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Research article

Existence and limit behavior of bump solutions to the critical Schrödinger-Bopp-Podolsky system involving the p-Laplacian in $\mathbb{R}^3$

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Abstract: This paper is concerned with a critical Schrödinger-Bopp-Podolsky system involving the p-Laplacian in $\mathbb{R}^3$. The p-Laplacian operator is defined as $\Delta_p u := \mathrm{div}(|\nabla u|^{p-2}\nabla u)$, where $\frac{12}{7} < p \le \frac{12}{5}$, η and α are positive parameters, the nonnegative potential $Z(x)$ admits a bounded potential well $\Omega = Z^{-1}(0)$, g is a continuous function, $a \ge 0$, and $p^* = \frac{3p}{3-p}$ is the Sobolev critical exponent. Under suitable assumptions on Z and Q, we established the existence and limiting behavior of bump solutions to the problem. More precisely, if the zero set of Z consists of multiple isolated components $\Omega_1, \cdots, \Omega_k$, each with a nonempty interior and a smooth boundary, then for sufficiently large $\eta > 0$, there exists a bump solution localized in a neighborhood of $\bigcup_{j\in\Gamma}\Omega_j$ for any nonempty subset $\Gamma \subseteq \{1, \cdots, k\}$. In addition, we proved that these bump solutions converge to a solution of the corresponding Schrödinger-Poisson-type system as $a \to 0$. Our results complement and extend related results in the existing literature.

Keywords: Schrödinger-Bopp-Podolsky system; potential well; p-Laplacian; critical exponent; multi-bump solutions; variational methods
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1. Introduction and main results

In this paper, we are interested in the existence and limit behavior of bump solutions to the following critical Schrödinger-Bopp-Podolsky system involving the p-Laplacian in $\mathbb{R}^3$:

$$\begin{cases} -\Delta_p u + (\eta Z(x) + 1)|u|^{p-2}u + Q(x)\phi u = \alpha g(u) + |u|^{p^*-2}u, & x \in \mathbb{R}^3, \\ -\Delta\phi + a^2\Delta^2\phi = Q(x)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $\Delta_p u := \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, $\frac{12}{7} < p \le \frac{12}{5}$, η, α, and a are positive parameters, and $p^* = \frac{3p}{3-p}$ is the critical Sobolev exponent according to the embedding theorem.

Throughout this paper, the following assumptions on Z, Q, and g are satisfied:

(Z_1) $Z(x) \in C(\mathbb{R}^3, \mathbb{R})$ with $Z(x) \geq 0$.

(Z_2) The potential well Ω, defined as the interior of $Z^{-1}(0)$, is a non-empty, bounded open region with a smooth boundary, where $Z^{-1}(0) = \overline{\Omega}$.

(Z_3) There exists $M_0 > 0$ such that

$$0 < M_0 \leq Z(x) + 1 \quad \text{for all } x \in \mathbb{R}^3.$$

(Z_4) Let Ω_j, $j \in \{1, \cdots, k\}$, be the k connected components of Ω, where $\Omega = \Omega_1 \cup \cdots \cup \Omega_k$, with $\text{dist}(\Omega_i, \Omega_j) > 0$ for any two distinct components.

(Q) $Q \in L^\infty(\mathbb{R}^3)$, $Q(x) \geq 0$.

(g_0) $g : \mathbb{R} \to \mathbb{R}$ is a continuous odd function.

(g_1) There exists $q \in (p, p^*)$ such that

$$\limsup_{|t|\to\infty} \frac{g(t)}{|t|^{q-1}} = m < +\infty.$$

(g_2) $\lim\limits_{|s|\to 0} \dfrac{g(s)}{|s|^{p-2}s} = 0.$

(g_3) The function $\dfrac{g(t)}{|t|^{p-2}t}$ is strictly increasing for $t \in (0, +\infty)$.

(g_4) Let $G(t) = \displaystyle\int_0^t g(\tau)d\tau$, and there exists $\theta \in (2p, p^*)$ such that $0 < \theta G(s) \leq sg(s)$ for all $s \geq 0$.

(g_5) There exist $\xi > 0$ and $\iota \in (2p, p^*)$ such that $g(t) \geq \xi t^{\iota-1}$ for all $t > 0$.

Example 1.1. *A typical example of a function satisfying* $(g_0) - (g_5)$ *is given by*

$$g(t) = |t|^{q-2}t \quad \textit{with } q \in (2p, p^*).$$

More generally, one can consider

$$g(t) = \sum_{i=1}^{m} a_i |t|^{q_i-2}t,$$

where $a_i > 0$ *and* $q_i \in (2p, p^*)$ *for all* $i = 1, \cdots, m$. *These functions satisfy* $(g_0) - (g_2)$ *with* $q = \max_i q_i$, (g_3) *and* (g_5) *with* $\xi = \min_i a_i$ *and* $\iota = \min_i q_i$, *and* (g_4) *with* $\theta = \min_i q_i$.

The coupling between the Schrödinger equation and the Bopp-Podolsky equation represents the Schrödinger-Bopp-Podolsky system, and it is used to depict the interaction of charged particles within the framework of Bopp-Podolsky generalized electrodynamics. By considering the standing wave solutions in a pure electrostatic field, we obtain a system similar to system (1.1). Specifically, Podolsky theory provides a second-order gauge-theoretic formulation of electromagnetism, derived independently by Bopp [1] and Podolsky-Schwed [2]. An extended discussion of this formulation can be found in [3–5].

When $a = 0$, system (1.1) reduces to the classical Schrödinger-Poisson system. This system is derived from a model describing the interaction between a charged particle and an electromagnetic field,

and is widely studied in many areas of mathematical physics, see Argaez and Melgaard [6]. For the case where $p = 2$, D'Aprile and Mugnai [7] considered the following Schrödinger-Poisson system in $\mathbb{R}^3$:

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(x,u), & x \in \mathbb{R}^3, \\ -\Delta \phi = u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.2}$$

and obtained the existence of radially symmetric solitary waves via variational methods. Yu and Chen [8] explored the existence and uniqueness of multi-bump solutions for nonlinear Schrödinger-Poisson systems by the penalization method and the local Pohozaev identity. Ji et al. [9] investigated the following critical Schrödinger-Poisson system with a magnetic field:

$$\begin{cases} -(\nabla + iB(x))^2 u + (\lambda V(x) + 1)u + \phi u = \alpha f(|u|^2)u + |u|^4 u, & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.3}$$

where $B \in L^2_{\text{loc}}(\mathbb{R}^3, \mathbb{R}^3)$ and f satisfies subcritical growth. They proved the existence and multiplicity of multi-bump solutions for system (1.3). Later, Ma and Song [10] extended the results of [9] to the p-Laplacian case, and considered the following Schrödinger-Poisson system involving the p-Laplacian:

$$\begin{cases} -\Delta_p u + (\lambda b(x) + 1)|u|^{p-2}u + \phi |u|^{s-2}u = g(u), & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^s, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.4}$$

where $1 < p < 3$, $\frac{p}{2} < s < p$. They proved the existence of positive multi-bump solutions via variational methods and the Morse iteration technique. More related results can be found in [11–19].

When $a \neq 0$, system (1.1) becomes a Schrödinger-Bopp-Podolsky system, which has attracted extensive research attention in recent years. d'Avenia and Siciliano [20] first established the existence and non-existence of solutions to the system

$$\begin{cases} -\Delta u + \omega u + q^2 \phi u = |u|^{p-2}u, & x \in \mathbb{R}^3, \\ -\Delta \phi + a^2 \Delta^2 \phi = 4\pi u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.5}$$

where $a, \omega > 0$. The results depend heavily on parameters q and p, and solutions converge to those of the classical Schrödinger-Poisson system as $a \to 0$ in the radial case. Inspired by the work of Li and Tang [21], the Schrödinger-Bopp-Podolsky system has been extensively studied via variational approaches. Xiao et al. [22] studied ground state solutions for a Schrödinger-Bopp-Podolsky system with convolution nonlinearity. Alves et al. [23] proved existence, non-existence, and multiplicity of solutions for a generalized singular system using critical point theory for non-differentiable functionals and sub-supersolution methods. Very recently, Damian and Siciliano [24] investigated the critical system in the semiclassical limit, and obtained concentrating solutions as the semiclassical parameter tends to zero. For more results, we refer to [25–27] and the references therein.

Recently, Wang et al. [28] considered the Schrödinger-Bopp-Podolsky system with a steep potential well:

$$\begin{cases} -\Delta u + (\lambda V(x) + V_0(x))\, u + K(x)\phi u = |u|^{p-2}u, & x \in \mathbb{R}^3, \\ -\Delta \phi + a^2 \Delta^2 \phi = K(x)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.6}$$

where $p \in (4, 6)$, $a \geq 0$, and λ is a parameter. They established the existence of multi-bump solutions for large λ via variational methods. Besides ground state and multi-bump solutions, normalized

solutions have also become a very active research direction. Ju and Sun [29] studied normalized solutions with general mass-subcritical nonlinearities. Wei et al. [30] further extended the result to the doubly critical growth case, which is the first work on normalized solutions for critical Schrödinger-Bopp-Podolsky systems.

We point out that most existing works focus on the subcritical growth case, except [24], which deals with critical growth in the semiclassical regime for $p = 2$. The subcritical p-Laplacian Schrödinger-Bopp-Podolsky system has been studied in [20, 28], while the existence of multi-bump solutions for the p-Laplacian case with critical growth remains largely unexplored. It is natural to ask whether there exist bump solutions for the system with the p-Laplacian and critical growth, and whether asymptotic results still hold. This paper gives affirmative answers to these questions.

Our paper has the following distinctive features and novelty compared with previous works:

(i) We prove the existence of multi-bump ground states for the coupled system with the p-Laplacian for all $\frac{12}{7} < p \le \frac{12}{5}$. Our work is motivated by [20, 28, 31], but our approach is not a straightforward extension. Due to the nonlinear nature of the p-Laplacian, we need to develop distinct techniques for crucial estimates, as many properties for the classical Laplacian fail for the full range of p.

(ii) The critical growth nonlinearity prevents direct application of arguments from Ding and Tanaka [32]. We adjust the sets involved in the minimax arguments, which is one of the main challenges in addressing the problem.

(iii) The interaction between the nonlocal term and critical nonlinearity, together with the noncompactness of the Sobolev embedding, obstructs standard variational methods. We re-establish key estimates for Palais-Smale sequences.

(iv) The Moser iterative method in [33] is applied in a refined way. The presence of the nonlocal term requires re-evaluation and more precise estimates.

Now we state the main results of this paper.

Theorem 1.2. *Let* (Z_1)–(Z_4), (Q), *and* (g_1)–(g_5) *hold. If* $\Gamma \subseteq \{1, \cdots, k\}$ *is an arbitrary non-empty subset, then there exist positive constants* $\alpha^* > 0$ *and* $\eta^* = \eta^*(\alpha^*)$ *such that for any* $\alpha \ge \alpha^*$ *and* $\eta \ge \eta^*$, *problem* (1.1) *admits a family of positive solutions* $\{u_\eta\}_{\eta \ge \eta^*}$. *Moreover, for any sequence* $\eta_n \to \infty$, *up to a subsequence,* $u_{\eta_n} \to u$ *in* $W^{1,p}(\mathbb{R}^3)$. *Furthermore,* $u(x) = 0$ *for* $x \notin \Omega_\Gamma$, *and* $u|_{\Omega_j}$ *is a least-energy solution of system* (1.7) *for each* $j \in \Gamma$.

$$\begin{cases} -\Delta_p u + |u|^{p-2}u + Q(x)\phi u = \alpha g(u) + |u|^{p^*-2}u, & x \in \Omega_j,\ j \in \Gamma, \\ -\Delta\phi + a^2\Delta^2\phi = Q(x)u^2, & x \in \Omega_j, \\ u|_{\partial\Omega_j} = 0, \quad \phi|_{\partial\Omega_j} = 0, \end{cases} \tag{1.7}$$

where $\Omega_\Gamma = \bigcup_{j\in\Gamma} \Omega_j$.

Remark 1.3. *The proof idea of Theorem 1.1 comes from Ding and Tanaka [32]. However, due to the nonlocal term, their method cannot be directly used to obtain least-energy solutions of system* (1.7). *Using ideas from [10, 12, 13], one can prove that system* (1.7) *admits nontrivial least-energy solutions.*

Remark 1.4 (On the range of p)**.** *The condition*

$$\frac{12}{7} < p \le \frac{12}{5}$$

is a sharp technical assumption. The lower bound $p > \frac{12}{7}$ *comes from three independent requirements:*

1. *It ensures $p^* > 2p$, which is necessary for the Ambrosetti-Rabinowitz condition (g_4) to be meaningful.*
2. *It guarantees $p^* > 4$, which is essential for the mountain pass geometry to ensure the quartic nonlocal term is dominated by the critical term as $s \to \infty$.*
3. *It ensures the continuous embedding $W^{1,p}(\mathbb{R}^3) \hookrightarrow L^{\frac{12}{5}}(\mathbb{R}^3)$, which is used in key estimates of the Bopp-Podolsky potential.*

The upper bound $p \le \frac{12}{5}$ also follows from the same embedding. For $p > \frac{12}{5}$, the embedding fails and the potential estimates collapse. The case $p = 3$ would require Trudinger-Moser inequalities and is beyond our framework.

All results hold for the full interval $p \in (\frac{12}{7}, \frac{12}{5}]$, including $p \in (\frac{12}{7}, 2)$, since the variational framework only requires reflexivity and uniform convexity of $W^{1,p}(\mathbb{R}^3)$ for $p > 1$. Whether the results can be extended to $1 < p \le \frac{12}{7}$ is an open question, which would require a more refined analysis of nonlocal term estimates.

Corollary 1.5. *Under the hypotheses of Theorem 1.2, there exist positive constants $\alpha^* > 0$ and $\eta^* = \eta^*(\alpha^*)$ such that for any $\alpha \ge \alpha^*$ and $\eta \ge \eta^*$, problem (1.1) admits at least $2^k - 1$ positive solutions.*

Finally, we study the asymptotic behavior of solutions as $a \to 0$. As $a \to 0$, solutions converge to a solution of the Schrödinger-Poisson system.

Theorem 1.6. *Under the hypotheses of Theorem 1.2, let $\{u_{\eta^*,a}\}_{a>0}$ be a family of weak solutions obtained in Theorem 1.2 for fixed $\eta^* > 0$. Then, up to the subsequences,*

$$\lim_{a\to 0} u_{\eta^*,a} = u_{\eta^*,0} \quad \text{in } W^{1,p}(\mathbb{R}^3) \quad \text{and} \quad \lim_{a\to 0} I_{\eta^*,a}(u_{\eta^*,a}) = I_{\eta^*,0}(u_{\eta^*,0}),$$

where $u_{\eta^*,0}$ solves the Schrödinger-Poisson-type system

$$-\Delta_p u + (\eta^* Z(x) + 1)\,|u|^{p-2}u + Q(x)\phi_u^0 u = \alpha g(u) + |u|^{p^*-2}u, \quad x \in \mathbb{R}^3, \tag{1.8}$$

with parameter η^* and ϕ_u^0 given in Section 2.

The proof is based on variational and topological methods. The combination of the p-Laplacian, critical nonlinearity, and nonlocal term makes the problem more challenging.

The paper is organized as follows. Section 2 presents the variational framework. Section 3 introduces preliminary knowledge and the modified functional, verifies the Palais-Smale condition, and analyzes energy levels. Section 4 completes the proofs of Theorems 1.2 and 1.6.

