 that to construct β_F^*, 6204 stocks have an order of 1. Orders 2, 3 and 4 are generally selected less often. This pattern is even more obvious when constructing

β_F^{-*} and β_F^{+*}, 8377 stocks produce the best estimates with order 1, 2293 stocks with order 2, and the number of stocks with order 3 and 4 are 1429 and 1458, respectively.

Furthermore, the relationships among stock returns and corresponding beta, downside beta and upside beta estimates for the CPPF model and the FFF models are examined. In order to uncover the relationship in a cross-sectional fashion, stocks at each point of time are cross-sectionally assigned into five portfolios according to the value of the estimate. Since the beta, upside beta and downside beta estimates for both models are not independent of each other, to distinguish the effects among them, the relative upside beta, denoted by *($\beta^+-\beta$)* and relative downside beta denoted by *($\beta^--\beta$)* are considered. To sort the portfolio, at each point of time, all stocks are sorted into five quintiles according to the value of the beta estimate. Therefore, portfolio 1 contains stocks with the lowest 20% of estimates, portfolio 2 contains stocks with the second lowest 20% of estimates, and accordingly. When stocks are sorted into 5 portfolios at each point of time,[8] the equally weighted average of the estimate for each portfolio and the corresponding average annualized stock returns are calculated. The results of both models are summarized in Table 4 and Table 5, respectively.

It can be seen from the CPPF model results in Table 4 that when sorting by β_S^*, portfolio 1 has an average β_S^* of −0.24 while on the other hand, portfolio 5 has an average β_S^* of 2.68. Consistent with the literature, the average annualized realized rates of return of each portfolio show an ascending order as the average β_S^* increases, portfolio 1 yields a return of 1.53% while portfolio 5 shows a return of 24.84%. The average β_S^{-*} and β_S^{+*} values of each portfolio follow the same trend as β_S^*, an average β_S^{-*} is 0.47 in portfolio 1 and increases to 2.18 in portfolio 5. Similarly, the average β_S^{+*} is −0.21 in portfolio 1 and increases to 2.4 in portfolio 5.

Interestingly, a different pattern in returns is demonstrated when stocks are sorted by β_S^{-*}. It is clear that average returns demonstrate a reversed trend while average β_S^* shows the same ascending trend from portfolio 1 to portfolio 5 along with the increase of β_S^{-*}. β_S^{-*} is −7.2 in portfolio 1 with an average β_S^* of 0.48 and an average return of 25%, while in portfolio 5, β_S^{-*} grows to 10.01 with an average β_S^* increasing to 1.86 and the average return drops to −2.67%. The pattern of β_S^{+*} is generally increasing but with a subtle variation in that, it drops from 0.85 to 0.62 from portfolio 1 to portfolio 2, and then keeps growing to portfolio 5 ending up with a value of 1.23.

Notably, although β_S^* and β_S^{+*} still have increasing trends in this panel, the difference between values for portfolio 1 and portfolio 5 (1.38 and 0.39, respectively) are narrower than the ones in Panel 1 (2.92 and 2.62 respectively).

When stocks are sorted by β_S^{+*}, a similar pattern appears to those in Panel 1. It can be seen from Panel 3 that, β_S^{+*} is −3.87 in portfolio 1 with an average β_S^* of 0.42 and an average return of −11.93%, and in portfolio 5, β_S^{+*} grows to 5.94 with average β_S^* increasing to 2.11 and average returns increasing to 35.58%. The pattern of β_S^{+*} is also generally increasing but with a sudden drop from 1.32 to 0.91 between portfolio 1 to portfolio 2, and then keeps increasing to portfolio 5 and ends up with a value of 1.42. Compared to Panel 1, the spread of β_S^*, and β_S^{-*} between portfolio 5 and portfolio 1 is less, however the spread of β_S^{+*} and average return is much higher (9.81 and 47.51% in Panel 3, while 2.62 and 23.3% in Panel 1). It is clear that β_S^{+*} has the same positive impact on portfolio returns as β_S^*.

