



## Research article

# Global existence and blow-up for a fourth-order $p$ -Laplacian dispersive wave equation with logarithmic nonlinearity and strong damping

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**Abstract:** We study the initial boundary value problem for fourth-order strongly damped  $p$ -Laplacian dispersive wave equations with logarithmic nonlinearity ( $1 < p \leq 2$ ,  $2 \leq r$ ). By the potential well theory, we prove global solution existence at a subcritical and critical initial energy, and characterize blow-up behaviors: blow-up at infinite time occurs at a subcritical or critical initial energy for  $1 < p < 2$ ,  $r = 2$ , and at a supercritical initial energy for  $p = r = 2$ ; and for  $1 < p \leq 2$ ,  $r > 2$ , we establish blow-up at finite time for a subcritical, critical, or supercritical initial energy, and explicit bounds for the blow-up time.

**Keywords:** wave equation; blow-up; strong damping; dispersive; logarithmic nonlinearity

**Mathematics Subject Classification:** 35L35, 35D30, 35B44, 35L77

## 1. Introduction

In this paper, we investigate the following initial boundary value problem (IBVP) for a fourth-order strongly damped  $p$ -Laplacian wave equation with logarithmic nonlinearity:

$$\begin{cases} u_{tt} - \Delta u_{tt} + \Delta^2 u - \Delta_p u - \Delta u_t + u_t = |u|^{r-2} u \ln |u|, & x \in \Omega, \ t > 0, \\ u(x, t) = 0, \ \Delta u(x, t) = 0, & x \in \partial\Omega, \ t > 0, \\ u(x, 0) = u_0(x), \ u_t(x, 0) = u_1(x), & x \in \Omega, \end{cases} \quad (1.1)$$

where  $\Omega \subset \mathbb{R}^n$  ( $n \in \mathbb{N}^+$ ) denotes a bounded domain with a smooth boundary  $\partial\Omega$ ,  $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u)$  represents the  $p$ -Laplacian operator,  $\ln |u|$  stands for the natural logarithm, and

$$1 < p \leq 2, \quad 2 \leq r < \begin{cases} \infty, & \text{if } p \geq n, \\ \frac{np + 2n - 2p}{2(n - p)}, & \text{if } p < n. \end{cases} \quad (1.2)$$

In particular,  $\Delta_p$  coincides with the standard Laplace operator  $\Delta$  for  $p = 2$ . While numerous results are available for  $p$ -Laplacian wave equations similar to Problem (1.1) for  $p > 2$  (see [1–8]), the case  $1 < p \leq 2$  remains largely unexplored. For example, Liu et al. [4] investigated a Kirchhoff-like plate equation with memory:

$$u_{tt} - \Delta u_{tt} + \Delta^2 u - \int_0^t g(t - \tau) \Delta^2 u(\tau) d\tau - \Delta_p u + u_t - \Delta u_t = |u|^{q-2} u, \quad x \in \Omega, \quad t > 0,$$

where  $p > 2$ . For more work concerning memory terms, see [9–11]. Thus, this work is devoted to the case where  $1 < p \leq 2$ . To clarify the physical origin of Equation (1.1), we outline its derivation from the Mindlin-Timoshenko plate model [12, 13] as follows:

$$\begin{cases} \rho \hbar u_{tt} - K \operatorname{div}(\nabla u + \Psi) = f, \\ \frac{\rho \hbar^3}{12} \Psi_{tt} - DS + K(\nabla u + \Psi) = H, \end{cases}$$

where  $\rho$  is the density,  $\hbar$  is the thickness,  $K = \frac{kY\hbar}{2(1+r)}$  is the shear modulus,  $D = \frac{Y\hbar^3}{12(1-r^2)}$  is the flexural rigidity,  $0 < r < \frac{1}{2}$  is Poisson's ratio,  $Y$  is Young's modulus, and  $k$  is the shear correction coefficient. In the two-dimensional case,  $\Psi = (\psi, \varphi)$ , where  $\psi = \psi(x_1, x_2, t)$  and  $\varphi = \varphi(x_1, x_2, t)$  correspond to rotation angles of the filament. According to the theory of elasticity, the stress tensor is given by  $S = A\Psi$  [14], where

$$A = \begin{bmatrix} \partial_{x_1 x_1} + \frac{1-r}{2} \partial_{x_2 x_2} & \frac{1+r}{2} \partial_{x_1 x_2} \\ \frac{1+r}{2} \partial_{x_1 x_2} & \frac{1-r}{2} \partial_{x_1 x_1} + \partial_{x_2 x_2} \end{bmatrix}.$$

For the nonconservative thin plate system with damping and nonlinear effects, we arrive at the following Mindlin-Timoshenko plate model:

$$\begin{cases} \rho \hbar u_{tt} - K \operatorname{div}(\nabla u + \Psi) = f(u) - \mu_2 u_t, \\ \frac{\rho \hbar^3}{12} \Psi_{tt} - DA\Psi + K(\nabla u + \Psi) = H(-\Psi) - \mu_3 \Psi_t, \end{cases} \quad (1.3)$$

where  $\mu_2, \mu_3 \geq 0$  are the damping coefficients,  $H$  stands for the nonlinear restoring force, and  $f(u)$  stands for the nonlinear external source. Substitution of (1.3)<sub>2</sub> into (1.3)<sub>1</sub> gives the following:

$$\rho \hbar u_{tt} + \frac{\rho \hbar^3}{12} \operatorname{div} \Psi_{tt} - D \operatorname{div}(A\Psi) - \operatorname{div} H(-\Psi) + \mu_2 u_t + \mu_3 \operatorname{div} \Psi_t = f(u). \quad (1.4)$$

Taking the Kirchhoff limit  $k \rightarrow \infty$ , we have  $\Psi = -\nabla u$ . Under a normalization of coefficients, (1.4) becomes the following:

$$u_{tt} - \mu_1 \Delta u_{tt} + \Delta^2 u - \operatorname{div} H(\nabla u) + \mu_2 u_t - \mu_3 \Delta u_t = f(u),$$

where  $\mu_1 = \frac{\hbar^2}{12}$ . We take  $\mu_1 = \mu_2 = \mu_3 = 1$  for the sake of convenience. Thus, by taking  $H(\nabla u) = |\nabla u|^{p-2} \nabla u$  and  $f(u) = |u|^{r-2} u \ln |u|$ , Problem (1.1) is derived.

The case  $p = 2$  of Problem (1.1) also originates from the modeling of second-grade Mooney-Rivlin viscoelastic solids. The one-dimensional fourth-order dispersive strongly damped wave equation was first derived by Hayes and Saccomandi [15]. It describes transverse homogeneous wave propagation. The dispersive term [16, 17] accounts for either the lateral or longitudinal inertia of the rod structure.

For the  $n$ -dimensional case ( $n = 1, 2, 3$ ), Shang [18] studied the well-posedness, asymptotic behavior, and blow-up dynamics of solutions to the following equation:

$$u_{tt} - \Delta u - \Delta u_t - \Delta u_{tt} = f(u), \quad x \in \Omega, \quad t > 0,$$

with  $f(u)$  satisfying the polynomial growth conditions. For more related results, also see [19–21].

Logarithmic nonlinearity appears in various areas of physics, including quantum mechanics and inflation cosmology [22–25]. Additionally, it arises in nuclear physics, optics, and geophysics [26–29]. Due to these special physical meanings, research on evolution equations with logarithmic nonlinearity has attracted great attention. Readers are referred to [1, 3, 5–8, 30–32] and the references therein. For example, Chen and Xu [33] considered the following fourth-order dispersive wave equation with nonlinear weak damping, linear strong damping, and logarithmic nonlinearity:

$$u_{tt} - \Delta u + \Delta^2 u - \omega (\Delta u_{tt} + \Delta u_t) + |u_t|^{r-1} u_t = u \ln |u|, \quad x \in \Omega, \quad t > 0, \quad (1.5)$$

where  $\omega \in \{0, 1\}$ . They established global existence and infinite time blow-up criteria under subcritical and critical initial energy levels via the potential well method [34], and further proposed two strategies for a high-energy blow-up analysis. Recently, Ding and Zhou [2] considered the following initial boundary value problem for a  $p$ -Laplacian type wave equation:

$$u_{tt} - \Delta u_t - \Delta_p u = |u|^{r-2} u \log |u|, \quad x \in \Omega, \quad t > 0. \quad (1.6)$$

Focusing on the rarely studied case  $1 < p \leq 2$ , they first proved the local existence of weak solutions via the Galerkin method. Using the potential well theory and differential inequality techniques, they established the global existence of solutions. Additionally, they proved finite time and infinite time blow-up results under subcritical, critical, and supercritical initial energy conditions. Furthermore, they derived upper and lower bounds for the blow-up time of finite time blow-up solutions.

Motivated by the aforementioned works, we aim to investigate the behavior of weak solutions to Problem (1.1) under different initial energy levels. Table 1 systematically summarizes our main results, existing findings in the literature, and unresolved problems. We refer the reader to this table for a clear overview of the connections among these three classes. The main contributions and technical challenges of this paper are presented as follows.

(i) In the regime  $p = r = 2$ , [33, Theorems 2.2 and 2.4] obtain infinite time blow-up results for both subcritical and critical energy cases. However, the authors do not demonstrate that solutions can be extended to infinity, which can be done by showing that the local solution can be extended over time (the whole half line). We fill this gap in the present paper by virtue of inequality estimation techniques.

(ii) In the regime  $p = r = 2$ , the problem of infinite time blow-up at a supercritical initial energy remains open in [33]. The principal difficulty lies in the coexistence of the strong damping term  $\Delta u_t$  and the dispersive term  $\Delta u_{tt}$ . This precludes the construction of invariant sets. Although new techniques for the high initial energy infinite time blow-up of strongly damped models have recently been established in [2, 35–38], as far as we are aware, the present work is among the first to prove such a result under the simultaneous presence of both the strong damping term  $\Delta u_t$  and the dispersive term  $\Delta u_{tt}$ .

(iii) In the regime  $1 < p < 2$  or  $p = 2$  and  $2 < r$ , we respectively establish criteria for a finite time blow-up at a supercritical initial energy. In addition, the corresponding criteria for a finite time blow-up with subcritical and critical initial energies are also obtained. These results have not been addressed in the existing work.

The remainder of this paper is organized as follows: Section 2 states the main results; Section 3 presents several key lemmas; and Section 4 provides the detailed proofs of the main results.

**Table 1.** Summary of results.

| $\Delta_p$  | Nonlinearity                     | Initial energy              | Global existence        | Infinite time blow-up               |
|-------------|----------------------------------|-----------------------------|-------------------------|-------------------------------------|
| $p = 2$     | $u \ln  u $ ( $r = 2$ )          | $E(0) \leq d$<br>$E(0) > d$ | [33]<br>Unsolved        | [33] and Theorem 2.4<br>Theorem 2.4 |
| $1 < p < 2$ | $u \ln  u $ ( $r = 2$ )          | $E(0) \leq d$<br>$E(0) > d$ | Theorem 2.2<br>Unsolved | Theorem 2.3<br>Unsolved             |
| $\Delta_p$  | Nonlinearity                     | Initial energy              | Global existence        | Finite time blow-up                 |
| $p = 2$     | $ u ^{r-2}u \ln  u $ ( $r > 2$ ) | $E(0) \leq d$<br>$E(0) > d$ | Theorem 2.2<br>Unsolved | Theorem 2.5<br>Theorem 2.6          |
| $1 < p < 2$ | $ u ^{r-2}u \ln  u $ ( $r > 2$ ) | $E(0) \leq d$<br>$E(0) > d$ | Theorem 2.2<br>Unsolved | Theorem 2.5<br>Theorem 2.7          |

## 2. Main results

Throughout this paper, the following standard notations are adopted. Write  $H := H^2(\Omega) \cap H_0^1(\Omega)$ . The standard inner product on  $L^2(\Omega)$  is denoted by  $(\cdot, \cdot)$ , (i.e.,  $(u, w) = \int_{\Omega} uw \, dx$ ); for  $1 \leq \gamma \leq \infty$ , the norm on  $L^\gamma(\Omega)$  is denoted by  $\|\cdot\|_\gamma$ . The inner product on  $H_0^1(\Omega)$  is denoted by  $(\cdot, \cdot)_*$ , defined as  $(u, w)_* := (u, w) + (\nabla u, \nabla w)$ , with the norm on  $H_0^1(\Omega)$  given by  $\|u\|_*^2 := \|u\|_2^2 + \|\nabla u\|_2^2$ . Let  $\lambda_1 > 0$  denote the first eigenvalue of the Laplace operator  $-\Delta$  subject to homogeneous Dirichlet boundary conditions. From (1.1), for all  $u \in H$ ,  $\lambda_1 \|\nabla u\|_2^2 \leq \|\Delta u\|_2^2$  and  $\lambda_1 \|u\|_2^2 \leq \|\nabla u\|_2^2$ .  $\langle \cdot, \cdot \rangle_*$  and  $\langle \cdot, \cdot \rangle_{**}$  stand for the duality pairings of  $L^p(\Omega)$  with  $L^{p'}(\Omega)$  and  $L^{r+\delta}(\Omega)$  with  $L^{\frac{r+\delta}{r+\delta-1}}(\Omega)$ , respectively, where a constant  $\delta > 0$  is chosen to satisfy the bound

$$0 < r + \delta < \begin{cases} \infty, & \text{if } p \geq n, \\ \frac{np + 2n - 2p}{2(n - p)}, & \text{if } p < n \text{ and Condition (1.2) is satisfied.} \end{cases} \quad (2.1)$$

**Definition 2.1.** Assume Condition (1.2) holds and  $0 < T \leq \infty$ . A function  $u = u(x, t)$  is called a weak solution to Problem (1.1) on  $\Omega \times [0, T]$ ; if  $u \in C([0, T]; H)$  with  $u_t \in L^2([0, T]; H_0^1(\Omega))$ , then the initial data satisfy  $u_0 \in H$ ,  $u_1 \in H_0^1(\Omega)$ , and

$$\begin{aligned} & (u_t(t), w)_* + \int_0^t (\Delta u(\tau), \Delta w) \, d\tau + \int_0^t \langle |\nabla u(\tau)|^{p-2} \nabla u(\tau), \nabla w \rangle_* \, d\tau + (u(t), w)_* \\ &= (u_1, w)_* + (u_0, w)_* + \int_0^t \langle |u(\tau)|^{r-2} u(\tau) \ln |u(\tau)|, w \rangle_{**} \, d\tau, \end{aligned} \quad (2.2)$$

for all test functions  $w \in H$  and all  $t \in (0, T)$ .

