



Research article

Compactness and multiplicity for a critical bi-nonlocal anisotropic Schrödinger–Kirchhoff problem with variable exponents

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Abstract: We study a critical bi-nonlocal anisotropic Schrödinger–Kirchhoff problem with variable exponents in a bounded Lipschitz domain of dimension $N \geq 2$. The equation combines a Kirchhoff coefficient depending on the full anisotropic modular, a second nonlocal reaction of the form $(\int_{\Omega} F(x, u) dx)^r f(x, u)$, and a reaction exponent that may reach the $p_m(\cdot)$ -based critical threshold in the interior of the domain while remaining subcritical on the boundary. We first establish the variational framework for general anisotropic Leray–Lions potentials with directional growth $1 < p_i^- \leq p_i(x) \leq p_i^+ < N$. The anisotropic concentration-compactness principle records the possible critical defect measures. Independently, testing the derivative with the weak difference yields a global monotonicity defect identity; the subcritical nonlocal source vanishes, the critical term contributes a non-negative defect, and the (S_+) property of the principal operator gives strong convergence. This convergence then identifies the recorded measures and eliminates every concentration mass. We then introduce a truncated even functional and apply Krasnosel’skii’s genus theory to obtain infinitely many distinct pairs of nontrivial weak solutions for sufficiently small values of the parameter. The abstract assumptions are verified for standard anisotropic $p_i(x)$ -growth operators, explicit power-type sources, and both polynomial and bounded Kirchhoff laws.

Keywords: Schrödinger–Kirchhoff equations; anisotropic problems; variable exponent spaces; Palais–Smale compactness; monotone operators; genus theory; critical embeddings

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1. Introduction

Nonlocal elliptic equations of the Kirchhoff type have attracted considerable attention in recent years because they combine a rich variational structure with important applications in mathematical physics, continuum mechanics, and nonlinear material theory. In Kirchhoff-type models, the coefficient multiplying the principal operator represents a nonlocal dependence of the effective tension or stiffness on the global deformation energy, a feature that is classical in the mechanics of elastic strings, membranes, and nonhomogeneous media. In parallel, anisotropic and variable-exponent models have become increasingly relevant for the description of heterogeneous media, image restoration, electrorheological fluids, and materials whose response depends both on the spatial position and on the direction of diffusion. From the modeling viewpoint, anisotropy and variable growth are precisely the mechanisms that allow one to encode direction-dependent and spatially inhomogeneous constitutive responses. In such contexts, the natural functional setting is no longer the standard Sobolev space but rather an anisotropic variable-exponent Sobolev space, where each partial derivative may exhibit a different growth law. We refer to [1–5] for the foundational theory of variable-exponent Lebesgue and Sobolev spaces, and to [6] for anisotropic problems with variable growth.

A further source of difficulty appears when the reaction term reaches the $p_m(\cdot)$ -based critical growth furnished by the embedding used here. In that case, the embedding $W_0^{1,\vec{p}(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is no longer compact. In the constant-exponent setting, the classical framework for describing the resulting concentration phenomena is the concentration-compactness principle of P.-L. Lions [7]. Variable-exponent and anisotropic versions were obtained in [8, 9], with sharper compactness and concentration-compactness refinements developed in [10]. In the present paper the anisotropic principle is used explicitly in the Palais–Smale proof to record the possible critical defect measures. We assume that the critical exponent is strictly subcritical on $\partial\Omega$, so the critical set is contained in the interior and the concentration-compactness theorem is used exactly in its established form. The resulting atoms are subsequently ruled out by a global identity specific to the sign structure of the equation: The critical term occurs with a positive coefficient and therefore contributes a non-negative monotonicity defect when the derivative is tested with $u_j - u$.

On the other hand, Kirchhoff-type equations and related nonlocal elliptic problems have been investigated by variational methods in several complementary directions. Foundational existence and minimax results for nonlocal Kirchhoff equations may be found in [11–13]. Variable-exponent Kirchhoff models were subsequently developed in [14–17], while critical or genuinely nonlocal variants are represented, for example, by [18, 19]. Recent anisotropic Schrödinger–Kirchhoff and concentration-compactness results closely related to the present setting are given in [9, 20, 21]; see also [22] for a variational treatment of a nonlocal elliptic reaction.

More specifically, the two earlier anisotropic Schrödinger–Kirchhoff studies by Chems Eddine et al. [20, 21] treated critical variable-exponent problems for more specific principal operators and source structures: The first established the existence of at least one weak solution, whereas the second obtained existence and multiplicity results in a two-parameter setting. The present problem extends those models in several coupled directions. First, the directional principal part is generated by a general family of anisotropic Leray–Lions potentials $A_i(x, \xi)$ rather than being restricted in the abstract theorem to a single explicit $\vec{p}(x)$ -Laplacian form. Second, the Kirchhoff coefficient depends on the full anisotropic modular, including the weighted lower-order $p_M(x)$ contribution. Third, the reaction contains the additional

global factor $\left(\int_{\Omega} F(x, u) dx\right)^r$, so the equation is genuinely bi-nonlocal. Finally, the exponent $q(x)$ is allowed to reach the $p_m(\cdot)$ -based critical threshold $p_m^*(x)$ on the critical set $\mathcal{A}$; the resulting concentration measures are first identified by the anisotropic concentration-compactness principle and then shown to have no atomic part by a global monotonicity defect argument, before a truncated Krasnosel'skii-genus construction is used to obtain infinitely many distinct pairs of solutions. Thus the generalization simultaneously concerns the operator structure, the two nonlocal couplings, the critical compactness mechanism, and the multiplicity conclusion.

The paper treats the Kirchhoff dependence, the nonlocal primitive factor, directional variable growth, and the $p_m(\cdot)$ -based critical concentration in one variational framework. The central analytical task is to obtain strong Palais–Smale compactness at the critical threshold in a form suitable for a genus-based multiplicity argument. This is achieved through the coercive monotonicity estimate $(\mathbf{A}_3)$, which converts the vanishing principal-operator defect into strong convergence in the anisotropic energy space.

In this work, we study the following critical bi-nonlocal anisotropic Schrödinger–Kirchhoff problem:

$$\begin{cases} -M(\rho_{\vec{p}(x)}(u)) \left(\sum_{i=1}^N \partial_{x_i} a_i(x, \partial_{x_i} u) - w(x) |u|^{p_M(x)-2} u \right) \\ \quad = \lambda \left(\int_{\Omega} F(x, u) dx \right)^r f(x, u) - \gamma |u|^{q(x)-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is a bounded Lipschitz domain, $\lambda > 0$, $\gamma > 0$, $r > 0$, and $w \in L^\infty(\Omega)$ satisfies

$$w(x) \geq w_0 > 0 \quad \text{for a.e. } x \in \Omega. \quad (1.2)$$

We seek weak solutions in the anisotropic energy space

$$X := W_0^{1, \vec{p}(x)}(\Omega),$$

whose precise definition and norm are recalled in Section 2. The quantity $\rho_{\vec{p}(x)}(u)$ is the full anisotropic energy modular defined in (1.10): It aggregates the directional energies $A_i(x, \partial_{x_i} u)$ and the lower-order weighted contribution $\frac{w(x)}{p_M(x)} |u|^{p_M(x)}$. Consequently, the factor $M(\rho_{\vec{p}(x)}(u))$ depends on the global state of u , rather than pointwise on x .

At a formal level, Problem (1.1) may be viewed as a stationary balance law for a nonhomogeneous anisotropic medium whose effective stiffness is governed by the global modular $\rho_{\vec{p}(x)}(u)$, while the exponents $p_i(x)$ describe directional growth laws that may vary from point to point. This is one of the reasons why the present analysis fits naturally within the analysis side of the analysis–mechanics interface addressed by Kirchhoff-type models.

The anisotropic exponents $p_i \in C(\overline{\Omega})$, $i = 1, \dots, N$, are assumed to satisfy

$$1 < p_i^- := \min_{x \in \overline{\Omega}} p_i(x) \leq p_i(x) \leq p_i^+ := \max_{x \in \overline{\Omega}} p_i(x) < N, \quad i = 1, \dots, N. \quad (1.3)$$

We introduce the pointwise extremal exponents

$$p_m(x) := \min_{1 \leq i \leq N} p_i(x), \quad p_M(x) := \max_{1 \leq i \leq N} p_i(x),$$

and the global bounds

$$p_m^- := \min_{1 \leq i \leq N} p_i^-, \quad p_M^- := \min_{x \in \overline{\Omega}} p_M(x), \quad p_M^+ := \max_{x \in \overline{\Omega}} p_M(x) = \max_{1 \leq i \leq N} p_i^+.$$

The $p_m(\cdot)$ -based critical exponent furnished by the embedding used below is

$$p_m^*(x) := \frac{N p_m(x)}{N - p_m(x)}.$$

Since the critical embedding in the anisotropic variable-exponent setting requires a logarithmic continuity assumption, we also impose

$$p_m \in C_+^{0, \log}(\overline{\Omega}). \quad (1.4)$$

In order for the zero-order term

$$\int_{\Omega} w(x) |u|^{p_M(x)} dx$$

to be well defined for every $u \in W_0^{1, \vec{p}(x)}(\Omega)$, we further assume the strict subcriticality condition

$$p_M(x) < p_m^*(x) \quad \text{for all } x \in \overline{\Omega}. \quad (1.5)$$

Under (1.4) and (1.5), the anisotropic Sobolev embedding yields

$$W_0^{1, \vec{p}(x)}(\Omega) \hookrightarrow \hookrightarrow L^{p_M(x)}(\Omega),$$

so the modular $\rho_{\vec{p}(x)}$ and the operator defined below are well posed on the whole energy space; see [9].

Let $\beta \in C(\overline{\Omega})$ denote the growth exponent appearing in assumption $(\mathbf{f}_1)$ below and set

$$\beta^- := \min_{x \in \overline{\Omega}} \beta(x), \quad \beta^+ := \max_{x \in \overline{\Omega}} \beta(x).$$

The critical reaction exponent $q \in C(\overline{\Omega})$ is assumed to satisfy

$$1 < \beta(x) < q(x) \leq p_m^*(x) \quad \text{for all } x \in \overline{\Omega}. \quad (1.6)$$

To place the critical set in the interior and apply the anisotropic concentration-compactness theorem in its established setting, we impose the boundary subcriticality condition

$$q(x) < p_m^*(x) \quad \text{for all } x \in \partial\Omega. \quad (1.7)$$

We also write

$$q^- := \min_{x \in \overline{\Omega}} q(x), \quad q^+ := \max_{x \in \overline{\Omega}} q(x).$$

We use

$$\mathcal{A} := \{x \in \Omega : q(x) = p_m^*(x)\} \quad (1.8)$$

to denote the critical set. By continuity and (1.7), $\mathcal{A}$ is a compact subset of Ω . Hence, the critical growth may occur in the interior, while a uniform subcritical gap holds in a neighborhood of the boundary.

The anisotropic principal part is driven by the family

$$\sum_{i=1}^N \partial_{x_i} a_i(x, \partial_{x_i} u), \quad a_i(x, \xi) = \partial_{\xi} A_i(x, \xi),$$

and we assume the following Leray–Lions type conditions.

(A₀) For each $i \in \{1, \dots, N\}$, the map $A_i : \Omega \times \mathbb{R} \rightarrow [0, \infty)$ is a Carathéodory function, $A_i(x, 0) = 0$, the section $A_i(x, \cdot)$ belongs to $C^1(\mathbb{R})$ for almost every (a.e.) $x \in \Omega$, and

$$A_i(x, -\xi) = A_i(x, \xi) \quad \text{for a.e. } x \in \Omega, \forall \xi \in \mathbb{R}.$$

In particular, $a_i(x, \xi) := \partial_\xi A_i(x, \xi)$ is well defined for a.e. (x, ξ) ;

(A₁) For each $i \in \{1, \dots, N\}$, $c_{a_i} > 0$ exists such that

$$|a_i(x, \xi)| \leq c_{a_i}(f_i(x) + |\xi|^{p_i(x)-1}) \quad \text{for a.e. } x \in \Omega, \forall \xi \in \mathbb{R},$$

where $f_i \in L^\infty(\Omega)$ is non-negative;

(A₂) For each $i \in \{1, \dots, N\}$, $k_i > 0$ exists such that

$$k_i |\xi|^{p_i(x)} \leq a_i(x, \xi) \xi \leq p_i(x) A_i(x, \xi) \quad \text{for a.e. } x \in \Omega, \forall \xi \in \mathbb{R};$$

(A₃) For each $i \in \{1, \dots, N\}$, $c_i > 0$ exists such that

$$(a_i(x, \xi) - a_i(x, \eta))(\xi - \eta) \geq c_i \mathcal{D}_{p_i(x)}(\xi, \eta) \quad \text{for a.e. } x \in \Omega, \forall \xi, \eta \in \mathbb{R},$$

where, for $p \in (1, \infty)$,

$$\mathcal{D}_p(\xi, \eta) := \begin{cases} |\xi - \eta|^p, & p \geq 2, \\ \frac{|\xi - \eta|^2}{(|\xi| + |\eta|)^{2-p}}, & 1 < p < 2. \end{cases}$$

with the convention that the second expression equals 0 when $\xi = \eta = 0$;

(A₄) For each $i \in \{1, \dots, N\}$, $K_i > 0$ exists such that

$$0 \leq A_i(x, \xi) \leq K_i |\xi|^{p_i(x)} \quad \text{for a.e. } x \in \Omega, \forall \xi \in \mathbb{R}.$$

Condition (A₃) implies strict monotonicity in both the subquadratic and superquadratic regimes, since $\mathcal{D}_p(\xi, \eta) > 0$ whenever $\xi \neq \eta$. Accordingly, (A₀)–(A₂) provide variational well-posedness and the boundedness estimates, (A₃) converts a vanishing monotonicity defect into strong convergence in X , and (A₄) provides the uniform small radius control of the anisotropic modular on finite-dimensional sets of the form $tS(E_n)$.

The formulation of (A₃) is compatible with the full range in (1.3). In particular, it covers the standard anisotropic variable-exponent class

$$A_i(x, \xi) = \frac{b_i(x)}{p_i(x)} |\xi|^{p_i(x)}, \quad a_i(x, \xi) = b_i(x) |\xi|^{p_i(x)-2} \xi,$$

provided $b_i \in L^\infty(\Omega)$ is uniformly positive, with

$$\underline{b}_i := \operatorname{ess\,inf}_{x \in \Omega} b_i(x) > 0.$$

Indeed, for every $p \in (1, \infty)$, we have

$$(|\xi|^{p-2} \xi - |\eta|^{p-2} \eta)(\xi - \eta) \geq \begin{cases} 2^{2-p} |\xi - \eta|^p, & p \geq 2, \\ (p-1) \frac{|\xi - \eta|^2}{(|\xi| + |\eta|)^{2-p}}, & 1 < p < 2. \end{cases}$$

Consequently, $(\mathbf{A}_3)$ holds for this model with the uniform constant

$$c_i = \underline{b}_i \min\{2^{2-p_i^+}, p_i^- - 1\} > 0.$$

Moreover, we have

$$0 \leq A_i(x, \xi) \leq \frac{\|b_i\|_{L^\infty(\Omega)}}{p_i^-} |\xi|^{p_i(x)},$$

so $(\mathbf{A}_4)$ is satisfied as well. Hence, this model covers the full interval $1 < p_i^- \leq p_i(x) \leq p_i^+ < N$.

