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Research article

Dynamical behavior of solutions to nonlinear evolution equations with time delay and memory

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Abstract: In this paper, we considered the nonlinear evolution equations with time delay and memory under the assumption that the nonlinearity satisfies critical exponent growth. We first proved the global well-posedness using the Galerkin approximation method. Second, we obtained the existence of a compact global attractor and exponential attractors via a stabilizability estimate. Finally, the upper semi-continuity of the global attractor was established as $\varepsilon \to 0^+$.

Keywords: exponential attractors; nonlinear evolution equation; memory; upper semi-continuity; time delay
Mathematics Subject Classification: 35B41, 35B40, 37B55

1. Introduction

We focus on the following equation with initial data and boundary conditions:

$$\begin{cases} u_{tt} - \alpha\Delta u - \varepsilon\Delta u_{tt} + \int_{-\infty}^{t} k(t-s)\Delta u(x,s)ds + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau(t)) + h(u) = g(x), \ (x,t) \in \Omega \times \mathbb{R}^+, \\ u(x,t) = 0, x \in \partial\Omega, t \geq 0, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega, \\ u_t(x,t) = h_0(x,t), x \in \Omega, t \in [-\tau(0), 0), \end{cases} \tag{1.1}$$

defined on a bounded region $\Omega \subset \mathbb{R}^N (N \geq 3)$, whose boundary $\partial\Omega$ possesses sufficient smoothness. Here, $\alpha > 0$, $\varepsilon > 0$, $\mu_1 > 0$, and $\mu_2 > 0$ denote real numbers, the time delay function is denoted by $\tau(t)$, and $\mathbb{R}^+ = [0, +\infty)$. The source term $h(u)$ and the term $g(x)$ are specified later.

Eq. (1.1), which appears as a class of nonlinear evolution equations like the strain solitary wave equation and dispersive dissipative wave equation, is used to describe the propagation of lengthwise waves in nonlinear elastic rods as well as ion-acoustic waves with spatial transformations under weak

nonlinear effects (see, e.g., [1–3]). For the class of nonlinear evolution equations, the existence of global solutions and attractors was studied in [4–7] and others. For instance, Xie and Zhong in [4, 5] systematically investigated the existence of global attractors for (1.1) with $k(\cdot) \equiv 0, \mu_1 \equiv \mu_2 \equiv 0$ on weak and strong Hilbert spaces, respectively. Based on the global well-posedness results given in [8], Sun, Yang, and Duan [6] constructed the uniformly asymptotic regularity of a solution with respect to $\varepsilon \in [0, 1]$ for (1.1) when $h \in L^2(\Omega)$ and $h \in H^{-1}(\Omega)$, respectively, and they also obtained the existence of exponential attractors as well as the upper semi-continuity of global attractors. In [7], the authors used the operator decomposition technique to discuss the existence of a time-dependent global attractor for a nonlinear evolution equation with linear memory $\int_0^\infty k'(s)\Delta u(t-s)ds$ within the theory of time-dependent space. Equations with different time delays have also been explored in depth by numerous scholars [9–11]. In the study of delayed wave equations, Nicaise and Pignotti [12] investigated the stabilization problem for time-varying delayed wave equations. They analyzed the impact of delay magnitude on system stability, discovering that the system loses stability when the delay exceeds a critical threshold.

The time-varying delay considered herein satisfies the condition $\tau'(t) \leq d < 1$, which builds upon the core assumption of bounded delay derivatives from prior work, further extending the scenario of coupled delay and memory effects. Liu et al. [13] focused on the wave equation incorporating time delay and hereditary memory, and demonstrated the existence and uniqueness of weak solutions by combining the Faedo-Galerkin method with energy estimates. In establishing the global well-posedness of weak solutions, this paper follows the Faedo-Galerkin approach. However, to address the coupled effects of nonlinear source and memory terms, it further employs refined Hölder and Young inequalities to rigorously ensure the existence and uniqueness of solutions. In investigating systems exhibiting both memory effects and time delays, Qin et al. [14] focused on viscoelastic Kirchhoff plate equations, investigating the coupled effects of memory terms and time delays. Researchers employed quasi-stability theory to prove the existence of global and exponential attractors. However, the attractors of evolution equations incorporating memory effects and time delays remain poorly understood, and several analytical difficulties arise when addressing such coupled systems:

(i) For (1.1), we first introduce a new variable to construct a relatively complex triple solution space. Moreover, (1.1) incorporates a dispersion term $-\Delta u_{tt}$, which essentially distinguishes it from the damped wave equation in [15].

(ii) To prove the existence of a bounded absorbing set, we use a different approach by demonstrating the system's gradient property as outlined in Lemma 4.2, which leads to the system's asymptotic smoothness. In the proof, we establish the relationship between ϵ_1 and ϵ_2 to ensure the desired estimates holds.

The remainder of this paper is organised as follows. Section 2 introduces function spaces and fundamental assumptions, along with some abstract results. Section 3 employs the Faedo-Galerkin method to provide a detailed proof of the global well-posedness of weak solutions. Section 4 constitutes the core of the proof. It first establishes the gradient structure of the system by constructing a modified energy functional, then derives the crucial quasi-stability estimates, and finally proves the existence and finite-dimension of the global attractor and the exponential attractor. Section 5 investigates the upper semi-continuity of the global attractor with respect to the rotational inertia coefficient $\varepsilon \to 0^+$.

2. Preliminaries

From [16], we introduce an auxiliary variable η^t to encode historical information within the system. This variable is defined by

$$\eta^t(x,s) = u(x,t) - u(x,t-s), \quad (x,s) \in \Omega \times \mathbb{R}^+, \ t \geq 0. \tag{2.1}$$

By differentiation, we obtain

$$\partial_t \eta^t(x,s) = -\partial_s \eta^t(x,s) + \partial_t u(x,t). \tag{2.2}$$

For the convolution integral term:

$$\begin{aligned}\int_{-\infty}^{t} k(t-s)\Delta u(s)\mathrm{d}s = \int_0^{+\infty} k(\sigma)\Delta u(t-\sigma)\mathrm{d}\sigma &= \int_0^{+\infty} k(\sigma)\left[\Delta u(t) - \Delta\eta^t(x,\sigma)\right]\mathrm{d}\sigma \\ &= \left(\int_0^{+\infty} k(\sigma)\mathrm{d}\sigma\right)\Delta u(t) - \int_0^{+\infty} k(\sigma)\Delta\eta^t(x,\sigma)\mathrm{d}\sigma.\end{aligned}$$

For simplicity, we set $\alpha \triangleq \int_0^{\infty} k(s)\mathrm{d}s + 1$, and equation (1.1) simplifies to the following system:

$$\begin{cases} u_{tt} - \Delta u - \varepsilon\Delta u_{tt} - \displaystyle\int_0^{\infty} k(s)\Delta\eta^t(x,s)\mathrm{d}s + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau(t)) + h(u) = g(x), x \in \Omega, \\ \partial_t\eta^t = -\partial_s\eta^t + \partial_t u, \ \ (x,t,s) \in \Omega \times \mathbb{R}^+ \times \mathbb{R}^+, \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \ \ \eta^t(x,0) = 0, x \in \Omega, \\ u_t(x,t) = h_0(x,t), \qquad x \in \Omega, \ t \in [-\tau(0), 0), \\ \eta^0(x,s) = \eta_0(x,s), \qquad (x,s) \in \Omega \times \mathbb{R}^+, \end{cases} \tag{2.3}$$

where the initial history function satisfies $\eta_0(x,s) = u_0(x) - \phi_0(s)$, with $\phi_0(s) = u(-s)$ for all $s > 0$.

2.1. Assumptions

We use the standard Banach space $L^p(\Omega)$ and the Sobolev space $H_0^1(\Omega)$ with their usual scalar products and norms.

(A1) Function spaces. We define the following Hilbert spaces

$$V_0 = L^2(\Omega), \quad V_1 = H_0^1(\Omega), \quad V_2 = H^2(\Omega) \cap H_0^1(\Omega).$$

The norm in $L^2(\Omega)$ is denoted by $\|\cdot\|$, and the inner product is $(\cdot,\cdot)$.
For the memory variable η, we introduce the weighted L^2-spaces

$$\mathcal{M}_i = L_k^2(\mathbb{R}^+; V_{i+1}) = \left\{\eta : \mathbb{R}^+ \to V_{i+1} \ \middle| \ \int_0^{\infty} k(s)\|\eta(s)\|_{V_{i+1}}^2 ds < \infty\right\}, \ i = 0, 1,$$

endowed with the inner product and norm

$$(\eta,\zeta)_{\mathcal{M}_i} = \int_0^{\infty} k(s)\left(\eta(s), \zeta(s)\right)_{V_{i+1}} ds, \quad \|\eta\|_{\mathcal{M}}^2 = \int_0^{\infty} k(s)\|\eta(s)\|_{V_{i+1}}^2 ds,$$

where the symbol i is always omitted whenever zero. The phase space for the problem is defined as

$$\mathcal{H} = V_1 \times V_1 \times \mathcal{M},$$

and the product norm is defined as

$$\|(u, u_t, \eta)\|^2_{\mathcal{H}_t} = \|\nabla u\|^2 + \varepsilon\|\nabla u_t\|^2 + \|u_t\|^2 + \|\eta\|^2_{\mathcal{M}}.$$

Moreover, the phase space $\mathcal{H}_1 = V_2 \times V_2 \times \mathcal{M}_1$ is also needed.

(A2) Hypotheses on h and g. Concerning the function g and the nonlinear term h, we assume

$$g \in L^2(\Omega) \quad \text{and} \quad h \in C^1(\mathbb{R}). \tag{2.4}$$

The nonlinear source term $h(u)$ is a nonlinear function such that $h(0) = 0$ and there exists $C_h > 0$ such that

$$|h'(s)| \leq C_h(1 + |s|^{\frac{4}{N-2}}). \tag{2.5}$$

A typical assumption implying the aforementioned conditions is

$$\liminf_{|s|\to\infty} \frac{h(s)}{s} > -\lambda_1, \tag{2.6}$$

where $\lambda_1 > 0$ is the first eigenvalue of $-\Delta$ under the Dirichlet boundary condition.

(A3) Hypotheses on $\tau(t)$. Concerning the time delay $\tau(t)$, assume there exist two normal constants τ_0 and τ_1 such that

$$0 < \tau_0 \leq \tau(t) \leq \tau_1 < \infty, \tag{2.7}$$

and further that

$$\tau(t) \in W^{2,\infty}(0, T) \ \text{ and } \ \tau'(t) \leq d < 1, \ \forall\, T, \ t > 0. \tag{2.8}$$

(A4) Hypotheses on k. Concerning the relaxation function k, assume $k : \mathbb{R}^+ \to \mathbb{R}^+$ is a differentiable function satisfying the following conditions:

$$k(0) > 0, \quad \int_0^\infty k(s)ds = l_0 > 0. \tag{2.9}$$

Furthermore, assume there exists a positive constant ν such that

$$k'(t) \leq -\nu k(t), \forall\, t \geq 0. \tag{2.10}$$

Lemma 2.1. *[14] From* (2.6)*, we further assume the existence of constants $\beta \in [0, \lambda_1)$ and $c_h > 0$ such that*

$$h(s)s \geq -\beta s^2 - c_h \quad \textit{and} \quad \hat{h}(s) \geq -\frac{\beta}{2}s^2 - c_h, \quad \forall s \in \mathbb{R}, \tag{2.11}$$

where $\hat{h}(z) = \int_0^z h(s)ds$.

2.2. Abstract results

We first revisit the definitions of global attractors, originally presented in classical literature such as [17–19], and further elaborated in the recent works of Chueshov and Lasiecka [20] and Miranville and Zelik [21].

Theorem 2.2. *[20, 21] Let $(H, S(t))$ be a dynamical system given by a C^0-semigroup $S(t)$ on a Banach space H. Then a global attractor for $S(t)$ is a compact set $\mathcal{A} \subset H$ that is fully invariant and uniformly attracting, that is, $S(t)\mathcal{A} = \mathcal{A}$ for all $t \geq 0$ and for every bounded subset $B \subset H$,*

$$\lim_{t\to\infty} dist_H(S(t)B, \mathcal{A}) = 0,$$

where $dist_H$ is the Hausdorff semi-distance in H. The fractal dimension of a compact set $M \subset H$ is defined by

$$\dim_f^H M = \limsup_{\varepsilon\to 0} \frac{\ln n(M, \varepsilon)}{\ln(1/\varepsilon)},$$

where $n(M, \varepsilon)$ is the minimal number of closed balls of radius 2ε that cover M.

Next, these preliminary concepts as well as the concept of quasi-stability are used as described by Chueshov and Lasiecka [20, 22].