In order to examine how β_S^{-*} is driving the return not considering the impact of β_S^*, a new estimate ($\beta_S^{-*}-\beta_S^*$) is employed in the analysis. Using this estimate to sort portfolios could discover the

[8]Since monthly data are used in this paper, and the whole sample is from January 1960 to December 2010, there are 612 time points.

unique property of β_S^{-*} after controlling for β_S^*. When stocks are sorted by (β_S^{-*}−β_S^*), an unfamiliar pattern appears in Panel 4. From portfolio 1 to portfolio 5, all average returns, β_S^* and β_S^{+*} are in descending order while only β_S^{-*} increases from −6.6 to 9.49. Although in Panel 1, Panel 2 and Panel 3, β_S^{-*} are also in ascending order, the spread of average returns in Panel 4 is the highest and reaches −48.22%. So controlling for β_S^*, it can be seen that β_S^{-*} shows a negative relationship with portfolio returns and β_S^*.

Table 4. Relationships between stock returns and CPPF factor loadings.

Portfolio	Return	β_S^*	β_S^{-*}	β_S^{+*}	Portfolio	Return	β_S^*	β_S^{-*}	β_S^{+*}
Panel 1 Stocks Sorted by β_S^*					Panel 2 Stocks Sorted by β_S^{-*}				
1 Low	1.53%	−0.24	0.47	−0.21	1 Low	25.00%	0.48	−7.2	0.85
2	6.76%	0.64	0.80	0.32	2	11.08%	0.77	0.68	0.68
3	8.95%	0.99	1.23	0.84	3	10.53%	1.03	1.1	0.83
4	11.59%	1.41	1.54	1.25	4	9.73%	1.33	1.61	0.99
5 High	24.84%	2.68	2.18	2.4	5 High	−2.67%	1.86	10.01	1.23
High − Low	23.30%	2.92	1.72	2.62	High − Low	−27.67%	1.38	17.21	0.39
Panel 3 Stocks Sorted by β_S^{+*}					Panel 4 Stocks Sorted by (β_S^{-*}−β_S^*)				
1 Low	−11.93%	0.42	1.32	−3.87	1 Low	35.87%	1.64	−6.6	2.03
2	6.67%	0.71	0.91	0.34	2	14.74%	1.11	0.88	1.09
3	9.85%	0.96	1.21	0.81	3	9.72%	0.96	1.05	0.72
4	13.43%	1.28	1.35	1.37	4	5.72%	0.95	1.38	0.6
5 High	35.58%	2.11	1.42	5.94	5 High	−12.35%	0.83	9.49	0.16
High − Low	47.51%	1.69	0.1	9.81	High − Low	−48.22%	−0.81	16.09	−1.87
Panel 5 Stocks Sorted by (β_S^{+*}−β_S^*)					Panel 6 Stocks Sorted by (β_S^{-*}−β_S^{+*})				
1 Low	−6.16%	1.35	1.88	−3.38	1 Low	36.79%	1.43	−6.32	5.09
2	7.24%	1.03	1.37	0.52	2	14.15%	1.05	0.89	1.2
3	9.94%	0.96	1.16	0.84	3	10.07%	0.96	1.05	0.83
4	13.12%	1	1.1	1.21	4	6.43%	0.98	1.35	0.53
5 High	29.51%	1.14	0.71	5.4	5 High	−13.76%	1.05	9.23	−3.06
High − Low	35.67%	−0.2	−1.18	8.78	High − Low	−50.55%	−0.38	15.55	−8.15

Note: This table presents the relationship between excess stock returns and factor loading of the CPPF model. The column labeled "return" reports the average stock returns over a one month T-bill rate. "High-Low" reports the difference between portfolio 5 and portfolio 1. Notably, in a perfect market, the average value of β_S^* is assumed to be 1.

As with (β_S^{-*}−β_S^*), (β_S^{+*}−β_S^*) is employed to uncover the unique property of β_S^{+*} after controlling for β_S^*. It can be seen from Panel 5 that from portfolio 1 to portfolio 5, both average return and β_S^{+*} are in ascending order, starting at −6.16% and −3.38, increasing to 29.51% and 5.4, respectively. β_S^{-*} exhibits a descending trend for the first time within these panels. It drops from −1.88 to 0.71. A U-shaped pattern in β_S^* is apparent, it starts at 1.35 in portfolio 1 and drops to 0.96 in portfolio 3, but restores to 1.14 in portfolio 5.