The nonlocal reaction is described through the primitive

$$F(x, t) := \int_0^t f(x, s) ds,$$

where the nonlinearity f is required to satisfy the following conditions:

$(\mathbf{f}_0)$ $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function;

$(\mathbf{f}_1)$ The constants $A_1, A_2 > 0$ and an exponent $\beta \in C(\overline{\Omega})$ exist with $\beta(x) > 1$ such that

$$A_1 |t|^{\beta(x)} \leq f(x, t) t \leq A_2 |t|^{\beta(x)} \quad \text{for a.e. } x \in \Omega, \forall t \in \mathbb{R};$$

$(\mathbf{f}_2)$

$$f(x, -t) = -f(x, t) \quad \text{for a.e. } x \in \Omega, \forall t \in \mathbb{R}.$$

Assumptions $(\mathbf{f}_0)$ – $(\mathbf{f}_2)$ yield both the sign condition $f(x, t) t \geq 0$ and the two-sided growth

$$A_1 |t|^{\beta(x)-1} \leq |f(x, t)| \leq A_2 |t|^{\beta(x)-1} \quad \text{for a.e. } x \in \Omega, \forall t \in \mathbb{R},$$

while $(\mathbf{f}_2)$ guarantees that the primitive $F(x, \cdot)$ is even. In particular, for a.e. fixed $x \in \Omega$ and every $t \in \mathbb{R}$, we have

$$\frac{A_1}{\beta^+} |t|^{\beta(x)} \leq F(x, t) \leq \frac{A_2}{\beta^-} |t|^{\beta(x)}. \quad (1.9)$$

Assumptions $(\mathbf{f}_0)$ – $(\mathbf{f}_2)$, together with $(\mathbf{A}_0)$ – $(\mathbf{A}_1)$, (1.5), (1.6), and the continuity of the Kirchhoff function M , ensure that the modular $\rho_{\vec{p}(x)}$, the nonlocal potential

$$u \mapsto \int_{\Omega} F(x, u) dx,$$

and the energy functional J_λ are well defined and of class C^1 on X ; see Proposition 2.7. Since (1.6) yields $\beta(x) < q(x) \leq p_m^*(x)$ on $\overline{\Omega}$, these properties also provide the precise subcritical control needed for the second nonlocal term and will be used repeatedly in Sections 3 and 4. In addition, the quantitative monotonicity hypothesis $(\mathbf{A}_3)$ will be invoked in the compactness part to convert the vanishing monotonicity defect into strong convergence in X .

The natural modular associated with Problem (1.1) is

$$\rho_{\vec{p}(x)}(u) := \int_{\Omega} \left(\sum_{i=1}^N A_i(x, \partial_{x_i} u) + \frac{w(x)}{p_M(x)} |u|^{p_M(x)} \right) dx, \quad (1.10)$$

which is finite for every $u \in W_0^{1, \vec{p}(x)}(\Omega)$ thanks to (1.5) and the anisotropic embedding with exponent $p_M(\cdot)$. For brevity, we shall often write

$$\rho(u) := \rho_{\vec{p}(x)}(u) \quad (u \in X)$$

whenever no confusion is possible.

Definition 1.1 (Weak solution). A function $u \in X$ is called a weak solution of Problem (1.1) if, for every $v \in X$, we have

$$\begin{aligned} M(\rho(u)) \int_{\Omega} \left[\sum_{i=1}^N a_i(x, \partial_{x_i} u) \partial_{x_i} v + w(x) |u|^{p_M(x)-2} uv \right] dx \\ = \lambda \left(\int_{\Omega} F(x, u) dx \right)^r \int_{\Omega} f(x, u) v dx - \gamma \int_{\Omega} |u|^{q(x)-2} uv dx. \end{aligned}$$

The corresponding energy functional is defined on $W_0^{1, \vec{p}(x)}(\Omega)$ by

$$J_{\lambda}(u) = \widehat{M}(\rho_{\vec{p}(x)}(u)) - \frac{\lambda}{r+1} \left(\int_{\Omega} F(x, u) dx \right)^{r+1} + \gamma \int_{\Omega} \frac{|u|^{q(x)}}{q(x)} dx, \quad (1.11)$$

where

$$\widehat{M}(t) := \int_0^t M(s) ds.$$

Under the hypotheses stated in Proposition 2.7, the functional J_{λ} belongs to $C^1(X, \mathbb{R})$. Accordingly, its critical points are precisely the weak solutions of Problem (1.1) in the sense of Definition 1.1.

The main analytical obstacle is the simultaneous presence of two nonlocal effects and one critical variable-exponent term. The first nonlocal effect comes from the Kirchhoff coefficient $M(\rho_{\vec{p}(x)}(u))$, which depends on the global modular of the solution. The second one comes from the factor $\left(\int_{\Omega} F(x, u) dx\right)^r$, which makes the reaction term itself nonlocal. In addition, the exponent $q(x)$ may reach the $p_m(\cdot)$ -based critical exponent $p_m^*(x)$, so the embedding into $L^{q(x)}(\Omega)$ is no longer compact. Consequently, the proof of the Palais–Smale condition cannot rely on standard compact embeddings alone.

The proof separates the description of possible defects from the mechanism that yields strong convergence. First, the anisotropic concentration-compactness principle is applied with $h = q$ to record the possible critical defect measures; Condition (1.7) keeps every possible concentration point in the interior. Next, the Palais–Smale identity is tested with $u_j - u$. The subcritical bi-nonlocal source vanishes by compact embedding, while the positive critical term decomposes into a non-negative monotonicity defect plus a weakly vanishing term. The resulting vanishing principal-operator defect and the (S_+) property supplied by (A_3) independently give strong convergence in X . This convergence identifies the previously recorded measures and forces every concentration mass to vanish. Finally, a modular truncation localizes the negative critical points and supports the Krasnosel'skii-genus construction; see [23, 24].

For the reader's convenience, let us isolate the main contributions of the paper.

- (i) We formulate the problem in a genuine anisotropic variable-exponent setting where the equation contains two distinct nonlocal mechanisms and a critical reaction term, and we show that the associated energy functional is well defined and of class C^1 on the natural energy space.
- (ii) We prove a Palais–Smale compactness result in the critical regime. The concentration-compactness principle records the possible defect measures, whereas the global monotonicity identity and the (S_+) property implied by (A_3) yield strong convergence; the latter convergence shows that every recorded concentration mass vanishes.
- (iii) We exploit this compactness input in a truncated even-functional framework and obtain infinitely many pairs of nontrivial weak solutions by means of Krasnosel'skii's genus theory.

- (iv) We verify the abstract assumptions for standard anisotropic $p_i(x)$ -growth operators throughout the full range $1 < p_i^- \leq p_i(x) \leq p_i^+ < N$ for explicit power-type nonlocal source terms and for both polynomial and bounded Kirchhoff laws.

This formulation also clarifies the level of generality of the result. The concentration-compactness decomposition describes the possible loss of mass when $q(x) = p_m^*(x)$, while the favorable sign of the critical term ensures that its duality contribution is a non-negative defect in the Palais–Smale identity. The remaining subcritical bi-nonlocal term is compact, and the principal operator converts the resulting vanishing defect into strong convergence. The measure decomposition then has no atomic part. This mechanism is combined with the truncated genus construction in the doubly nonlocal anisotropic setting considered here.

Our main result is the following.

Theorem 1.1. *Assume (1.3), (1.4), (1.5), (1.6), (1.7), (1.2), $(\mathbf{f}_0)$, $(\mathbf{f}_1)$, $(\mathbf{f}_2)$, and $(\mathbf{A}_0)$ – $(\mathbf{A}_4)$. Moreover, let $M : [0, \infty) \rightarrow (0, \infty)$ be continuous, and assume that*

$$\beta^+(r+1) < \min\{p_m^-, q^-\}. \quad (1.12)$$

Then the following conclusions hold.

(i) *If*

$$M(t) = a + bt^\eta, \quad a > 0, \quad b \geq 0, \quad \eta \geq 1,$$

then $\lambda^* > 0$ exists such that, for every $0 < \lambda < \lambda^*$, Problem (1.1) admits infinitely many distinct pairs of nontrivial weak solutions in $W_0^{1, \vec{p}(x)}(\Omega)$.

(ii) *If the constants $m_0, m_1 > 0$ exist such that*

$$m_0 \leq M(t) \leq m_1 \quad \text{for all } t \geq 0,$$

then $\tilde{\lambda} > 0$ exists such that, for every $0 < \lambda < \tilde{\lambda}$, Problem (1.1) admits infinitely many distinct pairs of nontrivial weak solutions in $W_0^{1, \vec{p}(x)}(\Omega)$.

Remark 1.1. The thresholds in Theorem 1.1 are genuine constants, independent of the genus index and of the minimax level. Once and for all, fix any radii

$$0 < R_0 < R_1.$$

Let $C_0 > 0$ be the coercivity constant in (3.1), let $C_G > 0$ be the growth constant in (4.2), and define

$$\mathcal{L}_{R_1} := \max \left\{ 1, \left(\frac{p_M^+ R_1}{C_0} \right)^{1/p_m^-} \right\}, \quad C_{R_1} := C_G \mathcal{L}_{R_1}^{\beta^+}. \quad (1.13)$$

Then every $u \in X$ satisfying $\rho(u) \leq R_1$ also satisfies

$$\|u\| \leq \mathcal{L}_{R_1}, \quad 0 \leq G(u) \leq C_{R_1}.$$

Consequently, one may take

$$\lambda^* = \frac{(r+1)aR_0}{2C_{R_1}^{r+1}} \quad \text{in Case (i),}$$

and

$$\tilde{\lambda} = \frac{(r+1)m_0R_0}{2C_{R_1}^{r+1}} \quad \text{in case (ii).}$$

These explicit sufficient thresholds determine the admissible parameter ranges $(0, \lambda^*)$ and $(0, \tilde{\lambda})$, respectively.

The operator assumptions in Theorem 1.1 have distinct roles. Conditions $(\mathbf{A}_0)$ – $(\mathbf{A}_2)$ establish the variational framework and the boundedness estimates, while the concentration-compactness decomposition follows from the anisotropic Sobolev structure. When the global Palais–Smale identity yields a vanishing principal-operator defect, $(\mathbf{A}_3)$ supplies the (S_+) property and upgrades weak convergence to strong convergence in X , thereby eliminating every atomic mass in the measure decomposition. Condition $(\mathbf{A}_4)$ supplies the small-radius estimate required by the negative-level genus geometry. Thus compactness follows from the global defect identity and the (S_+) property, whereas the genus geometry follows from homogeneous upper growth.

Equivalently, Theorem 1.1 is an abstract multiplicity statement combining a quantitative compactness hypothesis on the principal operator with a homogeneous upper growth hypothesis for the finite-dimensional minimax construction. Proposition 5.1 shows that both hypotheses are satisfied by the standard anisotropic variable-exponent model for the whole interval $1 < p_i^- \leq p_i(x) \leq p_i^+ < N$. The mixed form of $(\mathbf{A}_3)$ is essential in the compactness step, while $(\mathbf{A}_4)$ supplies the correct power $t^{p_m^-}$ near the origin in the genus argument.

Let us briefly emphasize the mathematical content of Theorem 1.1. The restriction (1.5) first guarantees that the lower-order potential term involving $|u|^{p_M(x)}$ remains genuinely subcritical with respect to the anisotropic Sobolev threshold, so the energy functional is well defined on the whole space $W_0^{1,\vec{p}(x)}(\Omega)$. The inequality (1.12) then expresses that the negative bi-nonlocal contribution generated by F has lower effective growth near the origin than both the anisotropic principal energy and the positive critical term. This imbalance is exactly what makes the genus construction possible at small energy levels. At the same time, the positive critical term contributes a non-negative monotonicity defect, so it can be retained rather than estimated by a compact embedding; this restores the compactness needed in the minimax argument through the (S_+) property of the principal operator.

The paper is organized as follows. Section 2 collects the functional analytic preliminaries on variable-exponent Lebesgue spaces, anisotropic Sobolev spaces, the principal anisotropic operator, and the anisotropic concentration-compactness principle in the established interior critical form. Section 3 applies that decomposition to Palais–Smale sequences and proves, through the global monotonicity defect identity, that all concentration masses vanish and that the sequence converges strongly. Section 4 introduces the truncated functional, verifies the negative-level Palais–Smale condition, and applies Krasnosel’skii’s genus theory to complete the proof of Theorem 1.1. Finally, Section 5 records the concrete classes of operators, nonlinearities, and Kirchhoff laws covered by our abstract assumptions.

2. Functional framework and preliminaries

In this section, we collect the functional analytic tools used throughout the remainder of the paper. Although some of these ingredients are standard in the variable-exponent literature, we record them here in the precise form needed later, because the doubly nonlocal critical argument developed in Sections 3 and 4 requires a self-contained combination of regularity, concentration, and monotonicity inputs. Unless

stated otherwise, we work here under the structural assumptions (1.3), (1.4), (1.5), (1.6), (1.7), (1.2), $(\mathbf{f}_0)$, $(\mathbf{f}_1)$, $(\mathbf{f}_2)$, and $(\mathbf{A}_0)$ – $(\mathbf{A}_2)$, and we also assume that the Kirchhoff function $M : [0, \infty) \rightarrow (0, \infty)$ is continuous. The quantitative monotonicity assumption $(\mathbf{A}_3)$ enters Proposition 2.6 and the compactness arguments of Section 3, whereas $(\mathbf{A}_4)$ enters the finite-dimensional estimate in Section 4. We keep the notation

$$p_m(x) := \min_{1 \leq i \leq N} p_i(x), \quad p_M(x) := \max_{1 \leq i \leq N} p_i(x),$$

and

$$p_m^*(x) := \frac{N p_m(x)}{N - p_m(x)}.$$

The basic facts on variable-exponent Lebesgue and Sobolev spaces may be found in [1–5]. The $p_m(\cdot)$ -based Sobolev embedding used below is obtained from the anisotropic framework of [9]. We use the anisotropic concentration-compactness principle in $\mathcal{M}^+(\bar{\Omega})$ in the precise form established in [9, Theorem 6]. The boundary gap (1.7) ensures that every possible concentration point lies in the interior. The separate assumption (1.5) places the lower-order exponent $p_M(\cdot)$ strictly below the $p_m(\cdot)$ -based critical threshold and guarantees the compact lower-order convergence required by the variational argument.

For convenience, we write

$$C_+(\bar{\Omega}) := \{h \in C(\bar{\Omega}) : h^- := \min_{x \in \Omega} h(x) > 1\},$$

and

$$C_+^{0,\log}(\bar{\Omega}) := \left\{ h \in C_+(\bar{\Omega}) : \exists C_h > 0 \text{ such that } |h(x) - h(y)| \leq \frac{C_h}{\log(e + |x - y|^{-1})} \quad \forall x \neq y \right\}.$$

Variable-exponent Lebesgue spaces

Let $p \in C_+(\bar{\Omega})$. We define

$$L^{p(x)}(\Omega) := \left\{ u : \Omega \rightarrow \mathbb{R} \text{ measurable} : \rho_{p(x)}(u) := \int_{\Omega} |u(x)|^{p(x)} dx < \infty \right\},$$

endowed with the Luxemburg norm

$$\|u\|_{L^{p(x)}(\Omega)} := \inf \left\{ \lambda > 0 : \int_{\Omega} \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}.$$

It is well known that $L^{p(x)}(\Omega)$ is a separable and reflexive Banach space; see [1, 2, 4, 5].