For any  $u \in H$ , we define the following functionals:

$$I(u) := \|\Delta u\|_2^2 + \|\nabla u\|_p^p - \int_{\Omega} |u|^r \ln |u| dx, \quad (2.3)$$

$$J(u) := \frac{1}{2} \|\Delta u\|_2^2 + \frac{1}{p} \|\nabla u\|_p^p + \frac{1}{r^2} \|u\|_r^r - \frac{1}{r} \int_{\Omega} |u|^r \ln |u| dx. \quad (2.4)$$

Then,

$$J(u) = \frac{r-2}{2r} \|\Delta u\|_2^2 + \frac{r-p}{pr} \|\nabla u\|_p^p + \frac{1}{r^2} \|u\|_r^r + \frac{1}{r} I(u). \quad (2.5)$$

Let  $d := \inf_{u \in \mathcal{N}} J(u)$  be the potential well depth, where the Nehari manifold  $\mathcal{N}$  is given by the following:

$$\mathcal{N} = \{u \in H \setminus \{0\} \mid I(u) = 0\}.$$

In addition, we introduce two auxiliary sets as follows:

$$\mathcal{W} := \{u \in H \mid I(u) > 0\} \cup \{0\},$$

$$\mathcal{V} := \{u \in H \mid I(u) < 0\}.$$

Additionally, we define the total energy functional as follows:

$$E(t) := \frac{1}{2} \|u_t(t)\|_2^2 + \frac{1}{2} \|\nabla u_t(t)\|_2^2 + J(u(t)) = \frac{1}{2} \|u_t(t)\|_*^2 + J(u(t)). \quad (2.6)$$

First, the local existence theorem is presented.

**Theorem 2.1.** Assume Condition (1.2) holds. If the initial data satisfy  $u_1 \in H_0^1(\Omega)$ ,  $u_0 \in H$ , and there exists a constant  $\widehat{\Theta} > 0$  such that  $\|\Delta u_0\|_2^2 + \|\nabla u_0\|_2^2 + \|u_1\|_*^2 \leq \widehat{\Theta}$ , then there exists a constant  $T_{\widehat{\Theta}} > 0$  depending only on  $\widehat{\Theta}$ , such that Problem (1.1) admits a weak solution  $u$  on  $[0, T_{\widehat{\Theta}}]$ , and the solution can be extended to the maximal existence time  $[0, T)$ . In addition, for any  $t \in [0, T)$ , the following inequality holds:

$$E(t) + \int_0^t \|u_{\tau}\|_*^2 d\tau \leq E(0). \quad (2.7)$$

Moreover, if the maximal existence time  $T < \infty$ , then the following blow-up criterion holds:

$$\lim_{t \rightarrow T^-} (\|u_t(t)\|_*^2 + \|\Delta u(t)\|_2^2 + \|\nabla u(t)\|_2^2) = \infty. \quad (2.8)$$

Next, we have the following result on global existence.

**Theorem 2.2.** Assume that either  $1 < p \leq 2$  with  $2 < r < \infty$  for  $n = 1, 2, \dots, p$ , and  $2 < r < \frac{np+2n-2p}{2(n-p)}$  for  $n > p$ , or  $1 < p < 2$  and  $r = 2$ . Let  $u_1 \in H_0^1(\Omega)$  and  $u_0 \in H$ . If  $I(u_0) > 0$  and  $E(0) \leq d$ , then Problem (1.1) admits a global weak solution  $u$ .

Subsequently, two results on an infinite time blow-up are presented.

**Theorem 2.3.** Assume  $1 < p \leq 2$  and  $r = 2$ . Let  $u_1 \in H_0^1(\Omega)$ ,  $u_0 \in H$ . If  $E(0) \leq d$  and  $I(u_0) < 0$ , then Problem (1.1) has a weak solution  $u$  that blows up at infinite time, satisfying  $\lim_{t \rightarrow \infty} \|u(t)\|_*^2 = \infty$ .

**Theorem 2.4.** Assume  $p = r = 2$ . Let  $u_1 \in H_0^1(\Omega)$ ,  $u_0 \in H$ . Then:

(i) If

$$\begin{cases} I(u_0) < \|u_1\|_*^2, \\ d < E(0) < \frac{\sqrt{\|u_0\|_*^2 + 2(u_0, u_1)_*}}{2\sqrt{e}\zeta}, \end{cases} \quad (2.9)$$

where

$$\zeta = \max \left\{ 2, \frac{1}{g_1(\mu_*)} \right\}, \quad (2.10)$$

where  $\mu_*$  is a constant satisfying (as shown in Remark 2.1)

$$g_1(\mu_*) = \sup_{\mu \in \{\mu > 0 : g_2(\mu) \geq 0\}} g_1(\mu) \geq \frac{2\pi\lambda_1(\lambda_1 + 1) + \pi - \lambda_1 e^{-2}}{8\pi(\lambda_1 + 1)} > 0,$$

and

$$g_1(\mu) = \frac{2\pi\lambda_1(\lambda_1 + 1) + \pi + \pi n(1 + \ln \mu) - \lambda_1 \mu^2}{8\pi(\lambda_1 + 1)}, \quad g_2(\mu) = \frac{\pi(2\lambda_1 + 1) - \mu^2 - \pi n(1 + \ln \mu)}{4\pi(\lambda_1 + 1)}, \quad (2.11)$$

then Problem (1.1) has a weak solution  $u$  that blows up at an infinite time, satisfying  $\lim_{t \rightarrow \infty} \|u(t)\|_*^2 = \infty$ .

(ii) For any  $Q > d$ , there exists a weak solution  $u$  to Problem (1.1) with initial data  $u_0^Q \in H$  and  $u_1^Q \in H_0^1(\Omega)$  satisfying the conditions in (i) and  $E(0; (u_0^Q, u_1^Q)) = Q$ , which blows up at an infinite time.

**Remark 2.1.** The positivity of  $g_1(\mu_*)$  is verified here. Indeed, a straightforward calculation shows that  $g_1(\mu)$  is increasing on  $(0, \mu_1)$  and decreasing on  $(\mu_1, \infty)$ , where  $\mu_1 := \sqrt{\frac{\pi n}{2\lambda_1}}$ . Since  $\lim_{\mu \rightarrow 0^+} g_1(\mu) = \lim_{\mu \rightarrow +\infty} g_1(\mu) = -\infty$  and  $g_1(e^{-1}) > 0$ , there exist constants  $0 < \mu_2 < \mu_1 < \mu_3$  such that  $g_1(\mu) > 0$  if and only if  $\mu \in (\mu_2, \mu_3)$ . In addition, since  $g_2(\mu)$  is strictly decreasing for all  $\mu > 0$  and  $\lim_{\mu \rightarrow 0^+} g_2(\mu) = \infty$ ,  $\lim_{\mu \rightarrow +\infty} g_2(\mu) = -\infty$ , and there exists a unique  $\mu_4 > 0$  such that  $g_2(\mu) > 0$  if and only if  $\mu \in (0, \mu_4)$ . Note that  $g_1(e^{-1}) > 0$  and  $g_2(e^{-1}) > 0$ , which yields  $e^{-1} \in I := (\mu_2, \mu_3) \cap (0, \mu_4)$ , hence  $I$  is nonempty. Moreover, it holds

$$\sup_{\mu \in \{\mu > 0 : g_2(\mu) \geq 0\}} g_1(\mu) = \sup_{\mu \in I} g_1(\mu) = g_1(\mu_*) \geq g_1(e^{-1}) = \frac{2\pi\lambda_1(\lambda_1 + 1) + \pi - \lambda_1 e^{-2}}{8\pi(\lambda_1 + 1)} > 0,$$

where  $\mu_* = \min\{\mu_1, \mu_4\}$ .

**Remark 2.2.** From (4.23),  $\lim_{t \rightarrow \infty} \|u(t)\|_*^2 = \infty$  holds under the condition  $I(u) < \|u_t\|_*^2$ . Therefore, we impose  $I(u_0) < \|u_1\|_*^2$  in (2.9). The inequality  $d < E(0) < \frac{\sqrt{\|u_0\|_*^2 + 2(u_0, u_1)_*}}{2\sqrt{e}\zeta}$  given in (2.9) ensures invariance of the set  $\{u \mid I(u) < \|u_t\|_*^2\}$ . The choice of  $\mu_*$  in Remark 2.1 minimizes  $\zeta$ . This enlarges the set of initial data satisfying (2.9).

Finally, we present results on finite time blow-up together with lower bounds for the blow-up time.

**Theorem 2.5.** Assume  $1 < p \leq 2$  and  $2 < r < \infty$  for  $n = 1, 2, \dots, p$  or  $2 < r < \frac{np+2n-2p}{2(n-p)}$  for  $n > p$ . Let  $u_1 \in H_0^1(\Omega)$  and  $u_0 \in H$ . If  $E(0) \leq d$  and  $I(u_0) < 0$ , then Problem (1.1) has a weak solution  $u$  that blows up in finite time  $T$ , satisfying  $\lim_{t \rightarrow T^-} \|u(t)\|_*^2 = \infty$ . Moreover, if  $E(0) < d$ , then

$$T \leq \frac{2\sqrt{B^2 + A(r-2)^2} - 2B}{(d - E(0))(r-2)^2},$$

where  $A = 2\|u_0\|_*^2(d - E(0))$ , and  $B = (r-2)(u_0, u_1)_* - 2\|u_0\|_*^2$ .

**Theorem 2.6.** Assume  $p = 2$  and  $2 < r < \infty$  for  $n = 1, 2, \dots, p$  or  $2 < r < \frac{np+2n-2p}{2(n-p)}$  for  $n > p$ . Let  $u_1 \in H_0^1(\Omega)$ ,  $u_0 \in H$ . Then:

(i) If

$$E(0) < \frac{k}{2r} [2(u_0, u_1)_* + \|u_0\|_*^2], \quad (2.12)$$

where  $k := (r-2) \min\left\{\frac{1}{2}, \frac{\lambda_1}{4}\right\}$ , then Problem (1.1) has a weak solution  $u$  that blows up in a finite time  $T$ , satisfying  $\lim_{t \rightarrow T^-} \|u(t)\|_*^2 = \infty$ . Moreover, the blow-up time satisfies

$$T \leq \frac{4(r+2) \left( \sqrt{D^2 + C(r-2)^2} - D \right)}{\eta(r-2)^2},$$

where  $\eta := k(2(u_0, u_1)_* + \|u_0\|_*^2) - 2rE(0)$ ,  $C := \frac{2\eta\|u_0\|_*^2}{r+2}$ , and  $D := (r-2)(u_0, u_1)_* - 4\|u_0\|_*^2$ .

(ii) For any  $Q \in \mathbb{R}$ , there exists a weak solution  $u$  to Problem (1.1) with initial data  $u_0^Q \in H$  and  $u_1^Q \in H_0^1(\Omega)$  satisfying the condition in (i) and  $E(0; (u_0^Q, u_1^Q)) = Q$ , which blows up at a finite time.

**Theorem 2.7.** Assume  $1 < p < 2$  and  $2 < r < \infty$  for  $n = 1, 2, \dots, p$  or  $2 < r < \frac{np+2n-2p}{2(n-p)}$  for  $n > p$ . Let  $u_1 \in H_0^1(\Omega)$ ,  $u_0 \in H$ . Then:

(i) If

$$E(0) < \frac{k_{**}}{2r} [2(u_0, u_1)_* + \|u_0\|_*^2], \quad (2.13)$$

where  $k_{**} := (r-2) \min\left\{\frac{1}{2}, \frac{\lambda_1^2}{4(\lambda_1+1)}\right\}$ , then Problem (1.1) has a weak solution  $u$  that blows up in a finite time  $T$ , satisfying  $\lim_{t \rightarrow T^-} \|u(t)\|_*^2 = \infty$ . Moreover, the blow-up time satisfies

$$T \leq \frac{4(r+2) \left( \sqrt{\mathcal{D}^2 + C(r-2)^2} - \mathcal{D} \right)}{\eta_{**}(r-2)^2},$$

where  $\eta_{**} := k_{**}(2(u_0, u_1)_* + \|u_0\|_*^2) - 2rE(0)$ ,  $C := \frac{2\eta_{**}\|u_0\|_*^2}{r+2}$  and  $\mathcal{D} := (r-2)(u_0, u_1)_* - 4\|u_0\|_*^2$ .