In Panel 6, ($\beta_S^{-*} - \beta_S^{+*}$) is adopted to sort the portfolio in order to control β_S^{+*} from β_S^{-*} and for the sake of precision. In can be seen from Panel 6 that both average returns and β_S^{+*} are in descending orders, starting at 36.79% and 5.09 and dropping to 13.76% and −3.06 respectively, while β_S^{-*} exhibits an ascending trend increasing from -6.32 to 9.23. As in Panel 5, a U-shaped pattern appears in β_S^*, starting at 1.43 in portfolio 1 dropping to 0.96 in portfolio 3 and recovering to 1.05 in portfolio 5. Notably, the spread of returns in Panel 6 is the highest among all 6 panels at −50.55%.

Regarding the FFF model, it can be seen from Table 5 that when sorting by β_F^*, portfolio 1 has an average β_F^* of −1.19 while on the other hand, portfolio 5 shows an average β_F^* of 1.22. Again, consistent with the literature, the average annualized rates of return to each portfolio are presented in an ascending order with the β_F^*. Portfolio 1 yields a return of 3.54%, while portfolio 5 has a return of 22.26%. Average β_F^{-*} and β_F^{+*} for each portfolio follow the same trend as β_F^*, with an average β_F^{-*} of −0.74 in portfolio 1 and 0.76 in portfolio 5. Similarly, the average β_F^{+*} is −0.87 in portfolio 1 and increases to 0.95 in portfolio 5. A different pattern was demonstrated when stocks are sorted by β_F^{-*}. It is clear that average returns exhibit a reversed trend while average β_F^* shows the same ascending trend from portfolio 1 to portfolio 5 along with the increase of β_F^{-*}. β_F^{-*} is −1.32 in portfolio 1 with an average β_F^* of −0.67 and average return of 21.88%, while in portfolio 5, β_F^{-*} grows to 1.33, average β_F^* increases to 0.69 and the average return drops to −0.03%. The pattern of β_F^{+*} is consistent with β_F^{-*}, it starts at −0.31 in portfolio 1, and then keeps growing to portfolio 5 ending up with a value of 0.33. Notably, although β_F^* and β_F^{+*} still have an increasing trend in this panel, the difference between values in portfolio 1 and portfolio 5 (1.36 and 0.64, respectively) are narrower than the ones in Panel 1 (2.41 and 1.82, respectively).

When sorting by β_F^{+*}, a similar pattern to that in Panel 1 appears. It can be seen from Panel 3 that β_F^{+*} is −1.28 in portfolio 1 with an average β_F^* of −0.81 and an average return of −7.72%, and in portfolio 5, β_F^{+*} grows to 1.33 with an average β_F^* increasing to 0.86 and an average return increasing to 32.33%. The pattern of β_F^{-*} is consistent with β_F^{+*}, it starts at −0.29 in portfolio 1, and then keeps growing to portfolio 5 ending up with a value of 0.3. Compared to Panel 1, the spread of β_F^*, and β_F^{-*} between portfolio 5 and portfolio 1 are less. However, the spread of β_F^{+*} and average returns is much higher (2.61 and 40.05% in Panel 3, and 1.82 and 18.72% in Panel 1, respectively).

Table 5. Relationships between stock returns and the FFF factor loadings.