Proposition 2.1. *Let $p \in C_+(\bar{\Omega})$ and $u \in L^{p(x)}(\Omega)$. Then the following conclusions hold:*

(i) $\|u\|_{L^{p(x)}(\Omega)} = \lambda$ if and only if

$$\rho_{p(x)}\left(\frac{u}{\lambda}\right) = 1;$$

(ii) $\|u\|_{L^{p(x)}(\Omega)} < 1$ (resp. $= 1$, resp. > 1) if and only if

$$\rho_{p(x)}(u) < 1 \quad (\text{resp. } = 1, \text{ resp. } > 1);$$

(iii) if $\|u\|_{L^{p(x)}(\Omega)} > 1$, then

$$\|u\|_{L^{p(x)}(\Omega)}^{p^-} \leq \rho_{p(x)}(u) \leq \|u\|_{L^{p(x)}(\Omega)}^{p^+};$$

(iv) if $\|u\|_{L^{p(x)}(\Omega)} < 1$, then

$$\|u\|_{L^{p(x)}(\Omega)}^{p^+} \leq \rho_{p(x)}(u) \leq \|u\|_{L^{p(x)}(\Omega)}^{p^-}.$$

Proof. This is standard in the theory of variable Lebesgue spaces; see, for instance, [5, Section 2] or [4, Section 2]. $\square$

Proposition 2.2. Let $q, h \in C_+(\overline{\Omega})$ with $q(x) \leq h(x)$ for all $x \in \overline{\Omega}$. Then the embedding

$$L^{h(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$$

is continuous. Moreover, if $p \in C_+(\overline{\Omega})$ and $p'(x)$ is defined by

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1,$$

then, for every $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have

$$\left| \int_{\Omega} u(x)v(x) dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{(p')^-} \right) \|u\|_{L^{p(x)}(\Omega)} \|v\|_{L^{p'(x)}(\Omega)} \leq 2 \|u\|_{L^{p(x)}(\Omega)} \|v\|_{L^{p'(x)}(\Omega)}.$$

Proof. See [1, 2, 4, 5]. $\square$

Anisotropic Sobolev spaces

Let $\vec{p}(x) = (p_1(x), \dots, p_N(x))$. We define

$$W^{1, \vec{p}(x)}(\Omega) := \left\{ u \in L^{p_m(x)}(\Omega) : \partial_{x_i} u \in L^{p_i(x)}(\Omega), i = 1, \dots, N \right\},$$

endowed with the norm

$$\|u\|_{W^{1, \vec{p}(x)}(\Omega)} := \|u\|_{L^{p_m(x)}(\Omega)} + \sum_{i=1}^N \|\partial_{x_i} u\|_{L^{p_i(x)}(\Omega)}.$$

We use $W_0^{1, \vec{p}(x)}(\Omega)$ to denote the closure of $C_c^\infty(\Omega)$ in $W^{1, \vec{p}(x)}(\Omega)$. Under (1.3), the spaces $W^{1, \vec{p}(x)}(\Omega)$ and $W_0^{1, \vec{p}(x)}(\Omega)$ are separable and reflexive.

Proposition 2.3 (Anisotropic Poincaré inequality). *There is a constant $C_P > 0$ such that, for every $u \in W_0^{1, \vec{p}(x)}(\Omega)$, we have*

$$\|u\|_{L^{p_m(x)}(\Omega)} \leq C_P \sum_{i=1}^N \|\partial_{x_i} u\|_{L^{p_i(x)}(\Omega)}.$$

Consequently

$$\|u\|_X := \sum_{i=1}^N \|\partial_{x_i} u\|_{L^{p_i(x)}(\Omega)}$$

defines a norm on $X := W_0^{1, \vec{p}(x)}(\Omega)$ that is equivalent to the usual Sobolev norm.

Proof. This is standard in anisotropic variable-exponent Sobolev spaces; see, for example, [6]. The equivalence of norms follows immediately. $\square$

Proposition 2.4. *The embedding*

$$W_0^{1,\vec{p}(x)}(\Omega) \hookrightarrow W_0^{1,p_m(x)}(\Omega)$$

is continuous.

Proof. Since $p_m(x) \leq p_i(x)$ for all i , Proposition 2.2 gives

$$L^{p_i(x)}(\Omega) \hookrightarrow L^{p_m(x)}(\Omega) \quad \text{for each } i = 1, \dots, N.$$

Hence, for every $u \in W_0^{1,\vec{p}(x)}(\Omega)$, we have

$$\|\partial_{x_i} u\|_{L^{p_m(x)}(\Omega)} \leq C_i \|\partial_{x_i} u\|_{L^{p_i(x)}(\Omega)}, \quad i = 1, \dots, N.$$

Combining these estimates with Proposition 2.3 yields the claim. $\square$

Proposition 2.5 ($p_m(\cdot)$ -based Sobolev embedding). *Assume (1.4). Let $h \in C_+(\overline{\Omega})$ satisfy*

$$h(x) \leq p_m^*(x) \quad \text{for all } x \in \overline{\Omega}.$$

Then the embedding

$$W_0^{1,\vec{p}(x)}(\Omega) \hookrightarrow L^{h(x)}(\Omega)$$

is continuous. If, in addition, we have

$$h(x) < p_m^*(x) \quad \text{for all } x \in \overline{\Omega},$$

then the embedding is compact. In particular, for every such exponent h , we have

$$S_h := \inf_{v \in W_0^{1,\vec{p}(x)}(\Omega) \setminus \{0\}} \frac{\|v\|_X}{\|v\|_{L^{h(x)}(\Omega)}} > 0.$$

Proof. By Proposition 2.4, one has $X \hookrightarrow W_0^{1,p_m(x)}(\Omega)$. The continuous and compact embeddings then follow from the variable-exponent Sobolev theorem under (1.4); see also [9, Theorem 3]. $\square$

Corollary 2.1. *Assume (1.5). Then*

$$W_0^{1,\vec{p}(x)}(\Omega) \hookrightarrow L^{p_M(x)}(\Omega).$$

In particular, a constant $C_{p_M} > 0$ exists such that

$$\|u\|_{L^{p_M(x)}(\Omega)} \leq C_{p_M} \|u\|_X \quad \text{for all } u \in X.$$

Proof. Since $p_M \in C(\overline{\Omega})$ and (1.5) holds pointwise, Proposition 2.5 applies with $h = p_M$ and yields the stated compact, and hence continuous, embedding. $\square$

Lemma 2.1 (Brezis–Lieb type lemma). *Let $q \in C_+(\overline{\Omega})$ and let (v_n) be a bounded sequence in $L^{q(x)}(\Omega)$ such that $v_n(x) \rightarrow v(x)$ almost everywhere in Ω . Then*

$$\int_{\Omega} |v_n|^{q(x)} dx - \int_{\Omega} |v_n - v|^{q(x)} dx \longrightarrow \int_{\Omega} |v|^{q(x)} dx.$$

Proof. See [25]. $\square$

The principal anisotropic operator

For $u, v \in X$, we define

$$\langle A(u), v \rangle := \int_{\Omega} \left(\sum_{i=1}^N a_i(x, \partial_{x_i} u) \partial_{x_i} v + w(x) |u|^{p_M(x)-2} uv \right) dx.$$

Proposition 2.6. Assume (1.3), (1.4), (1.5), (1.2), and $(\mathbf{A}_0)$ – $(\mathbf{A}_3)$. Then

$$A : X \rightarrow X^*$$

is bounded, demicontinuous, coercive, strictly monotone, and of type (S_+) . In particular, if $u_n \rightharpoonup u$ weakly in X and

$$\limsup_{n \rightarrow \infty} \langle A(u_n) - A(u), u_n - u \rangle \leq 0,$$

then $u_n \rightarrow u$ strongly in X .

Proof. We divide the proof into five steps.

Step 1: Well-definedness and boundedness. Let $u \in X$. Since Ω is bounded and $f_i \in L^\infty(\Omega)$, we have $f_i \in L^{p'_i(x)}(\Omega)$ for every i . Moreover

$$\rho_{p'_i(x)}(|\partial_{x_i} u|^{p_i(x)-1}) = \int_{\Omega} |\partial_{x_i} u|^{(p_i(x)-1)p'_i(x)} dx = \int_{\Omega} |\partial_{x_i} u|^{p_i(x)} dx < \infty,$$

so $|\partial_{x_i} u|^{p_i(x)-1} \in L^{p'_i(x)}(\Omega)$. By Proposition 2.1(iii)–(iv), $C_i > 0$ exists such that

$$\| |\partial_{x_i} u|^{p_i(x)-1} \|_{L^{p'_i(x)}(\Omega)} \leq C_i \max \{ \| \partial_{x_i} u \|_{L^{p_i(x)}(\Omega)}^{p_i^--1}, \| \partial_{x_i} u \|_{L^{p_i(x)}(\Omega)}^{p_i^+-1} \}.$$

Hence, by $(\mathbf{A}_1)$, we have

$$\| a_i(x, \partial_{x_i} u) \|_{L^{p'_i(x)}(\Omega)} \leq C \left(1 + \max \{ \| \partial_{x_i} u \|_{L^{p_i(x)}(\Omega)}^{p_i^--1}, \| \partial_{x_i} u \|_{L^{p_i(x)}(\Omega)}^{p_i^+-1} \} \right).$$

Similarly, Corollary 2.1 implies $u \in L^{p_M(x)}(\Omega)$ and

$$\| |u|^{p_M(x)-2} u \|_{L^{p'_M(x)}(\Omega)} \leq C \max \{ \| u \|_{L^{p_M(x)}(\Omega)}^{p_M^--1}, \| u \|_{L^{p_M(x)}(\Omega)}^{p_M^+-1} \}.$$

Therefore, by variable-exponent Hölder's inequality and Corollary 2.1, for every $u, v \in X$

$$|\langle A(u), v \rangle| \leq C \left(1 + \sum_{i=1}^N \max \{ \| \partial_{x_i} u \|_{L^{p_i(x)}(\Omega)}^{p_i^--1}, \| \partial_{x_i} u \|_{L^{p_i(x)}(\Omega)}^{p_i^+-1} \} + \max \{ \| u \|_{L^{p_M(x)}(\Omega)}^{p_M^--1}, \| u \|_{L^{p_M(x)}(\Omega)}^{p_M^+-1} \} \right) \| v \|_X.$$

Thus $A : X \rightarrow X^*$ is well defined and bounded on bounded subsets of X .

Step 2: Demicontinuity. Let $u_n \rightarrow u$ strongly in X . Then, for every $i \in \{1, \dots, N\}$, we have

$$\partial_{x_i} u_n \rightarrow \partial_{x_i} u \quad \text{strongly in } L^{p_i(x)}(\Omega),$$

and hence also in measure and, up to a subsequence, almost everywhere in Ω . By $(\mathbf{A}_1)$,

$$|a_i(x, \partial_{x_i} u_n)| \leq c_{a_i}(f_i(x) + |\partial_{x_i} u_n|^{p_i(x)-1}) \quad \text{for a.e. } x \in \Omega,$$

and therefore the sequence $(a_i(x, \partial_{x_i} u_n))_n$ is bounded in $L^{p'_i(x)}(\Omega)$. Since a_i is Carathéodory by (A_0) and $\partial_{x_i} u_n \rightarrow \partial_{x_i} u$ a.e. in Ω , we also have

$$a_i(x, \partial_{x_i} u_n) \rightarrow a_i(x, \partial_{x_i} u) \quad \text{for a.e. } x \in \Omega.$$

Since $L^{p'_i(x)}(\Omega)$ is reflexive and the sequence is bounded there, every subsequence admits a further subsequence converging weakly in $L^{p'_i(x)}(\Omega)$ to some g_i . Further along that subsequence, we still have

$$a_i(x, \partial_{x_i} u_n) \rightarrow a_i(x, \partial_{x_i} u) \quad \text{for a.e. } x \in \Omega,$$

so necessarily $g_i = a_i(x, \partial_{x_i} u)$ a.e. in Ω . By uniqueness of weak limits, the whole sequence satisfies

$$a_i(x, \partial_{x_i} u_n) \rightharpoonup a_i(x, \partial_{x_i} u) \quad \text{weakly in } L^{p'_i(x)}(\Omega).$$

The same argument applies to the zero-order term: Since $u_n \rightarrow u$ strongly in $L^{p_M(x)}(\Omega)$ by Corollary 2.1, we have

$$|u_n|^{p_M(x)-2} u_n \rightharpoonup |u|^{p_M(x)-2} u \quad \text{weakly in } L^{p'_M(x)}(\Omega).$$

Consequently, for every $v \in X$, we have

$$\langle A(u_n), v \rangle \rightarrow \langle A(u), v \rangle,$$

which proves that A is demicontinuous.

Step 3: Coercivity. By (A_2) and (1.2), if $k_0 := \min_{1 \leq i \leq N} k_i > 0$, then

$$\langle A(u), u \rangle \geq k_0 \sum_{i=1}^N \int_{\Omega} |\partial_{x_i} u|^{p_i(x)} dx + w_0 \int_{\Omega} |u|^{p_M(x)} dx.$$

Using Corollary 2.1, Proposition 2.3, and Proposition 2.1(iii)–(iv), we infer that

$$\frac{\langle A(u), u \rangle}{\|u\|_X} \rightarrow +\infty \quad \text{as } \|u\|_X \rightarrow \infty.$$

Hence, A is coercive.

Step 4: Strict monotonicity. Let $u, v \in X$. By (A_3) and the monotonicity of $t \mapsto |t|^{p_M(x)-2}t$, we have

$$\begin{aligned} \langle A(u) - A(v), u - v \rangle &\geq \sum_{i=1}^N c_i \int_{\Omega} \mathcal{D}_{p_i(x)}(\partial_{x_i} u, \partial_{x_i} v) dx \\ &\quad + \int_{\Omega} w(x)(|u|^{p_M(x)-2}u - |v|^{p_M(x)-2}v)(u - v) dx \geq 0. \end{aligned}$$

If equality holds, then

$$\mathcal{D}_{p_i(x)}(\partial_{x_i} u, \partial_{x_i} v) = 0 \quad \text{a.e. in } \Omega,$$

for every i , and therefore $\partial_{x_i} u = \partial_{x_i} v$ a.e. in Ω . Since $u - v \in W_0^{1, \vec{p}(x)}(\Omega)$, the anisotropic Poincaré inequality yields $u = v$. Hence, A is strictly monotone.