Theorem 2.3. *[20, 22] Let us denote $H = X \times Y \times Z$. Consider the dynamical system $(H, S(t))$ given by an evolution operator*

$$S(t)U_0 = (u, u_t, \eta), \quad U_0 = (u_0, u_1, \eta_0) \in H, \tag{2.12}$$

where the functions u and η possess the following regularity:

$$u \in C(\mathbb{R}^+; X) \cap C^1(\mathbb{R}^+; Y), \quad \eta \in C^1(\mathbb{R}^+, Z). \tag{2.13}$$

The dynamical system $(H, S(t))$ is quasi-stable on a set $B \subset H$ if there exists a compact seminorm n_X on X and nonnegative scalar functions $a(t)$ and $c(t)$, locally bounded in $[0, \infty)$. $b(t) \in L^1(\mathbb{R}^+)$ with $\lim_{t\to\infty} b(t) = 0$, such that

$$\left\|S(t)U^1 - S(t)U^2\right\|_H^2 \leq a(t)\left\|U^1 - U^2\right\|_H^2 \tag{2.14}$$

and

$$\left\|S(t)U^1 - S(t)U^2\right\|_H^2 \leq b(t)\left\|U^1 - U^2\right\|_H^2 + c(t)\sup_{0<s<t}\left[n_X\left(u^1(s) - u^2(s)\right)\right]^2, \tag{2.15}$$

for any $U^1, U^2 \in B$. Inequality (2.15) is often called the stabilizability inequality.

The property of quasi-stability can be employed to establish the asymptotic smoothness of the semigroup, as detailed in Chueshov and Lasiecka [22].

Theorem 2.4. *[22] Let the dynamical system $(H, S(t))$ be given by (2.12) and satisfy (2.13). Then the dynamical system $(H, S(t))$ is asymptotically smooth if system $(H, S(t))$ is quasi-stable on every bounded positively invariant set $B \subset H$.*

Theorem 2.5. *[22] Let $\mathcal{N}$ be the set of stationary points of $S(t)$. Then the unstable manifold $\mathbb{M}_+(\mathcal{N})$ is the family of $y \in H$ such that there exists a full trajectory $u(t)$ satisfying*

$$u(0) = y, \quad \lim_{t\to-\infty} \operatorname{dist}(u(t), S(t)) = 0.$$

The theorem below is essential for proving our principal result, as in the work of Chueshov and Lasiecka [22], as well as in Hale [18].

Definition 2.6. *[18, 22] A system $(X, S(t))$ is called gradient if it possesses a strict Lyapunov functional, that is, a functional $\Phi : X \to \mathbb{R}$ such that*

(i) the map $t \mapsto \Phi(S(t)z)$ is non-increasing for each $z \in X$,

(ii) if for some $z \in X$ one has $\Phi(S(t)z) = \Phi(z)$ for all t, then $S(t)z = z$ for all $t \geq 0$.

The above condition (ii) means that z must be a stationary point of $(X, S(t))$.

Theorem 2.7. *[18, 22] If the dynamical system $(H, S(t))$ is asymptotically smooth and gradient with Lyapunov functional Φ and further satisfies the following:*

(i) Φ is bounded from above on any bounded subset of H,

(ii) the set $\Phi_R = \{U \mid \Phi(U) \leq T\}$ is bounded for every R,

(iii) the set of stationary points $\mathcal{N}$ is bounded,

then the dynamical system $(H, S(t))$ has a compact global attractor characterized by $\mathcal{A} = \mathbb{M}_+(\mathcal{N})$.

In addition, quasi-stability implies that global attractors have finite fractal dimensions. This result is established in Chueshov and Lasiecka [22], and we recall it as the following theorem.

Theorem 2.8. *[22] Let $(H, S(t))$ be given by* (2.12) *and satisfy* (2.13)*. If $(H, S(t))$ possesses a compact global attractor $\mathcal{A}$ and is quasi-stable on $\mathcal{A}$, then the attractor $\mathcal{A}$ has finite fractal dimension.*

Theorem 2.9. *[22] Let $(H, S(t))$ be a dynamical system satisfying* (2.12) *and* (2.13)*, and let it be quasi-stable on some bounded absorbing set B. Assume further that there exists an extended space $\tilde{H} \supseteq H$ such that, for any $T > 0$,*

$$\|S(t_1)y - S(t_2)y\|_{\tilde{H}} \leq C_{BT} |t_1 - t_2|^{\gamma}, \quad t_1, t_2 \in [0, T], \quad y \in B, \tag{2.16}$$

where $C_{BT} > 0$ and $\gamma \in (0, 1]$ are constants. Then the dynamical system $(H, S(t))$ has a generalized exponential attractor $\mathcal{A}_{\exp} \subset H$ with finite fractal dimension in $\tilde{H}$.

3. Global well-posedness

In this section, we provide the definition of solutions to problem (2.3).

Definition 3.1. *(i) Given $T > 0$, we say that a function $U(t) = (u, u_t, \eta^t) \in C(0, T, \mathcal{H})$ is a weak solution to problem* (2.3) *if*

$$\begin{cases} \frac{d}{dt}(u_t, v) + (\nabla u, \nabla v) + \frac{d}{dt}(\varepsilon \nabla u_t, \nabla v) + \int_0^\infty k(s)(\nabla \eta^t(s), \nabla v) ds + \mu_1 (u_t, v) \\ \qquad\qquad + \mu_2 (u_t(t - \tau(t)), v) + (h(u) - g, v) = 0, \\ (\partial_t \eta^t + \partial_s \eta^t, \xi)_{\mathcal{M}} = -(\partial_s \eta^t, \xi)_{\mathcal{M}} + (u_t, \xi)_{\mathcal{M}}, \end{cases} \tag{3.1}$$

a.e. in $[0,T]$*, for all* $v \in V_1, \xi \in \mathcal{M}$*, and it satisfies the initial conditions*

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ \eta^0(x,s) = \eta_0(x,s), u_t(x,t) = h_0(x,t) \text{ for } x \in \Omega,\ t \in [-\tau(0), 0).$$

(ii) A strong solution to (2.3) *on* $[0,T]$ *with* $0 < T \leq \infty$ *is a continuous function* $U : [0,T] \to \mathcal{H}$ *with* $U(0) \in \mathcal{H}_1$*, differentiable a.e. on* $[0,T)$*, and for any* $t \in [0,T)$*,*

$$U = (u, u_t, \eta^t) \in C(0,T;\mathcal{H}) \cap L^\infty(0,T;\mathcal{H}_1).$$

With this notion, we can state the Hadamard well-posedness of problem (2.3) as follows.

Theorem 3.2. *Assume that* (2.4)–(2.10) *hold.*

(i) Given initial data $U(0) = (u_0, u_1, \eta_0) \in \mathcal{H}, h_0(x,t) \in L^2(\Omega \times (-\tau(0), 0))$*, and* $g(x) \in L^2(\Omega)$*, then problem* (2.3) *has a weak solution*

$$(u, u_t, \eta^t) \in C(0,T,\mathcal{H}), \forall\ T > 0,$$

satisfying

$$u \in L^\infty(0,T;V_1),\ u_t \in L^\infty(0,T;V_1),\ \eta^t \in L^\infty(0,T;\mathcal{M});\ \text{with}\ \ (I-\Delta)u_{tt} \in L^\infty(0,T;(V_1)'),$$

where $(V_i)'$ *stands for the topological dual of* V_i*.*

(ii) If $U(0) = (u_0, u_1, \eta_0) \in \mathcal{H}_1, h_0(x,t) \in H^1(\Omega \times (-\tau(0), 0)),\ g(x) \in H^1(\Omega)$*, then the above weak solution has higher regularity in the class*

$$(u, u_t, \eta^t) \in C(0,T;\mathcal{H}) \cap L^\infty(0,T;\mathcal{H}_1),\ \ \text{with}\ \ (I-\Delta)u_{tt} \in L^\infty(0,T;(V_0)').$$

(iii) In both case, the weak solution and strong solution depend continuously on the initial data in $\mathcal{H} \times L^2(\Omega \times (-\tau(0), 0))$*. In particular, problem* (2.3) *has a unique weak solution.*

Proof. (i)–(ii)**(existence)** The proof can be carried out via the Faedo-Galerkin method. The two proofs are similar, so only the proof for weak solutions is presented here.

Let $\{w_j\}_{j=1}^\infty$ be a complete orthonormal system in $H_0^1(\Omega)$, and $\{\varphi_j(x,s)\}_{j=1}^\infty$ be a complete orthonormal system in $\mathcal{M}$. We select sequences (u_0^m) belonging to $H_0^1(\Omega)$, (u_1^m) belonging to $H_0^1(\Omega)$, $h_0^m(t)(t \in [-\tau, 0))$ belonging to $L^2(\Omega \times [-\tau, 0))$, and η_0^m belonging to $\mathcal{M}$, such that $(u_0^m, u_1^m, \eta_0^m) \to (u_0, u_1, \eta_0)$ converges in $\mathcal{H}$, and $h_0^m \to h_0$ converges in $L^2(\Omega \times [-\tau, 0))$ as $m \to \infty$.
We shall seek the forms of the approximate solutions

$$u^m(x,t) = \sum_{j=1}^m g_{jm}(t) w_j(x) \quad \text{and} \quad \eta^{m,t}(x,s) = \sum_{j=1}^m h_{jm}(t)\varphi_j(x,s),$$

satisfying the following approximate equations:

$$\begin{cases} \frac{d}{dt}(u_t^m, w_j) + (\nabla u, \nabla w_j) + \frac{d}{dt}(\varepsilon \nabla u_t^m, \nabla w_j) + \int_0^\infty k(s)(\nabla \eta_m^t(s), \nabla w_j) ds + \mu_1(u_t^m, w_j) \\ + \mu_2\left(u_t^m(t-\tau(t)), w_j\right) + (h(u^m) - g, w_j) = 0, \\ (\partial_t \eta^m + \partial_s \eta^m, \varphi_j)_{\mathcal{M}} = (u_t^m(t), \varphi_j)_{\mathcal{M}}, \end{cases} \tag{3.2}$$

with initial data and the history value

$$(u^m(0), u_t^m(0), \eta^m(0)) = (u_0^m, u_1^m, \eta_0^m) \quad \text{and} \quad u_t^m(t) = h_0^m(t), \quad t \in [-\tau, 0). \tag{3.3}$$

According to the standard theory of ordinary differential equations, finite-dimensional problems (3.2) and (3.3) possess defined solutions on the interval $[0, t^m)$. Indeed, the prior estimates presented below indicate that $t^m = T$.
Let

$$E_\alpha(t) = \frac{1}{2}(\|u_t^m\|^2 + \|\nabla u^m\|^2 + \varepsilon\|\nabla u_t^m\|^2 + \|\eta^{t,m}\|_{\mathcal{M}}^2) + \int_\Omega h(u^m)u^m ds + \alpha(u_t^m, u^m) + \alpha\varepsilon(\nabla u_t^m, \nabla u^m). \tag{3.4}$$

Multiplying $(3.2)_1$ by $u_t^m + \alpha u^m$, and using $(3.2)_2$, we deduce that

$$\begin{aligned}\frac{d}{dt}E_\alpha(t) = {} & \alpha\|u_t^m\|^2 - \alpha\|\nabla u^m\|^2 + \alpha\varepsilon\|\nabla u_t^m\|^2 + \frac{1}{2}\int_0^\infty k'(s)\|\nabla\eta^{t,m}(s)\|^2 ds \\ & - \alpha(h(u^m), u^m) - \int_0^\infty k(s)(\nabla\eta^{t,m}(s), \alpha\nabla u^m)ds + (g, u_t^m + \alpha u_m) - \mu_1\|u_t^m\|^2 \\ & - \alpha\mu_1(u_t^m, u^m) - (\mu_2 u_t^m(t-\tau(t)), u_t^m) - \alpha(\mu_2 u_t^m(t-\tau(t)), u^m).\end{aligned} \tag{3.5}$$

Next, dealing with the cross-terms, from the Cauchy-Schwarz inequality, Young's inequality, and the Poincaré inequality, we have

$$|-\alpha\mu_1(u_t^m, u^m)| \le \alpha\mu_1\|u_t^m\|\|u^m\| \le \frac{\alpha|\mu_1|}{2}\|u^m\|^2 + \frac{\alpha|\mu_1|}{2}\|u_t^m\|^2 \le \frac{\alpha|\mu_1|C_{\lambda_1}}{2}\|\nabla u^m\|^2 + \frac{\alpha|\mu_1|}{2}\|u_t^m\|^2; \tag{3.6}$$

$$|-(\mu_2 u_t^m(t-\tau(t)), u_t^m)| \le \frac{|\mu_2|}{2}\|u_t^m(t-\tau(t))\|^2 + \frac{|\mu_2|}{2}\|u_t^m\|^2; \tag{3.7}$$

$$\begin{aligned}|-\alpha(\mu_2 u_t^m(t-\tau(t)), u^m)| &\le \frac{\alpha|\mu_2|}{2}\|u_t^m(t-\tau(t))\|^2 + \frac{\alpha|\mu_2|}{2}\|u^m\|^2 \\ &\le \frac{\alpha|\mu_2|}{2}\|u_t^m(t-\tau(t))\|^2 + \frac{\alpha|\mu_2|C_{\lambda_1}}{2}\|\nabla u^m\|^2;\end{aligned} \tag{3.8}$$

$$(g, u_t^m(t) + \alpha u^m(t)) = (g, u_t^m(t)) + \alpha(g, u^m(t)) \le \|g\|^2 + \frac{1}{2}\|u_t^m\|^2 + \frac{\alpha^2 C_{\lambda_1}}{2}\|\nabla u^m\|^2; \tag{3.9}$$

$$|-\alpha(h(u^m), u^m)| \le \alpha[\|u^m\|^2 + \int_\Omega |u^m|^{\frac{2N}{N-2}}dx] \le \alpha C_{\lambda_1}\|\nabla u^m\|^2 + C_h|\Omega|. \tag{3.10}$$