Portfolio	Return	$\beta_F{}^*$	$\beta_F{}^{-*}$	$\beta_F{}^{+*}$	Portfolio	Return	$\beta_F{}^*$	$\beta_F{}^{-*}$	$\beta_F{}^{+*}$
Panel 1 Stocks Sorted by $\beta_F{}^*$					Panel 2 Stocks Sorted by $\beta_F{}^{-*}$				
1 Low	3.54%	−1.19	−0.74	−0.87	1 Low	21.88%	−0.67	−1.32	−0.31
2	7.51%	−0.37	−0.26	−0.28	2	13.09%	−0.24	−0.38	−0.12
3	9.11%	−0.01	−0.01	−0.03	3	10.46%	−0.01	−0.01	−0.01
4	11.24%	0.36	0.24	0.23	4	8.22%	0.23	0.37	0.11
5 High	22.26%	1.22	0.76	0.95	5 High	−0.03%	0.69	1.33	0.33
High − Low	18.72%	2.41	1.49	1.82	High − Low	−21.91%	1.36	2.65	0.64
Panel 3 Stocks Sorted by $\beta_F{}^{+*}$					Panel 4 Stocks Sorted by $(\beta_F{}^{-*}-\beta_F{}^*)$				
1 Low	−7.72%	−0.81	−0.29	−1.28	1 Low	33.18%	0.52	−0.71	0.65
2	5.02%	−0.27	−0.13	−0.36	2	16.29%	0.11	−0.18	0.19
3	9.44%	−0.02	−0.01	−0.02	3	10.34%	−0.02	−0.02	−0.01
4	14.58%	0.24	0.13	0.33	4	4.66%	−0.14	0.17	−0.21
5 High	32.33%	0.86	0.3	1.33	5 High	−10.84%	−0.48	0.74	−0.61
High − Low	40.05%	1.67	0.59	2.61	High − Low	−44.02%	−1	1.45	−1.27
Panel 5 Stocks Sorted by $(\beta_F{}^{+*}-\beta_F{}^*)$					Panel 6 Stocks Sorted by $(\beta_F{}^{-*}-\beta_F{}^{+*})$				
1 Low	−4.27%	0.36	0.57	−0.63	1 Low	36.12%	0.17	−0.81	0.85
2	5.48%	0.12	0.21	−0.14	2	16.44%	−0.02	−0.24	0.17
3	9.74%	0	−0.01	−0.01	3	10.15%	−0.02	−0.01	−0.02
4	14.99%	−0.14	−0.23	0.12	4	4.34%	−0.02	0.22	−0.2
5 High	27.69%	−0.34	−0.54	0.66	5 High	−13.41%	−0.12	0.85	−0.81
High − Low	31.95%	−0.7	−1.1	1.29	High − Low	−49.53%	−0.3	1.66	−1.66

Note: This table presents the relationship between excess stock returns and factor loading of the FFF model. The column labeled "return" reports the average stock returns over one month T-bill rate. "High-Low" reports the difference between portfolio 5 and portfolio 1. Notably, in a perfect market, the average value of $\beta_F{}^*$ depends on the average value of the intercept.

As with the CPPF model, we consider $(\beta_F^{-*}-\ \beta_F^*)$ in the analysis. Using this estimate to sort portfolios could uncover further properties of β_F^{-*} after controlling for β_F^*. When stocks are sorted by $(\beta_F^{-*}-\beta_F^*)$, an unfamiliar pattern appears in Panel 4. From portfolio 1 to portfolio 5, average returns, β_F^* and β_F^{+*} all decrease only with β_F^{-*} increasing from −0.71 to 0.74. As in Panel 1, Panel 2 and Panel 3, β_F^{-*} are also in ascending order, the spread of average returns in Panel 4 is highest and reaches −44.02%. Referring back to when stocks are sorted by $(\beta_S^{-*}-\beta_S^*)$, a similar pattern appears.

Therefore, it can be concluded that when controlling for the effect of beta, the relative downside beta estimates of both models have a negative relationship with portfolio returns, which can be interpreted as stocks tend to suffer a loss if they have large downside betas.

When $(\beta_F^{+*} - \beta_F^{*})$ is employed (to uncover the unique property of β_F^{+*} after controlling for β_F^{*}), it can be seen from Panel 5 that from portfolio 1 to portfolio 5, both average returns and β_F^{+*} increase, starting at −6.16% and −3.38% and increasing to 29.51% and 5.4, respectively, while both β_F^{*} and β_F^{-*} decrease from 0.36 and 0.57 to −0.34 and −0.54, respectively.

Also in Panel 6, $(\beta_F^{-*} - \beta_F^{+*})$ is adopted to sort the portfolio to control β_F^{+*} from β_F^{-*} and for the sake of precision. It can be seen from this panel that the same pattern as in Panel 4 appears. From portfolio 1 to portfolio 5, average returns, β_F^{*} and β_F^{+*} all decrease while only β_F^{-*} increases from −0.81 to 0.85. Notably, the spread of returns in Panel 6 is the highest among all 6 panels at −49.53%.