Step 5: The (S_+) property. Assume that $u_n \rightharpoonup u$ weakly in X and

$$\limsup_{n \rightarrow \infty} \langle A(u_n) - A(u), u_n - u \rangle \leq 0.$$

By monotonicity

$$\langle A(u_n) - A(u), u_n - u \rangle \longrightarrow 0.$$

Every term in the expanded duality pairing is non-negative. Therefore, for every $i \in \{1, \dots, N\}$, we have

$$\int_{\Omega} \mathcal{D}_{p_i(x)}(\partial_{x_i} u_n, \partial_{x_i} u) dx \longrightarrow 0. \quad (2.1)$$

Set

$$\Omega_i^+ := \{x \in \Omega : p_i(x) \geq 2\}, \quad \Omega_i^- := \{x \in \Omega : 1 < p_i(x) < 2\}.$$

On Ω_i^+ , (2.1) directly gives

$$\int_{\Omega_i^+} |\partial_{x_i} u_n - \partial_{x_i} u|^{p_i(x)} dx \longrightarrow 0. \quad (2.2)$$

It remains to treat Ω_i^- . If $\Omega_i^- = \emptyset$, there is nothing to prove. Assume, therefore, that $\Omega_i^- \neq \emptyset$; then $p_i^- < 2$. Write

$$d_{i,n} := \partial_{x_i} u_n - \partial_{x_i} u, \quad s_{i,n} := |\partial_{x_i} u_n| + |\partial_{x_i} u|,$$

and

$$D_{i,n}(x) := \frac{|d_{i,n}(x)|^2}{s_{i,n}(x)^{2-p_i(x)}} \quad \text{on } \Omega_i^-,$$

with $D_{i,n}(x) = 0$ whenever $s_{i,n}(x) = 0$, and set

$$\alpha_i := \frac{p_i^-}{2 - p_i^-} > 0.$$

Since the function $s \mapsto s/(2 - s)$ is increasing on $(1, 2)$, one has

$$\frac{p_i(x)}{2 - p_i(x)} \geq \frac{p_i^-}{2 - p_i^-} = \alpha_i \quad \text{on } \Omega_i^-.$$

Consequently, we have

$$t^{-p_i(x)/(2-p_i(x))} \leq t^{-\alpha_i} \quad \text{for every } t > 1.$$

For every $t > 1$, the weighted Young inequality with the conjugate exponents $2/p_i(x)$ and $2/(2 - p_i(x))$ gives the following, pointwise on Ω_i^- :

$$\begin{aligned} |d_{i,n}|^{p_i(x)} &= (tD_{i,n})^{p_i(x)/2} (t^{-p_i(x)/(2-p_i(x))} s_{i,n}^{p_i(x)})^{(2-p_i(x))/2} \\ &\leq \frac{p_i(x)}{2} tD_{i,n} + \frac{2 - p_i(x)}{2} t^{-p_i(x)/(2-p_i(x))} s_{i,n}^{p_i(x)} \\ &\leq tD_{i,n} + t^{-\alpha_i} s_{i,n}^{p_i(x)}. \end{aligned}$$

Since $u_n \rightharpoonup u$ in X , the sequence (u_n) is bounded in X . Moreover

$$s_{i,n}^{p_i(x)} \leq 2^{p_i^+-1} (|\partial_{x_i} u_n|^{p_i(x)} + |\partial_{x_i} u|^{p_i(x)}),$$

and therefore

$$\sup_n \int_{\Omega_i^-} s_{i,n}^{p_i(x)} dx < \infty.$$

Integrating the preceding inequality and using (2.1), we obtain the following for every fixed $t > 1$:

$$\limsup_{n \rightarrow \infty} \int_{\Omega_i^-} |d_{i,n}|^{p_i(x)} dx \leq C_i t^{-\alpha_i}.$$

Letting $t \rightarrow \infty$ yields

$$\int_{\Omega_i^-} |\partial_{x_i} u_n - \partial_{x_i} u|^{p_i(x)} dx \rightarrow 0. \quad (2.3)$$

Combining (2.2) and (2.3), we conclude that

$$\int_{\Omega} |\partial_{x_i} u_n - \partial_{x_i} u|^{p_i(x)} dx \rightarrow 0, \quad i = 1, \dots, N.$$

By Proposition 2.1(ii), (iv), we have

$$\|\partial_{x_i} u_n - \partial_{x_i} u\|_{L^{p_i(x)}(\Omega)} \rightarrow 0, \quad i = 1, \dots, N.$$

Therefore, $u_n \rightarrow u$ strongly in X . This proves that A is of type (S_+) . $\square$

Remark 2.1. The mixed monotonicity density $\mathcal{D}_p$ introduced in $(\mathbf{A}_3)$ is precisely what permits the compactness argument to cover the entire range $1 < p_i^- \leq p_i(x) \leq p_i^+ < N$. In the subquadratic region, the monotonicity defect controls a weighted quadratic quantity, and the preceding Young's inequality argument converts this control into convergence in the variable-exponent modular. For the model

$$a_i(x, \xi) = b_i(x) |\xi|^{p_i(x)-2} \xi, \quad b_i \in L^\infty(\Omega), \quad b_i(x) \geq \underline{b}_i > 0,$$

Assumption $(\mathbf{A}_3)$ follows from the classical mixed monotonicity inequality stated above throughout the full range $1 < p_i^- \leq p_i(x) \leq p_i^+ < N$.

Proposition 2.7 (Differentiability of the energy functionals). *Assume (1.2), (1.3), (1.4), (1.5), (1.6), $(\mathbf{f}_0)$, $(\mathbf{f}_1)$, and $(\mathbf{A}_0)$ – $(\mathbf{A}_1)$, and let $M : [0, \infty) \rightarrow (0, \infty)$ be continuous. Define*

$$\rho(u) := \int_{\Omega} \left(\sum_{i=1}^N A_i(x, \partial_{x_i} u) + \frac{w(x)}{p_M(x)} |u|^{p_M(x)} \right) dx, \quad G(u) := \int_{\Omega} F(x, u) dx.$$

Then $\rho, G \in C^1(X, \mathbb{R})$, $G(u) \geq 0$ for every $u \in X$, and

$$\begin{aligned} \langle \rho'(u), v \rangle &= \int_{\Omega} \left(\sum_{i=1}^N a_i(x, \partial_{x_i} u) \partial_{x_i} v + w(x) |u|^{p_M(x)-2} uv \right) dx, \\ \langle G'(u), v \rangle &= \int_{\Omega} f(x, u) v dx, \end{aligned}$$

for every $u, v \in X$. Consequently, we have

$$J_\lambda(u) = \widehat{M}(\rho(u)) - \frac{\lambda}{r+1} G(u)^{r+1} + \gamma \int_{\Omega} \frac{|u|^{q(x)}}{q(x)} dx$$

belongs to $C^1(X, \mathbb{R})$, and

$$\langle J'_\lambda(u), v \rangle = M(\rho(u)) \langle A(u), v \rangle - \lambda G(u)^r \int_{\Omega} f(x, u) v \, dx + \gamma \int_{\Omega} |u|^{q(x)-2} uv \, dx$$

for every $u, v \in X$. In particular, $u \in X$ is a critical point of J_λ if and only if u is a weak solution of (1.1).

Proof. Fix $i \in \{1, \dots, N\}$. By (A_0) – (A_1) and $A_i(x, 0) = 0$, the fundamental theorem of calculus gives the following for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}$:

$$\begin{aligned} |A_i(x, \xi)| &= \left| \int_0^\xi a_i(x, s) \, ds \right| \\ &\leq c_{a_i} \left(f_i(x) |\xi| + \frac{|\xi|^{p_i(x)}}{p_i(x)} \right) \\ &\leq c_{a_i} \left(f_i(x) |\xi| + \frac{|\xi|^{p_i(x)}}{p_i^-} \right). \end{aligned} \quad (2.4)$$

Since Ω is bounded and $f_i \in L^\infty(\Omega)$, one has $f_i \in L^{p'_i(x)}(\Omega)$. Hence, the variable-exponent Hölder inequality shows that

$$\mathcal{A}_i(\zeta) := \int_{\Omega} A_i(x, \zeta(x)) \, dx$$

is finite for every $\zeta \in L^{p_i(x)}(\Omega)$.

The Carathéodory property and the growth estimate in (A_1) imply, by the standard continuity theorem for Nemytskii operators in variable-exponent spaces; see, for example, [2, 4], that

$$\mathcal{N}_i : L^{p_i(x)}(\Omega) \longrightarrow L^{p'_i(x)}(\Omega), \quad \mathcal{N}_i(\zeta) := a_i(\cdot, \zeta(\cdot)),$$

is well defined and continuous. For $\zeta, h \in L^{p_i(x)}(\Omega)$, one has

$$\begin{aligned} \mathcal{A}_i(\zeta + h) - \mathcal{A}_i(\zeta) &= \int_{\Omega} a_i(x, \zeta) h \, dx \\ &= \int_0^1 \int_{\Omega} [a_i(x, \zeta + \theta h) - a_i(x, \zeta)] h \, dx \, d\theta. \end{aligned}$$

Therefore, by the variable-exponent Hölder inequality, we have

$$\begin{aligned} &\left| \mathcal{A}_i(\zeta + h) - \mathcal{A}_i(\zeta) - \int_{\Omega} a_i(x, \zeta) h \, dx \right| \\ &\leq 2 \|h\|_{L^{p_i(x)}(\Omega)} \int_0^1 \|\mathcal{N}_i(\zeta + \theta h) - \mathcal{N}_i(\zeta)\|_{L^{p'_i(x)}(\Omega)} \, d\theta. \end{aligned}$$

Moreover

$$\sup_{\theta \in [0,1]} \|\zeta + \theta h - \zeta\|_{L^{p_i(x)}(\Omega)} \leq \|h\|_{L^{p_i(x)}(\Omega)}.$$

Thus the continuity of $\mathcal{N}_i$ at ζ yields

$$\sup_{\theta \in [0,1]} \|\mathcal{N}_i(\zeta + \theta h) - \mathcal{N}_i(\zeta)\|_{L^{p'_i(x)}(\Omega)} \longrightarrow 0 \quad \text{as } h \rightarrow 0.$$

Consequently

$$\mathcal{A}_i \in C^1(L^{p_i(x)}(\Omega), \mathbb{R}), \quad \langle \mathcal{A}'_i(\zeta), h \rangle = \int_{\Omega} a_i(x, \zeta) h \, dx.$$

Composing with the continuous linear map

$$X \longrightarrow L^{p_i(x)}(\Omega), \quad u \longmapsto \partial_{x_i} u,$$

we conclude that

$$\nu_i(u) := \int_{\Omega} A_i(x, \partial_{x_i} u) \, dx$$

belongs to $C^1(X, \mathbb{R})$ and

$$\langle \nu'_i(u), v \rangle = \int_{\Omega} a_i(x, \partial_{x_i} u) \partial_{x_i} v \, dx.$$

For the zero-order term, Corollary 2.1 gives $X \hookrightarrow L^{p_M(x)}(\Omega)$. Since $w \in L^{\infty}(\Omega)$, the standard variable-exponent modular formula yields

$$\nu_0(u) := \int_{\Omega} \frac{w(x)}{p_M(x)} |u|^{p_M(x)} \, dx \in C^1(X, \mathbb{R}),$$

with

$$\langle \nu'_0(u), v \rangle = \int_{\Omega} w(x) |u|^{p_M(x)-2} uv \, dx.$$

Summing the directional and zero-order contributions proves that $\rho \in C^1(X, \mathbb{R})$ and $\rho'(u) = A(u)$ in X^* .

We next treat G . For $t \neq 0$, Condition $(\mathbf{f}_1)$ gives

$$|f(x, t)| \leq A_2 |t|^{\beta(x)-1} \quad \text{for a.e. } x \in \Omega.$$

By $(\mathbf{f}_0)$, the function $f(x, \cdot)$ is continuous for a.e. x , and letting $t \rightarrow 0$ in the preceding estimate gives $f(x, 0) = 0$. Hence

$$|f(x, t)| \leq A_2 |t|^{\beta(x)-1} \quad \text{for a.e. } x \in \Omega, \quad \forall t \in \mathbb{R}. \quad (2.5)$$

Moreover, $(\mathbf{f}_1)$ implies $f(x, t)t \geq 0$ and therefore

$$F(x, t) \geq \frac{A_1}{\beta^+} |t|^{\beta(x)} \geq 0, \quad G(u) \geq 0.$$

By (1.3), (1.4), and (1.6), one has $\beta(x) < p_m^*(x)$ on $\overline{\Omega}$ and thus

$$X \hookrightarrow L^{\beta(x)}(\Omega).$$

The growth estimate (2.5) and the variable-exponent Nemytskii theorem imply that

$$\mathcal{N}_f : L^{\beta(x)}(\Omega) \longrightarrow L^{\beta'(x)}(\Omega), \quad \mathcal{N}_f(u) := f(\cdot, u(\cdot)),$$

is continuous. Repeating the preceding integral remainder argument, with p_i replaced by β , gives

$$G \in C^1(X, \mathbb{R}), \quad \langle G'(u), v \rangle = \int_{\Omega} f(x, u) v \, dx.$$

The functional

$$Q(u) := \int_{\Omega} \frac{|u|^{q(x)}}{q(x)} dx$$

is of class C^1 on $L^{q(x)}(\Omega)$, with

$$\langle Q'(u), v \rangle = \int_{\Omega} |u|^{q(x)-2} uv dx.$$

The continuous embedding $X \hookrightarrow L^{q(x)}(\Omega)$ supplied by (1.3), (1.4), and (1.6) therefore yields $Q \in C^1(X, \mathbb{R})$.

To justify the remaining compositions on open domains, define

$$\Phi(s) := |s|^{r+1}, \quad s \in \mathbb{R}.$$

Since $r > 0$, one has $\Phi \in C^1(\mathbb{R})$ and, because $G(u) \geq 0$, we also have

$$\frac{1}{r+1}(\Phi \circ G)'(u) = G(u)^r G'(u).$$

Also extend M continuously to $\mathbb{R}$ by

$$\widetilde{M}(t) := \begin{cases} M(t), & t \geq 0, \\ M(0), & t < 0, \end{cases}$$

and define

$$\mathcal{H}_M(t) := \int_0^t \widetilde{M}(s) ds.$$

Then $\mathcal{H}_M \in C^1(\mathbb{R})$ and agrees with $\widehat{M}$ on $[0, \infty)$. Since $\rho(u) \geq 0$, the chain rule gives

$$(\widehat{M} \circ \rho)'(u) = M(\rho(u))\rho'(u).$$

Combining the derivatives of $\widehat{M} \circ \rho$, $G^{r+1}/(r+1)$, and Q yields the displayed formula for J'_λ . The equivalence between $J'_\lambda(u) = 0$ and the weak formulation in Definition 1.1 follows immediately. $\square$

Anisotropic concentration-compactness

We write $\mathcal{M}^+(\overline{\Omega})$ for the cone of finite non-negative Radon measures on $\overline{\Omega}$. Weak-* convergence in this space is understood against test functions in $C(\overline{\Omega})$. The following statement is the anisotropic concentration-compactness theorem of [9, Theorem 6], as written in the notation of the present paper. The strict boundary gap places the critical set inside Ω and brings the problem directly within the scope of that theorem.

Proposition 2.8 (Anisotropic concentration-compactness principle). *Assume (1.3), (1.4), and (1.5). Let $h \in C(\overline{\Omega})$ satisfy*

$$1 < h^- := \min_{x \in \overline{\Omega}} h(x) \leq h(x) \leq p_m^*(x) \quad \text{for all } x \in \overline{\Omega},$$

and

$$h(x) < p_m^*(x) \quad \text{for all } x \in \partial\Omega.$$

Define

$$\mathcal{A}_h := \{x \in \Omega : h(x) = p_m^*(x)\}.$$

Let $(u_n) \subset X = W_0^{1, \vec{p}(x)}(\Omega)$ satisfy

$$u_n \rightharpoonup u \quad \text{weakly in } X.$$

Then, after passing to a subsequence, measures $\mu_i, \nu \in \mathcal{M}^+(\bar{\Omega})$ exist such that

$$|\partial_{x_i} u_n|^{p_i(x)} dx \xrightarrow{*} \mu_i, \quad i = 1, \dots, N,$$

and

$$|u_n|^{h(x)} dx \xrightarrow{*} \nu \quad \text{in } \mathcal{M}^+(\bar{\Omega}).$$

Set $\mu := \sum_{i=1}^N \mu_i$. There is an at most countable index set J , together with pairwise distinct points

$$\{x_j\}_{j \in J} \subset \mathcal{A}_h,$$

and positive numbers $\{\mu_j\}_{j \in J}$ and $\{\nu_j\}_{j \in J}$ such that

$$\nu = |u|^{h(x)} dx + \sum_{j \in J} \nu_j \delta_{x_j} \quad \text{in } \mathcal{M}^+(\bar{\Omega}), \quad (2.6)$$

and

$$\mu \geq \sum_{i=1}^N |\partial_{x_i} u|^{p_i(x)} dx + \sum_{j \in J} \mu_j \delta_{x_j} \quad \text{in } \mathcal{M}^+(\bar{\Omega}). \quad (2.7)$$

Moreover, for every $j \in J$, we have

$$N^{1-p_M^+} S_h \nu_j^{1/p_m^+(x_j)} \leq \max \left\{ \mu_j^{1/p_M^+}, \mu_j^{1/p_m^-} \right\}. \quad (2.8)$$

If $\mathcal{A}_h = \emptyset$, then

$$u_n \rightarrow u \quad \text{strongly in } L^{h(x)}(\Omega).$$

Proof. The weak-* measure convergences follow from the boundedness of (u_n) in X , the continuous anisotropic embedding into $L^{h(x)}(\Omega)$, and the weak-* compactness in $\mathcal{M}^+(\bar{\Omega})$. Under (1.5), Corollary 2.1 gives $X \hookrightarrow L^{p_M(x)}(\Omega)$. On $C_c^\infty(\Omega)$, the $L^{p_M(x)}$ -based anisotropic norm controls the present $L^{p_m(x)}$ -based norm by Proposition 2.2; conversely, Proposition 2.3 and Corollary 2.1 control both zero-order terms by the directional norm. Hence, the corresponding zero-boundary completions coincide with equivalent norms, and they agree with the space used in [9, Theorem 6]. If $\mathcal{A}_h \neq \emptyset$, all the hypotheses of that theorem are therefore satisfied. That theorem is formulated with measure convergence in $\mathcal{M}(\bar{\Omega})$ and yields exactly (2.6)–(2.8), with the concentration points $x_j \in \mathcal{A}_h \subset \Omega$.