From the Cauchy-Schwarz inequality and Young's inequality, it holds that

$$\begin{aligned}&|-\alpha\int_0^\infty k(s)\,(\nabla\eta^{t,m}(s), \nabla u^m)\,ds| \le \alpha\int_0^\infty k(s)\|\nabla\eta^{t,m}(s)\| \cdot \|\nabla u^m\|ds \\ &\le \frac{\alpha}{2}\int_0^\infty k(s)\|\nabla\eta^{t,m}(s)\|^2 ds + \frac{\alpha}{2}\left(\int_0^\infty k(s)ds\right)\|\nabla u^m\|^2 \\ &\le \frac{\alpha}{2}\int_0^\infty k(s)\|\nabla\eta^{t,m}(s)\|^2 ds + \frac{\alpha}{2}l_0\|\nabla u^m\|^2.\end{aligned} \tag{3.11}$$

Substituting (3.6)–(3.11) into (3.5), we have

$$\begin{aligned}\frac{d}{dt}E_\alpha(t) \leq &\left(\alpha+\frac{1}{2}-\mu_1+\frac{\alpha|\mu_1|}{2}+\frac{|\mu_2|}{2}\right)\|u_t^m\|^2+\alpha\varepsilon\|\nabla u_t^m\|^2\\ &+\left(\frac{\alpha l_0}{2}+\alpha C_{\lambda_1}+\frac{\alpha^2 C_{\lambda_1}}{2}+\frac{\alpha|\mu_1|C_{\lambda_1}}{2}+\frac{\alpha|\mu_2|C_{\lambda_1}}{2}-\alpha\right)\|\nabla u^m\|^2\\ &+\left(\frac{\alpha}{2}-\frac{\nu}{2}\right)\int_0^\infty k(s)\|\nabla\eta^{t,m}(s)\|^2ds+\left(\frac{|\mu_2|}{2}+\frac{\alpha|\mu_2|}{2}\right)\|u_t^m(t-\tau(t))\|^2+\|g\|^2+C_h|\Omega|. \quad (3.12)\end{aligned}$$

Choosing α small enough, such that $\alpha+\frac{1}{2}-\mu_1+\frac{\alpha|\mu_1|}{2}+\frac{|\mu_2|}{2}>0$, $M_0=-1+\frac{l_0}{2}+C_{\lambda_1}+\frac{|\mu_1|C_{\lambda_1}}{2}+\frac{|\mu_2|C_{\lambda_1}}{2}>0$. Let $\gamma=\max\{\alpha,\frac{\alpha+\nu}{2},\alpha M_0+\frac{\alpha^2C_{\lambda_1}}{2},\alpha+\frac{1}{2}-\mu_1+\frac{\alpha|\mu_1|}{2}+\frac{|\mu_2|}{2}\}$, and we obtain

$$\frac{d}{dt}E_\alpha(t)\leq\gamma E_\alpha(t)+C_1\|u_t^m(t-\tau(t))\|^2+C_2, \quad (3.13)$$

where $C_1=\frac{|\mu_2|}{2}+\frac{\alpha|\mu_2|}{2}$, $C_2=\|g\|^2+C_h|\Omega|$. Integrating (3.13) from 0 to t ($t<T$), we get

$$E_\alpha(t)\leq E_\alpha(0)+C_1\int_0^t\|u_t^m(s-\tau(t))\|^2ds+\int_0^t C_2ds+\gamma\int_0^t E_\alpha(s)ds. \quad (3.14)$$

Using the history values of u_t^m, $t\in[-\tau(0),0)$, we infer that

$$\begin{aligned}\int_0^t\|u_t^m(s-\tau(t))\|^2ds=\int_{-\tau(t)}^{t-\tau(t)}\|u_t^m(s)\|^2ds&=\int_{-\tau(t)}^0\|u_t^m(s)\|^2ds+\int_0^{t-\tau(t)}\|u_t^m(s)\|^2ds\\ &\leq\int_{-\tau(t)}^0\|h_0^m(s)\|^2ds+\int_0^{t-\tau(t)}\|u_t^m(s)\|^2ds\\ &\leq\int_{-\tau(t)}^0\|h_0^m(s)\|^2ds+\int_0^t\|u_t^m(s)\|^2ds. \quad (3.15)\end{aligned}$$

Energy function $E_\alpha(t)$ defined in (3.4) gives

$$\|u_t^m(s)\|^2\leq 2E_\alpha(s), \quad (3.16)$$

and inserting (3.15) and (3.16) into (3.14), we obtain that

$$E_\alpha(t)\leq\tilde{A}+\widetilde{B}\int_0^t E_\alpha(s)ds, \quad (3.17)$$

where $\tilde{A}=E_\alpha(0)+C_1\|h^m\|^2+C_2T$, $\widetilde{B}=2C_1+\gamma$. Applying Gronwall's inequality to (3.17), we conclude that for any $t\in[0,T]$,

$$E_\alpha(t)\leq\tilde{A}e^{\widetilde{B}t},$$

when $\widetilde{B}=0$, we have $E_\alpha(t)\leq\tilde{A}$, and when $\widetilde{B}\geq 0$, we have $E_\alpha(t)\leq\tilde{A}e^{\widetilde{B}t}\leq\tilde{A}e^{\widetilde{B}T}$, $\forall t\in[0,T]$. From our choice of initial data, we can obtain that for all $t\in[0,T]$,

$$E_\alpha(t)\leq K(T).$$

Thus, approximate solutions are defined on the whole range $[0, T]$, and it follows that

$$u^m \text{ is bounded in } L^\infty(0, T; H_0^1),$$

$$u_t^m \text{ is bounded in } L^\infty(0, T; H_0^1),$$

$$\eta^{t,m} \text{ is bounded in } L^\infty(0, T; \mathcal{M}).$$

In view of (3.17), passing to a subsequence if necessary, we have

$$u^m \to u \text{ is weakly-* in } L^\infty(0, T; H_0^1),$$

$$u_t^m \to u_t \text{ is weakly-* in } L^\infty(0, T; H_0^1),$$

$$\eta^{t,m} \to \eta \text{ is weakly-* in } L^\infty(0, T; \mathcal{M}).$$

The first two limits imply that

$$u^m \to u \text{ strongly in } L^2(0, T; H_0^1),$$

which, together with the Aubin-Lions lemma, yields

$$u^m \to u \text{ strongly in } C(0, T; H_0^1), as\ m \to \infty.$$

Therefore, for the approximation problems (3.2)–(3.3), the above convergence suffices for passing to the limit, and the desired weak solution is obtained.

Moreover, from $(2.3)_1$ and the above discussion, for any fixed $T > 0, t \in [0, T]$, we know there exists $R > 0$ such that

$$\|u_{tt} - \varepsilon\Delta u_{tt}\|_{H^{-1}} = \|\Delta u + \int_0^\infty k(s)\Delta\eta^t(x, s)\mathrm{d}s - \mu_1 u_t(x, t) - \mu_2 u_t(x, t - \tau(t)) - h(u) + g(x)\|_{H^{-1}} \le C(T, R).$$

(iii)**(uniqueness)** Let (u, η^t) and (v, ξ^t) be two weak solutions to system (2.3) corresponding to initial data $\{u_0, u_1, \eta_0, g_1, h_{01}\}$ and $\{v_0, v_1, \xi_0, g_2, h_{02}\}$, respectively. Set $\omega = u - v$ and $\zeta^t = \eta^t - \xi^t$. Then (w, ζ) is a weak solution of

$$\begin{cases} \omega_{tt} - \Delta\omega - \varepsilon\Delta\omega_{tt} - \int_0^\infty k(s)\Delta\zeta^t(s)ds + \mu_1\omega_t + \mu_2\omega_t(t - \tau(t)) + h(u) - h(v) = g_1 - g_2, \\ \zeta_t^t = -\zeta_s^t + w_t. \end{cases} \tag{3.18}$$

Moreover, the initial values of the difference function satisfy

$$\omega(0) = u_0 - v_0, \quad \omega_t(0) = u_1 - v_1, \quad \zeta^0 = \eta_0 - \xi_0.$$

Let

$$E_\beta(t) = \frac{1}{2}(\|\omega_t\|^2 + \|\nabla\omega\|^2 + \varepsilon\|\nabla\omega_t\|^2 + \|\zeta^t\|_{\mathcal{M}}^2).$$

Multiplying $(3.18)_1$ by $\varpi = \omega_t + \beta\omega$, integrating the result over Ω, and using $(3.18)_2$, we have

$$\frac{d}{dt}(E_\beta(t) + \beta(\omega_t, \omega) + \beta\varepsilon(\nabla\omega_t, \nabla\omega)) = \beta\|\omega_t\|^2 - \beta\|\nabla\omega\|^2 + \beta\varepsilon\|\nabla\omega_t\|^2 + \frac{1}{2}\int_0^\infty k'(s)\|\nabla\zeta^t(s)\|^2 ds$$

$$-\beta\int_0^\infty k(s)(\nabla\zeta^t(s),\nabla\omega)ds+(g_1-g_2,\varpi)-\mu_1\|\omega_t\|^2-\beta\mu_1(\omega_t,\omega)$$
$$-(\mu_2\omega_t(t-\tau(t)),\omega_t)-\beta(\mu_2\omega_t(t-\tau(t)),\omega)-(h(u)-h(v),\varpi). \tag{3.19}$$

The initial values satisfy $\|(u_0,u_1,\eta_0)\|_{\mathcal{H}}\le R_0$, $\|(v_0,v_1,\xi_0)\|_{\mathcal{H}}\le R_0$. From the energy estimate lemma 3.3 below, for any $T>0$, there exists a constant $M=M(T,R_0)$ such that

$$\sup_{t\in[0,T]}\|u(t)\|_{H_0^1(\Omega)}\le M,\quad \sup_{t\in[0,T]}\|v(t)\|_{H_0^1(\Omega)}\le M.$$

By the Sobolev embedding $H_0^1(\Omega)\subset L^{\frac{2N}{N-2}}(\Omega)$ $(N\ge 3)$, we have

$$\sup_{t\in[0,T]}\|u\|_{L^{\frac{2N}{N-2}}(\Omega)}\le CM,\quad \sup_{t\in[0,T]}\|v\|_{L^{\frac{2N}{N-2}}(\Omega)}\le CM.$$

Using Hölder's and Young's inequalities, we have

$$\begin{aligned}
&(h(u)-h(v),\varpi)=(h(u)-h(v),\omega_t)+\varsigma(h(u)-h(v),\omega)\\
&\le\|h'(\xi)\|_{L^{\frac{6}{4}}}\|\omega\|_{L^6}\|\omega_t\|_{L^6}+\|h'(\xi)\|_{L^{\frac{6}{4}}}\|\omega\|_{L^6}\|\omega\|_{L^6}\\
&\le C_h\left\{\int_\Omega(1+|u|^{\frac{4}{N-2}}+|v|^{\frac{4}{N-2}})^{\frac{6}{4}}dx\right\}^{\frac{4}{6}}\left[\|\omega\|_{L^6}\cdot\|\omega_t\|_{L^6}+\|\omega\|_{L^6}^2\right]\\
&\le C(1+\|u\|^{\frac{4}{N-2}}_{L^{\frac{6}{N-2}}}+\|v\|^{\frac{4}{N-2}}_{L^{\frac{6}{N-2}}})\left[\|\omega\|_{L^6}\cdot\|\omega_t\|_{L^6}+\|\omega\|_{L^6}^2\right]\\
&\le C_{\lambda_1}M\left[\|\nabla\omega\|\cdot\|\nabla\omega_t\|+\|\nabla\omega\|^2\right]\le 2C_{\lambda_1}ME_\beta(t),
\end{aligned}\tag{3.20}$$

where $\xi=\theta u+(1-\theta)v,\ 0\le\theta\le 1$, and the embeddings $H^1(\Omega)\subset L^6(\Omega),\ L^6(\Omega)\subset L^{\frac{6}{N-2}}(\Omega)$ $(N\ge 3)$ are used. For the force term,

$$(g_1-g_2,\varpi)\le C\|g_1-g_2\|^2+\frac{1}{2}\left(\|\omega_t\|^2+\|\nabla\omega\|^2\right)\le C\|g_1-g_2\|^2+E_\beta(t). \tag{3.21}$$

Substituting (3.20) and (3.21) into (3.19), and taking β sufficiently small as in (3.12), we conclude that

$$\begin{aligned}
&\frac{d}{dt}(E_\beta(t)+\beta(\omega_t,\omega)+\beta\varepsilon(\nabla\omega_t,\nabla\omega))\\
&\le\beta E_\beta(t)+C_1\int_0^t\|\omega_t(s-\tau(s))\|^2ds+C_3+2C_{R,\lambda_1}E_\beta(t)+C\|g_1-g_2\|^2.
\end{aligned}\tag{3.22}$$

Notice that

$$\int_0^t\int_\Omega\omega_s^2(s-\tau)\,dxds\le\int_{-\tau}^0\int_\Omega(h_{01}(\rho)-h_{02}(\rho))^2\,dxd\rho+\int_0^t\int_\Omega\omega_\rho^2(\rho)\,dxd\rho.$$

Integrating (3.22) from 0 to t, we get

$$E_\beta(t)+\beta(\omega_t,\omega)+\beta\varepsilon(\nabla\omega_t,\nabla\omega)\le\bar{A}+\bar{B}\int_0^t E_\beta(s)ds,$$

where

$$\tilde{A} = \|g_1 - g_2\|^2 T + E_\beta(0) + \|h_{01} - h_{02}\|^2_{L^2(\Omega\times(-\tau,0))} + C_3 T, \quad \bar{B} = 2C_{R,\lambda_1} + C_1 + \beta.$$

An application of Gronwall's inequality yields

$$E_\beta(t) + \beta(\omega_t, \omega) + \beta\varepsilon(\nabla\omega_t, \nabla\omega) \le \bar{A}e^{\bar{B}t}.$$

In particular, problem (2.3) has a unique solution. From the definition of $E_\beta(t)$, we deduce that

$$\begin{aligned} &\|u_t(t) - v_t(t)\|^2 + \|\nabla u(t) - \nabla v(t)\|^2 + \|\eta^t - \xi^t\|_{\mathcal{M}} \\ &\le C_R(T)\left(\|u_1 - v_1\| + \|\nabla u_0 - \nabla v_0\| + \|\eta_0 - \xi_0\|_{\mathcal{M}} + \|g_1 - g_2\| + \|h_{01} - h_{02}\|\right), \end{aligned} \tag{3.23}$$

for all $t \in [0, T]$, which shows that solutions of (2.3) depend continuously on the initial data. □

By Theorem 3.2, define the solution operator

$$S(t) : \mathcal{H} \to \mathcal{H}, S(t)(u_0, u_1, \eta_0) = (u, u_t, \eta), \quad \text{for all } t \ge 0,$$

where (u, u_t, η) is the unique solution of (3.1) corresponding to the initial data (u_0, u_1, η_0). The operator $S(t)$ satisfies the semigroup properties

$$S(0) = I, \quad S(t+s) = S(t) \circ S(s), \quad \text{for all } t, s \ge 0,$$

and defines a nonlinear C_0-semigroup, which is locally Lipschitz continuous on the phase space $\mathcal{H}$. Thus, we can study the long-time dynamics of problem (2.3) in the framework of the dynamical system $(\mathcal{H}, S(t))$.