To sum up, the results of the CPPF and the FFF models, β_S^{*} and β_F^{*} as classic risk estimates, still have a clear impact stock returns. Specifically, when stocks are sorted by β_S^{*} and β_F^{*}, average returns follow exactly the same increasing trend with the β_S^{*} and β_F^{*} presented, even the portfolio return is inversely related to the market return. More importantly, it can be seen from these panels that β_S^{-*}, β_S^{+*}, β_F^{-*} and β_F^{+*} do have an impact on stock returns. When stocks are sorted by β_S^{-*}, $(\beta_S^{-*} - \beta_S^{*})$, β_F^{-*} and $(\beta_F^{-*} - \beta_F^{*})$, clearly, downside related estimates have a negative relationship with the realized returns, this is even more obvious in Panel 6 when stocks are sorted by $(\beta_S^{-*} - \beta_S^{+*})$ and $(\beta_F^{-*} - \beta_F^{+*})$, while when stocks are sorted by β_S^{+*}, $(\beta_S^{+*} - \beta_S^{*})$, β_F^{+*} and $(\beta_F^{+*} - \beta_F^{*})$, positive relationships appear between upside related estimates and realized returns. Moreover, the classic estimates β_S^{*} and β_F^{*} appear to have a similar impact as the upside related estimates β_S^{+*}, $(\beta_S^{+*} - \beta_S^{*})$, β_F^{+*} and $(\beta_F^{+*} - \beta_F^{*})$. To rationalize that, when downside beta is calculated, the return of the market portfolio is below the average, and very likely to be negative. The stock expected excess return is the product of beta and excess returns to the market portfolio, so when stocks are sorted by downside beta into portfolios, the larger the downside beta, the lower the return. In addition to that, the panic on the falling market of investors' could also be a reason for aggravating the negative returns.

6. Fama-Macbeth regression

In this section, in order to uncover the premia of β_S^{*}, β_S^{-*}, β_S^{+*}, β_F^{*}, β_F^{-*} and β_F^{+*} on stock returns from a cross-sectional point of view, a series of Fama-Macbeth regressions are performed which employ various combinations of the above estimates as independent variables.

For the purposes of comparison and to demonstrate the importance of placing an appropriate number of knots for the CPPF model and choosing an appropriate order for the FFF model, additional variables β_{S0}, β_{S0}^{-}, β_{S0}^{+}, β_{F1}, β_{F1}^{-} and β_{F1}^{+} are introduced. For β_{S0}, β_{S0}^{-} and β_{S0}^{+}, they are the beta estimate, downside beta estimate and upside beta estimate, respectively, for each stock at each point in time estimated with the CPPF model without placing a knot. However for β_{F1}, β_{F1}^{-} and β_{F1}^{+}, they are the beta estimate, downside beta estimate and upside beta estimate, respectively, for each stock at each point in time estimated with the FFF model with order 1.

Table 6. Correlations of factor loadings without knot and in order one.

	β_{S0}	β_{S0}^{-}	β_{S0}^{+}	β_{F1}	β_{F1}^{-}	β_{F1}^{+}
β_{S0}	1.0000					
β_{S0}^{-}	0.3363	1.0000				
β_{S0}^{+}	0.4326	0.0365	1.0000			
β_{F1}	0.4397	0.1345	0.1770	1.0000		
β_{F1}^{-}	0.3209	0.2611	0.0458	0.7331	1.0000	
β_{F1}^{+}	0.4620	0.0336	0.2737	0.8093	0.2886	1.0000

Note: This table reports the correlation coefficients between factor loadings of both the CPPF model with zero knots and the FFF model in order one. To avoid unnecessary repetition, only the lower triangle of the matrix is shown.

The correlation coefficient matrices for groups of variables are presented in Table 6 and Table 7. It can be seen from Table 6 that β_{F1} is highly correlated with both β_{F1}^{-} and β_{F1}^{+} (the correlation coefficients are 0.7331 and 0.8093, respectively). The other pairs of variables are correlated with each other to some extent, but not as highly as the two mentioned pairs (above 0.5), for instance, β_{S0} and β_{F1}^{+} exhibits the highest correlation with a coefficient of 0.462 after the two peak values. Therefore, in the following Fama-Macbeth regression, β_{F1} will not appear in the same regression with β_{F1}^{-} or β_{F1}^{+}, and the other variables will form different combinations of independent variables.