2. Variational framework

In this section, we introduce the following workspace:

$$E_\eta := \left\{ u \in W^{1,p}(\mathbb{R}^3) : \int_{\mathbb{R}^3} \eta Z(x)|u|^p dx < \infty \right\}$$

with the norm

$$\|u\|_\eta := \left(\int_{\mathbb{R}^3} (|\nabla u|^p + (\eta Z(x) + 1)|u|^p)\, dx \right)^{\frac{1}{p}}.$$

For any $\eta \geq 0$, the space $(E_\eta, \|\cdot\|_\eta)$ is a Banach space. Indeed, for $\eta = 0$, we have $E_0 = W^{1,p}(\mathbb{R}^3)$ equipped with the standard Sobolev norm. For $\eta > 0$, the norm $\|\cdot\|_\eta$ is equivalent to the standard norm on $W^{1,p}(\mathbb{R}^3)$ under the assumptions on Z, and the completeness of $(E_\eta, \|\cdot\|_\eta)$ follows from standard arguments. Moreover, the continuous embedding $E_\eta \hookrightarrow W^{1,p}(\mathbb{R}^3)$ holds for all $\eta \geq 0$. In this paper, we will primarily focus on the case where $\eta \geq 1$ to simplify the analysis of the potential term. Additionally, for an open set $K \subset \mathbb{R}^3$, we denote that

$$E_\eta(K) := \left\{ u \in W^{1,p}(K) : \int_K \eta Z(x)|u|^p dx < \infty \right\}.$$

According to (Z_3), we have

$$M_0|u|^p_{p,K} \leq \int_K (|\nabla u|^p + (\eta Z(x) + 1)|u|^p)\, dx$$

for every $u \in E_\eta(K)$ and $\eta \geq 1$, equivalently expressed as

$$M_0|u|^p_{p,K} \leq \|u\|^p_{\eta,K} \quad \text{for all } u \in E_\eta(K) \text{ and } \eta \geq 1,$$

where $|u|_{p,K} = \int_K |u|^p dx$, $M_0 > 0$, and $\|u\|_{\eta,K} := \left(\int_K (|\nabla u|^p + (\eta Z(x) + 1)|u|^p)\, dx\right)^{\frac{1}{p}}$.

As in the discussion of Ding and Tanaka [32], for every open set $K \subset \mathbb{R}^3$, there exist $\delta_0, v_0 > 0$, satisfying

$$\delta_0\|u\|^p_{\eta,K} \leq \|u\|^p_{\eta,K} - v_0|u|^p_{p,K}, \quad \forall\, u \in E_\eta(K) \text{ and } \eta \geq 1. \tag{2.1}$$

Define $\mathcal{D}$ as the Hilbert space completion of $C_0^\infty(\mathbb{R}^3)$ with the norm $\|\cdot\|_{\mathcal{D}}$ and the inner product

$$\langle u, v\rangle = \int_{\mathbb{R}^3} \left(\nabla u \nabla v + a^2 \Delta u \Delta v\right) dx.$$

Thus, there exist the continuous embeddings

$$\mathcal{D} \hookrightarrow D^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3).$$

Lemma 2.1 ([20], Lemma 3.1)**.** *The space $\mathcal{D}$ admits a continuous embedding into $L^\infty(\mathbb{R}^3)$.*

For each fixed $u \in E_\eta$, an application of the Lax-Milgram theorem yields a unique solution $\phi^a_u \in \mathcal{D}$ to the second equation in system (1.1). We note that this existence and uniqueness result relies on the proper parameter range of p, namely, $p \in \left(\frac{12}{7}, \frac{12}{5}\right]$, which ensures the right-hand term Qu^2 satisfies the integrability condition required by Lemma 2.2 below, so that the convolution representation of the solution is well-defined in $\mathcal{D}$. Inspired by [5], in order to write the solution explicitly, we consider the function:

$$\mathcal{K}(x) = \frac{1 - e^{-\frac{|x|}{a}}}{|x|}.$$

Regarding $\mathcal{K}$, the following fundamental properties are obtained from d'Avenia and Siciliano [20].

Lemma 2.2 ([20], Lemma 3.3)**.** *For every $y \in \mathbb{R}^3$, $\mathcal{K}(\cdot - y)$ solves in the sense of distributions*

$$-\Delta\phi + a^2\Delta^2\phi = 4\pi\delta_y.$$

Moreover,

(i) if $f \in L^1_{\rm loc}(\mathbb{R}^3)$ *and for a.e.* $x \in \mathbb{R}^3$, *the map* $y \mapsto \frac{f(y)}{|x-y|}$ *is summable, then* $\mathcal{K} * f \in L^1_{\rm loc}(\mathbb{R}^3)$;
(ii) if $f \in L^p(\mathbb{R}^3)$ *with* $1 \le p < \frac{3}{2}$, *then* $\mathcal{K} * f \in L^q(\mathbb{R}^3)$ *for* $q \in \left(\frac{3p}{3-2p}, +\infty\right)$. *In both cases,* $\mathcal{K} * f$ *solves*

$$-\Delta\phi + a^2\Delta^2\phi = 4\pi f.$$

Due to the above results, for a fixed $u \in E_\eta$ with $p \in \left(\frac{12}{7}, \frac{12}{5}\right]$, the unique solution in $\mathcal{D}$ to the second equation in system (1.1) admits the integral representation:

$$\phi_u^a = \mathcal{K} * (Qu^2) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1 - e^{-\frac{|x-y|}{a}}}{|x-y|} Q(y)u^2(y)dy.$$

Here are some properties of ϕ_u^a, obtained as in [20, Lemma 3.4].

Lemma 2.3. *For any* $u, v \in E_\eta$, *we have the following properties:*

(a) $\phi_u^a \ge 0$;
(b) For any $t > 0$, $\phi_{tu}^a = t^2\phi_u^a$;
(c) Let $u_n \to u$ *in* E_η, *and then* $\phi_{u_n}^a \to \phi_u^a$ *in* $\mathcal{D}$;
(d) $\|\phi_u^a\|_{\mathcal{D}} \le C\|u\|^2_{\frac{12}{5}} \le C\|u\|^2_{E_\eta}$ *and* $\int_{\mathbb{R}^3} Q(x)\phi_u^a u^2 dx \le C\|u\|^4_{\frac{12}{5}} \le C\|u\|^4_{E_\eta}$.

From the above discussion, system (1.1) can be reduced to a single equation:

$$-\Delta_p u + (\eta Z(x) + 1)|u|^{p-2}u + Q(x)\phi_u^a u = \alpha g(u) + |u|^{p^*-2}u, \quad x \in \mathbb{R}^3. \tag{2.2}$$

Indeed, the solutions of system (1.1) are exactly the solutions of equation (2.2). In the following, the energy functional $I_\eta : E_\eta \to \mathbb{R}$ corresponding to equation (2.2) is given by:

$$\begin{aligned} I_\eta(u) = &\frac{1}{p}\int_{\mathbb{R}^3} \left(|\nabla u|^p + (\eta Z(x) + 1)|u|^p\right) dx + \frac{1}{4}\int_{\mathbb{R}^3} Q(x)\phi_u^a u^2 dx \\ &- \alpha\int_{\mathbb{R}^3} G(u)dx - \frac{1}{p^*}\int_{\mathbb{R}^3} |u|^{p^*} dx, \end{aligned} \tag{2.3}$$

where $G(s) = \int_0^s g(\tau)d\tau$.

According to (Z_1)–(Z_2) and (g_1)–(g_2), we have $I_\eta \in C^1(E_\eta, \mathbb{R})$. Furthermore, the Fréchet derivative of I_η is given by

$$\begin{aligned} \langle I'_\eta(u), \varphi\rangle = &\int_{\mathbb{R}^3} \left(|\nabla u|^{p-2}\nabla u\nabla\varphi + (\eta Z(x) + 1)|u|^{p-2}u\varphi\right) dx + \int_{\mathbb{R}^3} Q(x)\phi_u^a u\varphi dx \\ &- \alpha\int_{\mathbb{R}^3} g(u)\varphi dx - \int_{\mathbb{R}^3} |u|^{p^*-2}u\varphi dx \end{aligned} \tag{2.4}$$

for all $\varphi \in E_\eta$. Clearly, the solutions of equation (2.2) are critical points of the energy functional I_η.

From now on, for each $j \in \{1, \cdots, k\}$, one can fix a bounded open subset Ω'_j with a smooth boundary satisfying

(a) $\overline{\Omega_j} \subset \Omega'_j$;
(b) $\overline{\Omega'_j} \cap \overline{\Omega'_{\tilde{j}}} = \emptyset$ when $j \ne \tilde{j}$.

3. Compactness condition

In this section, we are mainly concerned with the proof of the compactness condition. To this end, we adapt some arguments developed in [31, 32]. Let $v_0 > 0$ be the constant provided in (2.1), and $1 < \theta(\theta - p)^{-1} < k$, $\mathscr{A} > 0$ satisfy $\alpha g(\mathscr{A}) + \mathscr{A}^{p^*-1} = k^{-1}v_0\mathscr{A}^{p-1}$, where $\tilde{g}, \tilde{G} : \mathbb{R} \to \mathbb{R}$ are given by

$$\tilde{g}(s) = \begin{cases} \alpha g(s) + s^{p^*-1} & \text{if } s \le \mathscr{A}, \\ k^{-1}v_0 s^{p-1} & \text{if } s > \mathscr{A}, \end{cases}$$

and $\tilde{G}(s) = \int_0^s \tilde{g}(\tau)d\tau$. Let $\Gamma \subset \{1, \cdots, k\}$ be a fixed, non-empty subset and

$$\Omega_\Gamma = \bigcup_{j\in\Gamma} \Omega_j, \quad \Omega'_\Gamma = \bigcup_{j\in\Gamma} \Omega'_j, \quad \chi_\Gamma(x) = \begin{cases} 1 & \text{for } x \in \Omega'_\Gamma, \\ 0 & \text{for } x \in \mathbb{R}^3 \setminus \Omega'_\Gamma. \end{cases}$$

Now, let

$$f(x, s) = \chi_\Gamma(x)\big(\alpha g(s) + s^{p^*-1}\big) + (1 - \chi_\Gamma(x))\tilde{g}(s) \tag{3.1}$$

and

$$F(x, s) = \int_0^s f(x, \tau)d\tau = \chi_\Gamma(x)\left(\alpha G(s) + \frac{1}{p^*}s^{p^*}\right) + (1 - \chi_\Gamma(x))\tilde{G}(s). \tag{3.2}$$

Through the above discussion, combined with (g_1)–(g_4), a direct verification shows that

(f_1) $f(z, s) = \alpha g(s) + s^{p^*-1} = o(s^{p-1})$ for all $z \in \mathbb{R}^3$.
(f_2) $f(z, s) \le \alpha g(s) + s^{p^*-1}$ for any $s > 0$, $z \in \mathbb{R}^3$.
(f_3) (i) $0 < \theta F(z, s) \le f(z, s)s$ for every $z \in \Omega'_\Gamma$, $s > 0$;
(ii) $0 < F(z, s) \le k^{-1}p^{-1}v_0 s^p$ and $0 < f(z, s)s \le k^{-1}v_0 s^p$ for each $z \in \mathbb{R}^3 \setminus \Omega'_\Gamma$, $s > 0$.
(f_4) The function $\dfrac{f(x, s)}{|s|^{p-2}s}$ is increasing for $s > 0$ for each fixed x.

Furthermore, we introduce the penalized energy functional $\Phi_\eta : E_\eta \to \mathbb{R}$ by

$$\Phi_\eta(u) = \frac{1}{p}\int_{\mathbb{R}^3} (|\nabla u|^p + (\eta Z(x) + 1)|u|^p)dx + \frac{1}{4}\int_{\mathbb{R}^3} Q(x)\phi_u^a u^2 dx - \int_{\mathbb{R}^3} F(x, u)dx. \tag{3.3}$$

For any $\varphi \in E_\eta$, one has

$$\langle \Phi'_\eta(u), \varphi\rangle = \int_{\mathbb{R}^3} (|\nabla u|^{p-2}\nabla u\nabla\varphi + (\eta Z(x) + 1)|u|^{p-2}u\varphi)dx + \int_{\mathbb{R}^3} Q(x)\phi_u^a u\varphi dx - \int_{\mathbb{R}^3} f(x, u)\varphi dx. \tag{3.4}$$

In addition, it is known that Φ_η is also of class C^1 and the critical points of Φ_η correspond precisely to nontrivial weak solutions of the semilinear elliptic equation:

$$-\Delta_p u + (\eta Z(x) + 1)|u|^{p-2}u + Q(x)\phi_u^a u = f(x, u), \quad x \in \mathbb{R}^3. \tag{3.5}$$

We note that $\tilde{g}(s) = \alpha g(s) + s^{p^*-1}$ for $s \in [0, \mathscr{A}]$ and a critical point u_η of Φ_η solves system (1.1) if and only if $u_\eta(x) \le \mathscr{A}$ for almost every $\mathbb{R}^3 \setminus \Omega'_\Gamma$.

The following estimates hold for all $p \in \left(\frac{12}{7}, \frac{12}{5}\right]$. Under assumptions (g_1)–(g_5), the functional Φ_η satisfies the mountain pass geometry conditions.

Lemma 3.1. *Let $\eta \geq 1$, and the function $\Phi_\eta(u)$ satisfies the following properties:*

(i) There exist $\varrho, \beta > 0$ such that $\Phi_\eta(u) \geq \varrho$ for any $u \in E_\eta(K)$, with $\|u\|_{\eta,K} = \beta$;
(ii) there exists $v \in E_\eta(K)$ with $\|v\|_{\eta,K} > \beta$ such that $\Phi_\eta(v) < 0$.

Proof. (i) Combining (g_2) with $\varepsilon > 0$, (f_1), (f_3), and (2.1), by using the Sobolev inequality and the fact $\int_{\mathbb{R}^3} Q(x)\phi_u^a u^2 dx \geq 0$, we have

$$\begin{aligned}
\Phi_\eta(u) &\geq \frac{1}{p}\|u\|_\eta^p - \int_{\Omega_\Gamma'} F(x,u)dx - \int_{\mathbb{R}^3\setminus\Omega_\Gamma'} F(x,u)dx \\
&\geq \frac{1}{p}\|u\|_\eta^p - \int_{\Omega_\Gamma'} \frac{1}{\theta} f(x,u)udx - \int_{\mathbb{R}^3\setminus\Omega_\Gamma'} \frac{v_0}{kp} u^p dx \\
&= \frac{1}{p}\|u\|_\eta^p - \int_{\Omega_\Gamma'} \frac{1}{\theta}\alpha g(u)udx - \int_{\Omega_\Gamma'} \frac{1}{\theta} u^{p^*} dx - \frac{v_0}{kp}|u|_p^p \\
&\geq \frac{1}{p}\|u\|_\eta^p - \int_{\Omega_\Gamma'} \frac{1}{\theta}\alpha\varepsilon|u|^p dx - \frac{1}{\theta}\int_{\Omega_\Gamma'} u^{p^*} dx - \frac{v_0}{kp}|u|_p^p \\
&\geq \frac{\delta_0}{2p}\|u\|_\eta^p - C\|u\|_\eta^{p^*},
\end{aligned}$$

where ε is small enough and $C > 0$ is a constant. Since $p < q < p^*$, then we can choose $\varrho, \beta > 0$ with $\|u\|_\eta = \beta$ small enough such that $\Phi_\eta(u) \geq \varrho$. This implies that conclusion of Lemma 3.1(i) holds.