If $\mathcal{A}_h = \emptyset$, the continuous function $p_m^* - h$ is strictly positive on the compact set $\bar{\Omega}$: It is positive in Ω because the critical set is empty, and it is positive on $\partial\Omega$ by hypothesis. Hence, $\delta > 0$ exists such that

$$h(x) \leq p_m^*(x) - \delta \quad \text{for all } x \in \bar{\Omega}.$$

The compact anisotropic embedding then gives $u_n \rightarrow u$ strongly in $L^{h(x)}(\Omega)$. In this case, (2.6) holds with $J = \emptyset$, while (2.7) follows from the weak lower semicontinuity of the directional modulars. $\square$

Remark 2.2. The boundary subcriticality assumption places the critical set strictly inside the domain and permits the direct application of the established concentration-compactness theorem. The weak-* convergences are taken in $\mathcal{M}^+(\bar{\Omega})$, exactly as in [9, Theorem 6]. When $h = q$, every possible concentration point belongs to the interior critical set $\mathcal{A}$ defined in (1.8).

Krasnosel'skii genus preliminaries

We finally recall the genus tools used in the multiplicity argument. Let X be a real Banach space and let

$$\Sigma := \{A \subset X \setminus \{0\} : A \text{ is closed and symmetric}\}.$$

Definition 2.1 (Krasnosel'skii genus). Let $A \in \Sigma$. The Krasnosel'skii genus $\gamma(A)$ is defined as the least integer $k \geq 1$ such that there is an odd continuous map

$$\varphi : A \rightarrow \mathbb{R}^k \setminus \{0\}.$$

If no such integer exists, we set $\gamma(A) = \infty$, and by convention $\gamma(\emptyset) = 0$.

Proposition 2.9. *Let $A, B \in \Sigma$. Then we have the following.*

(i) *If there is an odd continuous map $A \rightarrow B$, then*

$$\gamma(A) \leq \gamma(B);$$

(ii)

$$\gamma(A \cup B) \leq \gamma(A) + \gamma(B);$$

(iii) *If A is odd-homeomorphic to the Euclidean sphere $\mathbb{S}^{k-1}$, then*

$$\gamma(A) = k.$$

Proof. See [24, 26]. □

Theorem 2.1. *Let E be a k -dimensional real normed space and let*

$$S := \{u \in E : \|u\| = 1\}.$$

Then

$$\gamma(S) = k.$$

Proof. This is a standard consequence of Proposition 2.9; see [24, 26]. □

Corollary 2.2. *If X is infinite-dimensional and $S := \{u \in X : \|u\| = 1\}$ is its unit sphere, then*

$$\gamma(S) = \infty.$$

Proof. Apply Theorem 2.1 to arbitrary finite-dimensional subspaces of X . □

Theorem 2.2 (Clark–Krasnosel'skii theorem on negative levels). *Let X be a real Banach space and let $\Phi \in C^1(X, \mathbb{R})$ be even, bounded from below, and such that $\Phi(0) = 0$. Assume the following.*

(i) *For every level $c < 0$, Φ satisfies the Palais–Smale condition at level c ;*

(ii) *For every $k \in \mathbb{N}$, a compact set $K_k \in \Sigma$ exists such that*

$$\gamma(K_k) \geq k \quad \text{and} \quad \sup_{u \in K_k} \Phi(u) < 0.$$

Then Φ possesses infinitely many distinct pairs of nontrivial critical points, and every corresponding critical value is negative.

Proof. This is the standard Clark–Krasnosel'skii minimax theorem localized to the negative range; see [23, 24, 26]. Equivalently, if one defines

$$\Gamma_k := \{A \subset X \setminus \{0\} : A \text{ is compact, symmetric, and } \gamma(A) \geq k\}, \quad c_k := \inf_{A \in \Gamma_k} \sup_{u \in A} \Phi(u),$$

then Assumption (ii) yields $c_k < 0$ for every k , Assumption (i) gives the required deformation lemma at each level c_k , and the usual genus argument shows that every c_k is a negative critical value. If

$$c_k = c_{k+1} = \cdots = c_{k+\ell} =: c < 0,$$

then the critical set

$$K_c := \{u \in X : \Phi'(u) = 0, \Phi(u) = c\}$$

has a genus of at least $\ell + 1$. Thus repeated minimax levels already contain the required number of distinct critical pairs, while strictly decreasing levels provide distinct pairs directly. $\square$

3. Compactness of Palais–Smale sequences

In this section, we establish the compactness properties of the energy functional

$$J_\lambda(u) = \widehat{M}(\rho(u)) - \frac{\lambda}{r+1} \left(\int_{\Omega} F(x, u) dx \right)^{r+1} + \gamma \int_{\Omega} \frac{|u|^{q(x)}}{q(x)} dx, \quad u \in X := W_0^{1, \vec{p}(x)}(\Omega),$$

where

$$\rho(u) := \int_{\Omega} \left(\sum_{i=1}^N A_i(x, \partial_{x_i} u) + \frac{w(x)}{p_M(x)} |u|^{p_M(x)} \right) dx.$$

This quantity is well defined on X by Corollary 2.1. The boundedness analysis follows from the structural assumptions (A_0) – (A_2) together with the growth restriction $\beta^+(r+1) < p_m^-$. The quantitative monotonicity hypothesis (A_3) enters the final compactness step, where the vanishing monotonicity defect is upgraded to strong convergence in X .

The proof below explicitly applies Proposition 2.8 with $h = q$ to every bounded Palais–Smale sequence, thereby recording the possible critical defect measures. Strong convergence is then obtained independently from the global derivative identity and the (S_+) property: The subcritical source converges strongly and the positive critical term yields a non-negative monotonicity defect. The resulting strong convergence identifies the recorded measures and forces the entire concentration defect to vanish. Throughout this section, we endow X with the equivalent norm

$$\|u\| := \sum_{i=1}^N \|\partial_{x_i} u\|_{L^{p_i(x)}(\Omega)},$$

whose equivalence with the standard Sobolev norm follows from Proposition 2.3.

For later use, we also introduce

$$\Theta(u) := \langle A(u), u \rangle = \int_{\Omega} \left(\sum_{i=1}^N a_i(x, \partial_{x_i} u) \partial_{x_i} u + w(x) |u|^{p_M(x)} \right) dx,$$

where $A : X \rightarrow X^*$ is the operator of Proposition 2.6.

Preliminary estimates

We first collect the estimates used in the Palais–Smale analysis.

By (A_2) , (1.2), Proposition 2.1(iii)–(iv), and Proposition 2.3, a constant $C_0 > 0$ exists such that

$$\Theta(u) \geq C_0 \min\{\|u\|^{p_m^-}, \|u\|^{p_M^+}\} \quad \text{for all } u \in X. \quad (3.1)$$

In particular, we have

$$\Theta(u) \geq C_0 \|u\|^{p_m^-} \quad \text{whenever } \|u\| \geq 1. \quad (3.2)$$

Next, using (A_2) once again, we have

$$\Theta(u) \leq p_M^+ \rho(u) \quad \text{for all } u \in X. \quad (3.3)$$

Moreover, Corollary 2.1 provides

$$\int_{\Omega} |u|^{p_M(x)} dx \leq C_{p_M}^{\sharp} \max\{\|u\|^{p_M^-}, \|u\|^{p_M^+}\} \quad \text{for all } u \in X \quad (3.4)$$

for some constant $C_{p_M}^{\sharp} > 0$.

Since $\beta(x) < q(x) \leq p_m^*(x)$ in $\overline{\Omega}$, Proposition 2.5 gives the continuous embedding $X \hookrightarrow L^{\beta(x)}(\Omega)$. Therefore, by (f_1) , a constant $C_1 > 0$ exists such that

$$\left| \int_{\Omega} F(x, u) dx \right| \leq C_1 \int_{\Omega} |u|^{\beta(x)} dx \leq C_1 \max\{\|u\|^{\beta^-}, \|u\|^{\beta^+}\} \quad \text{for all } u \in X. \quad (3.5)$$

Similarly, again by (f_1) and Proposition 2.5, a constant $C_2 > 0$ exists such that

$$\left| \int_{\Omega} f(x, u) u dx \right| \leq C_2 \int_{\Omega} |u|^{\beta(x)} dx \leq C_2 \max\{\|u\|^{\beta^-}, \|u\|^{\beta^+}\} \quad \text{for all } u \in X. \quad (3.6)$$

Combining (3.5) and (3.6), we obtain

$$\left| \left(\int_{\Omega} F(x, u) dx \right)^r \int_{\Omega} f(x, u) u dx \right| \leq C_3 (1 + \|u\|^{\beta^+(r+1)}) \quad \text{for all } u \in X, \quad (3.7)$$

for some constant $C_3 > 0$.

Lemma 3.1. Assume (1.3), (1.4), (1.5), (1.6), (1.7), (1.2), (f_0) , (f_1) , (f_2) , and (A_0) – (A_2) , and let $M : [0, \infty) \rightarrow (0, \infty)$ be continuous. Let $(u_j) \subset X$ be a Palais–Smale sequence for J_{λ} , namely

$$J_{\lambda}(u_j) \rightarrow c \quad \text{and} \quad J'_{\lambda}(u_j) \rightarrow 0 \quad \text{in } X^*.$$

Assume moreover that one of the following two alternatives holds.

(i)

$$M(t) = a + bt^{\eta}, \quad a > 0, b \geq 0, \eta \geq 1,$$

and

$$\beta^+(r+1) < p_m^-;$$

(ii) The constants $m_0, m_1 > 0$ exist such that

$$m_0 \leq M(t) \leq m_1 \quad \text{for all } t \geq 0,$$

and again

$$\beta^+(r+1) < p_m^-.$$

Then (u_j) is bounded in X .

Proof. Since $J'_\lambda(u_j) \rightarrow 0$ in X^* , testing with u_j yields

$$\langle J'_\lambda(u_j), u_j \rangle = o_j(1) \|u_j\|. \quad (3.8)$$

Fix $\theta > q^+$. Then

$$J_\lambda(u_j) - \frac{1}{\theta} \langle J'_\lambda(u_j), u_j \rangle = c + o_j(1) + o_j(1) \|u_j\|. \quad (3.9)$$

Write

$$\rho_j := \rho(u_j), \quad \Theta_j := \Theta(u_j).$$

Case (i). Assume $M(t) = a + bt^\eta$, with $a > 0$, $b \geq 0$, and $\eta \geq 1$. Since $\Theta_j \leq p_M^+ \rho_j$ by (3.3), we get

$$\begin{aligned} \widehat{M}(\rho_j) - \frac{1}{\theta} M(\rho_j) \Theta_j &= a \rho_j + \frac{b}{\eta+1} \rho_j^{\eta+1} - \frac{a + b \rho_j^\eta}{\theta} \Theta_j \\ &\geq a \left(1 - \frac{p_M^+}{\theta}\right) \rho_j + b \left(\frac{1}{\eta+1} - \frac{p_M^+}{\theta}\right) \rho_j^{\eta+1}. \end{aligned}$$

Choose

$$\theta > \max\{q^+, (\eta+1)p_M^+\}.$$

Then $c_1 > 0$ exists such that

$$\widehat{M}(\rho_j) - \frac{1}{\theta} M(\rho_j) \Theta_j \geq c_1 \rho_j \quad \text{for all } j. \quad (3.10)$$

On the other hand, since

$$\gamma \int_{\Omega} \left(\frac{1}{q(x)} - \frac{1}{\theta} \right) |u_j|^{q(x)} dx \geq 0,$$

Identity (3.9) and Estimate (3.7) imply

$$c + o_j(1) + o_j(1) \|u_j\| \geq c_1 \rho_j - C_4 (1 + \|u_j\|^{\beta^+(r+1)})$$

for some $C_4 > 0$. Now (3.3) and (3.2) give the following whenever $\|u_j\| \geq 1$:

$$\rho_j \geq \frac{1}{p_M^+} \Theta_j \geq \frac{C_0}{p_M^+} \|u_j\|^{p_m^-}.$$

Hence

$$c + o_j(1) + o_j(1) \|u_j\| \geq C_5 \|u_j\|^{p_m^-} - C_4 (1 + \|u_j\|^{\beta^+(r+1)}). \quad (3.11)$$

Suppose, by contradiction, that $\|u_j\| \rightarrow \infty$ along a subsequence. Dividing (3.11) by $\|u_j\|^{p_m^-}$ gives

$$\frac{c + o_j(1)}{\|u_j\|^{p_m^-}} + o_j(1)\|u_j\|^{1-p_m^-} \geq C_5 - C_4 \left(\|u_j\|^{-p_m^-} + \|u_j\|^{\beta^+(r+1)-p_m^-} \right).$$

Since $p_m^- > 1$ and $\beta^+(r+1) < p_m^-$, the left-hand side tends to 0, whereas the right-hand side tends to $C_5 > 0$. This is a contradiction. Therefore, (u_j) is bounded in X .

Case (ii). Assume now that $m_0 \leq M(t) \leq m_1$ for all $t \geq 0$. Then

$$\widehat{M}(\rho_j) \geq m_0 \rho_j, \quad M(\rho_j)\Theta_j \leq m_1 \Theta_j \leq m_1 p_M^+ \rho_j$$

by (3.3). Therefore, we have

$$\widehat{M}(\rho_j) - \frac{1}{\theta} M(\rho_j)\Theta_j \geq \left(m_0 - \frac{m_1 p_M^+}{\theta} \right) \rho_j.$$

Choosing $\theta > \max\{q^+, m_1 p_M^+ / m_0\}$, we obtain

$$\widehat{M}(\rho_j) - \frac{1}{\theta} M(\rho_j)\Theta_j \geq c_2 \rho_j$$

for some $c_2 > 0$. Combining the preceding estimate with (3.9), (3.7), (3.3), and (3.2), we find the constants $C_6, C_7 > 0$ such that, whenever $\|u_j\| \geq 1$, we have

$$c + o_j(1) + o_j(1)\|u_j\| \geq C_6 \|u_j\|^{p_m^-} - C_7 (1 + \|u_j\|^{\beta^+(r+1)}).$$

If $\|u_j\| \rightarrow \infty$ along a subsequence, division by $\|u_j\|^{p_m^-}$ yields

$$\frac{c + o_j(1)}{\|u_j\|^{p_m^-}} + o_j(1)\|u_j\|^{1-p_m^-} \geq C_6 - C_7 \left(\|u_j\|^{-p_m^-} + \|u_j\|^{\beta^+(r+1)-p_m^-} \right).$$

The left-hand side tends to 0, while the right-hand side tends to $C_6 > 0$, because $p_m^- > 1$ and $\beta^+(r+1) < p_m^-$. Hence, (u_j) is bounded in X . $\square$

Remark 3.1. The hypothesis

$$\beta^+(r+1) < p_m^-$$

is precisely the condition ensuring boundedness of Palais–Smale sequences. The stronger assumption

$$\beta^+(r+1) < \min\{p_m^-, q^-\}$$

used in Theorem 1.1 supplies the additional exponent separation required by the multiplicity argument, while the boundedness step follows from $\beta^+(r+1) < p_m^-$.