It is clear from Table 7 that, as in Table 6, high correlations appear between the FFF based estimates, with β_{F}^{*} highly correlated with β_{F}^{-*} and β_{F}^{+*} (with correlation coefficients of 0.5239 and 0.6689, respectively). The remaining variables exhibit a weaker correlation with each other. Thus in the following Fama-Macbeth regression, β_{F}^{*} will not appear in the same regression with β_{F}^{-*} or β_{F}^{+*}, and the other variables will form different combinations of independent variables. Notably, in Table 8, for both the CPPF and the FFF models, the upside beta estimates β_{S}^{+*} and β_{F}^{+*} are negatively correlated with the downside beta estimates of the CPPF model β_{S}^{-*} with correlation coefficients of −0.0001 and −0.0018, respectively.

Table 7. Correlations of factor loading with knots and orders.

	β_{S}^{*}	β_{S}^{-*}	β_{S}^{+*}	β_{F}^{*}	β_{F}^{-*}	β_{F}^{+*}
β_{S}^{*}	1.0000					
β_{S}^{-*}	0.0130	1.0000				
β_{S}^{+*}	0.0237	-0.0001	1.0000			
β_{F}^{*}	0.3154	0.0029	0.0085	1.0000		
β_{F}^{-*}	0.2609	0.0082	0.0025	0.5239	1.0000	
β_{F}^{+*}	0.3099	-0.0018	0.0098	0.6689	0.1889	1.0000

Note: This table reports the correlation coefficients between factor loadings of both the CPPF model with appropriate number of knots and the FFF model in appropriate order, to avoid unnecessary repetition, only the lower triangle of the matrix is shown.

This phenomenon potentially shows that the downside beta estimates do have an opposite impact on stock returns compared to the upside beta estimates which complies with the conclusion made in the previous sections.

After deciding on the possible combinations of variables, the Fama-Macbeth regressions are performed on both groups of variables which are demonstrated in Table 8 and Table 9. Since the data are at monthly frequency from January 1960 to December 2010, there are 612 cross-sectional time points and 2,398,103 observations of each regression. Newey-West (1987) heterosecdasticity robust standard errors with 12 lags are employed to calculate the t-statistics and the adjusted R^2 values obtained from the cross-sectional regressions are provided.

As β_{F1} is excluded from the regression when β_{F1}^{-} or β_{F1}^{+} are employed, therefore, there are 11 possible combinations among β_{S0}, β_{S0}^{-}, β_{S0}^{+}, β_{F1}, β_{F1}^{-} and β_{F1}^{+} as independent variables. It can be seen from Table 8 that regression 1, 2 and 3 examine the impact of estimates of the CPPF model without a knot. Generally, these three regressions exhibit poor fit with the intercept in regression 1 and coefficients of β_{S0}^{-} and β_{S0}^{+} in regression 2 not significant at the 5% significance level and with adjusted R^2 values of 0.06 and 0.055, respectively. In regression 3, the coefficient on β_{S0}^{-} and the intercept are not significant even at the 10% significance level with an adjusted R^2 value of 0.098. Regression 4 and 5 examine the impact of estimates of the FFF model with order 1. Estimates of regression 4 and 5 are all significant at the 1% significance level, however both regressions present low adjusted R^2 values of 0.017 and 0.029, respectively.

The remaining regressions 6 to 11 employ variables from both the CPPF model and the FFF model to examine the impact of these variables on stock returns. It is clear from Table 8 that with the exception of the coefficient on β_{S0}^{-} and coefficients on β_{S0}^{-} and β_{S0}^{+} in regression 10, remaining estimates are all significant at the 5% level. Among the regressions in Table 8, regression 11 shows the highest adjusted R^2 value at 0.106 and it also contains the most variables. Notably, consistent with the literature, the estimated coefficients of β_{S0} and β_{F1} are always positive among regressions, and illustrate that the beta estimates for both models have a positive impact on stock returns. Moreover, the estimated coefficients on β_{F1}^{-} and β_{F1}^{+} are always significant at the 1% significance level, and their signs are constantly negative and positive, and show that the downside and upside risk estimates of the FFF model have negative and positive impacts on stock returns, respectively. However, the significance and sign of the estimated coefficients on β_{S0}^{-} and β_{S0}^{+} vary across regressions in Table 9, therefore, it is difficult to provide a definitive conclusion.