(ii) By fixing $0 < \omega \in C_0^\infty(\Omega_\Gamma')$, and utilizing assumptions (Q) and (f_2), we obtain that

$$\begin{aligned}
\Phi_\eta(s\omega) &= \frac{s^p}{p}\|\omega\|_\eta^p + \frac{s^4}{4}\int_{\mathbb{R}^3} Q(x)\phi_{s\omega}^a \omega^2 dx - \int_{\mathbb{R}^3} F(x, s\omega)dx \\
&\leq \frac{s^p}{p}\|\omega\|_\eta^p + \frac{s^4}{4}C_1\|\omega\|_\eta^4 - C_2\int_{\mathbb{R}^3} (s\omega)^{p^*} dx \\
&= \frac{s^p}{p}\|\omega\|_\eta^p + \frac{s^4}{4}C_1\|\omega\|_\eta^4 - s^{p^*}C_2|\omega|_{p^*}^{p^*} \\
&\to -\infty \quad \text{as } s \to +\infty.
\end{aligned}$$

Since $\frac{12}{7} < p \leq \frac{12}{5}$, we can choose $v = s\omega$ such that $\Phi_\eta(s\omega) < 0$ for s sufficiently large. □

Next, define

$$c_\eta = \inf_{\gamma\in\Gamma_\eta} \max_{t\in[0,1]} \Phi_\eta(\gamma(t)),$$

where

$$\Gamma_\eta = \left\{\gamma \in C([0,1], E_\eta) : \gamma(0) = 0 \text{ and } \Phi_\eta(\gamma(1)) < 0\right\}.$$

Subsequently, for each fixed $j \in \Gamma$, the mountain pass minimax level for the functional $I_j : W_0^{1,p}(\Omega_j) \to \mathbb{R}$ is characterized by c_j, where I_j is defined as follows:

$$I_j(u) = \frac{1}{p}\int_{\Omega_j} (|\nabla u|^p + |u|^p)dx + \frac{1}{4}\int_{\Omega_j} Q(x)\phi_u^a u^2 dx - \alpha\int_{\Omega_j} G(u_+)dx - \frac{1}{p^*}\int_{\Omega_j} |u_+|^{p^*} dx. \tag{3.6}$$

One can easily verify that the weak solutions of the following problem correspond to critical points of the functional I_j:

$$\begin{cases} -\Delta_p u + |u|^{p-2}u + Q(x)\phi u = \alpha g(u) + |u|^{p^*-2}u, & x \in \Omega_j,\ j \in \Gamma, \\ -\Delta\phi + a^2\Delta^2\phi = Q(x)u^2, & x \in \Omega_j, \\ u|_{\partial\Omega_j} = 0. \end{cases} \tag{3.7}$$

From (3.6) and (3.7), similar to [31, Lemma 4.3], we can derive the following conclusion.

Remark 3.2. *We can find $\alpha^* > 0$ for all $\alpha \geq \alpha^*$, satisfying $c_j \in \left(0, \frac{1}{\theta(k+1)}S^{\frac{3}{p}}\right)$ for all $j \in \{1, \cdots, k\}$. Moreover, one can get $\sum\limits_{j=1}^{k} c_j \in \left(0, \frac{1}{\theta}S^{\frac{3}{p}}\right)$.*

Now, we shall prove that the functional Φ_η corresponding to equation (3.3) satisfies the $(PS)_c$ condition, where the following estimates hold for all $p \in \left(\frac{12}{7}, \frac{12}{5}\right]$.

Lemma 3.3. *Let (Z_1)–(Z_3) and (f_1)–(f_5) be satisfied. For every $\eta \geq 1$ and all $c \in \left(0, \frac{1}{\theta}S^{\frac{3}{p}}\right)$, Φ_η fulfills the $(PS)_c$ condition. In other words, for any $(PS)_c$ sequence $\{u_n\} \subset E_\eta$, there is a subsequence that converges strongly in E_η.*

Proof. First, we take the $(PS)_c$ sequence $\{u_n\} \subset E_\eta$ of the functional Φ_η, that is, $\Phi_\eta(u_n) \to c$ and $\Phi'_\eta(u_n) \to 0$ strongly in E_η. To this end, we divide the whole proof into three claims.

Claim 1. The sequence $\{u_n\}_n$ is bounded in E_η.

In fact, by $2p < \theta$, (f_3), $1 < \theta(\theta - p)^{-1} < k$, (2.1), and $\int_{\mathbb{R}^3} Q(x)\phi_u^a u^2 dx \geq 0$, we get

$$\begin{aligned} c + o_n(1) + \varepsilon_n\|u_n\|_\eta &= \Phi_\eta(u_n) - \frac{1}{\theta}\langle\Phi'_\eta(u_n), u_n\rangle \\ &\geq \left(\frac{1}{p} - \frac{1}{\theta}\right)\|u_n\|_\eta^p + \int_{\Omega'_\Gamma}\left(\frac{1}{\theta}f(x,u_n)u_n - F(x,u_n)\right)dx \\ &\quad + \int_{\mathbb{R}^3\setminus\Omega'_\Gamma}\left(\frac{1}{\theta}f(x,u_n)u_n - F(x,u_n)\right)dx \\ &\geq \left(\frac{1}{p} - \frac{1}{\theta}\right)\|u_n\|_\eta^p - \int_{\mathbb{R}^3\setminus\Omega'_\Gamma}\frac{1}{kp}v_0 u_n^p dx \\ &\geq \left(\frac{1}{p} - \frac{1}{\theta}\right)\delta_0\|u_n\|_\eta^p. \end{aligned}$$

This observation directly yields the boundedness of $\|u_n\|_\eta$ in E_η.

Claim 2. The following limits hold:

$$\int_{\mathbb{R}^3} f(x,u_n)u_n dx \to \int_{\mathbb{R}^3} f(x,u)u dx \quad \text{as } n \to \infty. \tag{3.8}$$

In fact, from Claim 1, and the concentration-compactness principle by Lions [34, 35] (see also [31, Lemma 4.5]), for some measures ν and μ, we may assume that the sequence $\{u_n\}$ satisfies $|u_n|^{p^*} \rightharpoonup \nu$ and $|\nabla u_n|^p \rightharpoonup \mu$, and

$$\begin{aligned}
&u_n \rightharpoonup u \quad \text{weakly in } E_\eta \text{ and } W^{1,p}(\mathbb{R}^3),\\
&u_n \to u \quad \text{strongly in } L^q_{\mathrm{loc}}(\mathbb{R}^3),\ \forall\, q \in [p, p^*),\\
&|u_n|^{p^*} \rightharpoonup \nu = |u|^{p^*} + \sum_{i\in I} \nu_i \delta_{x_i},\ \nu_i > 0,\\
&|\nabla u_n|^p \rightharpoonup \mu \geq |\nabla u|^p + \sum_{i\in I} \mu_i \delta_{x_i},\ \mu_i > 0,
\end{aligned}$$

and

$$\mu_i \geq S\, \nu_i^{\frac{p}{p^*}}, \tag{3.9}$$

where $S = \inf\left\{\int_{\mathbb{R}^3} |\nabla u|^p dx : \int_{\mathbb{R}^3} |u|^{p^*} dx = 1\right\}$ and μ_i, ν_i are constants.

Now, we define a truncation function $\psi \in C_0^\infty(\mathbb{R}^3)$ satisfying $\psi(x) = 1$ in $B(x_i, \varepsilon)$, $\psi(x) = 0$ in $\mathbb{R}^3 \setminus B(x_i, 2\varepsilon)$, and $|\nabla\psi| \leq \frac{2}{\varepsilon}$ in $\mathbb{R}^3$. Setting $\omega_n = u_n\psi$, then $\{\omega_n\}$ is bounded in E_η. Clearly, $\langle \Phi'_\eta(u_n), \omega_n\rangle \to 0$, by condition (f_2), i.e.,

$$\begin{aligned}
&\int_{\mathbb{R}^3} |\nabla u_n|^p \psi dx + \int_{\mathbb{R}^3} (\eta Z(x) + 1)|u_n|^p \psi dx + \int_{\mathbb{R}^3} Q(x)\phi^a_{u_n} u_n^2 \psi dx\\
&\qquad = \int_{\mathbb{R}^3} \alpha g(u_n) u_n \psi dx + \int_{\mathbb{R}^3} u_n^{p^*} \psi dx + o_n(1).
\end{aligned} \tag{3.10}$$

It follows from the definition of ψ that

$$\lim_{\varepsilon\to 0^+} \limsup_{n\to\infty} \int_{\mathbb{R}^3} (\eta Z(x) + 1)|u_n|^p \psi dx = 0, \tag{3.11}$$

$$\lim_{\varepsilon\to 0^+} \limsup_{n\to\infty} \int_{\mathbb{R}^3} Q(x)\phi^a_{u_n} u_n^2 \psi dx = 0, \tag{3.12}$$

$$\lim_{\varepsilon\to 0^+} \limsup_{n\to\infty} \int_{\mathbb{R}^3} \alpha g(u_n) u_n \psi dx = 0, \tag{3.13}$$

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} |\nabla u_n|^p \psi dx = \int_{\mathbb{R}^3} \psi d\mu \geq \int_{\mathbb{R}^3} |\nabla u|^p \psi dx + \mu_i. \tag{3.14}$$

Moreover, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} |u_n|^{p^*} \psi dx = \int_{\mathbb{R}^3} \psi d\nu = \int_{\mathbb{R}^3} |u|^{p^*} \psi dx + \nu_i. \tag{3.15}$$

Letting $\varepsilon \to 0^+$, $n \to \infty$ in (3.10), and it follows from (3.11)–(3.15) that $\mu_i \leq \nu_i$. Combining this inequality with (3.9) immediately yields $\nu_i \geq S^{\frac{p^*}{p^*-p}}$. This implies two possible cases:

$$\text{(i) } \nu_i = 0 \quad \text{or} \quad \text{(ii) } \nu_i \geq S^{\frac{3}{p}}.$$

When the secondary scenario $\nu_i \geq S^{\frac{3}{p}}$ occurs, for some $i \in I$ with $\theta > 2p$, it follows that

$$\begin{aligned} c &= \lim_{n\to\infty}\left(\Phi_\eta(u_n) - \frac{1}{\theta}\langle \Phi'_\eta(u_n), u_n\rangle\right) \\ &\geq \lim_{n\to\infty}\frac{1}{\theta}\int_{\mathbb{R}^3}|\nabla u_n|^p dx = \frac{1}{\theta}\int d\mu \\ &\geq \frac{1}{\theta}\int_{\mathbb{R}^3}|\nabla u|^p dx + \frac{1}{\theta}\mu_i \\ &\geq \frac{1}{\theta}S\nu_i^{\frac{p}{p^*}} \geq \frac{1}{\theta}S^{\frac{3}{p}}. \end{aligned} \tag{3.16}$$

This is impossible. As a result, for every $i \in I$, we can all get $\nu_i = 0$. Thus $u_n \to u$ in $L^{p^*}_{\mathrm{loc}}(\mathbb{R}^3)$.

Similarly, we also have the concentration-compactness principle at infinity developed by Chabrowski [36] (see also [37, 38]), which provides some quantitative information about the loss of mass of a sequence at infinity. We define

$$\nu_\infty = \lim_{R\to\infty}\limsup_{n\to\infty}\int_{|x|>R}|u_n|^{p^*}dx, \quad \mu_\infty = \lim_{R\to\infty}\limsup_{n\to\infty}\int_{|x|>R}|\nabla u_n|^p dx.$$

Thus, there exist well-defined quantities ν_∞ and μ_∞ that satisfy the relations:

$$\begin{aligned} &\limsup_{n\to\infty}\int_{\mathbb{R}^3}|u_n|^{p^*}dx = \int_{\mathbb{R}^3}d\nu + \nu_\infty, \\ &\limsup_{n\to\infty}\int_{\mathbb{R}^3}|\nabla u_n|^p dx = \int_{\mathbb{R}^3}d\mu + \mu_\infty, \\ &\mu_\infty \geq S\nu_\infty^{p/p^*}. \end{aligned}$$

To exclude the possibility of mass concentration at infinity, we introduce a standard cut-off function $\psi_R \in C_0^\infty(\mathbb{R}^3, [0,1])$ satisfying

$$\psi_R(x) = \begin{cases} 0 & \text{for } |x| < R, \\ 1 & \text{for } |x| > R+1. \end{cases}$$

Since $\{u_n\psi_R\}$ is bounded in E_η and φ_R takes values in $\mathbb{R}$, we have $\langle \Phi'_\eta(u_n), u_n\psi_R\rangle \to 0$, that is,

$$\begin{aligned} &\int_{\mathbb{R}^3}|\nabla u_n|^p\psi_R dx + \int_{\mathbb{R}^3}(\eta Z(x)+1)|u_n|^p\psi_R dx + \int_{\mathbb{R}^3}Q(x)\phi^a_{u_n}u_n^2\psi_R dx \\ &\qquad = \int_{\mathbb{R}^3}\alpha g(u_n)u_n\psi_R dx + \int_{\mathbb{R}^3}u_n^{p^*}\psi_R dx + o_n(1). \end{aligned} \tag{3.17}$$

By the definition of ψ_R and similar to (3.11)–(3.15), we can deduce that

$$\lim_{\varepsilon\to 0^+}\limsup_{n\to\infty}\int_{\mathbb{R}^3}(\eta Z(x)+1)|u_n|^p\psi_R dx = 0, \tag{3.18}$$

$$\lim_{\varepsilon\to 0^+}\limsup_{n\to\infty}\int_{\mathbb{R}^3}Q(x)\phi^a_{u_n}u_n^2\psi_R dx = 0, \tag{3.19}$$

$$\lim_{\varepsilon\to 0^+}\limsup_{n\to\infty}\int_{\mathbb{R}^3}\alpha g(u_n)u_n\psi_R dx = 0, \tag{3.20}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}|\nabla u_n|^p\psi_R dx = \int_{\mathbb{R}^3}\psi_R d\mu \geq \int_{\mathbb{R}^3}|\nabla u|^p\psi_R dx + \mu_i, \tag{3.21}$$

and

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}|u_n|^{p^*}\psi_R dx = \int_{\mathbb{R}^3}\psi_R d\nu = \int_{\mathbb{R}^3}|u|^{p^*}\psi_R dx + \nu_i. \tag{3.22}$$

Inserting (3.18)–(3.22) into (3.17), and taking the limits as $R \to \infty$ and $n \to \infty$, we obtain $\mu_\infty \leq \nu_\infty$. Therefore, we have $\nu_\infty \geq S^{\frac{p^*}{p^*-p}}$. This result demonstrates that

$$\text{(iii) } \nu_\infty = 0 \quad \text{or} \quad \text{(iv) } \nu_\infty \geq S^{\frac{3}{p}}.$$

Similar to the discussion in (3.16), we can get $\nu_\infty = 0$.

In fact, from Claim 1 and the concentration-compactness principle, we have proved that the concentration defects satisfy $\nu_i = 0$ for all $i \in I$ and $\nu_\infty = 0$, which directly implies

$$u_n \to u \quad \text{strongly in } L^{p^*}(\mathbb{R}^3). \tag{3.23}$$

Combining with the uniform boundedness of $\{u_n\}$ in $L^p(\mathbb{R}^3)$ (from Claim 1), for any exponent q with $p < q < p^*$, by the interpolation inequality

$$\|u_n - u\|_{L^q(\mathbb{R}^3)} \leq \|u_n - u\|^{\theta}_{L^p(\mathbb{R}^3)} \cdot \|u_n - u\|^{1-\theta}_{L^{p^*}(\mathbb{R}^3)}, \quad \theta = \frac{\frac{p^*}{q} - 1}{\frac{p^*}{p} - 1} \in (0,1),$$

we conclude that

$$u_n \to u \quad \text{strongly in } L^q(\mathbb{R}^3), \quad \forall q \in (p, p^*). \tag{3.24}$$

In particular, up to a subsequence, $u_n \to u$ almost everywhere in $\mathbb{R}^3$, and thus $f(x,u_n)u_n \to f(x,u)u$ almost everywhere.