The boundedness lemma allows us to apply the concentration-compactness principle to a Palais–Smale sequence. The critical defect measures will then be retained throughout the proof and shown to vanish through the global derivative identity and the (S_+) property; the concentration measures are subsequently identified by strong convergence.

Theorem 3.1. Assume (1.3), (1.4), (1.5), (1.6), (1.7), (1.2), $(\mathbf{f}_0)$, $(\mathbf{f}_1)$, $(\mathbf{f}_2)$, and $(\mathbf{A}_0)$ – $(\mathbf{A}_3)$, and let $M : [0, \infty) \rightarrow (0, \infty)$ be continuous. Assume, moreover, that one of the following two alternatives holds.

(i)

$$M(t) = a + bt^\eta, \quad a > 0, \quad b \geq 0, \quad \eta \geq 1,$$

and

$$\beta^+(r+1) < p_m^-;$$

(ii) The constants $m_0, m_1 > 0$ exist such that

$$m_0 \leq M(t) \leq m_1 \quad \text{for all } t \geq 0,$$

and again

$$\beta^+(r+1) < p_m^-.$$

Let $(u_j) \subset X$ be a Palais–Smale sequence for J_λ . Then, up to a subsequence, $u \in X$ exists such that

$$u_j \rightarrow u \quad \text{strongly in } X.$$

Moreover, the concentration-compactness decomposition of Proposition 2.8, applied with $h = q$, has no atomic part. More precisely, if

$$|\partial_{x_i} u_j|^{p_i(x)} dx \xrightarrow{*} \mu_i, \quad |u_j|^{q(x)} dx \xrightarrow{*} \nu \quad \text{in } \mathcal{M}^+(\overline{\Omega}),$$

then

$$\mu_i = |\partial_{x_i} u|^{p_i(x)} dx \quad (i = 1, \dots, N), \quad \nu = |u|^{q(x)} dx.$$

Consequently, every concentration mass satisfies

$$\mu_k = \nu_k = 0.$$

Proof. By Lemma 3.1, (u_j) is bounded in X . In both alternatives (i) and (ii),

$$\underline{M} := \inf_{t \geq 0} M(t) > 0, \tag{3.12}$$

namely, $\underline{M} = a$ in Case (i) and $\underline{M} = m_0$ in Case (ii). The argument below uses this uniform positive lower bound.

After passing to a subsequence, $u \in X$ exists such that

$$u_j \rightharpoonup u \quad \text{weakly in } X. \tag{3.13}$$

We now apply Proposition 2.8 with $h = q$. Passing to a further subsequence if necessary, there are measures $\mu_i, \nu \in \mathcal{M}^+(\overline{\Omega})$, an at most countable set $\mathcal{J}$, points $x_k \in \mathcal{A}$, and positive masses μ_k, ν_k such that

$$|\partial_{x_i} u_j|^{p_i(x)} dx \xrightarrow{*} \mu_i, \quad i = 1, \dots, N, \tag{3.14}$$

$$|u_j|^{q(x)} dx \xrightarrow{*} \nu = |u|^{q(x)} dx + \sum_{k \in \mathcal{J}} \nu_k \delta_{x_k}, \tag{3.15}$$

while, with $\mu := \sum_{i=1}^N \mu_i$, we have

$$\mu \geq \sum_{i=1}^N |\partial_{x_i} u|^{p_i(x)} dx + \sum_{k \in \mathcal{J}} \mu_k \delta_{x_k}. \tag{3.16}$$

At this stage, concentration at the interior critical points is allowed; the purpose of the remaining argument is to show that all the masses in (3.15)–(3.16) are zero.

Because $\beta(x) < q(x) \leq p_m^*(x)$ on $\overline{\Omega}$, Proposition 2.5 gives

$$u_j \rightarrow u \quad \text{strongly in } L^{\beta(x)}(\Omega). \quad (3.17)$$

Moreover, the continuous embedding $X \hookrightarrow L^{q(x)}(\Omega)$ implies

$$u_j \rightharpoonup u \quad \text{weakly in } L^{q(x)}(\Omega). \quad (3.18)$$

Since $J'_\lambda(u_j) \rightarrow 0$ in X^* and $(u_j - u)$ is bounded in X , we have

$$\langle J'_\lambda(u_j), u_j - u \rangle = o_j(1). \quad (3.19)$$

Expanding the derivative yields

$$\begin{aligned} M(\rho(u_j)) \langle A(u_j), u_j - u \rangle + \gamma \int_{\Omega} |u_j|^{q(x)-2} u_j (u_j - u) dx \\ = \lambda \left(\int_{\Omega} F(x, u_j) dx \right)^r \int_{\Omega} f(x, u_j) (u_j - u) dx + o_j(1). \end{aligned} \quad (3.20)$$

We first treat the subcritical bi-nonlocal term. By $(\mathbf{f}_1)$, the sequence $f(\cdot, u_j)$ is bounded in $L^{\beta'(x)}(\Omega)$, since

$$|f(x, u_j)|^{\beta'(x)} \leq C |u_j|^{\beta(x)} \quad \text{a.e. in } \Omega.$$

Therefore, the variable-exponent Hölder inequality and (3.17) yield

$$\begin{aligned} \left| \int_{\Omega} f(x, u_j) (u_j - u) dx \right| &\leq 2 \|f(\cdot, u_j)\|_{L^{\beta'(x)}(\Omega)} \|u_j - u\|_{L^{\beta(x)}(\Omega)} \\ &\longrightarrow 0. \end{aligned}$$

Moreover, (3.5) shows that $\int_{\Omega} F(x, u_j) dx$ is bounded. Hence

$$\left(\int_{\Omega} F(x, u_j) dx \right)^r \int_{\Omega} f(x, u_j) (u_j - u) dx \longrightarrow 0. \quad (3.21)$$

For the critical term, set

$$B_q(x, t) := |t|^{q(x)-2} t.$$

For a.e. $x \in \Omega$, the map $t \mapsto B_q(x, t)$ is monotone because $q(x) > 1$. Hence

$$\begin{aligned} \int_{\Omega} B_q(x, u_j) (u_j - u) dx &= \int_{\Omega} (B_q(x, u_j) - B_q(x, u)) (u_j - u) dx \\ &\quad + \int_{\Omega} B_q(x, u) (u_j - u) dx. \end{aligned} \quad (3.22)$$

Since $B_q(\cdot, u) \in L^{q'(x)}(\Omega)$ and (3.18) holds, we have

$$\int_{\Omega} B_q(x, u) (u_j - u) dx \longrightarrow 0. \quad (3.23)$$

Therefore, with

$$D_j := \int_{\Omega} (B_q(x, u_j) - B_q(x, u))(u_j - u) dx \geq 0,$$

we have

$$\int_{\Omega} |u_j|^{q(x)-2} u_j (u_j - u) dx = D_j + o_j(1). \quad (3.24)$$

Combining (3.20), (3.21), and (3.24), we obtain

$$M(\rho(u_j)) \langle A(u_j), u_j - u \rangle + \gamma D_j = o_j(1).$$

Because $D_j \geq 0$, $\gamma > 0$, and $M(\rho(u_j)) \geq \underline{M} > 0$, it follows that

$$\limsup_{j \rightarrow \infty} \langle A(u_j), u_j - u \rangle \leq 0. \quad (3.25)$$

Since $A(u) \in X^*$ and $u_j \rightharpoonup u$ in X

$$\langle A(u), u_j - u \rangle \longrightarrow 0.$$

Thus

$$\limsup_{j \rightarrow \infty} \langle A(u_j) - A(u), u_j - u \rangle \leq 0.$$

By Proposition 2.6, A is of type (S_+) , and hence

$$u_j \rightarrow u \quad \text{strongly in } X. \quad (3.26)$$

It remains to return to the concentration measures retained in (3.14)–(3.16). From (3.26), we have

$$\partial_{x_i} u_j \rightarrow \partial_{x_i} u \quad \text{strongly in } L^{p_i(x)}(\Omega) \quad (i = 1, \dots, N).$$

The Nemytskii map $v \mapsto |v|^{p_i(\cdot)}$ is continuous from $L^{p_i(x)}(\Omega)$ into $L^1(\Omega)$. Therefore

$$|\partial_{x_i} u_j|^{p_i(x)} \rightarrow |\partial_{x_i} u|^{p_i(x)} \quad \text{strongly in } L^1(\Omega),$$

and hence

$$\mu_i = |\partial_{x_i} u|^{p_i(x)} dx \quad (i = 1, \dots, N). \quad (3.27)$$

Likewise, the continuous critical embedding and (3.26) yield $u_j \rightarrow u$ strongly in $L^{q(x)}(\Omega)$. The same Nemytskii continuity gives

$$|u_j|^{q(x)} \rightarrow |u|^{q(x)} \quad \text{strongly in } L^1(\Omega),$$

so

$$v = |u|^{q(x)} dx. \quad (3.28)$$

Comparing (3.27)–(3.28) with the concentration-compactness decomposition shows that $\mathcal{J} = \emptyset$; equivalently, every possible concentration mass satisfies

$$\mu_k = \nu_k = 0.$$

This proves both the Palais–Smale compactness and the absence of concentration. $\square$

Remark 3.2. The role of concentration-compactness in Theorem 3.1 is to retain a measure-level description of every possible loss of critical compactness on the interior critical set. The atomic masses are then excluded simultaneously by the global derivative identity: After the compact source term vanishes, the positive $q(x)$ -term contributes a non-negative defect and the principal-operator defect is forced to vanish. The (S_+) property gives strong convergence in X , which identifies the limiting measures exactly.

Theorem 3.1 supplies the compactness input for Section 4: Every Palais–Smale sequence for J_λ is relatively compact in X under the structural assumptions of the main theorem, with the boundedness condition $\beta^+(r+1) < p_m^-$ at this stage. The theorem also establishes that the concentration-compactness decomposition has no concentration atoms for a Palais–Smale sequence.

Corollary 3.1. *Under the assumptions of Theorem 3.1, the functional J_λ satisfies the Palais–Smale condition on X .*

Proof. This is immediate from Theorem 3.1. □

4. Truncation and genus-based multiplicity

In this section, we complete the proof of Theorem 1.1. Throughout Section 4, we assume (1.3), (1.4), (1.5), (1.6), (1.7), (1.2), $(\mathbf{f}_0)$, $(\mathbf{f}_1)$, $(\mathbf{f}_2)$, $(\mathbf{A}_0)$ – $(\mathbf{A}_2)$, $(\mathbf{A}_4)$, the smallness condition (1.12), and one of the Kirchhoff alternatives (i)–(ii) from Theorem 1.1. The quantitative compactness assumption $(\mathbf{A}_3)$ supplies the Palais–Smale compactness result of Section 3, while $(\mathbf{A}_4)$ controls the principal energy uniformly on the small finite-dimensional spheres used in Lemma 4.3. The truncation is defined under $(\mathbf{A}_0)$ – $(\mathbf{A}_2)$ and $(\mathbf{A}_4)$. Under the exponent separation (1.12), the same growth estimates used in Section 3 show that the original functional J_λ is even, coercive, and bounded from below. We introduce a modular truncation to obtain an explicit localization of every negative critical point inside a fixed small-energy region where the truncated and original functionals agree together with their derivatives. This localization also gives the explicit parameter thresholds used below. The compactness input is provided by Theorem 3.1 and Corollary 3.1.

Throughout this section, we set

$$X := W_0^{1,\vec{p}(x)}(\Omega), \quad \rho(u) := \int_{\Omega} \left(\sum_{i=1}^N A_i(x, \partial_{x_i} u) + \frac{w(x)}{p_M(x)} |u|^{p_M(x)} \right) dx,$$

which is well defined on X by Corollary 2.1, and

$$G(u) := \int_{\Omega} F(x, u) dx.$$

We continue to write

$$\|u\| := \sum_{i=1}^N \|\partial_{x_i} u\|_{L^{p_i(x)}(\Omega)} \quad (u \in X),$$

for the equivalent norm introduced in Proposition 2.3. Recall that

$$J_\lambda(u) = \widehat{M}(\rho(u)) - \frac{\lambda}{r+1} G(u)^{r+1} + \gamma \int_{\Omega} \frac{|u|^{q(x)}}{q(x)} dx.$$

By $(\mathbf{f}_1)$, $(\mathbf{f}_2)$, and (1.9), the primitive $F(x, \cdot)$ is even and non-negative. Setting

$$c_F := \frac{A_1}{\beta^+}, \quad C_F := \frac{A_2}{\beta^-},$$

we obtain

$$c_F |t|^{\beta(x)} \leq F(x, t) \leq C_F |t|^{\beta(x)} \quad \text{for a.e. } x \in \Omega \text{ and every } t \in \mathbb{R}. \quad (4.1)$$

Consequently, $G(u) \geq 0$ for all $u \in X$, and Proposition 2.5 yields

$$0 \leq G(u) \leq C_G \max\{\|u\|^{\beta^-}, \|u\|^{\beta^+}\} \quad \text{for all } u \in X \quad (4.2)$$

for some constant $C_G > 0$.

Fix two radii once and for all

$$0 < R_0 < R_1. \quad (4.3)$$

For every $R > 0$, define

$$\mathcal{L}_R := \max \left\{ 1, \left(\frac{p_M^+ R}{C_0} \right)^{1/p_m^-} \right\}, \quad C_R := C_G \mathcal{L}_R^{\beta^+}. \quad (4.4)$$

If $\rho(u) \leq R$, then (3.3) and (3.2) show that $\|u\| \leq \mathcal{L}_R$. Since $\mathcal{L}_R \geq 1$, Estimate (4.2) consequently gives

$$0 \leq G(u) \leq C_R \quad \text{whenever } \rho(u) \leq R. \quad (4.5)$$

Thus the bound on the nonlocal primitive over a modular sublevel is explicit in terms of the structural constants and the chosen radius.

Lemma 4.1. *Assume the standing hypotheses of Section 4, fix the radii R_0, R_1 in (4.3), and choose any function $\tau \in C^1([0, \infty); [0, 1])$ such that*

$$\tau(t) = 1 \quad \text{for } 0 \leq t \leq R_0, \quad \tau(t) = 0 \quad \text{for } t \geq R_1.$$

Set

$$\underline{M} := \begin{cases} a, & \text{in case (i),} \\ m_0, & \text{in case (ii),} \end{cases} \quad \lambda_0 := \frac{(r+1)\underline{M}R_0}{2C_{R_1}^{r+1}} > 0.$$

Then for every $0 < \lambda < \lambda_0$, the functional

$$I_\lambda(u) := \widehat{M}(\rho(u)) - \frac{\lambda}{r+1} \tau(\rho(u)) G(u)^{r+1} + \gamma \int_\Omega \frac{|u|^{q(x)}}{q(x)} dx \quad (4.6)$$

belongs to $C^1(X, \mathbb{R})$, is even, bounded from below, and coercive. Moreover, we have the following.