(ii) For any  $Q \in \mathbb{R}$ , there exists a weak solution  $u$  to Problem (1.1) with initial data  $u_0^Q \in H$  and  $u_1^Q \in H_0^1(\Omega)$  satisfying the condition in (i) and  $E(0; (u_0^Q, u_1^Q)) = Q$ , which blows up at finite time.

**Theorem 2.8.** Assume that all the conditions of Theorem 2.5, Theorem 2.6, or Theorem 2.7 hold. Then, a lower bound for the blow-up time  $T$  is estimated by

$$T \geq \int_{\mathcal{G}(0)}^{\infty} \frac{1}{C_7 y^{\frac{2(r+\delta-1)}{p}} + |\Omega|(e(r-1))^{-2} + y} dy,$$

where  $\mathcal{G}(0) = \|u_0\|_*^2 + \|u_1\|_*^2 + \|\Delta u_0\|_2^2 + \frac{2}{p}\|\nabla u_0\|_p^p$ ,  $C_7 = \frac{C_5^{2(r+\delta-1)}}{\delta^2} \left(\frac{p}{2}\right)^{\frac{2(r+\delta-1)}{p}}$ , with  $C_5 > 0$  being the best constant of the Sobolev embedding  $W_0^{1,p}(\Omega) \hookrightarrow L^{2(r+\delta-1)}(\Omega)$ .

### 3. Preliminaries

Similar to Lemma 2.2 in [33], we can obtain the following lemma.

**Lemma 3.1.** Assume Condition (1.2) holds. Let  $u \in H \setminus \{0\}$ . Then,

- (i)  $\lim_{\lambda \rightarrow 0} J(\lambda u) = 0$  and  $\lim_{\lambda \rightarrow +\infty} J(\lambda u) = -\infty$ ;
- (ii) there exists a unique  $\lambda^* = \lambda^*(u)$  such that  $\frac{d}{d\lambda} J(\lambda u)|_{\lambda=\lambda^*} = 0$ ; and
- (iii)  $J(\lambda u)$  is increasing on  $(0, \lambda^*)$ , decreasing on  $(\lambda^*, +\infty)$ , and attains its maximum at  $\lambda^*$ .

**Lemma 3.2.** [39] Let  $p \in (1, \infty)$ . For any  $\vartheta \in W_0^{1,p}(\Omega) \setminus \{0\}$  and  $\mu > 0$ , there holds

$$p \int_{\Omega} |\vartheta|^p \ln \left( \frac{|\vartheta|}{\|\vartheta\|_p} \right) dx + \frac{n}{p} \ln \left( \frac{p\mu e}{n\mathcal{L}_p} \right) \|\vartheta\|_p^p \leq \mu \|\nabla \vartheta\|_p^p,$$

where  $\mathcal{L}_p = \frac{pn^{-\frac{p}{2}}}{n-1} \left(\frac{p-1}{e}\right)^{p-1} \left[ \frac{\Gamma(\frac{p}{n}+1)}{\Gamma(\frac{n(p-1)}{n}+1)} \right]^{\frac{p}{n}}$ , and  $\Gamma$  denotes the Gamma function.

In particular, if  $p = 2$ , then for any  $\vartheta \in H_0^1(\Omega) \setminus \{0\}$  and  $\mu > 0$ , there holds

$$2 \int_{\Omega} |\vartheta|^2 \ln \left( \frac{|\vartheta|}{\|\vartheta\|_2} \right) dx + n(1 + \ln \mu) \|\vartheta\|_2^2 \leq \frac{\mu^2}{\pi} \|\nabla \vartheta\|_2^2. \quad (3.1)$$

Let Condition (1.2) hold. For any  $\varepsilon$  satisfying

$$0 < r + \varepsilon < \begin{cases} \infty, & \text{if } p \geq n, \\ \frac{np}{n-p}, & \text{if } p < n, \end{cases}$$

write  $r(\varepsilon) = \left(\frac{\varepsilon}{B_{r+\varepsilon}}\right)^{\frac{1}{r+\varepsilon-p}}$ , where  $B_{r+\varepsilon}$  is the best constant of the embedding  $W_0^{1,p}(\Omega) \hookrightarrow L^{r+\varepsilon}(\Omega)$ .

We note that if  $I(u) \leq 0$ , then by (2.3) and (2.4),

$$\|\nabla u\|_p^p \leq \int_{\Omega} |u|^r \ln |u| dx.$$

Furthermore,

$$J(u) \geq \frac{r-p}{pr} \|\nabla u\|_p^p + \frac{1}{r^2} \|u\|_r^r + \frac{1}{r} I(u).$$

Consequently, the following results from [2] remain valid for Problem (1.1).

**Lemma 3.3.** Assume Condition (1.2) holds. Let  $u \in H \setminus \{0\}$ . If  $I(u) \leq 0$ , then  $\|\nabla u\|_p \geq r_*$ . Here

$$\text{if } n > p, r_* := \sup_{\varepsilon \in (0, \frac{np}{n-p} - r)} r(\varepsilon); \text{ if } n \leq p, r_* := \sup_{\varepsilon \in (0, \infty)} r(\varepsilon).$$

**Lemma 3.4.** Assume Condition (1.2) holds. Then,

$$d \geq \begin{cases} \frac{r_*^p(r-p)}{rp}, & \text{if } p < r, \\ \frac{1}{p^2} \left( \frac{ep^2}{\mathcal{L}_p^n} \right)^{\frac{n}{p}}, & \text{if } p = r. \end{cases}$$

**Remark 3.1.** The case  $1 < p \leq r < 2$  is not considered as in [2], mainly because the non-negativity of the potential well depth  $d$  cannot be proved. Specifically, by (2.5) and  $I(u) = 0$ , we observe that

$$J(u) = \frac{r-2}{2r} \|\Delta u\|_2^2 + \frac{r-p}{pr} \|\nabla u\|_p^p + \frac{1}{r^2} \|u\|_r^r.$$

For  $r < 2$ , one cannot ensure  $J(u) \geq 0$ , indicating the inapplicability of the potential well method. Thus, some alternative methods may be required for this case of Problem (1.1).

**Lemma 3.5.** Assume Condition (1.2) holds. Let  $u$  be the weak solution of Problem (1.1) with maximal existence time  $[0, T)$ . Then,

- (i) If  $E(0) \leq d$  and  $I(u_0) > 0$ , then  $I(u(t)) > 0$  for all  $t \in [0, T)$  (i.e.,  $u(t) \in \mathcal{W}$ ); and
- (ii) If  $E(0) \leq d$  and  $I(u_0) < 0$ , then  $I(u(t)) < 0$  for all  $t \in [0, T)$  (i.e.,  $u(t) \in \mathcal{V}$ ).

*Proof.* We only prove (ii), as the proof of (i) is similar. Arguing by contradiction, suppose there exists  $t_0 \in (0, T)$  such that  $I(u(t)) < 0$  for  $t \in [0, t_0)$  and  $I(u(t_0)) = 0$ . Furthermore, Lemma 3.3 and the continuity of  $I(u)$  imply that  $u(t_0) \neq 0$ . By (2.7), we have the following:

$$J(u(t)) \leq E(t) = E(0) - \int_0^t \|u_\tau\|_*^2 d\tau \leq E(0), \quad \forall t \in [0, T). \quad (3.2)$$

If  $E(0) < d$ , since  $u(t_0) \in \mathcal{N}$ , then we have  $J(u(t_0)) \geq d$  by definition of  $d$ , which contradicts  $J(u(t_0)) \leq E(0) < d$  from (3.2).

If  $E(0) = d$ , then from (3.2), we get  $J(u(t_0)) \leq d$ . Combined with  $J(u(t_0)) \geq d$  (as  $u(t_0) \in \mathcal{N}$ ), we have  $J(u(t_0)) = d$ , which implies  $\int_0^{t_0} \|u_\tau\|_*^2 d\tau = 0$ . Thus,  $u_t \equiv 0$ ,  $u(t) \equiv u_0$  on  $[0, t_0]$ , so  $I(u_0) = I(u(t_0)) = 0$ , contradicting  $I(u_0) < 0$ . This completes the proof.  $\square$

**Lemma 3.6.** Assume Condition (1.2) holds. For any  $u \in H \setminus \{0\}$ , if  $I(u) < 0$ , it follows that

$$I(u) < r(J(u) - d). \quad (3.3)$$

*Proof.* Lemma 3.1 implies there exists  $\alpha^* \in (0, 1)$  such that  $I(\alpha^*u) = 0$ , i.e.,  $\alpha^*u \in \mathcal{N}$ . By the definition of  $d$ , we have  $J(\alpha^*u) \geq d$ . Define the following:

$$g(\alpha) := rJ(\alpha u) - I(\alpha u) = \alpha^2 \cdot \frac{r-2}{2} \|\Delta u\|_2^2 + \alpha^p \cdot \frac{r-p}{p} \|\nabla u\|_p^p + \frac{\alpha^r}{r} \|u\|_r^r, \quad \alpha > 0.$$

Differentiating  $g(\alpha)$  with respect to  $\alpha$  and applying Lemma 3.3 yields the following:

$$g'(\alpha) = \alpha(r-2)\|\Delta u\|_2^2 + \alpha^{p-1}(r-p)\|\nabla u\|_p^p + \alpha^{r-1}\|u\|_r^r \geq \alpha^{r-1}r_*^r > 0.$$

Thus,  $g'(\alpha) > 0$  for all  $\alpha > 0$ , meaning  $g(\alpha)$  is strictly increasing on  $(0, +\infty)$ . Then,

$$rJ(u) - I(u) = g(1) \geq g(\alpha^*) = rJ(\alpha^*u) - I(\alpha^*u) = rJ(\alpha^*u) \geq rd.$$

This completes the proof.  $\square$

**Lemma 3.7.** [40, 41] Suppose that a positive, twice-differentiable function  $\Psi(t)$  satisfies the inequality

$$\Psi''(t)\Psi(t) - (1 + \theta)(\Psi'(t))^2 \geq 0,$$

where  $\theta > 0$ . If  $\Psi(0) > 0$  and  $\Psi'(0) > 0$ , then  $\Psi(t) \rightarrow \infty$  as  $t \rightarrow t_* \leq t^* = \frac{\Psi(0)}{\theta\Psi'(0)}$ .

**Lemma 3.8.** Suppose  $p = r = 2$ . For any fixed  $\mathcal{O} > 0$ , we define the following:

$$\mathcal{U}_{\mathcal{O}} := \left\{ (\Phi, \Psi) \in H \times H_0^1(\Omega) : I(\Phi) < \|\Psi\|_*^2 - \mathcal{O}, d < E(\Phi, \Psi) \leq \frac{\sqrt{\|\nabla\Phi\|_2^2 + 2(\Phi, \Psi)_*}}{2\sqrt{e\zeta}} - \frac{\mathcal{O}}{2} \right\}, \quad (3.4)$$

where  $\zeta > 0$  is given in (2.10),  $I(\Phi) = \|\Delta\Phi\|_2^2 + \|\nabla\Phi\|_2^2 - \int_{\Omega} |\Phi|^2 \ln |\Phi| dx$ , and  $E(\Phi, \Psi) = \frac{1}{2}\|\Psi\|_*^2 + \frac{1}{2}\|\Delta\Psi\|_2^2 + \frac{1}{2}\|\nabla\Psi\|_2^2 + \frac{1}{4}\|\Psi\|_2^2 - \frac{1}{2}\int_{\Omega} |\Psi|^2 \ln |\Psi| dx$ . If  $(u_0, u_1) \in \mathcal{U}_{\mathcal{O}}$ , then  $I(u(t)) < \|u_t(t)\|_*^2 - \mathcal{O}$  for  $t \in [0, T)$  and

$$d < E(0) \leq \frac{\sqrt{\|u(t)\|_*^2 + 2(u(t), u_t(t))_*}}{2\sqrt{e\zeta}} - \frac{\mathcal{O}}{2}, \quad t \in [0, T). \quad (3.5)$$

*Proof.* Define  $\mathfrak{C}(t) := I(u(t)) - \|u_t(t)\|_*^2$  for  $t \in [0, T)$ . First, we prove that  $\mathfrak{C}(t) < -\mathcal{O}$  for all  $t \in [0, T)$ . Suppose otherwise. Since the initial condition  $(u_0, u_1) \in \mathcal{U}_{\mathcal{O}}$  yields  $\mathfrak{C}(0) = I(u_0) - \|u_1\|_*^2 < -\mathcal{O}$ , there exists  $t_0 \in (0, T)$  such that  $\mathfrak{C}(t_0) = -\mathcal{O}$  and  $\mathfrak{C}(t) < -\mathcal{O}$  for all  $t \in [0, t_0)$ . Let

$$\mathcal{B}(t) := \|u(t)\|_*^2 + \int_0^t \|u(\tau)\|_*^2 d\tau, \quad t \in [0, T).$$

Direct computation using (1.1) and (2.3) yields the following:

$$\mathcal{B}'(t) = 2(u(t), u_t(t))_* + \|u(t)\|_*^2,$$

$$\mathcal{B}''(t) = -2I(u(t)) + 2\|u_t(t)\|_*^2 = -2\mathfrak{C}(t) > 0, \quad t \in [0, t_0).$$

Thus, since  $(u_0, u_1) \in \mathcal{U}_{\mathcal{O}}$ , we obtain the following:

$$\begin{aligned} \|u(t_0)\|_*^2 + 2(u(t_0), u_t(t_0))_* &\geq \mathcal{B}'(t_0) \geq \mathcal{B}'(0) = 2(u_0, u_1)_* + \|u_0\|_*^2 \\ &\geq 4e\zeta \left( \frac{\mathcal{O}}{2} + E(0) \right)^2. \end{aligned}$$