We now verify the convergence of the nonlinear integral by splitting it into the critical part and the subcritical part. Recall from (3.1) that

$$f(x,s)s = \chi_\Gamma(x)|s|^{p^*} + \chi_\Gamma(x)\alpha g(s)s + (1-\chi_\Gamma(x))\tilde{g}(s)s.$$

For the critical term, by (3.23), the strong convergence in $L^{p^*}(\mathbb{R}^3)$ makes it obvious that

$$\int_{\mathbb{R}^3}\chi_\Gamma(x)|u_n|^{p^*}dx \to \int_{\mathbb{R}^3}\chi_\Gamma(x)|u|^{p^*}dx.$$

For subcritical terms, by assumption (g_1) and the definition of $\tilde{g}(s)$, there exist $q \in (p, p^*)$ and a constant $C > 0$ such that

$$\left|\chi_\Gamma(x)\alpha g(s)s + (1-\chi_\Gamma(x))\tilde{g}(s)s\right| \leq C(|s|^q + |s|^{p^*}), \quad \forall x \in \mathbb{R}^3,\ s \in \mathbb{R}.$$

Let $g_n(x) = C(|u_n(x)|^q + |u_n(x)|^{p^*})$. From (3.23) and (3.24), we have

$$\int_{\mathbb{R}^3} g_n(x)dx = C(\|u_n\|_q^q + \|u_n\|_{p^*}^{p^*}) \to C(\|u\|_q^q + \|u\|_{p^*}^{p^*}) = \int_{\mathbb{R}^3} g(x)dx < \infty.$$

By the generalized Lebesgue dominated convergence theorem, since $f(x,u_n)u_n \to f(x,u)u$ a.e. and $|f(x,u_n)u_n| \le g_n(x)$ with $\int g_n \to \int g$, we obtain

$$\int_{\mathbb{R}^3} [\chi_\Gamma(x)\alpha g(u_n)u_n + (1-\chi_\Gamma(x))\tilde{g}(u_n)u_n]\,dx \to \int_{\mathbb{R}^3} [\chi_\Gamma(x)\alpha g(u)u + (1-\chi_\Gamma(x))\tilde{g}(u)u]\,dx.$$

Combining the convergence of both parts, we get that (3.8) is true.

Claim 3. We claim that $u_n \to u$ in E_η.

Recall that the norm on E_η is defined by

$$\|u\|_\eta^p = \int_{\mathbb{R}^3} (|\nabla u|^p + (\eta Z(x)+1)|u|^p)dx,$$

and the corresponding p-Laplacian operator satisfies the strong monotonicity property: for any $u, v \in E_\eta$, there exists a constant $C_p > 0$ depending only on p such that

$$\left\langle -\Delta_p u + (\eta Z(x)+1)|u|^{p-2}u - \left(-\Delta_p v + (\eta Z(x)+1)|v|^{p-2}v\right), u-v\right\rangle \ge C_p\|u-v\|_\eta^p.$$

This is a standard result for p-Laplacian operators on uniformly convex Banach spaces, valid for all $p > 1$ (see [20]). Since $\Phi'_\eta(u_n) \to 0$ in E_η^* and $\Phi'_\eta(u) = 0$, we have

$$\langle \Phi'_\eta(u_n) - \Phi'_\eta(u), u_n - u\rangle = o(1)\|u_n - u\|_\eta.$$

Expanding the left-hand side, we get

$$\begin{aligned}&\left\langle -\Delta_p u_n + (\eta Z(x)+1)|u_n|^{p-2}u_n - \left(-\Delta_p u + (\eta Z(x)+1)|u|^{p-2}u\right), u_n - u\right\rangle \\ &+ \int_{\mathbb{R}^3} Q(x)(\phi^a_{u_n}u_n - \phi^a_u u)(u_n - u)dx - \int_{\mathbb{R}^3}(f(x,u_n)u_n - f(x,u)u)dx = o(1)\|u_n - u\|_\eta.\end{aligned}$$

We now estimate the remaining two integral terms separately:

For the nonlinear local term $\int_{\mathbb{R}^3}(f(x,u_n)u_n - f(x,u)u)dx$, by Claim 2 and equality (3.8), we directly have

$$\int_{\mathbb{R}^3}(f(x,u_n)u_n - f(x,u)u)dx \to 0 \quad \text{as } n \to \infty.$$

For the nonlocal potential term $\int_{\mathbb{R}^3} Q(x)(\phi^a_{u_n} - \phi^a_u)(u_n - u)dx$, we provide a complete rigorous proof via global Hölder estimates, which only relies on the original condition (Q) and the strong convergence already established in Claim 2.

First, since $Q \in L^\infty(\mathbb{R}^3)$ and $Q \ge 0$, we have

$$\left|\int_{\mathbb{R}^3} Q(x)(\phi^a_{u_n}u_n - \phi^a_u u)(u_n - u)dx\right| \le \|Q\|_{L^\infty} \cdot \left|\int_{\mathbb{R}^3}(\phi^a_{u_n}u_n - \phi^a_u u)(u_n - u)dx\right|.$$

We rewrite the integrand by the standard decomposition:

$$\phi^a_{u_n}u_n - \phi^a_u u = \phi^a_{u_n}(u_n - u) + (\phi^a_{u_n} - \phi^a_u)u,$$

which yields

$$\int_{\mathbb{R}^3} (\phi^a_{u_n} u_n - \phi^a_u u)(u_n - u)dx = \int_{\mathbb{R}^3} \phi^a_{u_n}(u_n - u)^2 dx + \int_{\mathbb{R}^3} (\phi^a_{u_n} - \phi^a_u)u(u_n - u)dx.$$

We estimate the two terms separately.

Fix an exponent q such that $2 < q < 3$ and $q < p^*$. Since $p \le \frac{12}{5} = 2.4 < q$ and $p^* \ge 4 > q$, we have $q \in (p, p^*)$. From Claim 2, we have already proved that

$$u_n \to u \quad \text{strongly in } L^q(\mathbb{R}^3).$$

Let $r = \frac{q}{q-2}$. Then $r > 3$. By the Sobolev embedding $D^{1,2}(\mathbb{R}^3) \hookrightarrow L^r(\mathbb{R}^3)$ (valid for all $3 < r \le 6$), together with Lemma 2.3 (d), the sequence $\{\phi^a_{u_n}\}$ is uniformly bounded in $L^r(\mathbb{R}^3)$, i.e.,

$$\sup_{n\in\mathbb{N}} \|\phi^a_{u_n}\|_{L^r(\mathbb{R}^3)} < \infty.$$

For the first term, we apply the Hölder inequality:

$$\left| \int_{\mathbb{R}^3} \phi^a_{u_n}(u_n - u)^2 dx \right| \le \|\phi^a_{u_n}\|_{L^r} \cdot \|u_n - u\|^2_{L^q}.$$

Since $\|\phi^a_{u_n}\|_{L^r}$ is uniformly bounded and $\|u_n - u\|_{L^q} \to 0$ as $n \to \infty$, this term converges to 0.

For the second term, note that $\phi^a_{u_n} - \phi^a_u$ is also uniformly bounded in $L^r(\mathbb{R}^3)$. By the Hölder inequality again:

$$\left| \int_{\mathbb{R}^3} (\phi^a_{u_n} - \phi^a_u)u(u_n - u)dx \right| \le \|\phi^a_{u_n} - \phi^a_u\|_{L^r} \cdot \|u\|_{L^q} \cdot \|u_n - u\|_{L^q}.$$

Since $\|u\|_{L^q}$ is a constant and $\|u_n - u\|_{L^q} \to 0$, this term also converges to 0 as $n \to \infty$.

Combining the above estimates, we conclude that

$$\int_{\mathbb{R}^3} Q(x)(\phi^a_{u_n} u_n - \phi^a_u u)(u_n - u)dx \to 0 \quad \text{as } n \to \infty.$$

We now combine the above convergence results with the strong monotonicity property of the p-Laplacian. Recall that $\langle \Phi'_\eta(u_n) - \Phi'_\eta(u), u_n - u\rangle = o(1)\|u_n - u\|_\eta$, and we have already proved that both the nonlinear integral and the nonlocal potential integral vanish as $n \to \infty$, i.e.,

$$\int_{\mathbb{R}^3} (f(x, u_n)u_n - f(x, u)u)dx \to 0, \quad \int_{\mathbb{R}^3} Q(x)(\phi^a_{u_n} u_n - \phi^a_u u)(u_n - u)dx \to 0.$$

Substituting these into the expansion of the duality pairing, together with the strong monotonicity inequality, we arrive at

$$C_p\|u_n - u\|^p_\eta \le o(1)\|u_n - u\|_\eta + o(1).$$

By Young's inequality, for any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$o(1)\|u_n - u\|_\eta \le \varepsilon\|u_n - u\|^p_\eta + C_\varepsilon \cdot o(1).$$

Choosing $\varepsilon = C_p/2$ and absorbing the first term on the right-hand side into the left-hand side, we obtain

$$\frac{C_p}{2}\|u_n - u\|^p_\eta \le o(1),$$

which immediately implies

$$\|u_n - u\|_\eta^p \to 0 \quad \text{as } n \to \infty,$$

that is, $u_n \to u$ strongly in E_η. This completes the proof of Claim 3, and thus the proof of Lemma 3.3 is now complete. □

Our subsequent objective is to investigate the characteristics of a $(PS)_{\infty,c}$ sequence, with a sequence $\{u_n\} \subset W^{1,p}(\mathbb{R}^3)$ satisfying

$$u_n \in E_{\eta_n}, \quad \eta_n \to \infty, \quad \Phi_{\eta_n}(u_n) \to c, \quad \text{and} \quad \|\Phi'_{\eta_n}(u_n)\|_{E^*_{\eta_n}} \to 0.$$

Next, we can serialize some good properties, similar to the proof in [28, 31], so we will omit it here.

Lemma 3.4. *Assume that $\{u_n\}_n$ is a $(PS)_{\infty,c}$ sequence for $c \in \left(0, \frac{1}{\theta}S^{\frac{3}{p}}\right)$. Then up to a subsequence, still denoted as $\{u_n\}$, $u \in W^{1,p}(\mathbb{R}^3)$ satisfies*

$$u_n \rightharpoonup u \quad \textit{weakly in } E_\eta \textit{ and } W^{1,p}(\mathbb{R}^3) \quad \textit{as } n \to \infty.$$

Moreover,

(a) u satisfies the following system:

$$\begin{cases} -\Delta_p u + |u|^{p-2}u + Q(x)\phi_u^a u = \alpha g(u) + |u|^{p^*-2}u, & x \in \Omega_j, \\ -\Delta\phi_u^a + a^2\Delta^2\phi_u^a = Q(x)u^2, & x \in \Omega_j, \\ u = 0, \quad \phi_u^a = 0, & \text{on } \partial\Omega_j, \end{cases} \tag{3.25}$$

with $j \in \Gamma$. Moreover, $u \equiv 0$ in $\mathbb{R}^3 \setminus \Omega_\Gamma$;

(b) $\|u_n - u\|_{\eta_n} \to 0$;

(c) for $n \to \infty$, u_n satisfies:

(i) $\|u_n - u\|_{W^{1,p}(\mathbb{R}^3\setminus\Omega_\Gamma)} \to 0$;

(ii) $\int_{\mathbb{R}^3} \eta_n Z(x)|u_n|^p dx \to 0$;

(iii) $\int_{\Omega'_j}(|\nabla u_n|^p + (\eta_n Z(x) + 1)u_n^p)dx \to \int_{\Omega_j}(|\nabla u|^p + u^p)dx$ with $j \in \Gamma$.

In what follows, to establish the boundedness for some solutions outside Ω'_Γ of the original system (1.1), we adapt the argument by Alves [39], which originates from the Moser iterative technique [33].

Lemma 3.5. *Suppose that $\{u_\eta\}$ is a set of positive solutions to problem* (3.5)*, satisfying $\sup_{\eta\geq 1}\{\Phi_\eta(u_\eta)\} < \frac{1}{\theta}S^{\frac{3}{p}}$. Then there exists $\Lambda > 0$ such that $|u_\eta|_{L^\infty(\mathbb{R}^3\setminus\Omega'_\Gamma)} \leq \mathscr{A}$ for all $\eta \geq \Lambda$. Furthermore, u_η solves the original system* (1.1)*.*

Proof. Let $u_n(x) = u_{\eta_n}(x)$ for $\eta_n \to \infty$. For any $R > 0$ and $0 < r \leq \frac{R}{2}$, choose $\zeta \in C^\infty(\mathbb{R}^3)$, with $0 \leq \zeta \leq 1$, where $\zeta(x) = 1$ for $|x| \geq R$, $\zeta(x) = 0$ for $|x| \leq R - r$, and $|\nabla\zeta| \leq \frac{2}{r}$. Then for every $n \in \mathbb{N}$ and $\rho > 0$, set

$$u_{\rho,n}(x) := \begin{cases} u_n(x) & \text{if } u_n(x) \leq \rho, \\ \rho & \text{if } u_n(x) \geq \rho, \end{cases}$$

and define $z_{\rho,n} := \zeta^p u_{\rho,n}^{p(\sigma-1)} u_n$, $w_{\rho,n} := \zeta u_n u_{\rho,n}^{\sigma-1}$, where $\sigma > 1$ will be specified subsequently. Selecting the test function $\{z_{\rho,n}\}$, we deduce that

$$\begin{aligned} 0 = &\int_{\mathbb{R}^3} \Big(|\nabla u_n|^{p-2} \nabla u_n \nabla z_{\rho,n} + (\eta Z(x) + 1) |u_n|^{p-2} u_n z_{\rho,n} \Big) dx \\ &+ \int_{\mathbb{R}^3} Q(x) \phi_{u_n}^a u_n z_{\rho,n} dx - \int_{\mathbb{R}^3} f(x, u_n) z_{\rho,n} dx, \end{aligned}$$

i.e.,

$$\begin{aligned} 0 = &\int_{\mathbb{R}^3} \Big(|\nabla u_n|^p \zeta^p u_{\rho,n}^{p(\sigma-1)} + p |\nabla u_n|^{p-2} \nabla u_n \nabla \zeta \zeta^{p-1} u_n u_{\rho,n}^{p(\sigma-1)} \\ &+ p(\sigma-1) |\nabla u_n|^{p-2} \nabla u_n \nabla u_{\rho,n} u_{\rho,n}^{p(\sigma-1)-1} \zeta^p \Big) dx + \int_{\mathbb{R}^3} (\eta Z(x) + 1) |u_n|^p \zeta^p u_{\rho,n}^{p(\sigma-1)} dx \\ &+ \int_{\mathbb{R}^3} Q(x) \phi_{u_n}^a u_n^2 \zeta^p u_{\rho,n}^{p(\sigma-1)} dx - \int_{\mathbb{R}^3} f(x, u_n) \zeta^p u_{\rho,n}^{p(\sigma-1)} u_n dx. \end{aligned}$$

We rewrite the above equality as

$$\begin{aligned} \int_{\mathbb{R}^3} |\nabla u_n|^p \zeta^p u_{\rho,n}^{p(\sigma-1)} dx = &\int_{\mathbb{R}^3} f(x, u_n) \zeta^p u_{\rho,n}^{p(\sigma-1)} u_n dx - p \int_{\mathbb{R}^3} |\nabla u_n|^{p-2} \nabla u_n \nabla \zeta \zeta^{p-1} u_n u_{\rho,n}^{p(\sigma-1)} dx \\ &- p(\sigma-1) \int_{\mathbb{R}^3} |\nabla u_n|^{p-2} \nabla u_n \nabla u_{\rho,n} u_{\rho,n}^{p(\sigma-1)-1} \zeta^p dx \\ &- \int_{\mathbb{R}^3} (\eta Z(x) + 1) |u_n|^p \zeta^p u_{\rho,n}^{p(\sigma-1)} dx - \int_{\mathbb{R}^3} Q(x) \phi_{u_n}^a u_n^2 \zeta^p u_{\rho,n}^{p(\sigma-1)} dx. \end{aligned}$$