We define, along a weak solution, the energy functional to problem (2.3) defined by

$$\mathcal{D}(t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\nabla u\|^2 + \frac{\varepsilon}{2}\|\nabla u_t\|^2 + \frac{1}{2}\|\eta^t\|^2_{\mathcal{M}} + \int_\Omega \hat{h}(u(t))dx - \int_\Omega g u dx. \tag{3.24}$$

Consequently, we derive the follow lemma related to the energy functional $\mathcal{D}(t)$.

Lemma 3.3. *The energy functional defined in* (3.24) *satisfies formally*

$$\frac{d}{dt}\mathcal{D}(t) = -\mu_1\|u_t\|^2 - \mu_2\int_\Omega u_t(t)u_t(t-\tau(t))dx + \frac{1}{2}\int_0^\infty k'(s)\|\nabla\eta^t\|^2 ds. \tag{3.25}$$

There exists δ_0, $C_D > 0$, independent of the initial date in $\mathcal{H}$, such that

$$\mathcal{D}(t) \ge \delta_0(\|\nabla u_t\|^2 + \|\nabla u\|^2 + \|\eta^t\|^2_{\mathcal{M}}) - C_D, t \ge 0, \tag{3.26}$$

where $C_D = c_h|\Omega| + C_{\delta_0}\|g\|^2$.

Proof. Multiplying equation $(2.3)_1$ by u_t, integrate over Ω, and using $(2.3)_2$, we obtain

$$\frac{d}{dt}\mathcal{D}(t) = -\mu_1\|u_t\|^2 - \mu_2\int_\Omega u_t(t)u_t(t-\tau(t))dx - (\eta^t, \eta^t_s)_{\mathcal{M}}. \tag{3.27}$$

By the condition $\eta^t(0) = 0$, we deduce

$$-(\eta^t, \eta^t_s)_{\mathcal{M}} = -\frac{1}{2}\int_\Omega \int_0^\infty k(s)\frac{d}{ds}\left|\nabla\eta^t(s)\right|^2 dsdx$$
$$= \frac{1}{2}\int_0^\infty k'(s)\left\|\nabla\eta^t(s)\right\|^2 ds \le 0. \tag{3.28}$$

Inserting (3.28) into (3.27), (3.25) can be obtained directly.
To prove (3.26), define the positive constant

$$\delta_0 = \frac{1}{4}\left(1 - \frac{\beta}{\lambda_1}\right). \tag{3.29}$$

By (2.11), we have

$$\int_\Omega \hat{h}(u)dx \ge -\frac{\beta}{2\lambda_1}\|\nabla u\|^2 - c_h|\Omega|. \tag{3.30}$$

Moreover, there exists a constant $C_{\delta_0} > 0$ such that

$$\int_\Omega g \cdot u\, dx \le \delta_0\|\nabla u\|^2 + C_{\delta_0}\|g\|^2. \tag{3.31}$$

Substituting (3.30) and (3.31) into (3.24) yields

$$\mathcal{D}(t) \ge \frac{1}{2}\|u_t\|^2 + \left(\frac{1}{2} - \frac{\beta}{2\lambda_1} - \delta_0\right)\|\nabla u\|^2 + \frac{1}{2}\|\eta^t\|^2_{\mathcal{M}} + \frac{\varepsilon}{2}\|\nabla u_t\|^2 - c_h|\Omega| - C_{\delta_0}\|g\|^2, \tag{3.32}$$

and combining (3.29) with (3.32), we have

$$\mathcal{D}(t) \ge \delta_0(\|\nabla u_t\|^2 + \|\nabla u\|^2 + \|\eta^t\|^2_{\mathcal{M}}) - C_D. \tag{3.33}$$

Taking

$$C_D = c_h|\Omega| + C_{\delta_0}\|g\|^2,$$

this yields the desired estimate (3.26). □

4. Proof of the main result

Based upon the aforementioned generalizations described in Theorems 2.2–2.9 of the fundamental principles of dynamical systems theory, this section shall prove Theorem 4.1.

Theorem 4.1. *Assume that* (2.4)–(2.10) *hold,* $g(x) \in L^2(\Omega)$, *and* $|\mu_2| < \sqrt{1-d}\mu_1$.

(i) The dynamical system $(\mathcal{H}, S(t))$ *generated by problem* (2.3) *is quasi-stable on any bounded positively invariant subset* $B \subset \mathcal{H}$.

(ii) This dynamical system $(\mathcal{H}, S(t))$ *has a compact global attractor* $\mathcal{A}$, *characterized by the unstable manifold* $\mathcal{A} = \mathbb{M}_+(\mathcal{N})$ *corresponding to the set* $\mathcal{N}$ *of stationary solutions of problem* (2.3).

(iii) The global attractor $\mathcal{A}$ *possesses a finite fractal dimension.*

(iv) There exists a generalized exponential attractor $\mathcal{A}_{\exp} \subset \mathcal{H}$ *for the dynamical system* $(\mathcal{H}, S(t))$, *which also has a finite fractal dimension in*

$$\tilde{\mathcal{H}} = V_0 \times V_1' \times \mathcal{M}_0,$$

where V_1' *is the topological dual of* V_1. *Additionally, it is noted that a generalized exponential attractor exists with a finite fractal dimension in* $\mathcal{H}^\delta$, *where* $\mathcal{H} \subset \mathcal{H}^\delta \subset \tilde{\mathcal{H}}$ *with* $\delta \in (0, 1]$.

4.1. Gradient system and stationary solutions

To demonstrate that the dynamical system $(\mathcal{H}, S(t))$ possesses gradient properties, we first define the modified energy for problem (2.3) as follows:

$$\mathcal{L}(t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\nabla u\|^2 + \frac{\varepsilon}{2}\|\nabla u_t\|^2 + \frac{1}{2}\left\|\eta^t\right\|_{\mathcal{M}}^2 + \int_\Omega (\hat{h}(u(t) - gu)dx + \frac{\varrho}{2}\int_{t-\tau(t)}^t \int_\Omega e^{\lambda(s-t)}u_t^2(x,s)dxds, \tag{4.1}$$

where $\varrho > 0$ will be established subsequently and the normal constant $\lambda > 0$. As previously introduced by Nicaise and Pignotti [12],

$$\lambda < \frac{1}{\tau_1}\left|\log\frac{|\mu_2|}{\sqrt{1-d}}\right|.$$

Lemma 4.2. *The dynamical system $(\mathcal{H}, S(t))$ is a gradient system, and the Lyapunov functional Φ possesses an upper bound on any bounded subset of $\mathcal{H}$. For any $R > 0$, the corresponding set $\Phi_R = \{U_0 \in \mathcal{H} : \Phi(U_0) \le R\}$ is finite in $\mathcal{H}$.*

Proof. Take the functional Φ as the modified energy in (4.1). Subsequently, for every initial condition $U_0 = (u_0, u_1, \eta_0) \in \mathcal{H}$, it follows from (4.1) that

$$\frac{d}{dt}\mathcal{L}(t) = -\mu_1\|u_t\|^2 - \mu_2\int_\Omega u_t(t)u_t(t-\tau(t))dx - (\eta^t, \eta_s^t)_{\mathcal{M}} + \frac{d}{dt}\left(\frac{\varrho}{2}\int_{t-\tau(t)}^t\int_\Omega e^{\lambda(s-t)}u_t^2(x,s)dxds\right). \tag{4.2}$$

Differentiating $\frac{\varrho}{2}\int_{t-\tau(t)}^t\int_\Omega e^{\lambda(s-t)}u_t^2(x,s)dxds$ with respect to t, we get

$$\begin{aligned}&\frac{\varrho}{2}\frac{d}{dt}\int_{t-\tau(t)}^t e^{\lambda(s-t)}\|u_t(s)\|^2 ds\\ &= \frac{\varrho}{2}\|u_t\|^2 - \frac{\varrho}{2}e^{-\lambda\tau(t)}(1-\tau'(t))\|u_t(t-\tau(t))\|^2 - \frac{\lambda\varrho}{2}\int_{t-\tau(t)}^t e^{\lambda(s-t)}\|u_t(s)\|^2 ds.\end{aligned} \tag{4.3}$$

Inserting (4.3) into (4.2), we have

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(t) = &-\mu_1\|u_t\|^2 - \mu_2\int_\Omega u_t(t)u_t(t-\tau(t))dx - (\eta^t, \eta_s^t)_{\mathcal{M}} + \frac{\varrho}{2}\|u_t\|^2\\ &- \frac{\varrho}{2}e^{-\lambda\tau(t)}(1-\tau'(t))\|u_t(t-\tau(t))\|^2 - \frac{\lambda\varrho}{2}\int_{t-\tau(t)}^t e^{\lambda(s-t)}\|u_t(s)\|^2 ds.\end{aligned} \tag{4.4}$$

From (3.28), we infer that

$$-(\eta^t, \eta_s^t)_{\mathcal{M}} = \frac{1}{2}\int_0^\infty k'(s)\left\|\nabla\eta^t(s)\right\|^2 ds. \tag{4.5}$$

Using (2.8) and (2.9), this yields

$$-\frac{\varrho}{2}e^{-\lambda\tau(t)}(1-\tau'(t))\|u_t(t-\tau(t))\|^2 \le -\frac{\varrho}{2}(1-d)e^{-\lambda\tau_1}\int_\Omega u_t^2(t-\tau(t))dx. \tag{4.6}$$

By Young's inequality, it holds that

$$-\mu_2 \int_\Omega u_t(t)\cdot u_t(t-\tau(t))dx \le \frac{|\mu_2|}{2\sqrt{1-d}}\|u_t\|^2 + \frac{|\mu_2|}{2}\sqrt{1-d}\int_\Omega u_t^2(t-\tau(t))dx. \tag{4.7}$$

Substituting (4.5)–(4.7) into (4.4), for any $t>0$, we deduce

$$\begin{aligned}\mathcal{L}'(t) \le &-\psi_1\|u_t\|^2 - \psi_2\|u_t(t-\tau(t))\|^2 + \frac{1}{2}\int_0^\infty k'(s)\left\|\nabla\eta^t(s)\right\|^2 ds\\ &-\frac{\lambda\varrho}{2}\int_{t-\tau(t)}^t\int_\Omega e^{-\lambda(t-s)}u_t^2(x,s)dxds,\end{aligned} \tag{4.8}$$

where ψ_1,ψ_2 are constants defined by

$$\psi_1 = -\frac{|\mu_2|}{2\sqrt{1-d}} + \mu_1 - \frac{\varrho}{2}, \psi_2 = -\frac{|\mu_2|}{2}\sqrt{1-d} + \frac{\varrho}{2}e^{-\lambda\tau_1}(1-d).$$

Note that when $e^{\lambda\tau_1}\to 1$ and $\lambda\to 0^+$, by the continuity of the set of real numbers, we select a sufficiently small $\lambda>0$ to ensure the existence of a constant ϱ such that

$$\frac{e^{\lambda\tau_1}|\mu_2|}{\sqrt{1-d}} < \varrho < \mu_1. \tag{4.9}$$

It follows from (4.9) that

$$\frac{|\mu_2|}{2\sqrt{1-d}} - \mu_1 + \frac{\varrho}{2} < 0, \frac{|\mu_2|}{2}\sqrt{1-d} - \frac{\varrho}{2}e^{-\lambda\tau_1}(1-d) < 0, \tag{4.10}$$

which yields

$$\mathcal{L}'(t) \le 0. \tag{4.11}$$

Hence, it means $\frac{d}{dt}\Phi(S(t)U) \le 0$, and therefore $\Phi(S(t)U)$ is monotonically decreasing. Now suppose that $\Phi(S(t)U) = \Phi(U)$ holds for all $t\ge 0$. Then

$$-\psi_1\|u_t\|^2 - \psi_2\|u_t(t-\tau(t))\|^2 + \frac{1}{2}\int_0^\infty k'(s)\left\|\nabla\eta^t(s)\right\|^2 ds - \frac{\lambda\varrho}{2}\int_{t-\tau(t)}^t\int_\Omega e^{-\lambda(t-s)}u_t^2(x,s)dxds = 0. \tag{4.12}$$