Moreover, by (2.7),  $\mathfrak{C}(t_0) = -\mathcal{O}$  together with the identity  $J(u) = \frac{1}{2}I(u) + \frac{1}{4}\|u\|_2^2$  yields the following:

$$\begin{aligned} E(0) &\geq E(t_0) = \frac{1}{2}\|u_t(t_0)\|_*^2 + J(u(t_0)) \\ &= \frac{1}{2}\|u_t(t_0)\|_*^2 + \frac{1}{2}I(u(t_0)) + \frac{1}{4}\|u(t_0)\|_2^2 \\ &= \|u_t(t_0)\|_*^2 - \frac{\mathcal{O}}{2} + \frac{1}{4}\|u(t_0)\|_2^2 \\ &\geq \frac{1}{4}\|u(t_0)\|_2^2 - \frac{\mathcal{O}}{2}. \end{aligned}$$

This yields  $2\mathcal{O} + 4E(0) \geq \|u(t_0)\|_2^2$ . Note that  $2\mathcal{O} + 4E(0) > 4E(0) > 4d \geq (2\pi e^2)^{\frac{n}{2}} > 1$ , as shown in Lemma 3.4. Then,

$$(2\mathcal{O} + 4E(0)) \ln(2\mathcal{O} + 4E(0)) \geq \|u(t_0)\|_2^2 \ln \|u(t_0)\|_2^2,$$

which, together with (3.1) and  $\|\nabla u\|_2^2 \leq \frac{1}{\lambda_1} \|\Delta u\|_2^2$ , implies the following:

$$\begin{aligned} E(0) &\geq E(t_0) = \frac{1}{2} \|u_t(t_0)\|_*^2 + \frac{1}{2} \|\Delta u(t_0)\|_2^2 + \frac{1}{2} \|\nabla u(t_0)\|_2^2 + \frac{1}{4} \|u(t_0)\|_2^2 - \frac{1}{2} \int_{\Omega} u(t_0)^2 \ln |u(t_0)| dx \\ &\geq \frac{1}{2} \|u_t(t_0)\|_*^2 + A(\mu) \|\nabla u(t_0)\|_2^2 + B(\mu) \|u(t_0)\|_2^2 - \left( \frac{\mathcal{O}}{2} + E(0) \right) \ln(2\mathcal{O} + 4E(0)), \end{aligned}$$

where  $A(\mu) = \frac{\lambda_1}{2} + \frac{2\pi - \mu^2}{4\pi}$ ,  $B(\mu) = \frac{1+n(1+\ln \mu)}{4}$ . It is not difficult to verify that

$$A(\mu) = 2g_1(\mu) + g_2(\mu), \quad 2g_1(\mu) = B(\mu) + \lambda_1 g_2(\mu),$$

where  $g_1(\mu)$  and  $g_2(\mu)$  are given by (2.11). Note that  $\|u\|_2^2 \leq \frac{1}{\lambda_1} \|\nabla u\|_2^2$ . For any  $\mu > 0$  satisfying  $g_2(\mu) \geq 0$ ,

$$\begin{aligned} E(0) &\geq \frac{1}{2} \|u_t(t_0)\|_*^2 + 2g_1(\mu) \|\nabla u(t_0)\|_2^2 + \lambda_1 g_2(\mu) \|u(t_0)\|_2^2 + B(\mu) \|u(t_0)\|_2^2 - \left( \frac{\mathcal{O}}{2} + E(0) \right) \ln(2\mathcal{O} + 4E(0)) \\ &= \frac{1}{2} \|u_t(t_0)\|_*^2 + g_1(\mu) \|u(t_0)\|_*^2 + g_1(\mu) \|u(t_0)\|_*^2 - \left( \frac{\mathcal{O}}{2} + E(0) \right) \ln(2\mathcal{O} + 4E(0)). \end{aligned}$$

In view of Remark 2.1, we set  $\mu = \mu_*$  in the preceding inequality. Together with (2.10), this yields the following:

$$\begin{aligned} \frac{\mathcal{O}}{2} + E(0) &\geq \frac{1}{\zeta} \left( \|u(t_0)\|_*^2 + 2(u(t_0), u_t(t_0))_* \right) - \left( \frac{\mathcal{O}}{2} + E(0) \right) \ln(2\mathcal{O} + 4E(0)) \\ &\geq 4e \left( \frac{\mathcal{O}}{2} + E(0) \right)^2 - \left( \frac{\mathcal{O}}{2} + E(0) \right) \ln(2\mathcal{O} + 4E(0)). \end{aligned}$$

This leads to  $2e\mathcal{O} + 4eE(0) \leq \ln(2e\mathcal{O} + 4eE(0))$ , which is a contradiction in view of  $x > \ln x$  for all  $x > 0$ . Consequently, for every  $t \in [0, T)$ , we have  $\mathfrak{E}(t) < -\mathcal{O}$ .

Next, we establish (3.5). From the above,  $\mathfrak{E}(t) < -\mathcal{O}$  for all  $t \in [0, T)$ , which implies  $\mathcal{B}''(t) = -2\mathfrak{E}(t) > 0$ . This yields the following:

$$\begin{aligned} \|u(t)\|_*^2 + 2(u(t), u_t(t))_* &= \mathcal{B}'(t) \geq \mathcal{B}'(0) = \|u_0\|_*^2 + 2(u_0, u_1)_* \\ &\geq 4e\zeta \left( \frac{\mathcal{O}}{2} + E(0) \right)^2, \quad t \in [0, T). \end{aligned}$$

This completes the proof of the invariant set. □

#### 4. Proof of the Theorems

##### Proof of Theorem 2.1

Let  $\{e_i\}_{i=1}^\infty$  be the eigenfunctions of the biharmonic operator  $\Delta^2$  subject to the boundary conditions  $u = 0$  and  $\Delta u = 0$  on  $\partial\Omega$ . Then,  $\{e_i\}_{i=1}^\infty$  forms an orthogonal basis of  $H = H^2(\Omega) \cap H_0^1(\Omega)$ . Define the Galerkin approximate solutions of problem (1.1) as follows:

$$u_k(x, t) = \sum_{i=1}^k h_{ik}(t) e_i(x), \quad k = 1, 2, 3, \dots$$

which satisfy, for each  $i = 1, 2, \dots, k$ ,

$$(u_{ktt}, e_i)_* + (\Delta u_k, \Delta e_i) + \langle |\nabla u_k|^{p-2} \nabla u_k, \nabla e_i \rangle_* + (u_{kt}, e_i)_* = \langle |u_k|^{r-2} u_k \ln |u_k|, e_i \rangle_{**}, \quad (4.1)$$

with initial conditions  $u_k(0) = u_{k0}$ ,  $u_{kt}(0) = u_{k1}$ , where  $u_{k0}$  and  $u_{k1}$  are chosen in  $\text{span}\{e_1, e_2, \dots, e_k\}$  so that

$$u_{k0} \rightarrow u_0 \quad \text{in } H \quad \text{as } k \rightarrow \infty, \quad (4.2)$$

$$u_{k1} \rightarrow u_1 \quad \text{in } H_0^1(\Omega) \quad \text{as } k \rightarrow \infty. \quad (4.3)$$

By the standard theory of ordinary differential equations, System (4.1) admits a local solution  $u_k(t)$  on  $[0, T_k]$  for some  $T_k > 0$ . By multiplying (4.1) by  $h'_{ik}(t)$  and summing over  $i = 1, 2, \dots, k$ , we obtain the following:

$$\begin{aligned} & \frac{1}{2} \|u_{kt}\|_*^2 + \frac{1}{2} \|\Delta u_k\|_2^2 + \frac{1}{p} \|\nabla u_k\|_p^p + \frac{1}{2} \int_0^t \|u_{k\tau}\|_*^2 d\tau \\ &= \int_0^t \int_\Omega |u_k|^{r-2} u_k u_{k\tau} \ln |u_k| dx d\tau + \frac{1}{2} \|u_{k1}\|_*^2 + \frac{1}{2} \|\Delta u_{k0}\|_2^2 + \frac{1}{p} \|\nabla u_{k0}\|_p^p. \end{aligned}$$

Following the same argument as in [2, (4.5)–(4.13)], there exists a constant  $T_3 > 0$  only depending on  $\widehat{\Theta}$  such that

$$\frac{1}{2} \|u_{kt}\|_*^2 + \frac{1}{2} \|\Delta u_k\|_2^2 + \frac{1}{p} \|\nabla u_k\|_p^p + \frac{1}{2} \int_0^t \|u_{k\tau}\|_*^2 d\tau \leq \Theta_1, \quad \forall t \in [0, T_3], \quad (4.4)$$

for some constant  $\Theta_1 > 0$  independent of  $k$ .

Therefore, there exists a function  $u$  and a subsequence of  $\{u_k\}_{k=1}^\infty$  (still denoted by  $\{u_k\}$ ) such that

$$u_k \xrightarrow{w^*} u \quad \text{in } L^\infty(0, T_3; H), \quad \text{as } k \rightarrow \infty, \quad (4.5)$$

$$u_k \xrightarrow{w} u \quad \text{in } L^2(0, T_3; H), \quad \text{as } k \rightarrow \infty, \quad (4.6)$$

$$u_{kt} \xrightarrow{w^*} u_t \quad \text{in } L^\infty(0, T_3; H_0^1(\Omega)), \quad \text{as } k \rightarrow \infty, \quad (4.7)$$

$$u_{kt} \xrightarrow{w} u_t \quad \text{in } L^2(0, T_3; H_0^1(\Omega)), \quad \text{as } k \rightarrow \infty. \quad (4.8)$$

By Lions' theorem [42], (4.5) and (4.7) imply

$$u_k \rightarrow u \quad \text{strongly in } L^2(0, T_3; H_0^1(\Omega)), \quad \text{as } k \rightarrow \infty,$$

and hence  $u_k \rightarrow u$  almost everywhere in  $\Omega \times [0, T_3]$ . Therefore,

$$|u_k|^{r-2} u_k \ln |u_k| \rightarrow |u|^{r-2} u \ln |u| \quad \text{a.e. in } \Omega \times [0, T_3], \quad \text{as } k \rightarrow \infty. \quad (4.9)$$

Furthermore, similar to the estimate in [2, (4.23)], we deduce from (4.4) that

$$\int_{\Omega} \left| |u_k|^{r-2} u_k \ln |u_k| \right|^{\frac{r+\delta}{r+\delta-1}} dx \leq \left( \frac{1}{e(r-1)} \right)^{\frac{r+\delta}{r+\delta-1}} |\Omega| + \frac{C_*^{r+\delta}}{\delta^{\frac{r+\delta}{r+\delta-1}}} (p\Theta_1)^{\frac{r+\delta}{p}} =: \tilde{C}, \quad (4.10)$$

where  $C_* > 0$  is the best constant of the embedding  $W_0^{1,p}(\Omega) \hookrightarrow L^{r+\delta}(\Omega)$ . By (4.9) and (4.10), there exists a subsequence such that

$$|u_k|^{r-2} u_k \ln |u_k| \xrightarrow{w^*} |u|^{r-2} u \ln |u| \quad \text{in } L^\infty \left( 0, T_3; L^{\frac{r+\delta}{r+\delta-1}}(\Omega) \right), \quad \text{as } k \rightarrow \infty,$$

$$|u_k|^{r-2} u_k \ln |u_k| \xrightarrow{w} |u|^{r-2} u \ln |u| \quad \text{in } L^2 \left( 0, T_3; L^{\frac{r+\delta}{r+\delta-1}}(\Omega) \right), \quad \text{as } k \rightarrow \infty. \quad (4.11)$$

In addition, from (4.4), we have  $\int_{\Omega} |\nabla u_k|^{p-2} \nabla u_k \cdot \frac{p}{p-1} dx = \|\nabla u_k\|_p^p \leq p\Theta_1 < \infty$ . Combined with the almost everywhere convergence of  $\nabla u_k$ , this implies the following:

$$|\nabla u_k|^{p-2} \nabla u_k \xrightarrow{w^*} |\nabla u|^{p-2} \nabla u \quad \text{in } L^\infty \left( 0, T_3; L^{\frac{p}{p-1}}(\Omega) \right), \quad \text{as } k \rightarrow \infty. \quad (4.12)$$

Let  $P = \left\{ \sum_{i=1}^k \zeta_i e_i : \zeta_i \in \mathbb{R}, k \in \mathbb{N} \right\}$ . Then,  $P$  is dense in  $H$ . Hence, for any  $\psi \in H$ , there exists a sequence  $\{\psi_i\}_{i=1}^\infty \subset P$  such that  $\lim_{i \rightarrow \infty} \psi_i = \psi$  in  $H$ . By the construction of Galerkin approximations, (4.1) holds with  $e_i$  replaced by  $\psi_i$ . Taking the limit  $i \rightarrow \infty$ , we obtain the following:

$$\begin{aligned} |\langle u_{ktt}, \psi \rangle_*| &\leq |(\Delta u_k, \Delta \psi)| + \left| \langle |\nabla u_k|^{p-2} \nabla u_k, \nabla \psi \rangle_* \right| + |(u_{kt}, \psi)_*| + \left| \langle |u_k|^{r-2} u_k \ln |u_k|, \psi \rangle_{**} \right| \\ &\leq \|\Delta u_k\|_2 \|\Delta \psi\|_2 + \|\nabla u_k\|_p^{p-1} \|\nabla \psi\|_p + \|u_{kt}\|_* \|\psi\|_* + \left\| |u_k|^{r-2} u_k \ln |u_k| \right\|_{\frac{r+\delta}{r+\delta-1}} \|\psi\|_{r+\delta}. \end{aligned}$$