Table 9 shows the 11 possible combinations among β_{S}^{*}, β_{S}^{-*}, β_{S}^{+*}, β_{F}^{*}, β_{F}^{-*} and β_{F}^{+*}. It can be concluded from Table 9 that, similar to Table 8, regressions 1 to 3 examine the impact of estimates of the CPPF model with appropriate numbers of knots according to the AIC. Moreover, regressions 4 and 5 examine the impact of best estimates of the FFF model (according to the AIC). The remaining regressions 6 to 11 employ variables from both the best CPPF and FFF models to examine the impact of these variables on stock returns. Unlike Table 8, all estimated coefficients except the intercept in regression 1 are significant at the 1% significance level. These best fit estimates are all highly significant, and the beta coefficients are consistently positive, which is consistent with the classic literature (as in Table 8). Moreover, the estimated coefficients of downside and upside beta estimates show negative and positive signs, respectively, over all regressions, which is consistent with the conclusions made regarding Table 4 and Table 5. Furthermore, showing better fit than classic CAPM, among all 11 regressions in Table 9, regression 11 exhibits the highest adjusted R^2 value at 0.153.[9]

However, since β_{F}^{*} is excluded from regressions employed β_{F}^{-*} or β_{F}^{+*}, checking the alterative regression (regression 8) shows that employing β_{F}^{*} instead of β_{F}^{-*} and β_{F}^{+*}, generates the second highest adjusted R^2 among all regressions at 0.15. Therefore, it can be concluded from Table 9 that the variables produce a much higher adjusted R^2 value than the variables used in Table 8, thus indicating that placing appropriate numbers of knots in the CPPF model and selecting the appropriate order in the FFF model produces better beta estimates. Moreover, all the variables in regression 8 and regression 11 show significant effects on excess stock returns, while the regression that employs β_{F}^{-*} and β_{F}^{+*} outperforming the one that employs β_{F}^{*}.

[9]The outcome of CAMP is not shown due to limited space.

Table 8. Fama-Macbeth regressions of factor loadings restricted estimates.

	1	2	3	4	5	6	7	8	9	10	11
Intercept	0.00155	0.00743***	0.00154	0.00852***	0.00907***	0.00212	0.00705***	0.00213*	0.00345***	0.00801***	0.00251**
	[1.14]	[3.82]	[1.21]	[3.48]	[3.90]	[1.57]	[3.88]	[1.70]	[2.89]	[4.06]	[2.24]
β_{S0}	0.00663***		0.00824***			0.00614***		0.00768***	0.00525***		0.00626***
	[3.25]		[3.68]			[3.02]		[3.46]	[2.73]		[2.98]
β_{S0}^{-}		0.0000637	−0.000721				0.00187***	−0.000678	−0.0119***	−0.000103	0.00147***
		[0.08]	[−1.28]				[2.99]	[−1.20]	[−13.04]	[−0.15]	[2.95]
β_{S0}^{+}		0.000937*	−0.000977***				−0.000884**	−0.000983**	0.0115***	0.000520	−0.00205***
		[1.90]	[−2.13]				[−2.21]	[−2.14]	[14.41]	[1.14]	[−4.72]
β_{F1}				0.00542***		0.00132**		0.00141***		0.00509***	
				[3.35]		[2.50]		[2.66]		[3.77]	
β_{F1}^{-}					−0.0113***		−0.0126***				−0.0131***
					[−9.43]		[−12.50]				[−15.27]
β_{F1}^{+}					0.0148***		0.0156***				0.0127***
					[10.83]		[12.60]				[17.27]
Number of obs	2398103	2398103	2398103	2398103	2398103	2398103	2398103	2398103	2398103	2398103	2398103
Average R^2	0.055	0.060	0.098	0.017	0.029	0.056	0.077	0.100	0.065	0.069	0.106

Note: This table reports the results of the Fama-Macbeth regression of factor loadings without knot and order. The t-statistics in the square brackets are calculated by using Newey-West (1987) heteroscedastic robust standard error with 12 lags. * denotes significance at the 10% level, ** denotes significance at the 5% level and ***denotes significance at the 1% level.