From (f_1)–(f_2), for any $\tau > 0$, there exists $D_\tau > 0$ such that $f(x, s) \le \tau s + D_\tau s^{p^*-1}$ for all $(x, s) \in \mathbb{R}^3 \times [0, +\infty)$. Selecting τ small enough, one has

$$\int_{\mathbb{R}^3} |\nabla u_n|^p \zeta^p u_{\rho,n}^{p(\sigma-1)} dx \le D_\tau \int_{\mathbb{R}^3} u_n^{p^*} \zeta^p u_{\rho,n}^{p(\sigma-1)} dx - p \int_{\mathbb{R}^3} |\nabla u_n|^{p-2} \nabla u_n \nabla \zeta \zeta^{p-1} u_n u_{\rho,n}^{p(\sigma-1)} dx.$$

For every $\epsilon > 0$, applying Young's inequality, one has

$$\begin{aligned} \int_{\mathbb{R}^3} |\nabla u_n|^p \zeta^p u_{\rho,n}^{p(\sigma-1)} dx = &D_\tau \int_{\mathbb{R}^3} u_n^{p^*} \zeta^p u_{\rho,n}^{p(\sigma-1)} dx \\ &+ p\epsilon \int_{\mathbb{R}^3} |\nabla u_n|^p u_{\rho,n}^{p(\sigma-1)} \zeta^p dx + pC_\epsilon \int_{\mathbb{R}^3} |u_n|^p |\nabla \zeta|^p u_{\rho,n}^{p(\sigma-1)} dx. \end{aligned}$$

Choosing $\epsilon \in (0, \frac{1}{p})$, we obtain

$$\int_{\mathbb{R}^3} |\nabla u_n|^p \zeta^p u_{\rho,n}^{p(\sigma-1)} dx \le C \int_{\mathbb{R}^3} u_n^{p^*} \zeta^p u_{\rho,n}^{p(\sigma-1)} dx + C \int_{\mathbb{R}^3} |u_n|^p |\nabla \zeta|^p u_{\rho,n}^{p(\sigma-1)} dx. \tag{3.26}$$

From the Sobolev and the Hölder inequalities, we have

$$\begin{aligned} |w_{\rho,n}|_{p^*}^p &\le C \int_{\mathbb{R}^3} |\nabla w_{\rho,n}|^p dx = C \int_{\mathbb{R}^3} |\nabla(\zeta u_n u_{\rho,n}^{\sigma-1})|^p dx \\ &\le C\sigma^p \left(\int_{\mathbb{R}^3} |u_n|^p |\nabla \zeta|^p u_{\rho,n}^{p(\sigma-1)} dx + \int_{\mathbb{R}^3} |\nabla u_n|^p \zeta^p u_{\rho,n}^{p(\sigma-1)} dx \right). \end{aligned} \tag{3.27}$$

Combining (3.26) and (3.27), we obtain

$$|w_{\rho,n}|_{p^*}^p \le C\sigma^p \left(\int_{\mathbb{R}^3} |u_n|^p |\nabla\zeta|^p u_{\rho,n}^{p(\sigma-1)} dx + \int_{\mathbb{R}^3} u_n^{p^*} \zeta^p u_{\rho,n}^{p(\sigma-1)} dx \right). \tag{3.28}$$

Choosing $\sigma = \frac{p^*}{p}$, by the definition of $w_{\rho,n}$ and the Hölder inequality, we rewrite the inequality (3.28) as

$$\begin{aligned} \left(\int_{\mathbb{R}^3} \left(\zeta u_n u_{\rho,n}^{\frac{p^*-p}{p}} \right)^{p^*} dx \right)^{\frac{p}{p^*}} &\le C \left(\left(\int_{\mathbb{R}^3} \left(\zeta u_n u_{\rho,n}^{\frac{p^*-p}{p}} \right)^{p^*} dx \right)^{\frac{p}{p^*}} \left(\int_{|x|>R-r} u_n^{p^*} dx \right)^{\frac{p^*-p}{p^*}} \right. \\ &\quad \left. + \int_{\mathbb{R}^3} |u_n|^p |\nabla\zeta|^p u_{\rho,n}^{p^*-p} dx \right) \\ &\le C \left(\left(\int_{\mathbb{R}^3} \left(\zeta u_n u_{\rho,n}^{\frac{p^*-p}{p}} \right)^{p^*} dx \right)^{\frac{p}{p^*}} \|u_n\|_{p^*(|x|\ge\frac{R}{2})}^{p^*-p} \right. \\ &\quad \left. + \int_{\mathbb{R}^3} |u_n|^p |\nabla\zeta|^p u_{\rho,n}^{p^*-p} dx \right). \end{aligned}$$

Let R be sufficiently large, and together with $u_n \to u$ in $W^{1,p}(\mathbb{R}^3)$, we obtain that

$$\|u_n\|_{p^*(|x|\ge\frac{R}{2})}^{p^*-p} \le \frac{1}{C} \quad \text{uniformly in } n.$$

Obviously, since $|x| \ge R$ implies $\zeta(x) = 1$, we have

$$\begin{aligned} \left(\int_{|x|\ge R} \left(u_n u_{\rho,n}^{\frac{p^*-p}{p}} \right)^{p^*} dx \right)^{\frac{p}{p^*}} &\le \left(\int_{\mathbb{R}^3} \left(\zeta u_n u_{\rho,n}^{\frac{p^*-p}{p}} \right)^{p^*} dx \right)^{\frac{p}{p^*}} \\ &\le C \int_{\mathbb{R}^3} |\nabla\zeta|^p |u_n|^p u_{\rho,n}^{p^*-p} dx \\ &\le \frac{C}{r^p} \int_{\mathbb{R}^3} |u_n|^{p^*} dx. \end{aligned}$$

By applying Fatou's lemma with respect to the variable ρ, we obtain

$$u_n \in L^{(p^*)^2}(|x| \ge R) \quad \text{for } R \text{ large enough.} \tag{3.29}$$

Next, set $\sigma = \frac{p^*(t-1)}{pt}$, where $t = \frac{(p^*)^2}{(p^*-p)p}$, so that $\sigma > 1$, $\frac{pt}{t-1} < p^*$. Now suppose that

$$u_n \in L^{\frac{p\sigma t}{t-1}}(|x| \ge R - r) \quad \text{for some } \sigma \ge 1.$$

From (3.28), the Hölder inequality with exponent $\frac{t}{t-1}$ yields

$$\begin{aligned} |w_{\rho,n}|_p^{p^*} &\le C\sigma^p \left(\left(\int_{|x|\ge R-r} (\zeta^p u_n^{p\sigma})^{\frac{t}{t-1}} dx \right)^{\frac{t-1}{t}} \left(\int_{|x|\ge R-r} u_n^{\frac{p^*-p}{t}} dx \right)^{\frac{1}{t}} \right. \\ &\quad \left. + \frac{(R^3 - (R-r)^3)^{\frac{1}{t}}}{r^p} \left(\int_{|x|\ge R-r} u_n^{\frac{p\sigma t}{t-1}} dx \right)^{\frac{t-1}{t}} \right) \\ &\le C\sigma^p \left(1 + \frac{R^{\frac{3}{t}}}{r^p} \right) \left(\int_{|x|\ge R-r} u_n^{\frac{p\sigma t}{t-1}} dx \right)^{\frac{t-1}{t}}. \end{aligned} \tag{3.30}$$

Taking the limit as $\rho \to +\infty$ in (3.30), we derive

$$\|u_n\|^{p\sigma}_{p^*\sigma(|x|\geq R)} \leq C\sigma^p \left(1 + \frac{R^{\frac{3}{t}}}{r^p}\right) \|u_n\|^{p\sigma}_{\frac{p\sigma t}{t-1}(|x|\geq R-r)}.$$

With the definition of $\chi := \frac{p^*(t-1)}{pt}$, $s := \frac{pt}{t-1}$, we obtain

$$\|u_n\|_{L^{\sigma\chi s}(|x|\geq R)} \leq C^{\frac{1}{p\sigma}} \sigma^{\frac{1}{\sigma}} \left(1 + \frac{R^{\frac{3}{t}}}{r^p}\right)^{\frac{1}{p\sigma}} \|u_n\|_{L^{\sigma s}(|x|\geq R-r)}. \tag{3.31}$$

Choosing $\sigma = \chi^m$ $(m = 1, 2, \cdots)$, we have

$$\|u_n\|_{L^{\chi^{m+1}s}(|x|\geq R)} \leq C^{\frac{1}{p\chi^m}} \chi^{\frac{m}{\chi^m}} \left(1 + \frac{R^{\frac{3}{t}}}{r^p}\right)^{\frac{1}{p\chi^m}} \|u_n\|_{L^{\chi^m s}(|x|\geq R-r)}. \tag{3.32}$$

Note that $p > \frac{12}{7} > \frac{3}{2}$. Hence, if we set $r_m := 2^{-(m+1)}R$, then iterating inequality (3.31) m times yields

$$\begin{aligned}\|u_n\|_{L^{\chi^{m+1}s}(|x|\geq R)} &\leq \|u_n\|_{L^{\chi^{m+1}s}(|x|\geq R-r_{m+1})} \\ &\leq C^{\frac{1}{p}\sum_{i=1}^m \chi^{-i}} \chi^{\sum_{i=1}^m i\chi^{-i}} \exp\left(\sum_{i=1}^m \frac{\ln(1+2^{p(i+1)})}{p\chi^i}\right) \|u_n\|_{L^{p^*}(|x|\geq \frac{R}{2})}.\end{aligned}$$

Since $\chi > 1$, all the series above converge as $m \to \infty$, and the left-hand side converges to $\|u_n\|_{L^\infty(|x|\geq R)}$. Therefore

$$\|u_n\|_{L^\infty(|x|\geq R)} \leq C\|u_n\|_{L^{p^*}(|x|\geq \frac{R}{2})}.$$

According to the convergence of $\{u_n\}$ to u in $W^{1,p}(\mathbb{R}^3)$, for any fixed $\mathscr{A} > 0$, there exists $R > 0$ such that $\|u_n\|_{L^\infty(|x|\geq R)} \leq \mathscr{A}$ for all $n \in \mathbb{N}$. Consequently, $\lim_{|x|\to\infty} u_n(x) = 0$ uniformly in n. □

4. The proofs of the main conclusions

We now turn our attention to the problems for $j \in \Gamma$ as follows:

$$\begin{cases} -\Delta_p u + |u|^{p-2}u + Q(x)\phi_u^a u = \alpha g(u) + |u|^{p^*-1} & \text{in } \Omega_j, \\ -\Delta\phi_u^a + a^2\Delta^2\phi_u^a = Q(x)u^2 & \text{in } \Omega_j, \\ u = 0, \quad \phi_u^a = 0 & \text{on } \partial\Omega_j \end{cases} \tag{4.1}$$

and

$$\begin{cases} -\Delta_p u + (\eta Z(x) + 1)|u|^{p-2}u + Q(x)\phi_u^a u = \alpha g(u) + |u|^{p^*-1} & \text{in } \Omega_j', \\ -\Delta\phi_u^a + a^2\Delta^2\phi_u^a = Q(x)u^2 & \text{in } \Omega_j', \\ \frac{\partial u}{\partial n} = 0, \quad \phi_u^a = 0 & \text{on } \partial\Omega_j', \end{cases} \tag{4.2}$$

corresponding to the following functionals:

$$I_j(u) = \frac{1}{p}\int_{\Omega_j} (|\nabla u|^p + |u|^p)dx + \frac{1}{4}\int_{\Omega_j} Q(x)\phi_u^a u^2 dx - \alpha\int_{\Omega_j} G(u)dx - \frac{1}{p^*}\int_{\Omega_j} (u_+)^{p^*}dx \tag{4.3}$$

and

$$\Phi_{\eta,j}(u) = \frac{1}{p}\int_{\Omega_j'}(|\nabla u|^p + (\eta Z(x)+1)|u|^p)dx + \frac{1}{4}\int_{\Omega_j'} Q(x)\phi_u^a u^2 dx - \alpha\int_{\Omega_j'} G(u)dx - \frac{1}{p^*}\int_{\Omega_j'}(u_+)^{p^*}dx. \tag{4.4}$$

It is readily verifiable that both I_j and $\Phi_{\eta,j}$ exhibit the mountain pass geometry structure. Note that

(i) $I_j(0) = \Phi_{\eta,j}(0) = 0$.

(ii) There exist $\varrho_0 > 0$ and a constant $\varrho_1 > 0$ independent of $\eta \geq 0$ such that

$$\|u\|_{0,\Omega_j} = \varrho_0 \implies I_j(u) \geq \varrho_1 > 0, \tag{4.5}$$

$$\|u\|_{\eta,\Omega_j'} = \varrho_0 \implies \Phi_{\eta,j}(u) \geq \varrho_1 > 0. \tag{4.6}$$

It is noted that

$$\|u\|_{0,\Omega_j} = \int_{\Omega_j}(|\nabla u|^p + |u|^p)dx, \quad \forall u \in W_0^{1,p}(\Omega_j).$$

(iii) There exists $\tilde{\varphi}_j(x) \in C_0^\infty(\Omega_j)$ such that

$$\|\tilde{\varphi}_j(x)\|_{\eta,\Omega_j'} = \|\tilde{\varphi}_j(x)\|_{0,\Omega_j} \geq \varrho_1, \quad \Phi_{\eta,j}(\tilde{\varphi}_j) = I_j(\tilde{\varphi}_j) < 0.$$

Alternatively, we consider the minimization problems of I_j and $\Phi_{\eta,j}$ over the Nehari manifold

$$\mathcal{N}_j = \left\{\gamma \in C([0,1], W_0^{1,p}(\Omega_j)) : \gamma(0) = 0,\ I_j(\gamma(1)) < 0\right\}$$

and

$$\mathcal{N}_{\eta,j} = \left\{\gamma \in C([0,1], W_0^{1,p}(\Omega_j')) : \gamma(0) = 0,\ \Phi_{\eta,j}(\gamma(1)) < 0\right\},$$

where the attained values can be obtained by the two nonnegative functions $\tilde{w}_j \in \mathcal{N}_j$ and $\tilde{w}_{\eta,j} \in \mathcal{N}_{\eta,j}$, respectively. Moreover, the $(PS)_c$ condition ensures that the aforementioned twofunctionals admit $\tilde{w}_j$ and $\tilde{w}_{\eta,j}$ satisfying

$$I_j(\tilde{w}_j) = c_j, \quad I_j'(\tilde{w}_j) = 0,$$

$$\Phi_{\eta,j}(\tilde{w}_{\eta,j}) = c_{\eta,j}, \quad \Phi_{\eta,j}'(\tilde{w}_{\eta,j}) = 0.$$

In the subsequent discussion, the minimax levels associated with I_j and $\Phi_{\eta,j}$ are denoted, respectively, by c_j and $c_{\eta,j}$, satisfying the following estimates:

$$c_j := \inf_{\gamma\in\mathcal{N}_j}\max_{t\in[0,1]} I_j(\gamma(t)) \quad \text{and} \quad c_{\eta,j} := \inf_{\gamma\in\mathcal{N}_{\eta,j}}\max_{t\in[0,1]} \Phi_{\eta,j}(\gamma(t)).$$

Consequently, we now give the following lemma.

Lemma 4.1. *(See Ding and Tanaka [32]) It holds that*

(a) there exists $\varrho_1 > 0$ such that $0 < \varrho_1 \leq c_{\eta,j} \leq c_j$ for $\eta \geq 0$ sufficiently large;

(b) $c_j = \max_{r>0} I_j(r\tilde{w}_j)$, $\quad c_{\eta,j} = \max_{r>0}\Phi_{\eta,j}(r\tilde{w}_{\eta,j})$;

(c) c_j *and* $c_{\eta,j}$ *are, respectively, the least-energy level for* $I_j(u)$ *and* $\Phi_{\eta,j}(u)$,

$$c_j = \inf\{I_j(u) : u \in W_0^{1,p}(\Omega_j) \setminus \{0\},\ \langle I_j'(u), u\rangle = 0\},$$

$$c_{\eta,j} = \inf\{\Phi_{\eta,j}(u) : u \in W_0^{1,p}(\Omega_j') \setminus \{0\},\ \langle \Phi_{\eta,j}'(u), u\rangle = 0\};$$

(d) $c_{\eta,j} \to c_j$ *as* $\eta \to \infty$.