Since  $H \hookrightarrow W_0^{1,p}(\Omega)$ , and  $W_0^{1,p}(\Omega) \hookrightarrow L^{r+\delta}(\Omega)$  holds under Condition (1.2), there exists a constant  $C_1 > 0$  such that for all  $\psi \in H$ ,  $\|\nabla \psi\|_p \leq C_1 \|\Delta \psi\|_2$ , and  $\|\psi\|_{r+\delta} \leq C_1 \|\Delta \psi\|_2$ . For any test function  $\psi \in H$ ,

$$|\langle u_{ktt}, \psi \rangle_*| \leq \widetilde{C}_1 \|\Delta \psi\|_2 \left( \|\Delta u_k\|_2 + \|\nabla u_k\|_p^{p-1} + \|u_{kt}\|_* + \left\| |u_k|^{r-2} u_k \ln |u_k| \right\|_{\frac{r+\delta}{r+\delta-1}} \right).$$

From (4.10), we immediately obtain  $\|u_{ktt}\|_{H'} \leq \widetilde{C}_1 \left( \|\Delta u_k\|_2 + \|\nabla u_k\|_p^{p-1} + \|u_{kt}\|_* + \widetilde{C}^{\frac{r+\delta-1}{r+\delta}} \right)$ , where  $H'$  denotes the dual space of  $H$ . From the uniform bound (4.4), we know  $\|\Delta u_k\|_2 \leq \sqrt{2\Theta_1}$  and  $\|\nabla u_k\|_p^{p-1} \leq (p\Theta_1)^{\frac{p-1}{p}}$  hold uniformly for all  $k$ . Then, the dual norm bound simplifies to the following:

$$\|u_{ktt}\|_{H'} \leq \widetilde{C}_1 \|u_{kt}\|_* + \widetilde{C}_2,$$

where  $\widetilde{C}_2 := \widetilde{C}_1 \left( \sqrt{2\Theta_1} + (p\Theta_1)^{\frac{p-1}{p}} + \widetilde{C}^{\frac{r+\delta-1}{r+\delta}} \right)$ . From (4.8), we immediately obtain the following:

$$u_{ktt} \xrightarrow{w} u_{tt} \quad \text{in } L^2(0, T_3; H'), \quad \text{as } k \rightarrow \infty. \quad (4.13)$$

Integrating (4.1) with respect to  $t$  from 0 to  $t$  and taking the limit  $k \rightarrow \infty$  via (4.8), (4.11), (4.12), and (4.13), we conclude that for all  $t \in [0, T_3]$ ,

$$\begin{aligned} (u_t(t), e_i)_* + \int_0^t (\Delta u(\tau), \Delta e_i) d\tau + \int_0^t \langle |\nabla u(\tau)|^{p-2} \nabla u(\tau), \nabla e_i \rangle_* d\tau + (u(t), e_i)_* \\ = (u_1, e_i)_* + (u_0, e_i)_* + \int_0^t \langle |u(\tau)|^{r-2} u(\tau) \ln |u(\tau)|, e_i \rangle_{**} d\tau, \end{aligned}$$

for all  $i = 1, 2, \dots$ . Since  $\{e_i\}$  is dense in  $H$ , the weak formulation in Definition 2.1 holds for all test functions  $\psi \in H$ . Moreover, from (4.5) and (4.7), we have  $u \in C([0, T_3]; H_0^1(\Omega))$ . From (4.7), (4.13), and [43, Proposition 1.2], we have  $u_t \in C([0, T_3]; L^2(\Omega))$ . Combined with (4.2) and (4.3), this yields  $u(0) = u_0$  and  $u_t(0) = u_1$ . The energy inequality (2.7) can also be derived in a standard way (for example, see [44]). Finally, we prove the blow-up criterion (2.8). Suppose, for contradiction, that  $T < \infty$  but

$$\limsup_{t \rightarrow T^-} (\|u_t(t)\|_*^2 + \|\Delta u(t)\|_2^2 + \|\nabla u(t)\|_2^2) < \infty.$$

Then, there exists a constant  $R_* > 0$  such that

$$\|u_t(t)\|_*^2 + \|\Delta u(t)\|_2^2 + \|\nabla u(t)\|_2^2 \leq R_*, \quad \forall t \in [0, T).$$

By the local existence result, there exists  $T_{R_*} > 0$  only depending on  $R_*$  such that for any  $t_0 \in [0, T)$ , the solution can be extended to  $[t_0, t_0 + T_{R_*}]$ . Taking  $t_0 = T - T_{R_*}/2$ , we see that the solution exists on  $[0, T + T_{R_*}/2]$ , which contradicts the maximality of  $T$ . This completes the proof.  $\square$

### Proof of Theorem 2.2

Only the case of  $E(0) < d$  is proved here, and the result for  $E(0) = d$  can be obtained by the same argument as in Theorem 2.3 of [33].

Case (I):  $1 < p \leq 2$  with  $2 < r < \infty$  for  $n = 1, 2, \dots, p$ , and  $2 < r < \frac{np+2n-2p}{2(n-p)}$  for  $n > p$ . By Lemma 3.5, we have  $u(t) \in \mathcal{W}$  and thus  $I(u(t)) \geq 0$  for all  $t \in [0, T)$ . From (2.5), (2.6), and (2.7), we obtain the following:

$$\begin{aligned} d > E(0) &= \int_0^t \|u_\tau\|_*^2 d\tau + \frac{1}{2} \|u_t(t)\|_*^2 + J(u(t)) \\ &= \int_0^t \|u_\tau\|_*^2 d\tau + \frac{1}{2} \|u_t(t)\|_*^2 + \frac{r-2}{2r} \|\Delta u(t)\|_2^2 + \frac{r-p}{pr} \|\nabla u(t)\|_p^p + \frac{1}{r^2} \|u(t)\|_r^r + \frac{1}{r} I(u(t)) \quad (4.14) \\ &\geq \int_0^t \|u_\tau\|_*^2 d\tau + \frac{1}{2} \|u_t(t)\|_*^2 + \frac{r-2}{2r} \|\Delta u(t)\|_2^2 + \frac{1}{r^2} \|u(t)\|_r^r, \quad t \in [0, T). \end{aligned}$$

By  $\|\nabla u\|_2^2 \leq \frac{1}{\lambda_1} \|\Delta u\|_2^2$  and (4.14), for  $t \in [0, T)$ ,

$$\|u_t\|_*^2 + \|\Delta u\|_2^2 + \|\nabla u\|_2^2 \leq 2d + \frac{2rd}{r-2} + \frac{2rd}{\lambda_1(r-2)} < \infty.$$

Hence, it follows from Theorem 2.1 that  $T = \infty$ .

Case (II):  $1 < p < 2$  and  $r = 2$ . In fact, we only need to supplement the proof by verifying that  $\|\Delta u\|_2^2 < \infty$ . By (3.1) ( $\mu = \sqrt{\frac{\pi\lambda_1}{2}}$ ),  $\|\nabla u\|_2^2 \leq \lambda_1^{-1} \|\Delta u\|_2^2$ , and  $\|u\|_2^2 \leq 4d$  (from (4.14) with  $r = 2$ ),

$$\int_\Omega |u|^2 \ln |u| dx \leq \frac{\mu^2}{2\pi} \|\nabla u\|_2^2 + \|u\|_2^2 \ln \|u\|_2 - \frac{n(1 + \ln \mu)}{2} \|u\|_2^2$$

$$\begin{aligned}
&\leq \frac{\lambda_1}{4} \|\nabla u\|_2^2 + \frac{1}{2}(4d+1) \ln(4d+1) + \frac{n}{2} \left| 1 + \frac{1}{2} \ln\left(\frac{\pi\lambda_1}{2}\right) \right| 4d \\
&\leq \frac{1}{4} \|\Delta u\|_2^2 + B_1,
\end{aligned}$$

where  $B_1 := \frac{1}{2}(4d+1) \ln(4d+1) + \frac{n}{2} \left| 1 + \frac{1}{2} \ln\left(\frac{\pi\lambda_1}{2}\right) \right| 4d \geq 0$ . By combining (2.4), (2.6), and (2.7) with the above estimate, we obtain the following:

$$\begin{aligned}
\|\Delta u\|_2^2 &= 2J(u) - \frac{2}{p} \|\nabla u\|_p^p - \frac{1}{2} \|u\|_2^2 + \int_{\Omega} u^2 \ln |u| dx \\
&\leq 2E(0) + \int_{\Omega} u^2 \ln |u| dx \\
&\leq 2d + \frac{1}{4} \|\Delta u\|_2^2 + B_1;
\end{aligned}$$

then,  $\|\Delta u\|_2^2 \leq \frac{4}{3} (2d + B_1)$ . Thus, we complete the proof.  $\square$

### Proof of Theorem 2.3

Step 1: Extension of  $u$  to infinite time.

By Lemma 3.5, we have  $u(t) \in \mathcal{V}$  and thus  $I(u(t)) < 0$  for all  $t \in [0, T)$ . Let  $\mathcal{H}(t) := \|u\|_2^2$  and  $y(t) := \sqrt{\mathcal{H}(t)} = \|u(t)\|_2$  for  $t \in [0, T)$ . Since  $I(u(t)) < 0$ ,  $u(t) \not\equiv 0$ , so  $y(t) > 0$  for all  $t \in [0, T)$ . Then,  $\mathcal{H}'(t) = 2(u, u_t)$ . By the Cauchy-Schwarz inequality,

$$2y(t)y'(t) = \mathcal{H}'(t) \leq 2\|u\|_2 \|u_t\|_2 \leq 2\|u\|_2 \|u_t\|_* \leq 2y(t)\|u_t\|_*.$$

Then,  $y'(t) \leq \|u_t(t)\|_*$ . By (3.1) ( $\mu = \sqrt{\pi\lambda_1}$ ) and  $\|\nabla u\|_2^2 \leq \lambda_1^{-1} \|\Delta u\|_2^2$ ,

$$\begin{aligned}
\int_{\Omega} |u|^2 \ln |u| dx &\leq \frac{\mu^2}{2\pi} \|\nabla u\|_2^2 + \mathcal{H}(t) \ln \|u\|_2 - \frac{n(1 + \ln \mu)}{2} \mathcal{H}(t) \\
&\leq \frac{\lambda_1}{2} \|\nabla u\|_2^2 + \mathcal{H}(t) \ln \|u\|_2 - \frac{n}{2} \left( 1 + \frac{1}{2} \ln(\pi\lambda_1) \right) \mathcal{H}(t) \\
&\leq \frac{1}{2} \|\Delta u\|_2^2 + \frac{1}{2} \mathcal{H}(t) \ln \mathcal{H}(t) + D_1 \mathcal{H}(t),
\end{aligned} \tag{4.15}$$

where  $D_1 := \frac{n}{2} \left| 1 + \frac{1}{2} \ln(\pi\lambda_1) \right| \geq 0$ . Then, by (2.4) and (4.15), we deduce the following:

$$\begin{aligned}
J(u) &= \frac{1}{2} \|\Delta u\|_2^2 + \frac{1}{p} \|\nabla u\|_p^p + \frac{1}{4} \mathcal{H}(t) - \frac{1}{2} \int_{\Omega} |u|^2 \ln |u| dx \\
&\geq \frac{1}{4} \|\Delta u\|_2^2 + \frac{1}{4} \mathcal{H}(t) - \frac{1}{2} \left( \frac{1}{2} \|\Delta u\|_2^2 + \frac{1}{2} \mathcal{H}(t) \ln \mathcal{H}(t) + D_1 \mathcal{H}(t) \right) \\
&= \left( \frac{1}{4} - \frac{D_1}{2} \right) \mathcal{H}(t) - \frac{1}{4} \mathcal{H}(t) \ln \mathcal{H}(t) \\
&\geq -\frac{1}{4} \mathcal{H}(t) \ln \mathcal{H}(t) - D \mathcal{H}(t),
\end{aligned} \tag{4.16}$$

where  $D := \max \left\{ 0, \left| \frac{D_1}{2} - \frac{1}{4} \right| \right\} \geq 0$ . It follows from (2.6), (2.7), and (4.16) that

$$\begin{aligned} \|u_t(t)\|_* &= \sqrt{2E(t) - 2J(u(t))} \leq \sqrt{2E(0) - 2J(u(t))} \\ &\leq \sqrt{2E(0) + \frac{1}{2}\mathcal{H}(t) \ln \mathcal{H}(t) + 2D\mathcal{H}(t)} \\ &= \sqrt{2E(0) + y(t)^2 \ln y(t) + 2Dy(t)^2}. \end{aligned} \quad (4.17)$$

This, together with  $y'(t) \leq \|u_t(t)\|_*$ , implies the following:

$$y'(t) \leq \sqrt{2E(0) + y(t)^2 \ln y(t) + 2Dy(t)^2}.$$

Define  $y_* := \max \left\{ \sqrt{2|E(0)|}, e^{2D+1} \right\}$ . Let  $y(t)$  be continuous on  $[0, T)$ . We consider two cases.

Case (I): There is no subinterval of  $[0, T)$  on which  $y(t) \geq y_*$ , which implies  $y(t) \leq y_* < \infty$  for all  $t \in [0, T)$ .