Table 9. Fama-Macbeth regressions of factor loadings with appropriate knots and orders.

	1	2	3	4	5	6	7	8	9	10	11
Intercept	0.00186	0.00898***	0.00538***	0.00826***	0.00943***	0.00279**	0.00965***	0.00618***	0.00369***	0.00947***	0.00623***
	[1.52]	[7.05]	[5.04]	[3.56]	[4.42]	[2.19]	[7.54]	[5.76]	[3.38]	[7.14]	[6.23]
β_S^*	0.00610***		0.00448***			0.00481***		0.00383***	0.00490***		0.00384***
	[3.42]		[3.82]			[3.00]		[3.54]	[3.20]		[3.42]
β_S^{-*}		−0.0120***	−0.00948***				−0.0106***	−0.00944***	−0.00994***		−0.00850***
		[−9.82]	[−11.19]				[−10.25]	[−11.19]	[−12.94]		[−11.30]
β_S^{+*}		0.0134***	0.00911***				0.0117***	0.00890***	0.0107***		0.00823***
		[14.47]	[14.41]				[15.25]	[14.63]	[15.74]		[14.44]
β_F^*				0.00673***		0.00334***		0.00238***		0.00342***	
				[4.16]		[5.67]		[4.73]		[3.91]	
β_F^{-*}					−0.0113***		−0.00419***			−0.0111***	−0.00485***
					[−9.13]		[−8.21]			[−10.53]	[−13.74]
β_F^{+*}					0.0155***		0.00647***			0.0121***	0.00586***
					[13.00]		[11.34]			[15.64]	[13.51]
Number of obs	2398103	2398103	2398103	2398103	2398103	2398103	2398103	2398103	2398103	2398103	2398103
Adjusted R^2	0.092	0.13	0.147	0.039	0.055	0.096	0.138	0.150	0.11	0.137	0.153

Note: This table reports the results of the Fama-Macbeth regression of factor loadings with appropriate knots and orders on stock excess returns. The t-statistics in the square brackets are calculated by using the Newey-West (1987) heteroscedastic robust standard error with 12 lags. * denotes significance at the 10% level, ** denotes significance at the 5% level and ***denotes significance at the 1% level.

7. Conclusions

It can be concluded from this paper that the beta, upside beta and downside beta estimates produced by the CPPF model and the FFF model do have a significant impact on cross-sectional stock returns. The beta estimates, whose role has been doubted in the literature for several decades, are significant in driving the stock returns for both models. Moreover, the downside and upside beta estimates of both models demonstrate reversed impacts on stock returns. The reason for that is when downside beta is calculated, the return of the market portfolio is below the average, and likely to be negative. The expected excess stock return is the product of beta and excess returns to the market portfolio, so when stocks are sorted by downside beta into portfolios, the larger downside beta, the lower the return, and vice versa. The former ones show negative impacts on stock returns, while the latter ones, consistent with the beta estimates, have positive effects (both are significant). For stocks with negative downside beta, they are inversely related with downside risk and more desirable in a downside market, therefore positive returns are rewarded. Moreover, placing the appropriate number of knots in the CPPF model and selecting the correct order of the FFF model are crucial procedures to generate the best fit estimates according to the AIC. It has been shown in this paper that estimates with the appropriate number of knots (or order) deliver more significant impacts on stock returns within the cross-sectional return regressions with respect to those based on non-optimal knots or orders. Furthermore, in order to avoid potential multicollinearity, beta estimates based on the FFF model β_F^* can be treated as an alternative variable of downside and upside beta estimates (β_F^{-*} and β_F^{+*}, respectively). However, employing β_F^{-*} and β_F^{+*} in the regression produces higher adjusted R^2 values than employing β_F^*.

The findings of this paper fill the gap in literature, specifically on both investigating and pricing the time variation and asymmetric nature of systematic risk. The methods and models proposed in this paper embed advanced mathematical techniques of data smoothing and widen the current options of asset pricing models. The application of proposed models provides high degree of explanatory power to capture and price risk in stock market. Nonetheless, as an experiment, there are certain limitations of this paper, e.g. the number of knots and orders are determined relatively arbitrarily, and the sample could be extended.

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Conflict of interest

The authors declare no conflict of interest.

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