All terms on the left-hand side of the above equation bear the same sign, from which we infer that

$$\nabla\eta^t = u_t = 0. \tag{4.13}$$

This implies that $u(t) = u_0$ holds for all $t\ge 0$. Thus, $U(t) = S(t)U_0 = (u_0, 0, 0)$ is a stationary solution, indicating that Φ acts as a strict Lyapunov functional; in other words, the dynamical system $(\mathcal{H}, S(t))$ possesses gradient properties. Furthermore, inequality (4.11) gives

$$\Phi\left(S(t)U_0\right) \le \Phi\left(U_0\right),$$

which implies the inequality

$$\frac{\varrho}{2}\int_{t-\tau(t)}^{t}\int_{\Omega}e^{\lambda(s-t)}u_t^2(x,s)dxds\le C_B,$$

where $C_B>0$ is a constant depending on any bounded subset B of $\mathcal{H}$. We can readily prove that Φ is bounded for any bounded subset of $\mathcal{H}$. Consider any weak solution $U(t)=(u,u_t,\eta)\in\mathcal{H}$ to equation (2.3) satisfying $\Phi(U(t))\le R$. From (3.26), we obtain

$$\bar{\beta}_0\left(\|u_t\|^2+\|\nabla u\|^2+\|\eta\|_{\mathcal{M}}^2\right)-\bar{C}_D\le\Phi(U(t))\le R,$$

which yields $\|U(t)\|_H^2\le(R+\bar{C}_D)\bar{\beta}_0^{-1}$. Here Φ_R is a bounded set in $\mathcal{H}$, and the proof is complete. □

Similar to [14], the result can be deduce directly:

Lemma 4.3. *The set $\mathcal{N}=\{(u,0,0)\in\mathcal{H}:-\Delta u+h(u)=g\}$ of stationary solutions of problem* (2.3) *is bounded in $\mathcal{H}$.*

4.2. Quasi-stability

Lemma 4.4. *Assuming the conditions of Theorem 4.1 hold, for a given bounded set B in $\mathcal{H}$, there exist constants γ, $b_0>0$ and $C_B>0$ such that*

$$\left\|S(t)U^1-S(t)U^2\right\|_{\mathcal{H}}^2\le b_0e^{-\gamma t}\left\|U^1-U^2\right\|_{\mathcal{H}}^2+C\int_0^t e^{-\gamma(t-s)}\|\nabla u\|^2ds,\tag{4.14}$$

where $S(t)U^i=(u^i,u_t^i)$ denotes the weak solution of equation (2.3) *with initial conditions U^i belonging to B for $i=1,2$, and $u=u^1-u^2$.*

Proof. Let $S(t)U^i=(u^i,u_t^i)$ denote a weak solution of problem (2.3) in B corresponding to the initial conditions U^i for $i=1,2$, and denote

$$u=u^1-u^2 \text{ and } \eta=\eta^1-\eta^2.$$

Then (u,u_t,η) is a weak solution of

$$\begin{cases} u_{tt}-\Delta u-\varepsilon\Delta u_{tt}-\int_0^\infty k(s)\Delta\eta^t(s)ds+\mu_1u_t+\mu_2u_t(t-\tau(t))+h\left(u^1\right)-h\left(u^2\right)=0,\\ \eta^t=-\eta_s^t+u_t,\\ u(0)=u^1(0)-u^2(0),\ u_t(0)=u_t^1(0)-u_t^2(0),\ \eta_0=\eta_0^1-\eta_0^2. \end{cases}\tag{4.15}$$

The energy functional of this system is defined as

$$\mathcal{F}(t)=\frac{1}{2}\|u_t\|^2+\frac{1}{2}\|\nabla u\|^2+\frac{1}{2}\|\eta\|_{\mathcal{M}}^2+\frac{\varepsilon}{2}\|\nabla u_t\|^2+\frac{\varrho}{2}\int_{t-\tau(t)}^{t}\int_{\Omega}e^{\lambda(s-t)}u_t^2(x,s)dxds.\tag{4.16}$$

To demonstrate this lemma, we will outline the following five steps.

Step 1. Let $\delta_1>0$. We assert that there exists a constant $C_{\delta_1}>0$ (which depends on B) such that

$$\mathcal{F}'(t)\le\delta_1\|\nabla u_t\|^2+C_{\delta_1}\|\nabla u\|^2+\frac{1}{2}\int_0^\infty k'(s)\|\nabla\eta^t\|^2ds-\frac{\mu_1}{2}\|u_t\|^2$$

$$-\left(\frac{\varrho}{2}(1-d)e^{-\lambda\tau_1}-\frac{|\mu_2|}{2}\sqrt{1-d}\right)\|u_t(t-\tau(t))\|^2-\frac{\lambda\varrho}{2}\int_{t-\tau(t)}^{t}e^{\lambda(s-t)}\|u_t(t)\|^2\,ds. \tag{4.17}$$

To demonstrate this point, similar to (3.27), taking the inner product of equation $(4.15)_1$ with u_t, and using $(4.15)_2$, we have

$$\begin{aligned}\mathcal{F}'(t)=&-\mu_1\|u_t\|^2-\mu_2\int_\Omega u_t(t)u_t(t-\tau(t))dx+\frac{\varrho}{2}\|u_t\|^2-\frac{\varrho}{2}e^{-\lambda\tau(t)}\left(1-\tau'(t)\right)\|u_t(t-\tau(t))\|^2\\&-\frac{\lambda\varrho}{2}\int_{t-\tau(t)}^{t}e^{\lambda(s-t)}\|u_t(s)\|^2\,ds-(\eta^t,\eta_s^t)_{\mathcal{M}}-\int_\Omega\left(h\left(u^1\right)-h\left(u^2\right)\right)u_t dx.\end{aligned} \tag{4.18}$$

Furthermore, based on the mean value theorem and assumption (2.5), similar to (3.20), it follows that

$$\int_\Omega\left(h\left(u^1\right)-h\left(u^2\right)\right)u_t dx\le\left\|h'\left(\theta u^1+(1-\theta)u^2\right)\right\|_{L^{\frac{6}{4}}}\|u^1-u^2\|_{L^6}\|u_t\|_{L^6},0\le\theta\le1, \tag{4.19}$$

so

$$\left|\int_\Omega\left(h\left(u^1\right)-h\left(u^2\right)\right)u_t dx\right|\le C_B\|\nabla u\|\|\nabla u_t\|\le\delta_1\|\nabla u_t\|^2+C_{\delta_1}\|\nabla u\|^2,\forall\delta_1>0, \tag{4.20}$$

where C_B denotes a bound associated with the bounded set B. Subsequently, inserting (4.6), (4.7), and (4.20) into (4.18), we get

$$\begin{aligned}\mathcal{F}'(t)\le&\left(\frac{|\mu_2|}{2\sqrt{1-d}}-\mu_1+\frac{\varrho}{2}\right)\|u_t\|^2+\left[\frac{|\mu_2|}{2}\sqrt{1-d}-\frac{\varrho}{2}e^{-\lambda\tau_1}(1-d)\right]\|u_t(t-\tau(t))\|^2\\&+\frac{1}{2}\int_0^\infty k'(s)\|\nabla\eta^t\|^2ds-\frac{\lambda\varrho}{2}\int_{t-\tau(t)}^{t}\int_\Omega e^{-\lambda(t-s)}u_t^2(x,s)dxds+\delta_1\|\nabla u_t\|^2+C_{\delta_1}\|\nabla u\|^2.\end{aligned} \tag{4.21}$$

In subsequent discussions, by selecting a constant $\xi>0$ satisfying the following conditions:

$$\frac{|\mu_2|\,e^{\lambda\tau_1}}{\sqrt{1-d}}<\varrho<\mu_1-\frac{|\mu_2|}{\sqrt{1-d}},$$

this gives

$$\frac{|\mu_2|}{2\sqrt{1-d}}-\mu_1+\frac{\varrho}{2}<-\frac{\mu_1}{2}\text{ and }\frac{|\mu_2|}{2}\sqrt{1-d}-\frac{\varrho}{2}e^{-\lambda\tau_1}(1-d)<0.$$

Ultimately, we obtain the expected estimate as described in (4.17).

Step 2. Define the functional

$$\Lambda(t)=\int_\Omega(u_t-\varepsilon\Delta u_t)\,udx.$$

Then we assert that there exist positive constants c_i $(i=1,2,3)$ such that

$$\begin{aligned}\Lambda'(t)\le&-\mathcal{F}(t)-\frac{1}{4}\|\nabla u\|^2+c_1\|u_t\|^2-c_2\int_0^\infty k'(s)\|\nabla\eta^t(s)\|^2ds+c_3\|u_t(t-\tau(t))\|^2\\&+\frac{\varrho}{2}\int_{t-\tau(t)}^{t}\int_\Omega e^{\lambda(s-t)}u_t^2(x,s)dxds+C_B\|\nabla u\|^2+\frac{3\varepsilon}{2}\|\nabla u_t\|^2.\end{aligned} \tag{4.22}$$

Through direct calculation, we obtain

$$\frac{d}{dt}\Lambda(t) = \int_\Omega (u_{tt} - \varepsilon\Delta u_{tt}(t))u dx + \int_\Omega (u_t - \varepsilon\Delta u_t)u_t dx. \tag{4.23}$$

From $(4.15)_1$, we can conclude

$$\begin{aligned}&\int_\Omega (u_{tt}(t) - \varepsilon\Delta u_{tt}(t))u(t)dx\\ &= \int_\Omega \left[\Delta u - \mu_1 u_t - \mu_2 u_t(t-\tau(t)) + \int_0^\infty k(s)\Delta\eta^t ds\right] u dx - \int_\Omega \left(h\left(u^1\right) - h\left(u^2\right)\right) u dx\end{aligned} \tag{4.24}$$

and

$$\int_\Omega \left(h\left(u^1\right) - h\left(u^2\right)\right) u dx \le C_B\|\nabla u\|^2. \tag{4.25}$$

Substituting (4.24) into (4.23), then from (4.16), we infer that

$$\begin{aligned}\Lambda'(t) + \mathcal{F}(t) \le \frac{3}{2}\|u_t\|^2 + \frac{3\varepsilon}{2}\|\nabla u_t\|^2 - \frac{1}{2}\|\nabla u\|^2 + \frac{1}{2}\|\eta\|_{\mathcal{M}}^2 + \frac{\varrho}{2}\int_{t-\tau(t)}^t\int_\Omega e^{\lambda(s-t)}u_t^2(x,s)dxds - (\mu_1 u_t, u)\\ - (\mu_2 u_t(t-\tau(t)), u) + \left(\int_0^\infty k(s)\Delta\eta^t(s)ds, u\right) + C_B\|\nabla u\|^2.\end{aligned} \tag{4.26}$$

By Hölder's and Young's inequalities, for any $\delta > 0$, based on assumption (2.10), we observe that

$$|-(\mu_1 u_t, u)| \le |\mu_1|\delta\|u\|^2 + \frac{|\mu_1|}{4\delta}\|u_t\|^2 \le \frac{|\mu_1|\delta}{\lambda_1}\|\nabla u\|^2 + \frac{|\mu_1|}{4\delta}\|u_t\|^2; \tag{4.27}$$

$$|-(\mu_2 u_t(t-\tau(t)), u)| \le \frac{|\mu_2|\delta}{\lambda_1}\|\nabla u\|^2 + \frac{|\mu_2|}{4\delta}\int_\Omega u_t^2(t-\tau(t))dx; \tag{4.28}$$

$$\left|-\left(\int_0^\infty k(s)\Delta\eta^t(s)ds, u\right)\right| \le \frac{1}{8}\|\nabla u\|^2 + 2l_0\|\eta^t\|_{\mathcal{M}}^2. \tag{4.29}$$

Substituting (4.27)–(4.29) into (4.26), for any $\delta > 0$, we deduce

$$\begin{aligned}\Lambda'(t) \le -\mathcal{F}(t) - \left(\frac{3}{8} - \frac{|\mu_1|\delta}{\lambda_1} - \frac{|\mu_2|\delta}{\lambda_1}\right)\|\nabla u\|^2 + \left(\frac{1}{2} + 2l_0\right)\|\eta^t\|_{\mathcal{M}}^2\\ + \left(\frac{3}{2} + \frac{|\mu_1|}{4\delta}\right)\|u_t\|^2 + \frac{|\mu_2|}{4\delta}\int_\Omega u_t^2(t-\tau(t))dx + \frac{3\varepsilon}{2}\|\nabla u_t\|^2\\ + \frac{\varrho}{2}\int_{t-\tau(t)}^t\int_\Omega e^{\lambda(s-t)}u_t^2(x,s)dxds + C_B\|\nabla u\|^2.\end{aligned} \tag{4.30}$$

Take $\delta > 0$ small enough, such that

$$\frac{|\mu_1|\delta}{\lambda_1} + \frac{|\mu_2|\delta}{\lambda_1} \le \frac{1}{8},$$

and we note that

$$\|\eta^t\|_{\mathcal{M}}^2 \le -\frac{1}{\nu}\int_0^\infty k'(s)\|\nabla\eta^t(s)\|^2 ds.$$

Then set $c_1 = \frac{3}{2} + \frac{|\mu_1|}{4\delta}$, $c_2 = \frac{1}{2k} + \frac{2l_0}{k}$, $c_3 = \frac{|\mu_2|}{4\delta}$, from which the required estimate (4.22) is derived.