Case (II): There exists a subinterval  $[\tau, \mathcal{T}) \subset [0, T)$  such that  $y(\tau) = y_*$  and  $y(t) \geq y_*$  for all  $t \in [\tau, \mathcal{T})$ . Moreover, we have the following:

$$y'(t) \leq \sqrt{y(t)^2 + y(t)^2 \ln y(t) + 2Dy(t)^2} = y(t) \sqrt{\ln y(t) + 2D + 1} \leq y(t) \sqrt{2 \ln y(t)}.$$

Then, for every  $t \in [\tau, \mathcal{T})$ , the following holds:

$$y(t) \leq \exp \left( \left( \sqrt{\ln y_*} + \frac{\sqrt{2}}{2}(t - \tau) \right)^2 \right).$$

Therefore, for  $T < \infty$ , there exists a constant  $M := \max \left\{ y_*, \exp \left( \left( \sqrt{\ln y_*} + \frac{\sqrt{2}}{2}T \right)^2 \right) \right\}$  such that  $\|u\|_2^2 \leq M_0 := M^2 < \infty$ . By combining this bound with (2.3) and (4.15), we deduce

$$\|\Delta u\|_2^2 + \|\nabla u\|_p^p \leq \int_{\Omega} |u|^2 \ln |u| \, dx \leq \frac{1}{2} \|\Delta u\|_2^2 + \frac{1}{2} (M_0 + 1) |\ln(M_0 + 1)| + D_1 M_0,$$

i.e.,

$$\frac{1}{2} \|\Delta u\|_2^2 + \|\nabla u\|_p^p \leq D_3 := \frac{1}{2} (M_0 + 1) |\ln(M_0 + 1)| + D_1 M_0,$$

which implies  $\|\Delta u\|_2^2 \leq 2D_3$ . Moreover, from (4.17), we obtain the following:

$$\begin{aligned} \|u_t\|_*^2 &\leq 2E(0) + \frac{1}{2}\mathcal{H}(t) \ln \mathcal{H}(t) + 2D\mathcal{H}(t) \\ &\leq D_4 := 2E(0) + \frac{1}{2} (M_0 + 1) |\ln(M_0 + 1)| + 2DM_0. \end{aligned}$$

Recalling that  $\|\nabla u\|_2^2 \leq \lambda_1^{-1} \|\Delta u\|_2^2$ , we conclude the following:

$$\|u_t\|_*^2 + \|\Delta u\|_2^2 + \|\nabla u\|_2^2 \leq D_4 + 2D_3 + \frac{2}{\lambda_1} D_3, \quad t \in [0, T).$$

Hence, it follows from Theorem 2.1 that  $T = \infty$ .

Step 2: Blow-up at infinite time. Let

$$\mathcal{F}(t) := \|u(t)\|_*^2 + \int_0^t \|u(\tau)\|_*^2 d\tau, \quad t \geq 0.$$

By (1.1), (2.3), and  $I(u) < 0$  for all  $t \in [0, T)$ , for  $t \geq 0$ ,

$$\mathcal{F}'(t) = 2(u, u_t)_* + \|u\|_*^2, \quad \mathcal{F}''(t) = 2\|u_t\|_*^2 - 2I(u) > 0. \quad (4.18)$$

Case (I):  $E(0) < d$ . It follows from (2.7), (4.18), and (3.3) that for  $t \geq 0$ ,

$$\mathcal{F}''(t) \geq -2I(u) > 2r(d - J(u)) \geq 2r(d - E(0)) =: C_4 > 0.$$

Since  $\mathcal{F}''(t) \geq C_4 > 0$ , integrating twice gives  $\mathcal{F}(t) > \frac{C_4}{2}t^2 + F'(0)t + F(0)$ . By contradiction, if  $\|u(t)\|_*^2 \leq L_1$  for some  $L_1 > 0$  and all  $t \geq 0$ , then  $\mathcal{F}(t) \leq L_1(1+t)$ , which contradicts the quadratic growth of  $\mathcal{F}(t)$ . Thus,  $\lim_{t \rightarrow \infty} \|u(t)\|_*^2 = \infty$ .

Case (II):  $E(0) = d$ . We claim that there exists  $t_0 > 0$  such that  $\int_0^{t_0} \|u_\tau\|_*^2 d\tau > 0$ . Suppose not, (i.e., for all  $t \in [0, T)$ ,  $\int_0^t \|u_\tau\|_*^2 d\tau \equiv 0$ ). This implies  $\|u_t(t)\|_*^2 = 0$  for a.e.  $t \in [0, T)$ . Thus,  $u_t \equiv 0$ ,  $u(t) \equiv u_0$  on  $[0, T)$ , so  $I(u_0) = 0$ , contradicting  $I(u_0) < 0$ . By (2.7),

$$E(t_0) = d - \int_0^{t_0} \|u_\tau\|_*^2 d\tau < d.$$

If we take  $t_0$  as the initial time, repeating the procedure of Case (I), similarly, then we can obtain the result.  $\square$

#### Proof of Theorem 2.4

Proof of (i). Step 1: Extension of  $u$  to infinite time.

Since  $(u_0, u_1) \in \mathcal{U}_\emptyset$ , from Lemma 3.8, we know  $I(u(t)) < \|u_t(t)\|_*^2 - \mathcal{O}$  for  $t \in [0, T)$  and

$$d < E(0) \leq \frac{\sqrt{\|u(t)\|_*^2 + 2(u(t), u_t(t))_*}}{2\sqrt{e}\zeta} - \frac{\mathcal{O}}{2}, \quad t \in [0, T), \quad (4.19)$$

which, along with (2.10) and Lemma 3.4, yields that for  $t \in [0, T)$ ,

$$\|u\|_*^2 + 2(u, u_t)_* \geq 8ed^2 \geq 2^{n-1}\pi^n e^{2n+1} > 1. \quad (4.20)$$

From (4.19) and (4.20), one obtains the following:

$$d < E(0) \leq \frac{\|u\|_*^2 + 2(u, u_t)_*}{2\sqrt{e}\zeta}, \quad t \in [0, T). \quad (4.21)$$

Let  $\mathfrak{D}(t) := \|u\|_*^2$  for  $t \in [0, T)$ . From (2.10), the Cauchy-Schwarz inequality, and Young's inequality, we deduce

$$\mathfrak{D}'(t) = 2(u, u_t)_* \leq \|u\|_*^2 + \left(2 - \frac{2}{\sqrt{e}\zeta}\right) \|u_t\|_*^2,$$

which, combined with (2.3), (2.6), (2.7),  $J(u) = \frac{1}{2}I(u) + \frac{1}{4}\|u\|_2^2$ , (3.1), and (4.21), implies that for all  $\mu > 0$ ,

$$(e^3 + \mathfrak{D}(t)) \ln(e^3 + \mathfrak{D}(t)) - \mathfrak{D}'(t)$$

$$\begin{aligned}
&\geq (\|u\|_2^2 + \|\nabla u\|_2^2) \ln(\mathfrak{D}(t) + e^3) - 2\|u_t\|_*^2 + \frac{2}{\sqrt{e\zeta}} \|u_t\|_*^2 - \|u\|_*^2 \\
&\geq \|u\|_2^2 \ln \|u\|_2^2 + 3\|\nabla u\|_2^2 + 4\left(\frac{1}{4}\|u\|_2^2 + \frac{1}{2}I(u) - E(0)\right) + \frac{2}{\sqrt{e\zeta}} \|u_t\|_*^2 - \|u\|_*^2 \\
&\geq \|u\|_2^2 \ln \|u\|_2^2 + 2\|\nabla u\|_2^2 + 2I(u) - 4E(0) + \frac{2}{\sqrt{e\zeta}} \|u_t\|_*^2 \\
&\geq \|u\|_2^2 \ln \|u\|_2^2 + 2\|\nabla u\|_2^2 + 2\|\nabla u\|_2^2 - 2 \int_{\Omega} u^2 \ln |u| dx - \frac{2\|u\|_*^2 + 4(u, u_t)_*}{\sqrt{e\zeta}} + \frac{2\|u_t\|_*^2}{\sqrt{e\zeta}} \\
&\geq 2\|\nabla u\|_2^2 + 2\|\nabla u\|_2^2 + n(1 + \ln \mu)\|u\|_2^2 - \frac{\mu^2}{\pi} \|\nabla u\|_2^2 - \frac{2\|u\|_*^2}{\sqrt{e\zeta}} - \frac{2\|u\|_*^2}{\sqrt{e\zeta}} - \frac{2\|u_t\|_*^2}{\sqrt{e\zeta}} + \frac{2\|u_t\|_*^2}{\sqrt{e\zeta}} \\
&\geq \left(2 - \frac{\mu^2}{\pi}\right) \|\nabla u\|_2^2 + \left[n(1 + \ln \mu) - \frac{4}{\sqrt{e\zeta}}\right] \|u\|_2^2 \\
&:= C(\mu)\|\nabla u\|_2^2 + D(\mu)\|u\|_2^2.
\end{aligned}$$

Take  $\mu \in \left(\exp\left(\frac{4}{n\sqrt{e\zeta}} - 1\right), \sqrt{2\pi}\right)$ . Since  $\zeta \geq 2$ , this interval is nonempty, whence it follows that  $C(\mu) > 0$  and  $D(\mu) > 0$ . Consequently, we obtain  $(e^3 + \mathfrak{D}(t)) \ln(e^3 + \mathfrak{D}(t)) \geq \mathfrak{D}'(t)$ . By Gronwall's inequality,

$$\|u\|_2^2 \leq (1 + \lambda_1)\|u\|_2^2 \leq \|u\|_*^2 = \mathfrak{D}(t) \leq \left(\|u_0\|_*^2 + e^3\right)^{e^t} - e^3 =: H_3(t). \quad (4.22)$$

By (2.6), (2.7), (3.1), and (4.22), we have for any  $\mu > 0$ ,

$$\begin{aligned}
|E(0)| \geq E(t) &\geq \frac{1}{2}\|u_t\|_*^2 + \frac{1}{2}\|\Delta u\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{4}\|u\|_2^2 \\
&\quad - \frac{1}{2}\left(\frac{\mu^2}{4\pi}\|\nabla u\|_2^2 + \frac{1}{2}\|u\|_2^2 \ln \|u\|_2^2 - \frac{n(1 + \ln \mu)}{4}\|u\|_2^2\right) \\
&\geq \frac{1}{2}\|u_t\|_*^2 + \frac{1}{2}\|\Delta u\|_2^2 + \left(\frac{1}{2} - \frac{\mu^2}{4\pi}\right)\|\nabla u\|_2^2 \\
&\quad + \frac{1}{4}\|u\|_2^2 - \frac{1}{4}(H_3(t) + 1) \ln(H_3(t) + 1) + \frac{n(1 + \ln \mu)}{4}\|u\|_2^2.
\end{aligned} \quad (4.23)$$

By substituting  $\mu = \sqrt{\pi}$  into (4.23), we obtain the following:

$$|E(0)| \geq \frac{1}{2}\|u_t\|_*^2 + \frac{1}{2}\|\Delta u\|_2^2 + \frac{1}{4}\|\nabla u\|_2^2 - \frac{1}{4}(H_3(t) + 1) \ln(H_3(t) + 1).$$

Then, for  $t \in [0, T)$ ,

$$\|u_t\|_*^2 + \|\Delta u\|_2^2 + \|\nabla u\|_2^2 \leq (H_3(t) + 1) \ln(H_3(t) + 1) + 4|E(0)|.$$

From Theorem 2.1, we conclude  $T = \infty$ .

Step 2: Polynomial growth.

By (2.9) and Lemma 3.8,  $I(u) < \|u_t\|_*^2 - \mathcal{O}$ . This, together with (4.18), yields  $\mathcal{F}''(t) = 2\|u_t\|_*^2 - 2I(u) > 2\mathcal{O}$ , for  $t \geq 0$ . Let  $\hat{\mathcal{O}} = \min\left\{\|u_1\|_*^2 - I(u_0), \frac{\sqrt{\|u_0\|_*^2 + 2(u_0, u_1)_*}}{\sqrt{e\zeta}} - 2E(0)\right\}$ . By (2.9),  $(u_0, u_1) \in \mathcal{U}_{\mathcal{O}}$  for all  $\mathcal{O} \in (0, \hat{\mathcal{O}})$ . As  $\mathcal{O} \rightarrow \hat{\mathcal{O}}$ , the same argument based on (4.18) yields  $\lim_{t \rightarrow \infty} \|u(t)\|_*^2 = \infty$ .