(a) If $\rho(u) \leq R_0$, then

$$I_\lambda(u) = J_\lambda(u) \quad \text{and} \quad I'_\lambda(u) = J'_\lambda(u);$$

(b) One has

$$I_\lambda(u) > \frac{\underline{M}R_0}{2} > 0 \quad \text{whenever } R_0 \leq \rho(u) \leq R_1,$$

and $I_\lambda(u) \geq 0$ whenever $\rho(u) \geq R_1$. In particular,

$$\{u \in X : I_\lambda(u) < 0\} \subset \{u \in X : \rho(u) < R_0\}; \quad (4.7)$$

(c) Every negative-level critical point u of I_λ is nonzero and satisfies

$$J'_\lambda(u) = 0, \quad J_\lambda(u) = I_\lambda(u) < 0.$$

Hence, u is a nontrivial weak solution of Problem (1.1).

Proof. By Proposition 2.7, the maps ρ and G belong to $C^1(X, \mathbb{R})$, and $G(u) \geq 0$ for every $u \in X$. Extend the cut-off to a function $\tilde{\tau} \in C^1(\mathbb{R}; [0, 1])$ by setting $\tilde{\tau}(t) = 1$ for $t < 0$ and $\tilde{\tau}(t) = \tau(t)$ for $t \geq 0$; this extension is C^1 because $\tau \equiv 1$ near the origin. With $\Phi(s) = |s|^{r+1} \in C^1(\mathbb{R})$, one has $\tilde{\tau}(\rho(u)) = \tau(\rho(u))$ and $\Phi(G(u)) = G(u)^{r+1}$. The chain rule on open domains therefore gives $I_\lambda \in C^1(X, \mathbb{R})$, with

$$\begin{aligned} \langle I'_\lambda(u), v \rangle &= M(\rho(u)) \langle \rho'(u), v \rangle - \frac{\lambda}{r+1} \tau'(\rho(u)) G(u)^{r+1} \langle \rho'(u), v \rangle \\ &\quad - \lambda \tau(\rho(u)) G(u)^r \langle G'(u), v \rangle + \gamma \int_{\Omega} |u|^{q(x)-2} uv \, dx. \end{aligned}$$

Assumption $(\mathbf{A}_0)$ gives $\rho(-u) = \rho(u)$, whereas $(\mathbf{f}_2)$ implies that $F(x, \cdot)$, and therefore G , is even. Thus, I_λ is even.

In both Kirchhoff alternatives, we have

$$M(t) \geq \underline{M} > 0 \quad \text{for all } t \geq 0, \quad (4.8)$$

and therefore

$$\widehat{M}(t) \geq \underline{M}t \quad \text{for all } t \geq 0.$$

If $\rho(u) \geq R_1$, then $\tau(\rho(u)) = 0$, and hence

$$I_\lambda(u) = \widehat{M}(\rho(u)) + \gamma \int_{\Omega} \frac{|u|^{q(x)}}{q(x)} \, dx \geq 0.$$

If $R_0 \leq \rho(u) \leq R_1$, then (4.5) and $0 \leq \tau \leq 1$ yield

$$\begin{aligned} I_\lambda(u) &\geq \underline{M}R_0 - \frac{\lambda}{r+1} C_{R_1}^{r+1} \\ &> \underline{M}R_0 - \frac{\lambda_0}{r+1} C_{R_1}^{r+1} = \frac{\underline{M}R_0}{2}. \end{aligned}$$

Moreover, whenever $\rho(u) \leq R_1$, Estimate (4.5) gives the global lower bound

$$I_\lambda(u) \geq -\frac{\lambda}{r+1} C_{R_1}^{r+1}.$$

Together with the non-negativity of I_λ on $\{\rho \geq R_1\}$, this proves directly that I_λ is bounded from below. It also proves (b), including the localization (4.7).

If $\rho(u) \leq R_0$, then $\tau(\rho(u)) = 1$ and $\tau'(\rho(u)) = 0$, since τ is constant on $[0, R_0]$. The definition and derivative formula therefore give

$$I_\lambda(u) = J_\lambda(u), \quad I'_\lambda(u) = J'_\lambda(u),$$

which proves (a).

Finally, (3.3) and (3.2) imply $\rho(u) \rightarrow \infty$ whenever $\|u\| \rightarrow \infty$. Hence, for any u with $\|u\|$ large enough, $\rho(u) \geq R_1$, and

$$I_\lambda(u) \geq \underline{M}\rho(u) \longrightarrow +\infty.$$

Thus I_λ is coercive; its boundedness from below was established above.

Now let u be a critical point of I_λ with $I_\lambda(u) < 0$. By (4.7), $\rho(u) < R_0$; hence, (a) gives

$$J'_\lambda(u) = I'_\lambda(u) = 0, \quad J_\lambda(u) = I_\lambda(u) < 0.$$

Since $I_\lambda(0) = 0$, one has $u \neq 0$. Proposition 2.7 then identifies u as a nontrivial weak solution of (1.1). This proves (c). $\square$

From now on, we fix a parameter $0 < \lambda < \lambda_0$, where λ_0 is the explicit constant in Lemma 4.1, together with the fixed radii $0 < R_0 < R_1$, the cut-off τ , and the truncated functional I_λ defined in (4.6). All subsequent results in this section refer to these objects.

The next lemma uses $(\mathbf{A}_3)$ through Corollary 3.1 and shows that negative Palais–Smale sequences for I_λ remain in the region where the truncation is inactive.

Lemma 4.2. *Assume the standing hypotheses of Section 4, and also assume $(\mathbf{A}_3)$. Then the truncated functional I_λ satisfies the Palais–Smale condition at every level $c < 0$.*

Proof. Let $(u_j) \subset X$ be a Palais–Smale sequence for I_λ at a level $c < 0$.

$$I_\lambda(u_j) \rightarrow c, \quad I'_\lambda(u_j) \rightarrow 0 \quad \text{in } X^*.$$

The coercivity of I_λ gives the boundedness of (u_j) in X . Since $c < 0$, j_0 exists such that

$$I_\lambda(u_j) < \frac{c}{2} < 0 \quad \text{for every } j \geq j_0.$$

The localization (4.7) therefore yields

$$\rho(u_j) < R_0 \quad \text{for every } j \geq j_0.$$

By Lemma 4.1(a),

$$I_\lambda(u_j) = J_\lambda(u_j), \quad I'_\lambda(u_j) = J'_\lambda(u_j) \quad (j \geq j_0).$$

Thus the tail of (u_j) is a Palais–Smale sequence for J_λ at the same level c . Corollary 3.1 gives the following after passing to a subsequence:

$$u_j \rightarrow u \quad \text{strongly in } X.$$

Hence I_λ satisfies the Palais–Smale condition at every negative level. $\square$

Corollary 4.1. *Assume the standing hypotheses of Section 4, and also assume $(\mathbf{A}_3)$. Then every sequence $(u_j) \subset X$ satisfying*

$$I_\lambda(u_j) \rightarrow c < 0, \quad I'_\lambda(u_j) \rightarrow 0 \quad \text{in } X^*,$$

admits a strongly convergent subsequence in X .

Proof. This is exactly Lemma 4.2. $\square$

We now construct compact symmetric subsets of an arbitrarily large genus contained in a negative sublevel of I_λ .

Lemma 4.3. *Under the standing assumptions of Section 4, for every $n \in \mathbb{N}$, a compact symmetric set $K_n \subset X \setminus \{0\}$ and a number $\varepsilon_n > 0$ exist such that*

$$\gamma(K_n) \geq n, \quad \sup_{u \in K_n} I_\lambda(u) \leq -\varepsilon_n < 0.$$

Proof. Fix $n \in \mathbb{N}$, and let $E_n \subset X$ be an n -dimensional subspace. We use

$$S(E_n) := \{z \in E_n : \|z\| = 1\}$$

to denote its unit sphere. Since E_n is finite-dimensional, $S(E_n)$ is compact.

We first justify the small-radius estimate for the full modular. By $(\mathbf{A}_4)$, for every $z \in S(E_n)$ and $0 < t < 1$, we have

$$\begin{aligned} \rho(tz) &= \int_{\Omega} \left(\sum_{i=1}^N A_i(x, t\partial_{x_i} z) + \frac{w(x)}{p_M(x)} |tz|^{p_M(x)} \right) dx \\ &\leq t^{p_m^-} \left[\sum_{i=1}^N K_i \int_{\Omega} |\partial_{x_i} z|^{p_i(x)} dx + \frac{\|w\|_{L^\infty(\Omega)}}{p_m^-} \int_{\Omega} |z|^{p_M(x)} dx \right]. \end{aligned}$$

The expression in brackets is a continuous function of $z \in E_n$. Since $S(E_n)$ is compact, it is uniformly bounded there. Hence, $C_n > 0$ exists such that

$$\rho(tz) \leq C_n t^{p_m^-} \quad \text{for every } z \in S(E_n), \quad 0 < t < 1. \quad (4.9)$$

Likewise, the continuity of the embedding $X \hookrightarrow L^{q(x)}(\Omega)$ and the compactness of $S(E_n)$ give the following after enlarging C_n if necessary:

$$\int_{\Omega} |tz|^{q(x)} dx \leq C_n t^{q^-} \quad \text{for every } z \in S(E_n), \quad 0 < t < 1. \quad (4.10)$$

Moreover by (4.1),

$$G(tz) = \int_{\Omega} F(x, tz) dx \geq c_F \int_{\Omega} |tz|^{\beta(x)} dx \geq c_F t^{\beta^+} \int_{\Omega} |z|^{\beta(x)} dx \quad (0 < t < 1).$$

Since the map

$$z \mapsto \int_{\Omega} |z|^{\beta(x)} dx$$

is continuous and strictly positive on the compact set $S(E_n)$, $m_n > 0$ exists such that

$$\int_{\Omega} |z|^{\beta(x)} dx \geq m_n \quad \text{for all } z \in S(E_n).$$

Therefore, we have

$$G(tz) \geq c_F m_n t^{\beta^+} \quad \text{for all } z \in S(E_n), \quad 0 < t < 1. \quad (4.11)$$

Choose $\delta_n \in (0, 1)$ small enough that

$$\rho(tz) \leq R_0 \quad \text{for all } z \in S(E_n) \text{ and all } 0 < t \leq \delta_n.$$

By Lemma 4.1(a), this implies

$$I_\lambda(tz) = J_\lambda(tz) \quad \text{for all } z \in S(E_n) \text{ and all } 0 < t \leq \delta_n.$$

We now estimate $J_\lambda(tz)$ for $z \in S(E_n)$ and $0 < t < 1$.

In Case (i), $M(t) = a + bt^\eta$, and hence

$$\widehat{M}(s) = as + \frac{b}{\eta + 1} s^{\eta+1},$$

and therefore, by (4.9), we have

$$\widehat{M}(\rho(tz)) \leq aC_n t^{p_m^-} + \frac{b}{\eta + 1} C_n^{\eta+1} t^{(\eta+1)p_m^-}.$$

In case (ii), using $M(s) \leq m_1$, we have

$$\widehat{M}(\rho(tz)) \leq m_1 \rho(tz) \leq m_1 C_n t^{p_m^-}.$$

Thus, in both cases, after enlarging $\widetilde{C}_n$ if necessary and using $t^{(\eta+1)p_m^-} \leq t^{p_m^-}$ for $0 < t < 1$ in Case (i), we obtain the uniform estimate

$$\widehat{M}(\rho(tz)) \leq \widetilde{C}_n t^{p_m^-} \quad \text{for all } z \in S(E_n), 0 < t < 1. \quad (4.12)$$

Combining (4.12), (4.11), and (4.10), we obtain

$$I_\lambda(tz) \leq \widetilde{C}_n t^{p_m^-} - \frac{\lambda}{r+1} (c_F m_n)^{r+1} t^{\beta^+(r+1)} + \frac{\gamma C_n}{q^-} t^{q^-}$$

for all $z \in S(E_n)$ and all sufficiently small $t \in (0, 1)$.

Since, by (1.12),

$$\beta^+(r+1) < \min\{p_m^-, q^-\},$$

the negative term dominates as $t \downarrow 0$. Hence, $t_n \in (0, \delta_n)$ and $\varepsilon_n > 0$ exist such that

$$I_\lambda(t_n z) \leq -\varepsilon_n \quad \text{for all } z \in S(E_n).$$

Define

$$K_n := t_n S(E_n).$$

Then K_n is compact, symmetric, and odd-homeomorphic to $S(E_n)$, and hence to the Euclidean sphere $\mathbb{S}^{n-1}$. By Theorem 2.1, we obtain

$$\gamma(K_n) = n.$$

This completes the proof. $\square$

We may now conclude by invoking Theorem 2.2, which matches exactly the compactness information available for I_λ .

Theorem 4.1. *Assume the standing hypotheses of Section 4 and also assume $(\mathbf{A}_3)$. Then the truncated functional I_λ possesses infinitely many distinct pairs of nontrivial critical points $\{\pm u_n\}_{n \geq 1} \subset X$, all with negative critical values. Moreover*

$$J'_\lambda(u_n) = 0, \quad J_\lambda(u_n) = I_\lambda(u_n) < 0 \quad \text{for every } n,$$

so every pair is a pair of nontrivial weak solutions of Problem (1.1).

Proof. The functional I_λ is even, belongs to $C^1(X, \mathbb{R})$, is bounded from below, and satisfies $I_\lambda(0) = 0$, by Lemma 4.1. Corollary 4.1 gives the Palais–Smale condition at every level $c < 0$. For each $n \in \mathbb{N}$, Lemma 4.3 provides a compact symmetric set $K_n \subset X \setminus \{0\}$ such that

$$\gamma(K_n) \geq n, \quad \sup_{u \in K_n} I_\lambda(u) < 0.$$

Equivalently, with

$$\Gamma_n := \{A \subset X \setminus \{0\} : A \text{ compact and symmetric, } \gamma(A) \geq n\}, \quad c_n := \inf_{A \in \Gamma_n} \sup_{u \in A} I_\lambda(u),$$

one has $c_n < 0$ for every n . Theorem 2.2 applies and, including its standard repeated-level alternative, yields infinitely many distinct critical pairs $\{\pm u_n\}_{n \geq 1}$ of I_λ with

$$I_\lambda(u_n) < 0.$$

In particular, every u_n is nonzero. By Lemma 4.1(c), every such critical point lies in $\{\rho < R_0\}$, where the truncated and original functionals agree together with their derivatives. Therefore

$$J'_\lambda(u_n) = 0, \quad J_\lambda(u_n) = I_\lambda(u_n) < 0,$$

and Proposition 2.7 shows that u_n is a nontrivial weak solution of (1.1). Distinct critical pairs of I_λ remain distinct pairs of solutions of the original problem. $\square$

Proof of Theorem 1.1. Fix the radii $0 < R_0 < R_1$ in (4.3) and the resulting constant C_{R_1} from (4.4).

In Case (i), one has $\underline{M} = a$. Define

$$\lambda^* := \frac{(r+1)aR_0}{2C_{R_1}^{r+1}} > 0.$$

For every $0 < \lambda < \lambda^*$, Lemma 4.1 and Theorem 4.1 yield infinitely many distinct pairs $\{\pm u_n\}_{n \geq 1}$ satisfying

$$J'_\lambda(u_n) = 0, \quad J_\lambda(u_n) < 0.$$

Hence, all these pairs are nontrivial weak solutions of Problem (1.1).