Step 3. Define the functional

$$\Upsilon(t) = -\int_\Omega (u_t(t) - \varepsilon\Delta u_t(t)) \cdot \left(\int_0^\infty k(s)\eta^t(s)ds\right)dx, \tag{4.31}$$

and then there exists a constant $c_4 > 0$ such that for any $\delta_2 > 0$,

$$\begin{aligned}\Upsilon'(t) \le &-\left(\frac{3}{4}l_0 - \delta_2\right)\|u_t\|^2 + \delta_2\|\nabla u\|^2 + \delta_2\int_\Omega u_t^2(t-\tau(t))dx \\ &- c_4\int_0^\infty k'(s)\left\|\nabla\eta^t(s)\right\|^2 ds - \frac{3l_0}{4}\|\nabla u_t\|^2 + C_B\|\nabla u\|^2.\end{aligned} \tag{4.32}$$

To estimate this, we differentiate $\Upsilon(t)$. Referring to $(4.15)_1$, one arrives at

$$\begin{aligned}\Upsilon'(t) = &-\int_\Omega (u_{tt}(t) - \varepsilon\Delta u_{tt}(t)) \cdot \left(\int_0^\infty k(s)\eta^t(s)ds\right)dx \\ &-\int_\Omega (u_t(t) - \varepsilon\Delta u_t(t)) \cdot \left(\int_0^\infty k(s)\eta_t^t(s)ds\right)dx \\ = &\int_\Omega\left(-\Delta u - \int_0^\infty k(s)\Delta\eta^t(s)ds + \mu_1 u_t + \mu_2 u_t(t-\tau(t))\right) \\ &\cdot\left(\int_0^\infty k(s)\eta^t(s)ds\right)dx - \int_\Omega (u_t - \varepsilon\Delta u_t)\cdot\left(\int_0^\infty k(s)\eta_t^t(s)ds\right)dx + Q_3,\end{aligned} \tag{4.33}$$

where

$$Q_3 = \int_\Omega \left(h\left(u^1\right) - h\left(u^2\right)\right)\cdot\left(\int_0^\infty k(s)\eta^t(s)ds\right)dx. \tag{4.34}$$

Applying Young's inequality, Hölder's inequality, and assumptions (2.4)–(2.10), we conclude that for any $\delta_2 > 0$,

$$\begin{aligned}-\int_\Omega \Delta u\cdot\left(\int_0^\infty k(s)\eta^t(s)\,\mathrm{d}s\right)\mathrm{d}x &\le \delta_2\|\nabla u\|^2 + \frac{l_0}{4\delta_2}\left\|\eta^t\right\|_{\mathcal{M}}^2 \\ &\le \delta_2\|\nabla u\|^2 - \frac{l_0}{4\nu\delta_2}\int_0^\infty k'(s)\left\|\nabla\eta^t(s)\right\|^2\mathrm{d}s;\end{aligned} \tag{4.35}$$

$$\begin{aligned}-\int_\Omega\left(\int_0^\infty k(s)\Delta\eta^t(s)\,\mathrm{d}s\right)\cdot\left(\int_0^\infty k(s)\eta^t(s)\,\mathrm{d}s\right)\mathrm{d}x &= \int_\Omega\left(\int_0^\infty k(s)\nabla\eta^t(s)ds\right)^2 dx \\ &\le l_0\left\|\eta^t\right\|_{\mathcal{M}}^2 \\ &\le -\frac{l_0}{\nu}\int_0^\infty k'(s)\left\|\nabla\eta^t(s)\right\|^2\mathrm{d}s;\end{aligned} \tag{4.36}$$

$$\mu_1\int_\Omega u_t\cdot\left(\int_0^\infty k(s)\eta^t(s)\,\mathrm{d}s\right)\mathrm{d}x \le \delta_2\|u_t\|^2 + \frac{\mu_1^2 l_0}{4\delta_2\lambda_1}\int_0^\infty k(s)\left\|\nabla\eta^t(s)\right\|^2\mathrm{d}s$$

$$\leq \delta_2 \|u_t\|^2 - \frac{\mu_1^2 l_0}{4\delta_2\lambda_1\nu}\int_0^\infty k'(s)\left\|\nabla\eta^t(s)\right\|^2 \mathrm{d}s; \tag{4.37}$$

$$\mu_2\int_\Omega u_t(t-\tau(t))\cdot\left(\int_0^\infty k(s)\eta^t(s)\,\mathrm{d}s\right)\mathrm{d}x \leq \delta_2\int_\Omega u_t^2(t-\tau(t))\mathrm{d}x + \frac{\mu_2^2 l_0}{4\delta_2\lambda_1}\int_0^\infty k(s)\left\|\nabla\eta^t\right\|^2\mathrm{d}s$$
$$\leq \delta_2\int_\Omega u_t^2(t-\tau(t))\mathrm{d}x - \frac{\mu_2^2 l_0}{4\delta_2\lambda_1\nu}\int_0^\infty k'(s)\left\|\nabla\eta^t\right\|^2\mathrm{d}s. \tag{4.38}$$

Taking into account the following facts:

$$\int_0^\infty k(s)\eta_t^t(s)ds = \int_0^\infty k'(s)\eta^t(s)ds + l_0 u_t, \tag{4.39}$$

we find that

$$-\int_\Omega u_t\cdot\left(\int_0^\infty k(s)\eta_t^t(s)ds\right)dx \leq -l_0\|u_t\|^2 + \frac{l_0}{4}\|u_t\|^2 + \frac{1}{l_0}\int_\Omega\left(\int_0^\infty k'(s)\eta^t(s)ds\right)^2 dx$$
$$\leq -\frac{3}{4}l_0\|u_t\|^2 - \frac{k(0)}{l_0\lambda_1}\int_0^\infty k'(s)\left\|\nabla\eta^t(s)\right\|^2 ds, \tag{4.40}$$

$$\int_\Omega \varepsilon\Delta u_t\cdot\left(\int_0^\infty k(s)\eta_t^t(s)ds\right)dx = \int_\Omega \varepsilon\Delta u_t\cdot\left(\int_0^\infty k'(s)\eta^t(s)ds + l_0 u_t\right)dx$$
$$\leq -\int_\Omega \varepsilon\nabla u_t\cdot\left(\int_0^\infty k'(s)\nabla\eta^t(s)ds + l_0\nabla u_t\right)dx$$
$$\leq -l_0\varepsilon\|\nabla u_t\|^2 + \frac{l_0}{4}\varepsilon\|\nabla u_t\|^2 + \frac{\varepsilon}{l_0}\int_\Omega\left(\int_0^\infty k'(s)\nabla\eta^t(s)ds\right)^2 dx$$
$$\leq -\frac{3\varepsilon}{4}l_0\|\nabla u_t\|^2 - \frac{\varepsilon k(0)}{l_0\lambda_2}\int_0^\infty k'(s)\left\|\nabla\eta^t(s)\right\|^2 ds. \tag{4.41}$$

Moreover,

$$Q_3 \leq C_B\|\nabla u\|\cdot\left\|\int_0^\infty k(s)\nabla\eta^t(s)ds\right\| \leq C_B\|\nabla u\|^2 + \frac{1}{4}\int_\Omega\left(\int_0^\infty k(s)\nabla\eta^t(s)ds\right)^2 dx$$
$$\leq C_B\|\nabla u\|^2 - \frac{l_0}{4\nu}\int_0^\infty k'(s)\left\|\nabla\eta^t(s)\right\|^2 ds, \tag{4.42}$$

where the constant c_4 is defined as

$$c_4 = \frac{l_0}{4\nu\delta_2} + \frac{l_0}{\nu} + \frac{\mu_1^2 l_0}{4\delta_2\lambda_1^2\nu} + \frac{\mu_2^2 l_0}{4\delta_2\lambda_1^2\nu} + \frac{k(0)}{l_0\lambda_1^2} + \frac{k(0)}{l_0\lambda_2} + \frac{l_0}{4\nu}.$$

Taking into account the fact above and inserting (4.35)–(4.42) into (4.33), the estimate (4.32) is obtained.

Step 4. Define the Lyapunov functional

$$\mathcal{X}(t) := \mathcal{F}(t) + \epsilon_1\Lambda(t) + \epsilon_2\Upsilon(t), \tag{4.43}$$

where ϵ_1 and ϵ_2 are normal constants. Then, for sufficiently small $\epsilon_1 > 0$ and $\epsilon_2 > 0$, it follows that for any $t > 0$,

$$\frac{1}{2}\mathcal{F}(t) \leq \mathcal{X}(t) \leq \frac{3}{2}\mathcal{F}(t), \tag{4.44}$$

and the aforementioned equivalent constant is independent of ε. Indeed, by Hölder's and Young's inequalities, we find that for any $\epsilon > 0$,

$$\begin{aligned}
|\mathcal{X}(t) - \mathcal{F}(t)| &= |\epsilon_1\Lambda(t) + \epsilon_2\Upsilon(t)| \\
&\leq \epsilon_1\left(\epsilon\|u_t\|^2 + \left(\frac{1}{4\epsilon\lambda_1} + \frac{1}{4\epsilon}\right)\|\nabla u\|^2 + \epsilon\varepsilon\|\nabla u_t\|^2\right) \\
&\quad + \epsilon_2\left(\epsilon\|u_t\|^2 + \frac{l_0}{4\epsilon}\int_0^\infty k(s)\|\eta^t(s)\|^2 ds + \epsilon\varepsilon\|\nabla u_t\|^2 + \frac{l_0}{4\epsilon}\int_0^\infty k(s)\|\nabla\eta^t(s)\|^2 ds\right) \\
&\leq \epsilon_1\left(\epsilon\|u_t\|^2 + \left(\frac{1}{4\epsilon\lambda_1} + \frac{1}{4\epsilon}\right)\|\nabla u\|^2 + \epsilon\varepsilon\|\nabla u_t\|^2\right) \\
&\quad + \epsilon_2\left(\epsilon\|u_t\|^2 + \left(\frac{l_0}{4\epsilon\lambda_1} + \frac{l_0}{4\epsilon}\right)\|\eta^t\|_{\mathcal{M}}^2 + \epsilon\varepsilon\|\nabla u_t\|^2\right) \\
&\leq \epsilon(\epsilon_1 + \epsilon_2)\|u_t\|^2 + \left(\frac{\epsilon_1}{4\epsilon\lambda_1} + \frac{\epsilon_1}{4\epsilon}\right)\|\nabla u\|^2 \\
&\quad + \epsilon(\epsilon_1 + \epsilon_2)\varepsilon\|\nabla u_t\|^2 + \varepsilon_2 l_0\left(\frac{1}{4\epsilon\lambda_1} + \frac{l}{4\epsilon}\right)\|\eta^t\|_{\mathcal{M}}. \qquad (4.45)
\end{aligned}$$

Therefore, we may readily conclude that there exists a normal number $\epsilon > 0$ satisfying the following condition:

$$|\mathcal{X}(t) - \mathcal{F}(t)| \leq \epsilon\mathcal{F}(t),$$

which leads to (4.44) by choosing sufficiently small $\epsilon_1 > 0$ and $\epsilon_2 > 0$.