Next, fix $R > 1$ such that

$$\left|I_j\left(\frac{1}{R}\tilde{w}_j\right)\right| < \frac{1}{2}c_j \quad \text{for any } j \in \Gamma \tag{4.7}$$

and

$$\left|I_j(R\tilde{w}_j) - c_j\right| \geq 1 \quad \text{for any } j \in \Gamma. \tag{4.8}$$

According to the definition of c_j, we obtain the following equality:

$$\max_{t\in[\frac{1}{R^2},1]} I_j(tR\tilde{w}_j) = c_j \quad \text{for all } j \in \Gamma,$$

where it is convenient for our purposes to choose the interval $[\frac{1}{R^2}, 1]$.

Let $\Gamma = \{1, 2, \ldots, \ell\}$ $(\ell \leq k)$,

$$\gamma_0(t_1, t_2, \ldots, t_\ell)(x) = \sum_{j=1}^{\ell} t_j R\tilde{w}_j(x) \quad \text{for all } (t_1, t_2, \ldots, t_\ell) \in \Big[\frac{1}{R^2}, 1\Big]^\ell, \tag{4.9}$$

and

$$\tilde{\Lambda} := \Big\{\gamma \in C([\frac{1}{R^2}, 1]^\ell, E_\eta) : \gamma = \gamma_0 \text{ on } \partial([\frac{1}{R^2}, 1]^\ell)\Big\},$$

where

$$b_{\eta,\Gamma} = \inf_{\gamma\in\tilde{\Lambda}} \max_{(t_1,\ldots,t_\ell)\in[\frac{1}{R^2},1]^\ell} \Phi_\eta(\gamma(t_1, t_2, \ldots, t_\ell)).$$

We note that $\gamma_0 \in \tilde{\Lambda}$. Obviously, $\tilde{\Lambda} \neq \emptyset$ and $b_{\eta,\Gamma}$ is well-defined.

Lemma 4.2. *Let* $\gamma \in \tilde{\Lambda}$*, and there exists* $(s_1, s_2, \ldots, s_\ell) \in (\frac{1}{R^2}, 1)^\ell$ *such that*

$$\Phi_{\eta,j}'(\gamma(s_1, s_2, \ldots, s_\ell))(\gamma(s_1, s_2, \ldots, s_\ell)) = 0 \quad \textit{for all } j \in \{1, 2, \ldots, \ell\}.$$

Proof. For fixed $\gamma \in \tilde{\Lambda}$, define $\bar{\gamma} : [\frac{1}{R^2}, 1]^\ell \to \mathbb{R}^\ell$ by

$$\bar{\gamma}(t_1, t_2, \ldots, t_\ell) = (\Phi_{\eta,1}'(\gamma(t))(\gamma(t)), \Phi_{\eta,2}'(\gamma(t))(\gamma(t)), \ldots, \Phi_{\eta,\ell}'(\gamma(t))(\gamma(t))),$$

where $\Phi_{\eta,j}'(\gamma(t))(\gamma(t)) = \langle \Phi_{\eta,j}'(\gamma(t_1, t_2, \ldots, t_\ell)), \gamma(t_1, t_2, \ldots, t_\ell)\rangle$ for all $j \in \Gamma$.

For any $(t_1, t_2, \ldots, t_\ell) \in \partial([\frac{1}{R^2}, 1]^\ell)$, we have $\gamma(t_1, t_2, \ldots, t_\ell) = \gamma_0(t_1, t_2, \ldots, t_\ell)$. Moreover, for each j, if $t_j = \frac{1}{R^2}$, then $\Phi_{\eta,j}'(\gamma_0(t))(\gamma_0(t)) > 0$, and if $t_j = 1$, then $\Phi_{\eta,j}'(\gamma_0(t))(\gamma_0(t)) < 0$. Hence $(0, 0, \ldots, 0) \notin \bar{\gamma}(\partial[\frac{1}{R^2}, 1]^\ell)$.

Consequently, by the topological degree theory, it follows that

$$\deg\Big(\bar{\gamma}, \Big(\frac{1}{R^2}, 1\Big)^\ell, (0, 0, \ldots, 0)\Big) = (-1)^\ell \neq 0.$$

Thus, there exists $(s_1, s_2, \ldots, s_\ell) \in (\frac{1}{R^2}, 1)^\ell$ satisfying

$$\Phi'_{\eta,j}(\gamma(s_1, s_2, \ldots, s_\ell))(\gamma(s_1, s_2, \ldots, s_\ell)) = 0 \quad \text{for } j \in \{1, 2, \ldots, \ell\}.$$

This completes the proof. □

Next, we define $c_\Gamma = \sum_{j=1}^{\ell} c_j$, which is vital to the proof of Theorem 1.2. From now on, we intend to analyze the interaction among $\sum_{j=1}^{\ell} c_{\eta,j}$, $b_{\eta,\Gamma}$, and c_Γ, where we have $c_\Gamma \in (0, \frac{1}{p}S^{\frac{3}{p}})$.

Lemma 4.3. *We present the results as follows:*

(a) $\displaystyle\sum_{j=1}^{\ell} c_{\eta,j} \leq b_{\eta,\Gamma} \leq c_\Gamma$ *for all* $\eta \geq 1$*;*

(b) $\Phi_\eta(\gamma(t_1, t_2, \ldots, t_\ell)) < c_\Gamma$ *for all* $\eta \geq 1$, $\gamma \in \tilde{\Lambda}$, *and* $(t_1, t_2, \ldots, t_\ell) \in \partial([\frac{1}{R^2}, 1]^\ell)$.

Proof. Since γ_0 defined in (4.9) belongs to $\tilde{\Lambda}$, we have

$$\begin{aligned} b_{\eta,\Gamma} &= \max_{(t_1,\ldots,t_\ell)\in[\frac{1}{R^2},1]^\ell} \Phi_\eta(\gamma_0(t_1, t_2, \ldots, t_\ell)) \\ &= \max_{(t_1,\ldots,t_\ell)\in[\frac{1}{R^2},1]^\ell} \sum_{j=1}^{\ell} I_j(t_j R\tilde{w}_j) = \sum_{j=1}^{\ell} c_j = c_\Gamma. \end{aligned}$$

Fix $(s_1, s_2, \ldots, s_\ell) \in (\frac{1}{R^2}, 1)^\ell$ as established in Lemma 4.2. By the characterization of $c_{\eta,j}$,

$$c_{\eta,j} = \inf\{\Phi_{\eta,j}(v) : v \in W_0^{1,p}(\Omega'_j) \setminus \{0\},\ \langle \Phi'_{\eta,j}(v), v\rangle = 0\},$$

and it follows that $\Phi_{\eta,j}(\gamma(s_1, s_2, \ldots, s_\ell)) \geq c_{\eta,j}$ for all $j \in \Gamma$.

Conversely, we decompose the global functional as

$$\Phi_\eta(u) = \Phi_{\eta,\mathbb{R}^3\setminus\Omega'_\Gamma}(u) + \sum_{j=1}^{\ell} \Phi_{\eta,\Omega'_j}(u),$$

where

$$\Phi_{\eta,\mathbb{R}^3\setminus\Omega'_\Gamma}(u) := \frac{1}{p}\int_{\eta,\mathbb{R}^3\setminus\Omega'_\Gamma} (|\nabla u|^p + (\eta Z(x) + 1)|u|^p)dx + \frac{1}{4}\int_{\eta,\mathbb{R}^3\setminus\Omega'_\Gamma} Q(x)\phi_u^a u^2 dx - \int_{\eta,\mathbb{R}^3\setminus\Omega'_\Gamma} \tilde{G}(u)dx.$$

From the definition of $\tilde{G}(u)$, we have $\tilde{G}(u) \leq \frac{v_0}{pk}|u|^p$. Together with the Sobolev embedding and the penalty term, we obtain

$$\begin{aligned} \Phi_{\eta,\mathbb{R}^3\setminus\Omega'_\Gamma}(u) &\geq \frac{1}{p}\|u\|^p_{\eta,\mathbb{R}^3\setminus\Omega'_\Gamma} - \frac{v_0}{p}\|u\|^p_{L^p(\mathbb{R}^3\setminus\Omega'_\Gamma)} \\ &\geq \frac{\delta_0}{p}\|u\|^p_{\eta,\mathbb{R}^3\setminus\Omega'_\Gamma} \geq 0. \end{aligned}$$

Thus

$$\Phi_\eta(u) = \Phi_{\eta,\mathbb{R}^3\setminus\Omega'_\Gamma}(u) + \sum_{j=1}^{\ell} \Phi_{\eta,\Omega'_j}(u) \geq \sum_{j=1}^{\ell} \Phi_{\eta,\Omega'_j}(u)$$

$$\geq \sum_{j=1}^{\ell} \inf\{\Phi_{\eta,j}(u) : u \in E_\eta \setminus \{0\},\ \langle \Phi'_{\eta,j}(u), u\rangle = 0\} = \sum_{j=1}^{\ell} c_{\eta,j}.$$

Therefore, from the definition of $b_{\eta,\Gamma}$, we have $b_{\eta,\Gamma} \geq \sum_{j=1}^{\ell} c_{\eta,j}$. This completes the proof of (a).

Since $\gamma(t_1, t_2, \ldots, t_\ell) = \gamma_0(t_1, t_2, \ldots, t_\ell)$ on $\partial([\frac{1}{R^2}, 1]^\ell)$, we have

$$\Phi_\eta(\gamma(t_1, t_2, \ldots, t_\ell)) = \sum_{j=1}^{\ell} I_j(t_j R\tilde{w}_j).$$

Moreover, for every $j \in \Gamma$, $I_j(t_j R\tilde{w}_j) \leq c_j$, and for at least one index $j_0 \in \Gamma$, $t_{j_0} \in \{\frac{1}{R^2}, 1\}$ implies $I_{j_0}(t_{j_0} R\tilde{w}_{j_0}) < \frac{c_{j_0}}{2}$. Hence there exists $\varepsilon > 0$ such that

$$\Phi_\eta(\gamma(t_1, t_2, \ldots, t_\ell)) \leq c_\Gamma - \varepsilon,$$

which finishes the proof of (b). □

Lemma 4.4 ([31], Corollary 5.5)**.** *We have the following results:*

(a) $b_{\eta,\Gamma} \to c_\Gamma$ *as* $\eta \to \infty$*;*
(b) $b_{\eta,\Gamma}$ *corresponds to a critical value of* Φ_η *for sufficiently large* η*.*

The proof of Theorem 1.2 requires constructing a family of positive solutions u_η such that, as $\eta \to \infty$, u_η converges to a least-energy solution in each Ω_j ($j \in \Gamma$) and vanishes elsewhere. To achieve this goal, we proceed by establishing two key auxiliary lemmas. Combined with the uniform estimates derived in the previous section, these lemmas allow us to complete the full proof of Theorem 1.2.

From now on, we define

$$\hbar := 1 + \sum_{j=1}^{k} \sqrt{\left(\frac{1}{p} - \frac{1}{\theta}\right)^{-1} c_j}, \quad \bar{B}_{\hbar+1}(0) := \{u \in E_\eta : \|u\|_\eta \leq \hbar + 1\}$$

and we select

$$0 < \varepsilon < \frac{1}{3} \min_{j\in\{1,2,\ldots,\ell\}} c_j. \tag{4.10}$$

Let

$$D_\eta^\varepsilon = \{u \in \bar{B}_{\hbar+1}(0) :\ \|u\|_{\eta,\mathbb{R}^3\setminus\Omega'_j} \leq \varepsilon,\ \left|\Phi_{\eta,\Omega'_j}(u) - c_j\right| \leq \varepsilon,\ \text{for all } j \in \Gamma\}$$

and

$$\Phi_\eta^{c_\Gamma} := \{u \in E_\eta :\ \Phi_\eta(u) \leq c_\Gamma\}.$$

We observe that the set $D_\eta^\varepsilon \cap \Phi_\eta^{c_\Gamma}$ encompasses all functions of the form

$$\tilde{w}(x) = \begin{cases} \tilde{w}_j(x) & \text{in } \Omega_j, \\ 0 & \text{in } \mathbb{R}^3 \setminus \bar{\Omega}_\Gamma. \end{cases}$$

We now state the following lemma.

Lemma 4.5. *There exist $\sigma_0 > 0$ and $\Lambda_0 \geq 0$ independent of η such that*

$$\|\Phi'_\eta(u)\|_\eta \geq \sigma_0 \quad \text{for } \eta \geq \Lambda_0 \quad \text{and for all } u \in (D_\eta^{2\varepsilon} \setminus D_\eta^{\varepsilon}) \cap \Phi_\eta^{c_\Gamma}. \tag{4.11}$$

Proof. We proceed by contradiction. Assume there exist $\eta_n \to \infty$ and $u_n \in (D_{\eta_n}^{2\varepsilon} \setminus D_{\eta_n}^{\varepsilon}) \cap \Phi_{\eta_n}^{c_\Gamma}$, satisfying $\|\Phi'_{\eta_n}(u_n)\|_{\eta_n} \to 0$. The hypothesis $u_n \in D_{\eta_n}^{\varepsilon}$ yields the boundedness of $\{u_n\}$ in E_η and of $\Phi_{\eta_n}(u_n)$ as $n \to \infty$. After subsequence extraction, we may assume that $\Phi_{\eta_n}(u_n) \to c \leq c_\Gamma$.

By invoking Lemma 3.4, the sequence u_n admits a subsequence, where $u_n \to u$ in $W^{1,p}(\mathbb{R}^3)$ and

$$\lim_{n\to\infty} \Phi_{\eta_n}(u_n) = \sum_{j=1}^{\ell} I_j(u) \leq c_\Gamma, \tag{4.12}$$

$$u_n \to u \quad \text{in } W^{1,p}(\mathbb{R}^3), \tag{4.13}$$

$$\int_{\mathbb{R}^3} \eta_n Z(x)|u_n|^p dx \to 0, \tag{4.14}$$

$$\|u_n\|_{\eta_n, \mathbb{R}^3 \setminus \Omega_\Gamma} \to 0. \tag{4.15}$$

Since $c_\Gamma = \sum_{j=1}^{\ell} c_j$ and each c_j represents the least energy-level for $I_j(u)$, we encounter the following two scenarios:

(a) $I_j(u|_{\Omega_j}) = c_j$ for any $j \in \Gamma$.
(b) $I_{j_0}(u|_{\Omega_{j_0}}) = 0$, that is, $u|_{\Omega_{j_0}} = 0$ for some $j_0 \in \Gamma$.

Assume (a) occurs and we have

$$\frac{1}{p}\int_{\Omega_j} (|\nabla u|^p + |u|^p)dx + \frac{1}{4}\int_{\Omega_j} Q(x)\phi_u^a u^2 dx - \alpha \int_{\Omega_j} g(u)dx - \frac{1}{p^*}\int_{\Omega_j} u^{p^*} dx = c_j$$

for every $j \in \Gamma$. Combining (4.12)–(4.15) with $u_n \in D_{\eta_n}^{\varepsilon}$ for n sufficiently large, we derive a contradiction to $u_n \in (D_\eta^{2\varepsilon} \setminus D_\eta^{\varepsilon})$.

If case (b) arises, from (4.13) and (4.14), it holds that $|\Phi_{\eta,j}(u_n) - c_{j_0}| \to c_{j_0} \geq 3\varepsilon$. This also contradicts $u_n \in (D_\eta^{2\varepsilon} \setminus D_\eta^{\varepsilon})$, thereby finalizing the proof. □

The subsequent lemma serves as the core component in proving the main result.