In Case (ii), one has $\underline{M} = m_0$. Define

$$\tilde{\lambda} := \frac{(r+1)m_0R_0}{2C_{R_1}^{r+1}} > 0.$$

For every $0 < \lambda < \tilde{\lambda}$, the same argument gives infinitely many distinct pairs of nontrivial weak solutions. The radii and the constant C_{R_1} are fixed before the genus construction; therefore, neither threshold depends on the genus index or on a critical level. $\square$

5. Concrete examples and model applications

The abstract framework developed in the previous sections covers standard anisotropic Kirchhoff models with variable exponents. This section verifies the structural conditions $(\mathbf{A}_0)$ – $(\mathbf{A}_4)$ and $(\mathbf{f}_0)$ – $(\mathbf{f}_2)$ for natural classes of anisotropic diffusion and nonlocal reaction terms, and derives explicit multiplicity statements for the corresponding boundary-value problems. These applications connect the abstract existence theory directly with concrete doubly nonlocal anisotropic models.

More specifically, this section identifies the effective model class covered by the abstract theorem. The abstract argument from Sections 2–4 is stated for general anisotropic operators satisfying $(\mathbf{A}_3)$ and $(\mathbf{A}_4)$, and the verification below shows that these assumptions are realized by the standard anisotropic variable-exponent $p_i(x)$ -growth law with uniformly positive coefficients throughout the complete range $1 < p_i^- \leq p_i(x) \leq p_i^+ < N$.

Throughout this section, we assume that the exponent functions p_i , β , and q satisfy (1.3), (1.4), (1.5), (1.6), (1.7), and (1.12), and that $w \in L^\infty(\Omega)$ satisfies (1.2). We also keep the notation

$$X = W_0^{1, \vec{p}(x)}(\Omega), \quad \rho(u) = \int_{\Omega} \left(\sum_{i=1}^N A_i(x, \partial_{x_i} u) + \frac{w(x)}{p_M(x)} |u|^{p_M(x)} \right) dx.$$

A first natural class is given by the standard anisotropic variable-exponent growth law

$$A_i(x, \xi) = \frac{b_i(x)}{p_i(x)} |\xi|^{p_i(x)}, \quad a_i(x, \xi) = b_i(x) |\xi|^{p_i(x)-2} \xi, \quad i = 1, \dots, N, \quad (5.1)$$

where the coefficients b_i are measurable and uniformly positive.

Proposition 5.1. *Assume that, for each $i \in \{1, \dots, N\}$, we have*

$$b_i \in L^\infty(\Omega), \quad \underline{b}_i := \operatorname{ess\,inf}_{x \in \Omega} b_i(x) > 0, \quad 1 < p_i^- \leq p_i(x) \leq p_i^+ < N.$$

Define A_i and a_i by (5.1). Then the family $\{A_i\}_{i=1}^N$ satisfies $(\mathbf{A}_0)$ – $(\mathbf{A}_4)$. More precisely, one may take

$$k_i = \underline{b}_i, \quad c_i = \underline{b}_i \min\{2^{2-p_i^+}, p_i^- - 1\}, \quad K_i = \frac{\|b_i\|_{L^\infty(\Omega)}}{p_i^-}$$

in assumptions $(\mathbf{A}_2)$, $(\mathbf{A}_3)$, and $(\mathbf{A}_4)$, respectively.

Proof. For a.e. fixed $x \in \Omega$, the map $\xi \mapsto A_i(x, \xi)$ belongs to $C^1(\mathbb{R})$, is even, non-negative, and satisfies $A_i(x, 0) = 0$; hence, $(\mathbf{A}_0)$ holds. Moreover, we have

$$|a_i(x, \xi)| = b_i(x) |\xi|^{p_i(x)-1} \leq \|b_i\|_{L^\infty(\Omega)} (1 + |\xi|^{p_i(x)-1}),$$

so $(\mathbf{A}_1)$ follows. Since

$$a_i(x, \xi) \xi = b_i(x) |\xi|^{p_i(x)} = p_i(x) A_i(x, \xi),$$

we have

$$\underline{b}_i |\xi|^{p_i(x)} \leq a_i(x, \xi) \xi = p_i(x) A_i(x, \xi),$$

which proves $(\mathbf{A}_2)$ with $k_i = \underline{b}_i$.

Finally, apply the classical mixed monotonicity inequality stated after $(\mathbf{A}_3)$ with $p = p_i(x)$. If $p_i(x) \geq 2$, then

$$(a_i(x, \xi) - a_i(x, \eta))(\xi - \eta) \geq \underline{b}_i 2^{2-p_i^+} |\xi - \eta|^{p_i(x)}.$$

If $1 < p_i(x) < 2$, then

$$(a_i(x, \xi) - a_i(x, \eta))(\xi - \eta) \geq \underline{b}_i (p_i^- - 1) \frac{|\xi - \eta|^2}{(|\xi| + |\eta|)^{2-p_i(x)}}.$$

Both inequalities are summarized by

$$(a_i(x, \xi) - a_i(x, \eta))(\xi - \eta) \geq c_i \mathcal{D}_{p_i(x)}(\xi, \eta),$$

with the stated value of c_i . Hence, $(\mathbf{A}_3)$ holds throughout the full interval $1 < p_i^- \leq p_i(x) \leq p_i^+ < N$. Finally, we have

$$0 \leq A_i(x, \xi) = \frac{b_i(x)}{p_i(x)} |\xi|^{p_i(x)} \leq \frac{\|b_i\|_{L^\infty(\Omega)}}{p_i^-} |\xi|^{p_i(x)},$$

which proves $(\mathbf{A}_4)$ with the stated constant K_i . $\square$

Thus the quantitative monotonicity and upper-growth conditions needed in Sections 3 and 4, respectively, are automatically satisfied by the usual anisotropic variable-exponent $p_i(x)$ -Laplacian for every $1 < p_i^- \leq p_i(x) \leq p_i^+ < N$, provided that the diffusion coefficients remain uniformly positive. Proposition 5.1 therefore supplies the concrete bridge between the abstract operator hypothesis in Theorem 1.1 and both the subquadratic and superquadratic anisotropic model classes. We next turn to the nonlocal source term and specialize it to the most natural variable-exponent power law.

Proposition 5.2. *Assume that*

$$c \in L^\infty(\Omega), \quad 0 < c_0 \leq c(x) \leq c_1 \quad \text{for a.e. } x \in \Omega,$$

and define

$$f(x, t) = c(x) |t|^{\beta(x)-2} t, \quad \text{for a.e. } x \in \Omega \text{ and every } t \in \mathbb{R}. \quad (5.2)$$

Then $(\mathbf{f}_0)$ – $(\mathbf{f}_2)$ are satisfied with

$$A_1 = c_0, \quad A_2 = c_1.$$

Moreover, we have

$$F(x, t) = \int_0^t f(x, s) ds = \frac{c(x)}{\beta(x)} |t|^{\beta(x)} \quad \text{for a.e. } x \in \Omega, \quad \forall t \in \mathbb{R}, \quad (5.3)$$

and therefore

$$\frac{c_0}{\beta^+} |t|^{\beta(x)} \leq F(x, t) \leq \frac{c_1}{\beta^-} |t|^{\beta(x)} \quad \text{for a.e. } x \in \Omega, \quad \forall t \in \mathbb{R}.$$

Proof. The map f in (5.2) is Carathéodory, so $(\mathbf{f}_0)$ holds. Since

$$f(x, t)t = c(x) |t|^{\beta(x)},$$

one immediately obtains

$$c_0|t|^{\beta(x)} \leq f(x, t)t \leq c_1|t|^{\beta(x)} \quad \text{for a.e. } x \in \Omega, \quad \forall t \in \mathbb{R},$$

which is exactly $(\mathbf{f}_1)$. The oddness relation $(\mathbf{f}_2)$ follows from the identity

$$|-t|^{\beta(x)-2}(-t) = -|t|^{\beta(x)-2}t.$$

Finally, formula (5.3) follows by direct integration. $\square$

Combining Propositions 5.1 and 5.2, we are led to the explicit problem

$$\begin{cases} M(\rho(u)) \left(- \sum_{i=1}^N \partial_{x_i} (b_i(x) |\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u) + w(x) |u|^{p_M(x)-2} u \right) \\ \quad = \lambda \left(\int_{\Omega} \frac{c(x)}{\beta(x)} |u|^{\beta(x)} dx \right)^r c(x) |u|^{\beta(x)-2} u - \gamma |u|^{q(x)-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (5.4)$$

where

$$\rho(u) = \int_{\Omega} \left(\sum_{i=1}^N \frac{b_i(x)}{p_i(x)} |\partial_{x_i} u|^{p_i(x)} + \frac{w(x)}{p_M(x)} |u|^{p_M(x)} \right) dx. \quad (5.5)$$

We may now formulate the corresponding multiplicity consequences under the two Kirchhoff alternatives appearing in Theorem 1.1.

Corollary 5.1. *Assume the standing hypotheses of this section and the hypotheses of Propositions 5.1 and 5.2, and let*

$$M(t) = a + bt^{\eta}, \quad a > 0, \quad b \geq 0, \quad \eta \geq 1.$$

Then $\lambda^* > 0$ exists such that for every $0 < \lambda < \lambda^*$, Problem (5.4) admits infinitely many distinct pairs of nontrivial weak solutions in X .

Proof. By Proposition 5.1, the family (5.1) satisfies $(\mathbf{A}_0)$ – $(\mathbf{A}_4)$. By Proposition 5.2, the nonlinearity (5.2) satisfies $(\mathbf{f}_0)$ – $(\mathbf{f}_2)$. Since the standing assumptions in the present section include (1.3), (1.4), (1.5), (1.6), (1.7), (1.2), and (1.12), all the hypotheses of Theorem 1.1(i) are fulfilled. The conclusion therefore follows directly from Theorem 1.1. $\square$

In particular, Corollary 5.1 applies to the classical Kirchhoff choice

$$M(t) = a + bt, \quad a > 0, \quad b \geq 0,$$

which corresponds to a linear dependence of the stiffness on the anisotropic modular $\rho(u)$. The same framework also covers bounded or saturating response laws.

Corollary 5.2. *Assume the standing hypotheses of this section and the hypotheses of Propositions 5.1 and 5.2, and let $M : [0, \infty) \rightarrow (0, \infty)$ be continuous with*

$$m_0 \leq M(t) \leq m_1 \quad \text{for all } t \geq 0,$$

for some constants $m_0, m_1 > 0$. Then $\tilde{\lambda} > 0$ exists such that, for every $0 < \lambda < \tilde{\lambda}$, Problem (5.4) admits infinitely many distinct pairs of nontrivial weak solutions in X .

Proof. Exactly as above, Propositions 5.1 and 5.2 verify the operator and reaction assumptions of Theorem 1.1. The boundedness condition on M is precisely the alternative in Theorem 1.1(ii). Hence, Theorem 1.1(ii) applies and yields the claimed multiplicity statement. $\square$

A representative saturating law is

$$M(t) = m_0 + \frac{m_1 t^\eta}{1 + t^\eta}, \quad m_0 > 0, \quad m_1 > 0, \quad \eta > 0, \quad (5.6)$$

for which

$$m_0 \leq M(t) \leq m_0 + m_1 \quad \text{for all } t \geq 0.$$

Consequently, (5.6) falls under Corollary 5.2.

A further useful specialization is obtained when all directional exponents and coefficients coincide. Assume that

$$p_1(x) = \cdots = p_N(x) = p(x) \quad \text{for all } x \in \overline{\Omega},$$

and

$$b_1(x) = \cdots = b_N(x) = b(x) \quad \text{for a.e. } x \in \Omega,$$

with $b \in L^\infty(\Omega)$ and $b(x) \geq \underline{b} > 0$ a.e. in Ω . Since the potentials in (5.1) are separable in the coordinate derivatives, the resulting operator is the equal-exponent coordinatewise $p(x)$ -operator, and (5.4) reduces to

$$\begin{cases} M(\rho(u)) \left(- \sum_{i=1}^N \partial_{x_i} (b(x) |\partial_{x_i} u|^{p(x)-2} \partial_{x_i} u) + w(x) |u|^{p(x)-2} u \right) \\ \quad = \lambda \left(\int_{\Omega} \frac{c(x)}{\beta(x)} |u|^{\beta(x)} dx \right)^r c(x) |u|^{\beta(x)-2} u - \gamma |u|^{q(x)-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (5.7)$$

where

$$\rho(u) = \int_{\Omega} \left(\frac{b(x)}{p(x)} \sum_{i=1}^N |\partial_{x_i} u|^{p(x)} + \frac{w(x)}{p(x)} |u|^{p(x)} \right) dx.$$

This is the equal-exponent reduction of the separable anisotropic framework. For $p(x) \neq 2$, it differs from the Euclidean gradient operator $-\operatorname{div}(b(x) |\nabla u|^{p(x)-2} \nabla u)$, which corresponds to the nonseparable potential $\frac{b(x)}{(x)} |\xi|^{p(x)}$ on $\mathbb{R}^N$ and lies outside the scalar directional-potential formulation adopted here. The reduction nevertheless provides a direct bridge with coordinatewise variable-exponent Kirchhoff models; also compare [20, 21].

Corollary 5.3. *Assume the structural hypotheses above, together with*

$$p \in C_+^{0, \log}(\overline{\Omega}), \quad 1 < p^- := \min_{x \in \overline{\Omega}} p(x) \leq p(x) \leq p^+ := \max_{x \in \overline{\Omega}} p(x) < N \quad \text{for all } x \in \overline{\Omega},$$

and let f be given by (5.2). If either

$$M(t) = a + bt^\eta, \quad a > 0, \quad b \geq 0, \quad \eta \geq 1,$$

or M is continuous and satisfies

$$m_0 \leq M(t) \leq m_1 \quad \text{for all } t \geq 0,$$

for some $m_0, m_1 > 0$, then $\lambda_* > 0$ exists such that, for every $0 < \lambda < \lambda_*$, Problem (5.7) admits infinitely many distinct pairs of nontrivial weak solutions.

Proof. When $p_i \equiv p$ and $b_i \equiv b$, the coordinatewise anisotropic framework of Sections 2–4 applies verbatim. In particular, Proposition 5.1 verifies $(\mathbf{A}_0)$ – $(\mathbf{A}_4)$ for the operator in (5.7), while Proposition 5.2 gives $(\mathbf{f}_0)$ – $(\mathbf{f}_2)$ for the source term. The conclusion follows from Theorem 1.1, according to the corresponding assumption on M . $\square$

The examples above show that Theorem 1.1 applies to standard anisotropic variable-exponent diffusion laws, explicit nonlocal power-type sources, polynomial and bounded Kirchhoff responses, and the equal-exponent coordinatewise reduction. The assumptions $(\mathbf{A}_3)$ and $(\mathbf{A}_4)$ have distinct and complementary roles: The first yields compactness throughout the full range $1 < p_i^- \leq p_i(x) \leq p_i^+ < N$, while the second provides the homogeneous small-radius control required by the genus construction. Consequently, the abstract result gives a concrete multiplicity principle for doubly nonlocal problems with directional variable growth and critical reaction. It includes models with mixed subquadratic and superquadratic directions. In dimension $N = 2$, it covers the full subquadratic directional range permitted by the structural assumptions.

Author contributions

All authors contributed to the work and to the preparation and revision of the manuscript. All authors have read and approved the final version of the manuscript.

Use of Generative-AI tools declaration

The authors declare that Generative Artificial Intelligence (AI) tools were used during the preparation and revision of this article. AI tool used: ChatGPT (OpenAI). The tool was used to assist with language refinement. All mathematical content, proofs, results, and conclusions were reviewed and verified by the authors, who take full responsibility for the final content of the article.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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