Step 5. Conclusion. It follows from (4.17), (4.22), and (4.32) that for any $t > 0$,

$$\begin{aligned}
\mathcal{X}'(t) &= \mathcal{F}(t)'(t) + \epsilon_1\Lambda'(t) + \epsilon_2\Upsilon'(t) \\
&\leq -\epsilon_1\mathcal{F}(t) - \left[\frac{\mu_1}{2} - c_1\epsilon_1 + \epsilon_2\left(\frac{3l_0}{4} - \delta_2\right)\right]\|u_t\|^2 \\
&\quad - \left[-\frac{|\mu_2|}{2}\sqrt{1-d} + \frac{\varrho}{2}e^{-\lambda t}(1-d) - c_3\epsilon_1 - \epsilon_2\delta_2\right]\|u_t(t-\tau(t))\|^2 \\
&\quad + \left(\frac{1}{2} - c_2\epsilon_1 - c_4\epsilon_2\right)\int_0^\infty k'(s)\|\nabla\eta^t(s)\|^2 ds + \left(-\frac{\epsilon_1}{4} + \epsilon_2\delta_2 + C_{\delta_1} + \epsilon_1 C_B + \epsilon_2 C_B\right)\|\nabla u\|^2 \\
&\quad - \left(\frac{\lambda\varrho}{2} - \frac{\varrho\epsilon_1}{2}\right)\int_{t-\tau(t)}^t\int_\Omega e^{-\lambda(t-s)}u_t^2(x,s)dxds - \left(-\delta_1 - \frac{3\varepsilon\epsilon_1}{2} + \frac{3l_0\varepsilon\epsilon_2}{4}\right)\|\nabla u_t\|^2. \qquad (4.46)
\end{aligned}$$

First, we take $\delta_2 > 0$ small enough such that

$$\frac{3l_0}{4} - \delta_2 > \frac{l_0}{2}.$$

Subsequently, for any fixed positive quantity $\delta_2 > 0$ satisfying $\delta_2 < \frac{l_0}{33}$, define

$$\epsilon_2 = \left(\frac{1}{16\delta_2} + \frac{3}{2l_0}\right)\epsilon_1$$

and $\delta_1 = \frac{l_0}{4}\delta_2$. Then we choose $\epsilon_1 > 0$ sufficiently small to ensure that (4.44) holds, and

$$\begin{aligned}
&\frac{\lambda\varrho}{2} - \frac{\varrho\epsilon_1}{2} > \frac{\lambda\varrho}{4} > 0, \ \ \frac{1}{2} - c_2\epsilon_1 > \frac{1}{4}, \ \ \frac{\mu_1}{2} - c_1\epsilon_1 > \frac{\mu_1}{8} > 0,\\
&-\frac{|\mu_2|}{4}\sqrt{1-d} + \frac{\varrho}{4}e^{-\lambda t}(1-d) - c_3\epsilon_1 > \frac{\varrho}{4}e^{-\lambda t}(1-d) - \frac{|\mu_2|}{4}\sqrt{1-d} > 0,\\
&\frac{1}{4} - c_4\epsilon_2 > \frac{1}{8}, \ \ \frac{\varrho}{4}e^{-\lambda t}(1-d) - \frac{|\mu_2|}{4}\sqrt{1-d} - \delta_2\epsilon_2 > \frac{1}{8},
\end{aligned} \tag{4.47}$$

which implies

$$\epsilon_1 < \min\left\{\frac{\lambda}{2}, \ \frac{1}{4c_2}, \ \frac{1}{4c_3}[\varrho e^{-\lambda\tau_1}(1-d) - |\mu_2|\sqrt{1-d}], \ \frac{3\mu_1}{8c_1}, \ \frac{1}{8c_4}, \ \frac{1}{8N\delta_2}[\varrho e^{-\lambda\tau_1}(1-d) - |\mu_2|\sqrt{1-d}]\right\},$$

where $N = \frac{1}{16\delta_1} + \frac{3}{4\delta_2}$. For any fixed $\delta_2 > 0, \epsilon_1 > 0, \epsilon_2 > 0, \delta_1 > 0$, we select a sufficiently small positive number $C_{\delta_1} > 0$, and we have

$$\epsilon_2\delta_2 + C_{\delta_1} + (\epsilon_1 C_B - \frac{\epsilon_1}{4}) + \epsilon_2 C_B < C_B.$$

As discussed previously, there exists a constant $\gamma_0 > 0$ such that

$$\mathcal{X}'(t) \le -\gamma_0\mathcal{F}(t) + C\|\nabla u\|^2.$$

From (4.44), this yields

$$\mathcal{X}'(t) \le -2\gamma_0\mathcal{X}(t) + C\|\nabla u\|^2.$$

Then we deduce that

$$\mathcal{X}(t) \le \mathcal{X}(0)e^{-2\gamma_0 t} + C\int_0^t e^{-2\gamma_0(t-s)}\|\nabla u\|^2 ds, \tag{4.48}$$

and by revisiting (4.44), it follows that

$$\mathcal{F}(t) \le 3\mathcal{F}(0)e^{-2\gamma_0 t} + C\int_0^t e^{-2\gamma_0(t-s)}\|\nabla u\|^2 ds. \tag{4.49}$$

Note that

$$\mathcal{F}(t) \ge \|(u, u_t, \eta)\|^2_{\mathcal{H}} = \left\|S(t)U^1 - S(t)U^2\right\|^2_{\mathcal{H}},$$

and then (4.14) can be deduced by appropriately redefining the constants present in (4.49). □

Remark 4.1. *From the above estimates derived in* (4.20)*, the constant C_{δ_1} depends solely on diam(B), the diameter of the bounded set B.*

Proof of Theorem 4.1

(i) (Quasi-stability) Considering the system dynamics $(\mathcal{H}, S(t))$ as the solution operator for problem (2.3), it follows that when $X = V_1$, $Y = V_1$, and $Z = \mathcal{M}$, (2.12) and (2.13) hold. Furthermore, by Theorem 3.2, it follows that equation (2.14) also holds.

Let $B \subset \mathcal{H}$ be a bounded set that remains positive under the action of $S(t)$. For U^1, $U^2 \in B$, set $S(t)U^i = (u^i, u_t^i, \eta^i), i = 1, 2$, and $u = u^1 - u^2$. Define the semi-norm

$$n_X(u) = \|\nabla u\| = \|u\|_{V_1},$$

which is compact on $X = V_1$, owing to the compact embedding $V_1 \hookrightarrow V_0$. Moreover, it follows from (4.14) that

$$\|S(t)U^1 - S(t)U^2\|_{\mathcal{H}}^2 \leq b(t)\|U^1 - U^2\|_{\mathcal{H}}^2 + c(t) \sup_{0<s<t} \left[n_X\left(u^1(s) - u^2(s)\right)\right]^2.$$

Evidently,

$$b(t) \in L^1(\mathbb{R}) \quad \text{and} \quad \lim_{t\to\infty} b(t) = 0.$$

Since $B \subset \mathcal{H}$ is bounded, $c(t)$ is locally bounded on $[0, \infty)$. Therefore, inequality (2.15) holds, which yields quasi-stability for all bounded positively invariant sets.

(ii) By using Theorem 2.4, we deduce the asymptotic smoothness of the dynamical system $(\mathcal{H}, S(t))$. Furthermore, Lemmas 4.2–4.3 and Theorem 2.7 confirm the existence of a global attractor $A = \mathbb{M}_+(\mathcal{N})$ for this system.

(iii) From (i) and (ii), it is evident that the dynamical system $(\mathcal{H}, S(t))$ exhibits quasi-stability on the attractor $\mathcal{A}$. Hence, it follows that $\mathcal{A}$ possesses a finite fractal dimension.

(iv) Consider the set $B = \{U : \Phi(U) \leq R\}$. As stated in Lemma 4.2, the strict Lyapunov functional is denoted by Φ. When R is sufficiently large, it follows that the set B is a positive invariant absorbing set. This condition implies that the dynamical system $(\mathcal{H}, S(t))$ exhibits quasi-stability on B. Furthermore, for the given initial conditions $(u_0, u_1, \eta_0) \in B$, applying (2.3) yields:

$$(u_t, u_{tt}, \eta_t) \in L^2_{loc}\left(\mathbb{R}^+, \tilde{\mathcal{H}}\right).$$

By employing the interpolation theorem, we can delineate smaller extended spaces so that

$$\mathcal{H} = \mathcal{H}^0 \subset \mathcal{H}^\delta \subset \mathcal{H}^1 = \tilde{\mathcal{H}}, \quad 0 < \delta \leq 1.$$

Given a solution $U(t)$ with the initial condition $y = U(0) \in B$, positive invariance of B ensures a constant $C_B > 0$ such that for any $0 \leq t \leq T$,

$$\int_0^T \|U_t(s)\|_{\tilde{\mathcal{H}}}^2 ds \leq C_B^2,$$

which yields, for any $0 \leq t_1 < t_2 \leq T$,

$$\|S(t_1)y - S(t_2)y\|_{\tilde{\mathcal{H}}} \leq \int_{t_1}^{t_2} \|U_t(s)\|_{\tilde{\mathcal{H}}} ds \leq C_B|t_1 - t_2|.$$

This implies that for every $y \in B$, the trajectory $t \mapsto S(t)y$ is Hölder continuous of order $\delta = 1$ in the extended space $\tilde{\mathcal{H}}$. It follows that identity (2.16) holds, which verifies the existence of a generalized exponential attractor of finite fractal dimension in $\tilde{\mathcal{H}}$.

The existence of the exponential attractor in $\mathcal{H}^\delta$ (where $\delta \in (0, 1)$) is consistent with the discussions in the literature [23, 24]. This completes the proof of Theorem 4.1.

5. Upper semi-continuity of the global attractor

In this section, we study the upper semi-continuity of the global attractor $\mathcal{A}_\varepsilon$ as the rotational inertia coefficient $\varepsilon \to 0^+$. The main strategy is to establish a uniform estimate for the difference between solutions of the ε-system and the limiting system ($\varepsilon = 0$).

The following lemma provides a key estimate for the difference between solutions with $\varepsilon > 0$ and $\varepsilon = 0$. To obtain the result, it is required that $l > 0$, such that

$$h'(u) \le l, \quad \forall u \in \mathbb{R}. \tag{5.1}$$

Moreover, in order to perform energy estimates, we shall assume that weak solutions $U(t) = (u, u_t, \eta^t) \in \mathcal{H}$ have higher regularity $\mathcal{H}_1$ in Lemma 5.1. Then by density, the estimates also hold for the weak solution in $\mathcal{H}$.

Lemma 5.1. *Assume that the initial data $(u_0, u_1, \eta_0) \in \mathcal{H}_1$ satisfy $\|(u_0, u_1, \eta_0)\|_{\mathcal{H}_1} \le R$ for some constant $R > 0$, and assumptions (2.4)–(2.10) hold. Then, for any fixed $T > 0$, there exist constants $C_T > 0$ such that the solution (u, u_t, η^t) of system (2.3) satisfies*

$$\varepsilon \int_0^t \|\Delta u_{tt}(s)\|^2 ds \le C_T, \quad \forall t \in [0, T].$$

Proof. Similar to Lemma 3.3, taking the inner product with $-\Delta u_t$ yields

$$\frac{d}{dt}\Pi(t) - \int_0^\infty k'(s)\|\nabla\eta^t\|^2 ds \le C((l+1)\|\nabla u_t\|^2 + l\|\nabla u\|^2 + \|g\|^2 + |\mu_1|^2\|\nabla u_t\|^2 + |\mu_2|^2\|\nabla u_t(t-\tau(t))\|^2),$$

where $\Pi(t) = \|\nabla u_t\|^2 + \|\Delta u\|^2 + \varepsilon\|\Delta u_t\|^2 + \|\eta^t\|^2_{\mathcal{M}_1}$. Integrating it between 0 and t gives

$$\Pi(t) \le \Pi(0) + C\int_0^t ((l+1)\|\nabla u_t(s)\|^2 + l\|\nabla u(s)\|^2 + \|g\|^2 + |\mu_1|^2\|\nabla u_t\|^2 + |\mu_2|^2\|\nabla u_t(t-\tau(t))\|^2)ds. \tag{5.2}$$

Since the initial data is bounded in $\mathcal{H}_1$ and all terms on the right-hand side are integrable and uniformly bounded for $t \in [0, T]$, there exists a positive constant $\tilde{C}_B$ independent of ε such that

$$\Pi(t) \le \tilde{C}_B. \tag{5.3}$$

Multiplying $(2.3)_1$ by $-\Delta u_{tt}$, using $(2.3)_2$, and integrating over $L^2(\Omega)$, we have

$$\begin{aligned}\frac{d}{dt}\Pi_1(t) + \|\nabla u_{tt}\|^2 + \varepsilon\|\Delta u_{tt}\|^2 - \|\Delta u_t\|^2 = &\left(\int_0^\infty k(s)\Delta\eta^t ds, -\Delta u_{tt}\right) \\ &- \mu_1(u_t, -\Delta u_{tt}) - \mu_2(u_t(t-\tau(t)), -\Delta u_{tt}) - (h(u), -\Delta u_{tt}),\end{aligned} \tag{5.4}$$

where $\Pi_1(t) = (\Delta u, \Delta u_t) + (g, -\Delta u_t)$. By the Cauchy-Schwarz inequality,

$$(\Delta u(t), \Delta u_t(t)) \le \frac{1}{2}\|\Delta u(t)\|^2 + \frac{1}{2}\|\Delta u_t(t)\|^2,$$

and there exist $C_3 > 0, C_4 > 0$ large enough, such that

$$\frac{1}{4}[\|\Delta u\|^2 + \|\Delta u_t\|^2] - C_3 \le \Pi_1(t) \le \|\Delta u\|^2 + \|\Delta u_t\|^2 + C_4. \tag{5.5}$$

Next, by Hölder's and Young's inequalities and (5.1), we estimate the cross-terms

$$(\mu_1 u_t, -\Delta u_{tt}) \le \frac{1}{4}\|\nabla u_{tt}\|^2 + |\mu_1|^2\|\nabla u_t\|^2, \tag{5.6}$$

$$(\mu_2 u_t(t-\tau(t)), -\Delta u_{tt}) \le \frac{1}{4}\|\nabla u_{tt}\|^2 + |\mu_2|^2\|\nabla u_t(t-\tau(t))\|^2, \tag{5.7}$$

$$\left(\int_0^\infty k(s)\Delta\eta^t ds, -\Delta u_{tt}\right) \le C\int_0^\infty k(s)\|\Delta\eta^t\|^2 ds + \frac{l_0}{4}\|\Delta u_{tt}\|^2, \tag{5.8}$$

$$(h(u), -\Delta u_{tt}) = \int_\Omega h'(u)\nabla u\cdot\nabla u_{tt}\,dx \le l\int_\Omega |\nabla u|\cdot|\nabla u_{tt}|\,dx \le \frac{1}{4}\|\Delta u_{tt}\|^2 + C_{B,\lambda_1}. \tag{5.9}$$