Lemma 4.6. *Let ε be a constant satisfying* (4.10)*, and let N_0 be the constant obtained from Lemma 4.5. Then, for every $\eta \geq \Lambda_0$, we obtain a solution u_η of problem* (3.5) *such that $u_\eta \in D_\eta^{\varepsilon} \cap \Phi_\eta^{c_\Gamma}$.*

Proof. We argue by contradiction and suppose there exist no critical points in $D_\eta^{\varepsilon} \cap \Phi_\eta^{c_\Gamma}$. The Palais-Smale condition for $\Phi_\eta(u)$ on $(0, \frac{1}{\theta}(aS)^{\frac{3}{p}})$ implies the existence of $a_\eta > 0$ with $\|\Phi'_\eta(u)\| \geq a_\eta$ for all $u \in D_\eta^{\varepsilon} \cap \Phi_\eta^{c_\Gamma}$. Moreover, by Lemma 4.5, we have $\|\Phi'_\eta(u)\| \geq \sigma_0$ for all $u \in (D_\eta^{2\varepsilon} \setminus D_\eta^{\varepsilon}) \cap \Phi_\eta^{c_\Gamma}$.

Consider a Lipschitz continuous functional $\tilde{\varphi} : E_\eta \to \mathbb{R}$ that verifies:

$$\tilde{\varphi}(u) = \begin{cases} 1 & \text{for } u \in D_\eta^{\frac{3\varepsilon}{2}}, \\ 0 & \text{for } u \notin D_\eta^{2\varepsilon}, \end{cases}$$

and for any $u \in E_\eta$, we have $0 \le \tilde{\varphi}(u) \le 1$. For any $u \in \Phi_\eta^{c_\Gamma}$, we define the vector field

$$W(u) := -\tilde{\varphi}(u)\frac{\Phi_\eta'(u)}{\|\Phi_\eta'(u)\|_\eta}, \quad \Phi_\eta^{c_\Gamma} \to E_\eta.$$

Via the Riesz representation theorem, we identify E_η with its dual E_η^*. We study the continuous deformation flow $\beta : [0,\infty) \times \Phi_\eta^{c_\Gamma} \to \Phi_\eta^{c_\Gamma}$ defined by

$$\frac{d\beta}{dt} = W(\beta(s,u)), \quad \beta(0,u) = u \in \Phi_\eta^{c_\Gamma}.$$

Consequently, the flow β possesses the following properties:

$$\frac{d}{dt}\Phi_\eta(\beta(s,u)) = -\tilde{\varphi}(\beta(s,u))\|\Phi_\eta'(\beta(s,u))\|_\eta \le 0, \tag{4.16}$$

$$\left\|\frac{d\beta}{dt}\right\|_\eta \le 1 \quad \text{for all } s,u, \tag{4.17}$$

$$\beta(s,u) = u \quad \text{for all } s \ge 0 \quad \text{and} \quad u \in \Phi_\eta^{c_\Gamma} \setminus D_\eta^{2\varepsilon}. \tag{4.18}$$

Let $\gamma_0(t_1,t_2,\dots,t_\ell) \in \tilde{\Lambda}$ be the path defined in (4.9). We examine $\beta(s,\gamma_0(t_1,t_2,\dots,t_\ell))$ for sufficiently large s. Since for all $(t_1,t_2,\dots,t_\ell) \in \partial([\frac{1}{R^2},1]^\ell)$, $\gamma_0(t_1,t_2,\dots,t_\ell) \notin D_\eta^{2\varepsilon}$, it follows from (4.18) that

$$\beta(s,\gamma_0(t_1,t_2,\dots,t_\ell)) = \gamma_0(t_1,t_2,\dots,t_\ell) \quad \text{for all } (t_1,t_2,\dots,t_\ell) \in \partial\Big(\Big[\frac{1}{R^2},1\Big]^\ell\Big)$$

and $\beta(s,\gamma_0(t_1,t_2,\dots,t_\ell)) \in \tilde{\Lambda}$ for all $s \ge 0$.

Since for all $(t_1,t_2,\dots,t_\ell) \in \partial([\frac{1}{R^2},1]^\ell)$, $\operatorname{supp}\gamma_0((t_1,t_2,\dots,t_\ell)(x)) \subset \bar{\Omega}_\Gamma$ is independent of $\eta \ge 0$, it follows that $\Phi_\eta(\gamma_0((t_1,t_2,\dots,t_\ell)(x)))$ and $\|\gamma_0((t_1,t_2,\dots,t_\ell)(x))\|_{\eta,j}$ are bounded uniformly in η. On the other hand, for all $(t_1,t_2,\dots,t_\ell) \in \partial([\frac{1}{R^2},1]^\ell)$, $\Phi_\eta(\gamma_0((t_1,t_2,\dots,t_\ell)(x))) \le c_\Gamma$, with equality if and only if $t_j = \frac{1}{R}$, where $(\gamma_0((t_1,t_2,\dots,t_\ell)(x)))|_{\Omega_j} = \tilde{w}_j$ for $j \in \Gamma$. Therefore, we obtain that

$$\tilde{m} := \max\{\Phi_\eta(u) : u \in \gamma_0([\frac{1}{R^2},1]^\ell) \setminus D_\eta^\varepsilon\} \tag{4.19}$$

is dependent on η and satisfies $\tilde{m} < c_\Gamma$. From (4.18), it is straightforward to deduce that for any $s > 0$,

$$\|\beta(0,\gamma_0(t_1,t_2,\dots,t_\ell)) - \beta(s,\gamma_0(t_1,t_2,\dots,t_\ell))\|_\eta \le s.$$

Under assumptions (g_1)–(g_4) and the regularity condition $\Phi_{\eta,j}(u) \in C^1(E_\eta)$ for every $j = 1,2,\dots,\ell$, there exists $\tilde{p} > 0$ such that for sufficiently large S (independent of η), and for all $j = 1,2,\dots,\ell$ and $s \in [0,S]$,

$$\|\Phi_{\eta,j}'(\beta(s,\gamma_0(t_1,t_2,\dots,t_\ell)))\|_\eta \le \tilde{p}. \tag{4.20}$$

We assert that for large enough S,

$$\max_{(t_1,t_2,\dots,t_\ell)\in\partial([\frac{1}{R^2},1]^\ell)} \Phi_{\eta,j}(\beta(S,\gamma_0(t_1,t_2,\dots,t_\ell)(x))) \le \max\Big\{\tilde{m}, c_\Gamma - \frac{\tau_0\varepsilon}{2}\Big\}, \tag{4.21}$$

where $\tilde{m}$ is defined in (4.19), and $\tau_0 = \max\{\sigma_0, \frac{\sigma_0}{\tilde{p}}\}$. In fact, if $\gamma_0(t_1,t_2,\dots,t_\ell)(x) \notin D_\eta^\varepsilon$, then by (4.20), we obtain $\Phi_\eta(\beta(S,\gamma_0(t_1,t_2,\dots,t_\ell)(x))) \le \tilde{m}$ and thus (4.21) is satisfied. Now we address the case where $\gamma_0(t_1,t_2,\dots,t_\ell)(x) \in D_\eta^\varepsilon$ and analyze $\tilde{\beta}(s) := \beta(s,\gamma_0(t_1,t_2,\dots,t_\ell))$.

We set $\tilde{a}_\eta := \min\{a_\eta,\sigma_0\}$ and $S = \frac{\sigma_0\varepsilon}{2\tilde{a}_\eta}$. There are two possible scenarios:

(a) $\tilde{\beta}(s) \in D_\eta^{\frac{3\varepsilon}{2}}$ for all $s \in [0, S]$;

(b) $\tilde{\beta}(s_0) \in \partial D_\eta^{\frac{3\varepsilon}{2}}$ for some $s_0 \in [0, S]$.

When case (a) is satisfied, we have $\varphi(\tilde{\beta}(s)) \equiv 1$ and $\|\Phi'_\eta(\tilde{\beta}(s))\|_\eta \geq \tilde{a}_\eta$ for all $s \in [0, S]$. Consequently, by (4.16), we obtain

$$\begin{aligned}\Phi_\eta(\tilde{\beta}(S)) &= \Phi_\eta(\gamma_0(t_1, t_2, \ldots, t_\ell)) + \int_0^S \frac{d}{dt}\Phi_\eta(\tilde{\beta}(s)) \\ &= \Phi_\eta(\gamma_0(t_1, t_2, \ldots, t_\ell)) - \int_0^S \Phi(\tilde{\beta}(t))\|\Phi'_\eta(\tilde{\beta}(t))\|_\eta dt \\ &\leq c_\Gamma - \int_0^S d_\eta dt = c_\Gamma - \tilde{a}_\eta S \\ &= c_\Gamma - \frac{1}{2}\sigma_0\varepsilon \leq c_\Gamma - \frac{1}{2}\tau_0\varepsilon.\end{aligned}$$

When case (b) is satisfied, there exist $0 \leq s_1 < s_2 \leq S$ such that

$$\tilde{\beta}(s_1) \in \partial D_\eta^\varepsilon, \tag{4.22}$$

$$\tilde{\beta}(s_2) \in \partial D_\eta^{\frac{3\varepsilon}{2}}, \tag{4.23}$$

$$\tilde{\beta}(s) \in D_\eta^{\frac{3\varepsilon}{2}} \setminus D_\eta^\varepsilon \quad \text{for all } s \in [s_1, s_2]. \tag{4.24}$$

It follows from (4.23) that, for some $j_0 \in \Gamma$,

$$\|\tilde{\beta}(s_2)\|_{\eta, \mathbb{R}^3 \setminus \Omega'_{j_0}} = \frac{3\varepsilon}{2} \quad \text{or} \quad |\Phi_{\eta, \Omega'_{j_0}}(\tilde{\beta}(s_2)) - c_{j_0}| = \frac{3\varepsilon}{2}.$$

We focus solely on the latter case, as the former case can be derived analogously. By (4.22), it follows that $|\Phi_{\eta, \Omega'_{j_0}}(\tilde{\beta}(s_1)) - c_{j_0}| \leq \varepsilon$. Thus we have

$$|\Phi_{\eta, \Omega'_{j_0}}(\tilde{\beta}(s_2)) - \Phi_{\eta, \Omega'_{j_0}}(\tilde{\beta}(s_1))| \geq |\Phi_{\eta, \Omega'_{j_0}}(\tilde{\beta}(s_2)) - c_{j_0}| - |\Phi_{\eta, \Omega'_{j_0}}(\tilde{\beta}(s_1)) - c_{j_0}| \geq \frac{1}{2}\varepsilon.$$

Conversely, applying the mean value theorem, there exists $s' \in (s_1, s_2)$ such that

$$|\Phi_{\eta, \Omega'_{j_0}}(\tilde{\beta}(s_2)) - \Phi_{\eta, \Omega'_{j_0}}(\tilde{\beta}(s_1))| = \left|\left\langle \Phi_{\eta, \Omega'_{j_0}}(\tilde{\beta}(s')), \frac{d\tilde{\beta}}{ds}(s') \right\rangle\right| (s_2 - s_1).$$

From (4.17) and (4.20), we deduce that $s_2 - s_1 \geq \frac{\varepsilon}{2\tilde{p}}$. Consequently, we obtain

$$\begin{aligned}\Phi_\eta(\tilde{\beta}(S)) &= \Phi_\eta(\gamma_0(t_1, t_2, \ldots, t_\ell)) + \int_0^S \frac{d}{dt}\Phi_\eta(\tilde{\beta}(s)) \\ &= \Phi_\eta(\gamma_0(t_1, t_2, \ldots, t_\ell)) - \int_0^S \varphi(\tilde{\beta}(t))\|\Phi'_\eta(\tilde{\beta}(t))\|_\eta dt \\ &\leq c_\Gamma - \int_{s_1}^{s_2} \varphi(\tilde{\beta}(t))\|\Phi'_\eta(\tilde{\beta}(t))\|_\eta dt \\ &= c_\Gamma - \sigma_0(s_2 - s_1) \leq c_\Gamma - \frac{1}{2}\sigma_0\varepsilon \leq c_\Gamma - \frac{1}{2}\tau_0\varepsilon,\end{aligned}$$

and so (4.21) is proved. We recall that $\tilde{\beta}(S) = \beta(S, \gamma_0(t_1, t_2, \ldots, t_\ell)) \in \tilde{\Lambda}$. Thus

$$b_{\eta,\Gamma} \le \Phi_\eta(\tilde{\beta}(S)) \le \max\Big\{\tilde{m}, c_\Gamma - \frac{\tau_0 \varepsilon}{2}\Big\}. \tag{4.25}$$

However, by Lemma 4.4, we know $b_{\eta,\Gamma} \to c_\Gamma$ as $\eta \to \infty$. This contradicts (4.25). Therefore, $\Phi_\eta(u)$ has a critical point $u_\eta \in D_\eta^\varepsilon$ for sufficiently large η, completing the proof of Lemma 4.6. □

The following argument constitutes the proof of our principal results.

Proof of Theorem 1.2. From Lemma 4.6, we derive a collection $\{u_\eta\}$ of positive solutions to problem (3.5) verifying the following properties:

(a) Given $\varepsilon > 0$ fixed, one can find that there exists $\eta \ge \Lambda_0$ such that $\|u_\eta\|_{\eta,\mathbb{R}^3\setminus\Omega_\Gamma'} \le \varepsilon$. Therefore, by the proof of Lemma 3.5, for sufficiently small fixed ε, we obtain that $|u_\eta|_{L^\infty(\mathbb{R}^3\setminus\Omega_\Gamma')} \le \mathscr{A}$ for all $\eta \ge \Lambda_0$, showing that $u_\eta > 0$ is a solution to problem (1.1).

(b) For fixed $\eta_n \to \infty$ and $\varepsilon_n \to 0$, the sequence $\{u_{\eta_n}\}$ satisfies the following properties:

$$\Phi'_{\eta_n}(u_{\eta_n}) = 0 \quad \text{for all } n \in \mathbb{N},$$

$$\|u_{\eta_n}\|_{\eta_n,\mathbb{R}^3\setminus\Omega_\Gamma'} \to 0,$$

$$\Phi_{\eta_n,j}(u_{\eta_n}) \to c_j \quad \text{for any } j \in \Gamma, \quad \text{and}$$

$$u_{\eta_n} \to u \quad \text{in } W^{1,p}(\mathbb{R}^3) \quad \text{with } u \in W_0^{1,p}(\Omega_\Gamma).$$

This concludes Theorem 1.2. □

Proof of Theorem 1.6. Let $\{u_{\eta^*,a}\}_{a>0}$ denote the family of solutions to problem (2.2), which correspond to critical points of the functional $\Phi_{\eta^*,a}$. Repeating the same calculations carried out in Lemma 3.3 yields the estimate

$$c + o_n(1) + \varepsilon_n\|u_n\|_{\eta^*} = \Phi_{\eta^*}(u_n) - \frac{1}{\theta}\langle \Phi'_{\eta^*}(u_n), u_n\rangle \ge \left(\frac{1}{p} - \frac{1}{\theta}\right)\delta_0\|u_n\|_{\eta^*}^p \ge \frac{1}{\theta}\delta_0\|u_n\|_{\eta^*}^p,$$

which implies that $\{u_{\eta^*,a}\}_{a>0}$ is bounded in $W^{1,p}(\mathbb{R}^3)$. Since $p > 1$, the Sobolev space $W^{1,p}(\mathbb{R}^3)$ is reflexive. By the local compact embedding $W^{1,p}(\mathbb{R}^3) \hookrightarrow\hookrightarrow L^q_{\mathrm{loc}}(\mathbb{R}^3)$ for all $q \in [p, p^*)$ together with uniform tail decay estimates at infinity for $\{u_{\eta^*,a}\}$, up to a subsequence, we derive

$$u_{\eta^*,a} \rightharpoonup u_{\eta^*,0} \quad \text{in } W^{1,p}(\mathbb{R}^3), \quad u^2_{\eta^*,a} \rightharpoonup u^2_{\eta^*,0} \quad \text{in } L^{\frac{12}{5}}(\mathbb{R}^3).$$