Proof of (ii). Take nonzero  $\phi \in H$ ,  $\psi \in H_0^1(\Omega)$  with  $(\phi, \psi)_* = 0$ . For any  $Q > d$ , set

$$u_0^Q = \hat{m}\phi, \quad u_1^Q = \hat{m}\phi + \hat{h}\psi \quad (\hat{m}, \hat{h} > 0 \text{ to be determined}).$$

By orthogonality,

$$(u_0^Q, u_1^Q)_* = \hat{m}^2 \|\phi\|_*^2, \quad \|u_1^Q\|_*^2 = \hat{h}^2 \|\psi\|_*^2 + \hat{m}^2 \|\phi\|_*^2.$$

From (2.4) and (2.6), the initial energy reads  $E(0) = \frac{\hat{h}^2}{2} \|\psi\|_*^2 + \mathcal{H}(\hat{m})$ , where

$$\mathcal{H}(\hat{m}) = \frac{\hat{m}^2}{2} \left( \|\Delta\phi\|_2^2 + 2\|\nabla\phi\|_2^2 + \frac{3}{2}\|\phi\|_2^2 - \int_{\Omega} |\phi|^2 \ln |\phi| dx - \|\phi\|_2^2 \ln \hat{m} \right).$$

Now, we verify (2.9) and  $E(0) = Q$  for suitable  $\hat{m}, \hat{h}$ . By (2.3),

$$I(u_0^Q) - \|u_1^Q\|_*^2 = \hat{m}^2 \left( \|\Delta\phi\|_2^2 + \|\nabla\phi\|_2^2 - \int_{\Omega} |\phi|^2 \ln |\phi| dx - \|\phi\|_2^2 \ln \hat{m} - \|\phi\|_2^2 \right) - \hat{h}^2 \|\psi\|_*^2,$$

which is negative for sufficiently large  $\hat{m}$ . As  $\lim_{\hat{m} \rightarrow \infty} \mathcal{H}(\hat{m}) = -\infty$ , take  $\hat{m}$  large enough such that  $Q > \mathcal{H}(\hat{m})$  and  $Q < \frac{\hat{m} \sqrt{3\|\phi\|_*^2}}{2\sqrt{e\xi}} = \frac{\sqrt{\|u_0^Q\|_*^2 + 2(u_0^Q, u_1^Q)_*}}{2\sqrt{e\xi}}$ . Setting  $\hat{h} = \frac{\sqrt{2(Q - \mathcal{H}(\hat{m}))}}{\|\psi\|_*}$  yields  $E(0) = Q$ . Thus,  $(u_0^Q, u_1^Q)$  meets the blow-up condition in Theorem 2.4 (i).  $\square$

### Proof of Theorem 2.5

Case (I):  $E(0) < d$ . For any  $\tilde{T} \in (0, T)$ , define the auxiliary functional

$$\mathcal{K}(t) := \|u(t)\|_*^2 + \int_0^t \|u(\tau)\|_*^2 d\tau + (\tilde{T} - t) \|u_0\|_*^2 + \sigma(t + \rho)^2, \quad t \in [0, \tilde{T}],$$

where constants  $\sigma, \rho > 0$  will be chosen later. Compute the first derivative as follows:

$$\mathcal{K}'(t) = 2(u(t), u_t(t))_* + 2 \int_0^t (u(\tau), u_t(\tau))_* d\tau + 2\sigma(t + \rho).$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned} (\mathcal{K}'(t))^2 &= 4 \left( (u(t), u_t(t))_* + \int_0^t (u(\tau), u_t(\tau))_* d\tau + \sigma(t + \rho) \right)^2 \\ &\leq 4\mathcal{K}(t) \left( \int_0^t \|u_\tau\|_*^2 d\tau + \sigma + \|u_t(t)\|_*^2 \right). \end{aligned} \quad (4.24)$$

Next, compute the second derivative. Using (3.3) and (2.7), we get the following:

$$\begin{aligned} \mathcal{K}''(t) &= -2I(u(t)) + 2\|u_t(t)\|_*^2 + 2\sigma \\ &\geq 2r(d - J(u(t))) + 2\|u_t(t)\|_*^2 + 2\sigma \\ &= 2r \int_0^t \|u_\tau\|_*^2 d\tau + (r + 2)\|u_t(t)\|_*^2 + 2r(d - E(0)) + 2\sigma. \end{aligned} \quad (4.25)$$

Combining (4.24) with (4.25), it follows that

$$\begin{aligned} & \mathcal{K}(t)\mathcal{K}''(t) - \frac{r+2}{4}(\mathcal{K}'(t))^2 \\ & \geq \mathcal{K}(t) \left[ 2r \int_0^t \|u_\tau\|_*^2 d\tau + (r+2)\|u_t\|_*^2 + 2r(d - E(0)) + 2\sigma - (r+2) \left( \int_0^t \|u_\tau\|_*^2 d\tau + \sigma + \|u_t\|_*^2 \right) \right] \\ & = r\mathcal{K}(t) [2(d - E(0)) - \sigma]. \end{aligned}$$

Choose  $\sigma$  such that  $\sigma \in (0, 2(d - E(0)))$ ; then, we obtain

$$\mathcal{K}(t)\mathcal{K}''(t) - \frac{r+2}{4}(\mathcal{K}'(t))^2 \geq 0, \quad \forall t \in [0, \tilde{T}].$$

Take  $\rho$  large enough such that  $\rho > \max \left\{ -\frac{(u_0, u_1)_*}{\sigma}, 0 \right\}$  so that  $\mathcal{K}'(0) > 0$ . By Lemma 3.7,

$$\tilde{T} \leq \frac{4\mathcal{K}(0)}{(r-2)\mathcal{K}'(0)} = \frac{2(\|u_0\|_*^2 + \tilde{T}\|u_0\|_*^2 + \sigma\rho^2)}{(r-2)((u_0, u_1)_* + \sigma\rho)}.$$

Then,  $\tilde{T} \left[ (r-2)((u_0, u_1)_* + \sigma\rho) - 2\|u_0\|_*^2 \right] \leq 2(\|u_0\|_*^2 + \sigma\rho^2)$ . Choose  $\rho$  such that  $\rho > \frac{2\|u_0\|_*^2 - (r-2)(u_0, u_1)_*}{(r-2)\sigma}$ . Let  $\hat{\xi} := \sigma\rho$ , which satisfies the following:

$$\hat{\xi} > \max \left\{ \frac{2\|u_0\|_*^2 - (r-2)(u_0, u_1)_*}{r-2}, 0 \right\}.$$

From  $\sigma \leq 2(d - E(0))$ , we have  $\rho = \frac{\hat{\xi}}{\sigma} \geq \frac{\hat{\xi}}{2(d-E(0))}$ . The upper bound of  $\tilde{T}$  can be rewritten as follows:

$$\tilde{T} \leq f(\hat{\xi}) := \frac{A + \hat{\xi}^2}{(B + (r-2)\hat{\xi})(d - E(0))},$$

where  $A := 2\|u_0\|_*^2(d - E(0))$ , and  $B := (r-2)(u_0, u_1)_* - 2\|u_0\|_*^2$ . A simple calculation shows  $f(\hat{\xi})$  attains its minimum at  $\hat{\xi}_1 = \frac{\sqrt{A(r-2)^2 + B^2} - B}{r-2}$ . By substituting  $\hat{\xi}_1$  into  $f(\hat{\xi})$ , we get the following explicit upper bound:

$$\tilde{T} \leq \frac{2\sqrt{B^2 + A(r-2)^2} - 2B}{(d - E(0))(r-2)^2}.$$

Letting  $\tilde{T} \rightarrow T$ , the upper bound of the blow-up time  $T$  holds.

Case (II):  $E(0) = d$ . Identical to the proof of Case (II) in Theorem 2.3.  $\square$

### Proof of Theorem 2.6

Proof of (i). Using (2.5), (2.6), (2.7),  $\lambda_1\|\nabla u\|_2^2 \leq \|\Delta u\|_2^2$ ,  $\lambda_1\|u\|_2^2 \leq \|\nabla u\|_2^2$ , the Cauchy-Schwarz inequality, and Young's inequality, we obtain the following:

$$\begin{aligned} I(u_0) & \leq rE(0) - \frac{r}{2}\|u_1\|_*^2 - \frac{r-2}{2}(\|\Delta u_0\|_2^2 + \|\nabla u_0\|_2^2) \\ & \leq rE(0) - \frac{r}{2}\|u_1\|_*^2 - \frac{\lambda_1(r-2)}{2}\|u_0\|_*^2 \\ & \leq rE(0) - k\|u_1\|_*^2 - 2k\|u_0\|_*^2 \\ & \leq rE(0) - k[2(u_0, u_1)_* + \|u_0\|_*^2] < 0. \end{aligned}$$

where  $k := (r - 2) \min \left\{ \frac{1}{2}, \frac{\lambda_1}{4} \right\}$ . Define  $L(t) := 2(u(t), u_t(t))_* + \|u(t)\|_*^2$ ; then,

$$L'(t) = 2\|u_t(t)\|_*^2 - 2I(u(t)). \quad (4.26)$$

We claim that  $I(u(t)) < 0$  throughout  $[0, T)$ . Assuming the contrary and utilizing  $I(u_0) < 0$ , we find there exists  $t_* \in (0, T)$  with  $I(u(t_*)) = 0$  and  $I(u(t)) < 0$  for  $t \in [0, t_*)$ . This fact, together with (4.26), yields  $L'(t) > 0$  for  $t \in [0, t_*)$ . By virtue of (2.12), we get  $E(0) < \frac{k}{r}L(0) \leq \frac{k}{r}L(t_*)$ . Nevertheless, from the Cauchy-Schwarz inequality, Young's inequality,  $\lambda_1\|\nabla u\|_2^2 \leq \|\Delta u\|_2^2$ ,  $\lambda_1\|u\|_2^2 \leq \|\nabla u\|_2^2$ , and (2.7), one has

$$\begin{aligned} \frac{k}{r}L(t_*) &\leq \frac{k}{r}\|u_t(t_*)\|_*^2 + \frac{2k}{r}\|u(t_*)\|_*^2 \\ &\leq \frac{1}{2}\|u_t(t_*)\|_*^2 + \frac{(r-2)\lambda_1}{2r}\|u(t_*)\|_*^2 \\ &\leq \frac{1}{2}\|u_t(t_*)\|_*^2 + \frac{r-2}{2r}(\|\Delta u(t_*)\|_2^2 + \|\nabla u(t_*)\|_2^2) \\ &\leq E(t_*) \leq E(0), \end{aligned}$$

which is a contradiction. Consequently,  $I(u) < 0$  for  $t \in [0, T)$  and  $L(t)$  is increasing on  $[0, T)$  from (4.26).

For any  $\tilde{T} \in (0, T)$ , define the following:

$$M(t) := \|u\|_*^2 + \int_0^t \|u\|_*^2 d\tau + (\tilde{T} - t)\|u_0\|_*^2 + \omega(t + \nu)^2, \quad t \in [0, \tilde{T}], \quad (4.27)$$

with positive constants  $\omega, \nu > 0$  to be fixed later on. Then,

$$M'(t) = 2(u_t, u)_* + \int_0^t \|u\|_*^2 d\tau + 2\omega(t + \nu).$$

Together with (1.1), (2.3), (2.5), (2.6), and (2.7), we deduce the following:

$$\begin{aligned} M''(t) &= 2\|u_t\|_*^2 - 2I(u) + 2\omega \\ &= (r+2)\|u_t\|_*^2 + (r-2)(\|\Delta u\|_2^2 + \|\nabla u\|_2^2) + \frac{2}{r}\|u\|_r^r + 2r \int_0^t \|u_\tau\|_*^2 d\tau - 2rE(0) + 2\omega. \end{aligned} \quad (4.28)$$

By the Cauchy-Schwarz inequality and (4.27),

$$\begin{aligned} (M'(t))^2 &\leq 4 \left( \|u\|_*^2 + \int_0^t \|u\|_*^2 d\tau + \omega(t + \nu)^2 \right) \left( \|u_t\|_*^2 + \int_0^t \|u_\tau\|_*^2 d\tau + \omega \right) \\ &\leq 4M(t) \left( \|u_t\|_*^2 + \int_0^t \|u_\tau\|_*^2 d\tau + \omega \right). \end{aligned} \quad (4.29)$$

By combining (4.28) and (4.29), we obtain the following:

$$M(t)M''(t) - \frac{r+6}{8}(M'(t))^2 \geq M(t) \left[ M''(t) - \frac{r+6}{2} \left( \|u_t\|_*^2 + \int_0^t \|u_\tau\|_*^2 d\tau + \omega \right) \right] := M(t)H(t). \quad (4.30)$$

By (4.28),  $\lambda_1 \|\nabla u\|_2^2 \leq \|\Delta u\|_2^2$ ,  $\lambda_1 \|u\|_2^2 \leq \|\nabla u\|_2^2$ , Young's inequality, and the Cauchy-Schwarz inequality,

$$\begin{aligned}
 H(t) &= \frac{r-2}{2} \|u_t\|_*^2 + (r-2) (\|\Delta u\|_2^2 + \|\nabla u\|_2^2) + \frac{2}{r} \|u\|_r^r + \frac{3(r-2)}{2} \int_0^t \|u_\tau\|_*^2 d\tau - 2rE(0) - \frac{r+2}{2} \omega \\
 &\geq \frac{r-2}{2} \|u_t\|_*^2 + (r-2) \lambda_1 \|u\|_*^2 - 2rE(0) - \frac{r+2}{2} \omega \\
 &\geq k (\|u_t\|_*^2 + 2\|u\|_*^2) - 2rE(0) - \frac{r+2}{2} \omega \\
 &\geq k (2(u_t, u)_* + \|u\|_*^2) - 2rE(0) - \frac{r+2}{2} \omega \\
 &\geq kL(0) - 2rE(0) - \frac{r+2}{2} \omega.
 \end{aligned} \tag{4.31}$$

Let  $\eta := kL(0) - 2rE(0) > 0$ . Choosing  $\omega \in \left(0, \frac{2\eta}{r+2}\right]$ , we obtain the following:

$$M(t)M''(t) - \frac{r+6}{8} (M'(t))^2 \geq 0. \tag{4.32}$$

Next, choose  $\nu > 0$  sufficiently large such that  $\nu > \max\left\{-\frac{(u_0, u_1)_*}{\omega}, 0\right\}$ , which ensures  $M'(0) = 2(u_0, u_1)_* + 2\omega\nu > 0$ . By Lemma 3.7, we have the following:

$$\tilde{T} \leq \frac{8M(0)}{(r-2)M'(0)} = \frac{4(\|u_0\|_*^2 + \omega\nu^2 + \tilde{T}\|u_0\|_*^2)}{(r-2)((u_0, u_1)_* + \omega\nu)}.$$

Then,  $\tilde{T}[(r-2)((u_0, u_1)_* + \omega\nu) - 4\|u_0\|_*^2] \leq 4\|u_0\|_*^2 + 4\omega\nu^2$ . To ensure that the coefficient of  $\tilde{T}$  is positive, we impose  $\omega\nu > \frac{4\|u_0\|_*^2}{r-2} - (u_0, u_1)_*$ . Letting  $\beta := \omega\nu$ , we immediately get  $\beta > \beta_0 := \max\left\{\frac{4\|u_0\|_*^2}{r-2} - (u_0, u_1)_*, 0\right\}$ . Under this condition,

$$\tilde{T} \leq \frac{4\|u_0\|_*^2 + \frac{4\beta^2}{\omega}}{(r-2)((u_0, u_1)_* + \beta) - 4\|u_0\|_*^2}.$$

Taking the maximum admissible  $\omega = \frac{2\eta}{r+2}$  to obtain the tightest upper bound, we define  $C := \frac{2\eta\|u_0\|_*^2}{r+2}$ ,  $D := (r-2)(u_0, u_1)_* - 4\|u_0\|_*^2$ . Then, the inequality can be rewritten as follows:

$$\tilde{T} \leq \frac{2(r+2)}{\eta} \cdot \frac{C + \beta^2}{(r-2)\beta + D}.$$

Define the function  $\mathcal{E}(\beta) := \frac{C + \beta^2}{(r-2)\beta + D}$  for  $\beta > \beta_0$ . By a simple computation,  $\mathcal{E}(\beta)$  attains its global minimum at  $\beta_1 = \frac{\sqrt{C(r-2)^2 + D^2} - D}{r-2}$ . Thus, we obtain the explicit upper bound of the blow-up time as follows:

$$\tilde{T} \leq \frac{2(r+2)}{\eta} \cdot \mathcal{E}(\beta_1) = \frac{4(r+2) \left( \sqrt{D^2 + C(r-2)^2} - D \right)}{\eta(r-2)^2}.$$

Letting  $\tilde{T} \rightarrow T$ , the above upper bound holds for the maximal existence time  $T$ .