Substituting (5.6)–(5.9) into (5.4), we obtain

$$\begin{aligned}&\frac{d}{dt}\Pi_1(t) + \frac{1}{4}\|\nabla u_{tt}\|^2 + \left(\varepsilon - \frac{l_0}{4}\right)\|\Delta u_{tt}\|^2\\ &\le \|\Delta u_t\|^2 + C_1\left(\|\nabla u_t\|^2 + \|\nabla u_t(t-\tau(t))\|^2 + \|\eta^t\|^2_{\mathcal{M}_1}\right) + \|g\|^2 + C_B,\end{aligned} \tag{5.10}$$

where $C_1 = \max\left\{\frac{|\mu_1|^2}{2}, \frac{|\mu_2|^2}{2}, C\right\}$ and $\varepsilon > \frac{l_0}{2}$. Integrate both sides over $[0,t]$,

$$\Pi_1(t) + \frac{\varepsilon}{2}\int_0^t \|\Delta u_{tt}(s)\|^2 ds + \frac{1}{4}\int_0^t \|\nabla u_{tt}(s)\|^2 ds \le \Pi_1(0) + \int_0^t \sigma(s)ds + \int_0^t \|\nabla u_t(s-\tau(s))\|^2 ds,$$

where $\sigma(s) = C\left(\|\Delta u_t\|^2 + \|\nabla u_t\|^2 + \|\eta^t\|^2_{\mathcal{M}_1}\right) + \|g\|^2$. Using the delay term estimate,

$$\int_0^t \|\nabla u_t(s-\tau(s))\|^2 ds \le \int_{-\tau(0)}^0 \|\nabla u_t(s)\|^2 ds + \int_0^t \|\nabla u_t(s)\|^2 ds. \tag{5.11}$$

Associating with (5.3) and (5.5)–(5.11), one concludes that for some $C_T \doteq \tilde{C}_B T > 0$, which are independent of ε, t, it holds that

$$\varepsilon\int_0^t \|\Delta u_{tt}(s)\|^2 ds \le \tilde{C}_B T. \tag{5.12}$$

□

Theorem 5.2. *Let the initial data $z_0 = (u_0, u_1, \eta_0) \in \mathcal{H}$ satisfy $\|z_0\|_{\mathcal{H}} \le R$, and conditions* (2.4)–(2.10) *hold. Then for any $T > 0$, there exists a constant $C_T > 0$ independent of ε such that for all $t \in [0,T]$,*

$$\|S_\varepsilon(t)z_0 - S_0(t)z_0\|_{\mathcal{H}} \le C_T\varepsilon.$$

Proof. Let $z = (u, u_t, \eta^t) = S_\varepsilon(t)z_0$ and $z^* = (u^*, u_t^*, \eta^{t*}) = S_0(t)z_0$. Clearly, z and z^* are two solutions to problem (2.3). Define

$$\omega = u - u^*, \quad \omega_t = u_t - u_t^*, \quad \xi^t(x,s) = \omega(x,t) - \omega(x,t-s).$$

Then $\omega(0), \omega_t(0), \xi(0)$ are all zero, and ω satisfies

$$\begin{cases}\omega_{tt} - \Delta\omega - \varepsilon\Delta u_{tt} + \int_0^\infty k(s)\Delta\xi^t(s)\,ds + \mu_1\omega_t\\ \qquad + \mu_2\left[u_t(t-\tau(t)) - u_t^*(t-\tau(t))\right] + h(u) - h(u^*) = 0,\\ \omega_t = \xi_s + \xi_t.\end{cases} \tag{5.13}$$

Multiplying (5.13) by ω_t, using $(2.3)_2$, and integrating over $L^2(\Omega)$ yields

$$\frac{1}{2}\frac{d}{dt}\left(\|\omega_t\|^2+\|\nabla\omega\|^2\right)-\varepsilon\left(\Delta u_{tt},\omega_t\right)+\mu_1\|\omega_t\|^2-\int_0^\infty k(s)\left(\Delta\xi^t(s),\xi_s+\xi_t\right)ds$$
$$+\mu_2\left(u_t(t-\tau(t))-u_t^*(t-\tau(t)),\omega_t\right)+\left(h(u)-h(u^*),\omega_t\right)=0. \tag{5.14}$$

Define the functional

$$\mathcal{Y}(t)=\frac{1}{2}\left[\|\nabla\omega\|^2+\|\omega_t\|^2+\|\xi^t\|_{\mathcal{M}}^2\right]. \tag{5.15}$$

Clearly $\mathcal{Y}(0)=0$. Differentiating (5.15) and using (5.14), we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{Y}(t)&=\int_\Omega(\nabla\omega,\nabla\omega_t)\,dx+\int_\Omega(\omega_t,\omega_{tt})\,dx-\int_\Omega\int_0^\infty k(s)\left(\nabla\xi(s),\nabla\xi_t(s)\right)dsdx\\&=-\frac{1}{2}\int_\Omega\int_0^\infty k(s)\frac{d}{dt}|\nabla\xi^t(s)|^2dsdx-\mu_1\|\omega_t\|^2-\mu_2\int_\Omega\left(u_t(t-\tau(t))-u_t^*(t-\tau(t)),\omega_t\right)dx\\&\quad+\varepsilon(\Delta u_{tt},\omega_t)-(h(u)-h(u^*),\omega_t).\end{aligned} \tag{5.16}$$

By (2.5), the Poincaré inequality, and the Hölder inequality, there exists $0<\theta<1$ such that

$$\begin{aligned}-(\omega_t,h(u)-h(u^*))&=-\int_\Omega\omega_t\cdot h'(\theta u+(1-\theta)u^*)\omega dx\\&\leqslant\frac{l}{2}\|\omega_t\|^2+\frac{l}{2}\|\omega\|^2\leqslant\frac{l}{2}\|\omega_t\|^2+\frac{l}{2\lambda_1}\|\nabla\omega\|^2.\end{aligned} \tag{5.17}$$

Using Young's inequality,

$$\varepsilon(\Delta u_{tt},\omega_t)\leq\varepsilon\|\Delta u_{tt}\|\|\omega_t\|\leq C\varepsilon^2\|\Delta u_{tt}\|^2+\frac{1}{4}\|\omega_t\|^2; \tag{5.18}$$

$$-\mu_2\left(\omega_t,u_t(t-\tau(t))-u_t^*(t-\tau(t))\right)=-\mu_2\left(\omega_t,\omega_t(t-\tau(t))\right)\leq\frac{|\mu_2|}{2}\|\omega_t\|^2+\frac{|\mu_2|}{2}\|\omega_t(t-\tau(t))\|^2. \tag{5.19}$$

Integration by parts in s and (2.11) give

$$-\frac{1}{2}\int_\Omega\int_0^\infty k(s)\frac{d}{ds}\|\nabla\xi^t(s)\|^2dsdx\leq\frac{\nu}{2}\int_0^\infty k(s)\|\nabla\xi^t(s)\|^2ds. \tag{5.20}$$

Substituting (5.17)–(5.20) into (5.16), we have

$$\begin{aligned}\frac{d}{dt}\mathcal{Y}(t)&\leq\frac{l}{2\lambda_1}\|\nabla\omega\|^2+\left(\frac{1}{4}-\mu_1+\frac{|\mu_2|+l}{2}\right)\|\omega_t\|^2+\frac{|\mu_2|}{2}\|\omega_t(t-\tau(t))\|^2\\&\quad+\frac{\nu}{2}\int_0^\infty k(s)\|\nabla\xi^t(s)\|^2ds+C\varepsilon^2\|\Delta u_{tt}\|^2.\end{aligned} \tag{5.21}$$

Since $\|\nabla\omega\|^2\leq2\mathcal{Y}(t)$, $\|\omega_t\|^2\leq2\mathcal{Y}(t)$, and $\|\xi^t\|_{\mathcal{M}}^2\leq2\mathcal{Y}(t)$, we obtain

$$\frac{d}{dt}\mathcal{Y}(t)\leq C_0\mathcal{Y}(t)+\frac{|\mu_2|}{2}\|\omega_t(t-\tau(t))\|^2+C\varepsilon^2\|\Delta u_{tt}\|^2, \tag{5.22}$$

where $C_0 = \max\{\frac{l}{\lambda_1}, |\frac{1}{2} - 2\mu_1 + |\mu_2| + l|, \nu\}$. Now define

$$\mathcal{Z}(t) = \mathcal{Y}(t) + \hat{\beta} \int_{t-\tau(t)}^{t} \|\omega_t(s)\|^2 ds, \tag{5.23}$$

with $\hat{\beta} > 0$. Then

$$\frac{d}{dt}\mathcal{Z}(t) \leq C_1 \mathcal{Z}(t) + C\varepsilon^2 \|\Delta u_{tt}\|^2, \tag{5.24}$$

where $C_1 = C_0 + 2\hat{\beta}$. Since $\mathcal{Z}(0) = 0$, Gronwall's inequality and (5.3) imply that

$$\mathcal{Z}(t) \leq C\varepsilon^2 \int_0^t e^{-C_1(t-s)} \|\Delta u_{tt}\|^2 ds \leq C \cdot C_T \varepsilon. \tag{5.25}$$

Hence

$$\mathcal{Y}(t) \leq \mathcal{Z}(t) \leq C_T \varepsilon,$$

and finally

$$\|S_\varepsilon(t)z_0 - S_0(t)z_0\|_{\mathcal{H}} = \sqrt{2\mathcal{Y}(t)} \leq C_T \varepsilon.$$

□

From the energy estimates in Section 4, it follows that there exists a constant $\rho > 0$, independent of $\varepsilon \in [0, \varepsilon_0]$, such that

$$\mathcal{A}_\varepsilon \subset B_{\mathcal{H}}(0, \rho) \quad \text{for all } \varepsilon \in [0, \varepsilon_0],$$

where $B_{\mathcal{H}}(0, \rho)$ denotes the ball of radius ρ centered at the origin in $\mathcal{H}$. This uniform boundedness is crucial for the proof of upper semi-continuity.

Theorem 5.3. *Under the assumptions of Theorem 2.9, the family of global attractors $\{\mathcal{A}_\varepsilon\}_{\varepsilon \geq 0}$ is upper semi-continuous at $\varepsilon = 0$ in the topology of $\mathcal{H}$:*

$$\lim_{\varepsilon \to 0^+} \operatorname{dist}_{\mathcal{H}}(\mathcal{A}_\varepsilon, \mathcal{A}_0) = 0, \tag{5.26}$$

where $\operatorname{dist}_X(A, B) = \sup_{a \in A} \inf_{b \in B} \|a - b\|_X$ denotes the Hausdorff semi-distance.

Proof. For the uniform boundedness, we can see that there exists an $R = \rho > 0$ such that $\mathcal{A}_\varepsilon \subset B_{\mathcal{H}}(0, R)$ for all $\varepsilon \in [0, \varepsilon_0]$.

For the semigroup convergence, Theorem 4.1 implies that for any fixed $T > 0$ and any $z_0 \in B_{\mathcal{H}}(0, R)$, we have

$$\sup_{t \in [0,T]} \|S_\varepsilon(t)z_0 - S_0(t)z_0\|_{\mathcal{H}} \leq C_T \varepsilon \to 0 \quad \text{as } \varepsilon \to 0^+.$$

Obviously, the limiting system ($\varepsilon = 0$) possesses a global attractor $\mathcal{A}_0$ in $\mathcal{H}$.
We need to prove that

$$\lim_{\varepsilon \to 0^+} \operatorname{dist}_{\mathcal{H}}(\mathcal{A}_\varepsilon, \mathcal{A}_0) = 0.$$

Let $\eta > 0$ be arbitrary. Since $\mathcal{A}_0$ is a global attractor for $S_0(t)$, there exists $T > 0$ such that

$$\operatorname{dist}_{\mathcal{H}}(S_0(T)B, \mathcal{A}_0) < \frac{\eta}{2},$$

where $B = B_{\mathcal{H}}(0, R)$. Using the invariance of $\mathcal{A}_\varepsilon$ and Theorem 4.1, we have

$$\begin{aligned} \operatorname{dist}_{\mathcal{H}}(\mathcal{A}_\varepsilon, \mathcal{A}_0) &\le \operatorname{dist}_{\mathcal{H}}(\mathcal{A}_\varepsilon, S_0(T)\mathcal{A}_\varepsilon) + \operatorname{dist}_{\mathcal{H}}(S_0(T)\mathcal{A}_\varepsilon, \mathcal{A}_0) \\ &\le \sup_{z\in\mathcal{A}_\varepsilon} \|S_\varepsilon(T)z - S_0(T)z\|_{\mathcal{H}} + \frac{\eta}{2} \\ &\le C_T\varepsilon + \frac{\eta}{2}. \end{aligned} \tag{5.27}$$

Choosing ε small enough so that $C_T\varepsilon < \frac{\eta}{2}$, we obtain $\operatorname{dist}_{\mathcal{H}}(\mathcal{A}_\varepsilon, \mathcal{A}_0) < \eta$. Since η is arbitrary, this completes the proof. □

Author contributions

Tingting Liu: Writing–review and editing, methodology, conceptualization, funding acquisition; Xiyao Liu: Writing–original draft, formal analysis, methodology.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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