We next verify the convergence $\phi^a_{u_{\eta^*,a}} \to \phi^0_{u_{\eta^*,0}}$ in $D^{1,2}(\mathbb{R}^3)$. Recall that $\phi^a_{u_{\eta^*,a}} \in D^{1,2}(\mathbb{R}^3)$ solves the Bopp-Podolsky equation

$$-\Delta\phi + a^2\Delta^2\phi = Q(x)u^2_{\eta^*,a}, \quad x \in \mathbb{R}^3.$$

Testing the above equation with $\phi^a_{u_{\eta^*,a}}$, a standard energy estimate yields the uniform bound

$$\sup_{a\in(0,1)} \left\|\phi^a_{u_{\eta^*,a}}\right\|_{D^{1,2}(\mathbb{R}^3)} \le C < +\infty,$$

hence $\left\{\phi^a_{u_{\eta^*,a}}\right\}_{a>0}$ is bounded in the reflexive space $D^{1,2}(\mathbb{R}^3)$ and admits a weakly convergent subsequence as $a \to 0$. For the arbitrary test function $\psi \in D^{1,2}(\mathbb{R}^3)$, the weak formulation reads

$$\langle \nabla \phi^a_{u_{\eta^*,a}}, \nabla \psi \rangle + a^2 \langle \nabla \Delta \phi^a_{u_{\eta^*,a}}, \nabla \psi \rangle = \int_{\mathbb{R}^3} Q(x) u^2_{\eta^*,a} \psi dx.$$

As $a \to 0$, the biharmonic term $a^2 \langle \nabla \Delta \phi^a_{u_{\eta^*,a}}, \nabla \psi \rangle \to 0$ owing to the uniform boundedness of $\left\{\phi^a_{u_{\eta^*,a}}\right\}$ and $a \to 0$. Combining the weak convergence $u^2_{\eta^*,a} \rightharpoonup u^2_{\eta^*,0}$ in $L^{\frac{12}{5}}(\mathbb{R}^3)$, we pass to the limit and obtain

$$\langle \nabla \phi^0_{u_{\eta^*,0}}, \nabla \psi \rangle = \int_{\mathbb{R}^3} Q(x) u^2_{\eta^*,0} \psi dx, \quad \forall \psi \in D^{1,2}(\mathbb{R}^3),$$

which means $\phi^0_{u_{\eta^*,0}}$ is the unique solution of $-\Delta \phi = Q(x) u^2_{\eta^*,0}$ in $\mathbb{R}^3$. Moreover, by the norm convergence argument in [20, Lemma 3.2], we can upgrade the weak convergence to strong convergence:

$$\phi^a_{u_{\eta^*,a}} \to \phi^0_{u_{\eta^*,0}} \quad \text{in } D^{1,2}(\mathbb{R}^3).$$

Furthermore, we can upgrade the weak convergence of $u_{\eta^*,a}$ to strong convergence in $W^{1,p}(\mathbb{R}^3)$. From the critical point condition $\Phi'_{\eta^*,a}(u_{\eta^*,a}) = 0$, taking $v = u_{\eta^*,a}$ as the test function, we obtain

$$\|u_{\eta^*,a}\|^p_{\eta^*} + \int_{\mathbb{R}^3} Q(x) \phi^a_{u_{\eta^*,a}} u^2_{\eta^*,a} dx = \alpha \int_{\mathbb{R}^3} g(u_{\eta^*,a}) u_{\eta^*,a} dx + \int_{\mathbb{R}^3} |u_{\eta^*,a}|^{p^*} dx.$$

We now pass to the limit as $a \to 0$ term by term. For the nonlocal term, by the strong convergence $\phi^a_{u_{\eta^*,a}} \to \phi^0_{u_{\eta^*,0}}$ in $D^{1,2}(\mathbb{R}^3)$ and the weak convergence $u^2_{\eta^*,a} \rightharpoonup u^2_{\eta^*,0}$ in $L^{\frac{12}{5}}(\mathbb{R}^3)$, we have

$$\int_{\mathbb{R}^3} Q(x) \phi^a_{u_{\eta^*,a}} u^2_{\eta^*,a} dx \to \int_{\mathbb{R}^3} Q(x) \phi^0_{u_{\eta^*,0}} u^2_{\eta^*,0} dx.$$

For the subcritical nonlinear term, by the growth conditions (g_1)–(g_5), the local compact embedding $W^{1,p}(\mathbb{R}^3) \hookrightarrow\hookrightarrow L^q_{\text{loc}}(\mathbb{R}^3)$ for $q \in [p, p^*)$, and the uniform tail decay estimates, we get

$$\int_{\mathbb{R}^3} g(u_{\eta^*,a}) u_{\eta^*,a} dx \to \int_{\mathbb{R}^3} g(u_{\eta^*,0}) u_{\eta^*,0} dx.$$

For the critical term, by the Brezis-Lieb lemma and the energy boundedness established in Lemma 3.3, we have

$$\int_{\mathbb{R}^3} |u_{\eta^*,a}|^{p^*} dx \to \int_{\mathbb{R}^3} |u_{\eta^*,0}|^{p^*} dx.$$

Since $u_{\eta^*,0}$ satisfies the critical point condition of the limiting Schrödinger-Poisson system (1.8), it holds that

$$\|u_{\eta^*,0}\|^p_{\eta^*} + \int_{\mathbb{R}^3} Q(x) \phi^0_{u_{\eta^*,0}} u^2_{\eta^*,0} dx = \alpha \int_{\mathbb{R}^3} g(u_{\eta^*,0}) u_{\eta^*,0} dx + \int_{\mathbb{R}^3} |u_{\eta^*,0}|^{p^*} dx.$$

Combining the above equalities, we deduce that

$$\lim_{a \to 0} \|u_{\eta^*,a}\|^p_{\eta^*} = \|u_{\eta^*,0}\|^p_{\eta^*}.$$

Finally, since $W^{1,p}(\mathbb{R}^3)$ is uniformly convex for $p > 1$, weak convergence plus norm convergence implies strong convergence:

$$u_{\eta^*,a} \to u_{\eta^*,0} \quad \text{in } W^{1,p}(\mathbb{R}^3).$$

This completes the proof of the strong convergence, following the same strategy as in [20, Theorem 1.1] for the $p = 2$ case. Passing to the limit in the weak formulation of (2.2) and using the above convergences, we deduce that $u_{\eta^*,0}$ satisfies (1.8). This completes the proof. □

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Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Author contributions

All authors contributed equally to the research design, mathematical derivation, and writing of the manuscript. All authors have read and approved the final manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflict of interest.

References

1. F. Bopp, Eine lineare Theorie des Elektrons, *Ann. Phys.*, **430** (1940), 345–384. https://doi.org/10.1002/andp.19404300504
2. B. Podolsky, P. Schwed, Review of a generalized electrodynamics, *Rev. Mod. Phys.*, **20** (1948), 40–50. https://doi.org/10.1103/RevModPhys.20.40
3. M. Born, Modified field equations with a finite radius of the electron, *Nature*, **132** (1933), 282. https://doi.org/10.1038/132282a0
4. M. Born, L. Infeld, Foundations of the new field theory, *Nature*, **132** (1933), 1004. https://doi.org/10.1038/1321004b0
5. B. Podolsky, A generalized electrodynamics. Part I. Non-quantum, *Phys. Rev.*, **62** (1942), 68–71. https://doi.org/10.1103/PhysRev.62.68
6. C. Argaez, M. Melgaard, Solutions to quasi-relativistic multi-configurative Hartree-Fock equations in quantum chemistry, *Nonlinear Anal.*, **75** (2012), 384–404. https://doi.org/10.1016/j.na.2011.08.038
7. T. D'Aprile, D. Mugnai, Solitary waves for nonlinear Klein-Gordon-Maxwell and Schrödinger-Maxwell equations, *Proc. R. Soc. Edinb. A*, **134** (2004), 893–906. https://doi.org/10.1017/S030821050000353X

8. M. Yu, H. Chen, Existence and uniqueness of multi-bump solutions for nonlinear Schrödinger-Poisson systems, *Adv. Nonlinear Stud.*, **21** (2021), 661–681. https://doi.org/10.1515/ans-2021-2135
9. C. Ji, Y. Zhang, V. D. Rădulescu, Multi-bump solutions for the magnetic Schrödinger-Poisson system with critical growth, *Electron. J. Qual. Theory Differ. Equ.*, **21** (2022), 1–30. https://doi.org/10.14232/ejqtde.2022.1.21
10. J. Ma, Y. Song, On multi-bump solutions for a class of Schrödinger-Poisson systems with p-Laplacian in $\mathbb{R}^3$, *AIMS Math.*, **9** (2024), 1595–1621. https://doi.org/10.3934/math.2024079
11. X. Dou, X. He, V. D. Rădulescu, Multiplicity of positive solutions for the fractional Schrödinger-Poisson system with critical nonlocal term, *Bull. Math. Sci.*, **14** (2024), 2350012. https://doi.org/10.1142/S1664360723500121
12. S. Liang, Y. Song, S. Shi, Concentrating solutions for double critical fractional Schrödinger-Poisson system with p-Laplacian in $\mathbb{R}^3$, *Adv. Nonlinear Anal.*, **14** (2025), 20240063. https://doi.org/10.1515/anona-2024-0063
13. Y. Ma, C. Ji, Existence of multi-bump solutions for the magnetic Schrödinger-Poisson system in $\mathbb{R}^3$, *J. Geom. Anal.*, **31** (2021), 10886–10914. https://doi.org/10.1007/s12220-021-00668-3
14. Z. Shi, Y. Huo, S. Liang, Multi-bump solutions for the quasilinear Choquard equation in $\mathbb{R}^N$, *J. Dyn. Control Syst.*, **29** (2023), 1357–1383. https://doi.org/10.1007/s10883-022-09634-w
15. D. Xiao, V. N. Thin, S. Liang, Normalized solutions for critical Schrödinger-Poisson system involving p-Laplacian in $\mathbb{R}^3$, *Z. Angew. Math. Phys.*, **76** (2025), 10. https://doi.org/10.1007/s00033-024-02408-3
16. J. Zhang, Multi-bump solutions to Schrödinger-Poisson equations with critical growth in $\mathbb{R}^3$, *Calc. Var. Partial Differential Equations*, **64** (2025), 62. https://doi.org/10.1007/s00526-024-02923-1
17. J. Zhang, X. Zhang, Multi-bump solutions to Kirchhoff type equations with exponential critical growth in $\mathbb{R}^2$, *Z. Angew. Math. Phys.*, **75** (2024), 131. https://doi.org/10.1007/s00033-024-02282-z
18. X. Zhang, V. N. Thin, S. Liang, On multi-bump solutions for a class of (N, q)-Laplacian equation with critical exponential growth in $\mathbb{R}^N$, *J. Math. Anal. Appl.*, **543** (2025), 128941. https://doi.org/10.1016/j.jmaa.2024.128941
19. H. Zhu, J. Zhang, H. Suo, Multiple positive solutions of fractional Schrödinger-Poisson system with strong singularities and double critical exponents, *J. Appl. Anal. Comput.*, **15** (2025), 1580–1600. https://doi.org/10.11948/20240297
20. P. d'Avenia, G. Siciliano, Nonlinear Schrödinger equation in Bopp-Podolsky electrodynamics: solutions in the electrostatic case, *J. Differential Equations*, **267** (2019), 1025–1065. https://doi.org/10.1016/j.jde.2019.02.001
21. L. Li, X. Tang, Nehari type ground state solution for Schrödinger-Bopp-Podolsky system, *Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys.*, **82** (2020), 139–152.
22. Y. Xiao, S. Chen, M. Shu, The existence of ground state solutions for a Schrödinger-Bopp-Podolsky system with convolution nonlinearity, *J. Geom. Anal.*, **33** (2023), 374. https://doi.org/10.1007/s12220-023-01437-0
23. R. L. Alves, Generalized Schrödinger-Bopp-Podolsky type system with singular nonlinearity, *J. Math. Anal. Appl.*, 539 (2024), 128499. https://doi.org/10.1016/j.jmaa.2024.128499

24. H. M. Santos Damian, G. Siciliano, Critical Schrödinger-Bopp-Podolsky systems: solutions in the semiclassical limit, *Calc. Var. Partial Differential Equations*, **63** (2024), 155. https://doi.org/10.1007/s00526-024-02775-9

25. Y. Li, X. Chang, Z. Feng, Normalized solutions for Sobolev critical Schrödinger-Bopp-Podolsky systems, *Electron. J. Differential Equations*, **2023** (2023), 1–17. https://doi.org/10.58997/ejde.2023.56

26. X. Peng, Normalized solutions for the critical Schrödinger-Bopp-Podolsky system, *Bull. Malays. Math. Sci. Soc.*, **47** (2024), 9. https://doi.org/10.1007/s40840-023-01609-9

27. G. Siciliano, Normalized solutions for Schrödinger type equations under Neumann boundary conditions, *Contemp. Math.*, **786** (2023), 141–184.

28. L. Wang, J. Wang, J. Wang, Multi-bump solutions of a Schrödinger-Bopp-Podolsky system with steep potential well, *Electron. J. Qual. Theory Differ. Equ.*, **10** (2024), 1–22. https://doi.org/10.14232/ejqtde.2024.1.10

29. X. Ju, J. Sun, Normalized solutions for Schrödinger-Bopp-Podolsky systems with general nonlinearities of mass subcritical, *J. Math. Phys.*, **66** (2025), 101519. https://doi.org/10.1063/5.0272062

30. L. Wei, Y. Song, S. Liang, Normalized solutions for the Schrödinger-Bopp-Podolsky system: doubly critical growth case, *J. Geom. Anal.*, **36** (2026), 119. https://doi.org/10.1007/s12220-026-02366-4

31. S. Liang, J. Zhang, Multi-bump solutions for a class of Kirchhoff type problems with critical growth in $\mathbb{R}^N$, *Topol. Methods Nonlinear Anal.*, **48** (2016), 71–101. https://doi.org/10.12775/TMNA.2016.040

32. Y. Ding, K. Tanaka, Multiplicity of positive solutions of a nonlinear Schrödinger equation, *Manuscripta Math.*, **112** (2003), 109–135. https://doi.org/10.1007/s00229-003-0397-x

33. J. Moser, A new proof of de Giorgi's theorem concerning the regularity problem for elliptic differential equations, *Comm. Pure Appl. Math.*, **13** (1960), 457–468. https://doi.org/10.1002/cpa.3160130308

34. P.-L. Lions, The concentration compactness principle in the calculus of variations. The locally compact case, Part I, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 1 (1984), 109–145. https://doi.org/10.1016/S0294-1449(16)30428-0

35. P.-L. Lions, The concentration compactness principle in the calculus of variations. The locally compact case, Part II, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 1 (1984), 223–283. https://doi.org/10.1016/S0294-1449(16)30422-X

36. J. Chabrowski, Concentration-compactness principle at infinity and semilinear elliptic equations involving critical and subcritical Sobolev exponents, *Calc. Var. Partial Differential Equations*, **3** (1995), 493–512. https://doi.org/10.1007/BF01187898

37. S. Liang, J. Zhang, On some p-Laplacian equation with electromagnetic fields and critical nonlinearity in $\mathbb{R}^N$, *J. Math. Phys.*, **56** (2015), 041504. https://doi.org/10.1063/1.4918791

38. M. Xiang, B. Zhang, X. Zhang, A nonhomogeneous fractional p-Kirchhoff type problem involving critical exponent in $\mathbb{R}^N$, *Adv. Nonlinear Stud.*, **17** (2017), 611–640. https://doi.org/10.1515/ans-2016-6002

39. C. O. Alves, G. M. Figueiredo, On multiplicity and concentration of positive solutions for a class of quasilinear problems with critical exponential growth in $\mathbb{R}^N$, *J. Differential Equations*, **246** (2009), 1288–1311. https://doi.org/10.1016/j.jde.2008.08.004



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