Proof of (ii). As in the proof of Theorem 2.4 (ii), we take nonzero  $\phi \in H$ ,  $\psi \in H_0^1(\Omega)$  with  $(\phi, \psi)_* = 0$ . For any  $Q \in \mathbb{R}$ , set  $u_0^Q = m\phi, u_1^Q = m\phi + h\psi$ , for  $m, h > 0$ . By orthogonality, it follows

that  $(u_0^Q, u_1^Q)_* = m^2 \|\phi\|_*^2$ ,  $\|u_1^Q\|_*^2 = m^2 \|\phi\|_*^2 + h^2 \|\psi\|_*^2$ . For  $p = 2$ , the energy functional yields  $E(0) = \mathcal{H}(m) + \frac{h^2}{2} \|\psi\|_*^2$ , with

$$\mathcal{H}(m) = \frac{m^2}{2} (\|\phi\|_*^2 + \|\Delta\phi\|_2^2 + \|\nabla\phi\|_2^2) + m^r \left( \frac{\|\phi\|_r^r}{r^2} - \frac{\|\phi\|_r^r}{r} \ln m - \frac{1}{r} \int_{\Omega} |\phi|^r \ln |\phi| dx \right).$$

Since  $r > 2$ ,  $\lim_{m \rightarrow \infty} \mathcal{H}(m) = -\infty$ . For a fixed  $Q$ , choose sufficiently large  $m > 0$  such that  $\mathcal{H}(m) < Q$  and  $Q < \frac{3k\|\phi\|_*^2}{2r} m^2 = \frac{k(2(u_0^Q, u_1^Q)_* + \|u_0^Q\|_*^2)}{2r}$ . Let  $h = \sqrt{\frac{2(Q - \mathcal{H}(m))}{\|\psi\|_*^2}} > 0$ . Thus,  $(u_0^Q, u_1^Q)$  meets the blow-up condition in Theorem 2.6 (i) with  $E(0) = Q$ .  $\square$

### Proof of Theorem 2.7

Proof of (i). By  $\|\Delta u\|_2^2 \geq \lambda_1 \|\nabla u\|_2^2 \geq \lambda_1^2 \|u\|_2^2$ ,

$$\begin{aligned} \|\Delta u\|_2^2 &\geq \frac{1}{\lambda_1 + 1} \cdot \lambda_1^2 \|u\|_2^2 + \left(1 - \frac{1}{\lambda_1 + 1}\right) \lambda_1 \|\nabla u\|_2^2 \\ &= \frac{\lambda_1^2}{\lambda_1 + 1} \|u\|_2^2 + \frac{\lambda_1^2}{\lambda_1 + 1} \|\nabla u\|_2^2 \\ &= \frac{\lambda_1^2}{\lambda_1 + 1} (\|u\|_2^2 + \|\nabla u\|_2^2) = \frac{\lambda_1^2}{\lambda_1 + 1} \|u\|_*^2. \end{aligned} \quad (4.33)$$

Combining (2.5), (2.6), (2.7), (4.33), the Cauchy-Schwarz inequality, and Young's inequality, we obtain the following:

$$\begin{aligned} I(u_0) &\leq rE(0) - \frac{r}{2} \|u_1\|_*^2 - \frac{r-2}{2} \|\Delta u_0\|_2^2 - \frac{r-p}{p} \|\nabla u_0\|_p^p \\ &\leq rE(0) - \frac{r}{2} \|u_1\|_*^2 - \frac{r-2}{2} \|\Delta u_0\|_2^2 \\ &\leq rE(0) - \frac{r}{2} \|u_1\|_*^2 - \frac{(r-2)\lambda_1^2}{2(\lambda_1 + 1)} \|u_0\|_*^2 \\ &\leq rE(0) - k_{**} \|u_1\|_*^2 - 2k_{**} \|u_0\|_*^2 \\ &\leq rE(0) - k_{**} [2(u_0, u_1)_* + \|u_0\|_*^2] < 0, \end{aligned}$$

where  $k_{**} := (r-2) \min \left\{ \frac{1}{2}, \frac{\lambda_1^2}{4(\lambda_1 + 1)} \right\}$ . We assert that  $I(u(t)) < 0$  throughout  $[0, T)$ . Assuming the contrary and utilizing  $I(u_0) < 0$ , we find there exists  $t_* \in (0, T)$  with  $I(u(t_*)) = 0$  and  $I(u(t)) < 0$  for  $t \in [0, t_*)$ . This fact, together with (4.26), yields  $L'(t) > 0$  for  $t \in [0, t_*)$ . By virtue of (2.13), we deduce that  $E(0) < \frac{k_{**}}{r} L(0) \leq \frac{k_{**}}{r} L(t_*)$ . Nevertheless, from the Cauchy-Schwarz inequality, Young's inequality, (4.33), and (2.7), one has the following:

$$\begin{aligned} \frac{k_{**}}{r} L(t_*) &\leq \frac{k_{**}}{r} \|u_t(t_*)\|_*^2 + \frac{2k_{**}}{r} \|u(t_*)\|_*^2 \\ &\leq \frac{1}{2} \|u_t(t_*)\|_*^2 + \frac{(r-2)\lambda_1^2}{2r(\lambda_1 + 1)} \|u(t_*)\|_*^2 \\ &\leq \frac{1}{2} \|u_t(t_*)\|_*^2 + \frac{r-2}{2r} \|\Delta u(t_*)\|_2^2 \\ &\leq E(t_*) \leq E(0), \end{aligned}$$

which is a contradiction. Consequently,  $I(u) < 0$  for  $t \in [0, T)$  and  $L(t)$  is increasing on  $[0, T)$ .

Recalling (4.27)-(4.31), by (4.33), Young's inequality, and the Cauchy-Schwarz inequality, we have the following:

$$\begin{aligned} H(t) &\geq \frac{r-2}{2} \|u_t\|_*^2 + (r-2) \|\Delta u\|_2^2 + \frac{2}{r} \|u\|_r^r + \frac{3(r-2)}{2} \int_0^t \|u_\tau\|_*^2 d\tau - 2rE(0) - \frac{r+2}{2} \omega \\ &\geq \frac{r-2}{2} \|u_t\|_*^2 + (r-2) \frac{\lambda_1^2}{\lambda_1 + 1} \|u\|_*^2 - 2rE(0) - \frac{r+2}{2} \omega \\ &\geq k_{**} (\|u_t\|_*^2 + 2\|u\|_*^2) - 2rE(0) - \frac{r+2}{2} \omega \\ &\geq k_{**} (2(u_t, u)_* + \|u\|_*^2) - 2rE(0) - \frac{r+2}{2} \omega \\ &\geq k_{**} L(0) - 2rE(0) - \frac{r+2}{2} \omega. \end{aligned}$$

Let  $\eta_{**} := k_{**} L(0) - 2rE(0) > 0$ . Choosing  $\omega \in (0, \frac{2\eta_{**}}{r+2}]$ , we see that (4.32) holds. The remaining proofs of (i) and (ii) are entirely analogous to those of Theorem 2.6 and are therefore omitted.  $\square$

### Proof of Theorem 2.8

Define the auxiliary functional as follows:

$$\mathcal{G}(t) := \|u(t)\|_*^2 + \|u_t(t)\|_*^2 + \|\Delta u(t)\|_2^2 + \frac{2}{p} \|\nabla u(t)\|_p^p, \quad t \in [0, T).$$

By applying (1.1), the Cauchy-Schwarz inequality, and Young's inequality, we have the following:

$$\begin{aligned} \mathcal{G}'(t) &= 2(u(t), u_t(t))_* + 2(u_t(t), u_{tt}(t))_* + 2(\Delta u(t), \Delta u_t(t)) + 2 \langle |\nabla u|^{p-2} \nabla u, \nabla u_t \rangle_* \\ &= 2(u(t), u_t(t))_* - 2\|u_t(t)\|_*^2 + 2 \int_{\Omega} |u|^{r-2} u u_t \ln |u| dx \\ &\leq \|u(t)\|_*^2 - \|u_t(t)\|_*^2 + 2 \int_{\Omega} |u|^{r-2} u u_t \ln |u| dx \\ &\leq \mathcal{G}(t) - \|u_t(t)\|_*^2 + 2 \int_{\Omega} |u|^{r-2} u u_t \ln |u| dx. \end{aligned} \tag{4.34}$$

Split the domain into  $\Omega_1 = \{x \in \Omega : |u| > 1\}$  and  $\Omega_2 = \{x \in \Omega : |u| \leq 1\}$ . Using  $|\ln |u||^\delta \leq |u|^\delta$  for  $|u| > 1$  and  $\|u\|^{r-2} \ln |u| \leq 1/(e(r-1))$  for  $|u| \leq 1$  and Young's inequality, we get the following:

$$\begin{aligned} 2 \int_{\Omega} |u|^{r-2} u u_t \ln |u| dx &\leq \|u_t(t)\|_*^2 + \int_{\Omega_1} |u|^{r-2} u \ln |u|^2 dx + \int_{\Omega_2} |u|^{r-2} u \ln |u|^2 dx \\ &\leq \|u_t(t)\|_*^2 + \frac{1}{\delta^2} \int_{\Omega_1} |u|^{2(r+\delta-1)} dx + |\Omega|(e(r-1))^{-2} \\ &\leq \|u_t(t)\|_*^2 + \frac{C_5^{2(r+\delta-1)}}{\delta^2} \|\nabla u(t)\|_p^{2(r+\delta-1)} + |\Omega|(e(r-1))^{-2} \\ &\leq \|u_t(t)\|_*^2 + C_7 \mathcal{G}(t)^{\frac{2(r+\delta-1)}{p}} + |\Omega|(e(r-1))^{-2}, \end{aligned}$$

where  $C_7 := \frac{C_5^{2(r+\delta-1)}}{\delta^2} \left(\frac{p}{2}\right)^{\frac{2(r+\delta-1)}{p}}$ . Combine with (4.34) to obtain the following differential inequality:

$$\mathcal{G}'(t) \leq C_7 \mathcal{G}(t)^{\frac{2(r+\delta-1)}{p}} + |\Omega|(e(r-1))^{-2} + \mathcal{G}(t).$$

By separating variables and integrating from 0 to  $t$ , we obtain the following:

$$\int_{\mathcal{G}(0)}^{\mathcal{G}(t)} \frac{1}{C_7 y^{\frac{2(r+\delta-1)}{p}} + |\Omega|(e(r-1))^{-2} + y} dy \leq t.$$

Let  $t \rightarrow T^-$ , so  $\mathcal{G}(t) \rightarrow \infty$ . Hence, we get the lower bound of the blow-up time.  $\square$

## 5. Conclusion

Our work makes three improvements over previous papers. First, we complete the proof that infinite time blow-up solutions can be extended globally in time for  $p = r = 2$ . Second, we first prove high energy infinite time blow-up for fourth-order equations with both dispersion and strong damping. Third, we give full finite time blow-up results for  $r > 2$ , which no earlier work covered.

We list several open problems for later study.

(i) We do not cover  $1 < p \leq r < 2$ . The potential well method fails here. Some alternative methods may be required for this parameter range.

(ii) Our model has no memory or nonlinear weak damping. Adding these terms will create more dynamic behaviors.

(iii) As far as we know, all existing infinite time blow-up results for high initial energy are established for semilinear equations. Can similar results be derived for quasilinear equations? For example, consider the case  $1 < p < 2$  and  $r = 2$  in Problem (1.1).

## Author contributions

Guangyang Fu: Writing-original draft, formal analysis; Jiang Zhou: Writing-review & editing, project administration, conceptualization.

## Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare there is no conflict of interest.

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