



Research article

On strain flow solutions to the Navier-Stokes equations

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Abstract: In this paper, we constructed solutions to the Cauchy problem of the n -dimensional incompressible Navier-Stokes equations, which were classical strain flows and extended ones presented in Majda and Bertozzi's classical book *Vorticity and Incompressible Flow*. Then we studied the asymptotic behaviors on these solutions. More precisely, under specific constraints of the initial data, we proved that the constructed solutions converged to the corresponding ones of the Euler equations uniformly on the time $t \in [0, +\infty)$ as the viscosity vanished. This was very different from the previous result which held on $t \in [0, T]$ for any $T > 0$. Furthermore, we showed that for some specific initial data, the constructed solutions converged to the corresponding ones of the Euler equations when $t \rightarrow \infty$ for any fixed viscosity and the convergence rate was exponential one.

Keywords: strain flows; the Navier-Stokes equations; asymptotic behaviors

Mathematics Subject Classification: 35Q30, 76D05

1. Introduction

The Cauchy problem of n -dimensional incompressible Navier-Stokes equations reads as

$$\begin{cases} \partial_t v + v \cdot \nabla v = -\nabla p + \nu \Delta v, \\ \operatorname{div} v = 0, \\ v|_{t=0} = v_0, \end{cases} \quad (1.1)$$

where $v = v(t, x) = (v^1(t, x), \dots, v^n(t, x))$ is the velocity, $p = p(t, x)$ is the pressure and $\nu > 0$ is the viscosity. When $\nu = 0$, (1.1) becomes the incompressible Euler equations.

There have been numerous investigations on exact solutions of the unsteady incompressible Navier-Stokes equations (see [1–4] and references therein). In [5], the author established the regularity and asymptotic behavior of a class of exact solutions. For the steady Navier-Stokes equations, Galdi (see [6]) obtained some results concerning the properties of certain exact solutions.

The solutions of the incompressible and compressible Navier–Stokes equations depend crucially on the viscosity. When the viscosity $\nu = 0$, the Navier–Stokes equations reduce to the Euler equations. The vanishing viscosity limit has been extensively studied (see [7–11]). For the incompressible case, Clopeau and Mikelić (see [8]), Xiao and Xin (see [11]), and Fjordholm, Mishra and Weber (see [9]) investigated this limit under various boundary conditions, such as the slip boundary condition. For the compressible case, Chen and Perepelitsa (see [7]) obtained the vanishing viscosity limit to the compressible Euler ideal fluid. For the general Navier–Stokes slip boundary condition, Wang, Xin, and Yong (see [10]) studied the convergence in suitable function spaces.

The precise rate of convergence in the vanishing viscosity limit and the associated long-time behavior are subtle. To investigate the convergence rate, we extend the result in [1] to more general cases. More precisely, we construct solutions (v, p) of (1.1) of the form

$$\begin{cases} v(t, x) = (t, a_1 x_1, \dots, a_{n-1} x_{n-1}, v^n(t, x_1, \dots, x_{n-1})), \\ p(t, x) = -\frac{1}{2}(a_1^2 x_1^2 + \dots + a_{n-1}^2 x_{n-1}^2), \end{cases} \quad (1.2)$$

where the scalar function $v^n = v^n(t, x_1, \dots, x_{n-1})$ is to be determined, and the coefficients $a_1, \dots, a_{n-1}$ are constants. When $n = 3$ and v^3 depends only on t and the spatial variable x_1 , the flow becomes a classical strain flow (see [1]), which is a special case of our assumption, i.e., $\partial_{x_2} v^3 = 0$. The divergence-free condition on v implies that $a = (a_1, \dots, a_{n-1})$ satisfies the compatibility condition

$$\sum_{j=1}^{n-1} a_j = 0. \quad (1.3)$$

1.1. Main results

Before stating our main results, we introduce some notations. We denote by $\hat{f} = \mathcal{F}(f)$ the Fourier transform of a function f . Moreover, some notations for the standard homogeneous and inhomogeneous Sobolev spaces are

$$\begin{aligned} D^{k,r} &= \{u : \|\nabla^k u\|_{L^r} < \infty\}, \\ W^{k,r} &= \{u : \|(\text{Id} - \Delta)^{\frac{s}{2}} u\|_{L^r} < \infty\}, \\ \|u\|_{D^{k,r}} &= \|\nabla^k u\|_{L^r}, \quad \|u\|_{W^{s,r}} = \|(\text{Id} - \Delta)^{\frac{s}{2}} u\|_{L^r}, \end{aligned}$$

where $k \in \mathbb{N}$, $r \in [1, +\infty]$ and $s \in \mathbb{R}$.

Our first result is about the case $a = (a_1, \dots, a_{n-1}) = 0$, which is the n -dimensional case of [1].

Theorem 1.1. *Assume that $V_0 \in D^{1,\infty}(\mathbb{R}^{n-1})$ and $a = 0$. Then there exists a constant $C = C(n) > 0$ such that*

$$\|v^E(\cdot, t) - v^v(\cdot, t)\|_{L^\infty} \leq C \|V_0\|_{D^{1,\infty}} (\nu t)^{1/2}. \quad (1.4)$$

To state our further results for the case $a \neq 0$, we make some assumptions and notations on the constants $a = (a_1, \dots, a_{n-1})$ as follows:

$$\begin{cases} a_j \neq 0, \quad j = 1, \dots, n-1; \\ a_{j_k} > 0, \quad k = 1, \dots, s, \quad \Sigma = \{1, \dots, n-1\} \setminus \{j_1, \dots, j_s\}; \\ a_+ = \min_{1 \leq k \leq s} a_{j_k}, \quad a_- = \min_{j \in \Sigma} a_j. \end{cases} \quad (1.5)$$

Then we have the following

Theorem 1.2. Assume that the constants $a = (a_1, \dots, a_{n-1})$ satisfy (1.3) and (1.5), and the initial data

$$V_0 \in W^{\kappa,1}(\mathbb{R}^{n-1}), \quad (1.6)$$

with $\kappa \in (n-1, +\infty)$. Then for any $p \in [1, +\infty)$, there exists a constant $C = C(n, p, a, \kappa) > 0$ such that

$$\|v^\nu(\cdot, t) - v^E(\cdot, t)\|_{L^{p'}(\mathbb{R}^n)} \leq C \nu^{\frac{\kappa-n+1}{\kappa+2p}} \|V_0\|_{W^{k,1}}^{\frac{1}{p}} \left(\frac{(e^{2|a_-|t} - 1)^{\kappa/2}}{(e^{2a_+t} - 1)^{s/2}} \right)^{\frac{1}{p}} \quad (1.7)$$

holds true for any $t > 0$, where $p' = \frac{p}{p-1}$.

Theorem 1.3. Assume that the conditions of Theorem 1.2 hold. Then, for any $\kappa \in (n-1, n+1)$ and $p \in [1, +\infty)$, there exists a constant $C = C(n, p, a, \kappa) > 0$ such that

$$\|v^\nu(\cdot, t) - v^E(\cdot, t)\|_{L^{p'}(\mathbb{R}^n)} \leq C \nu^{\min\{p, \frac{\kappa-n+1}{2}\}} \|V_0\|_{W^{k,1}}^{\frac{1}{p}} \left(\frac{(e^{2|a_-|t} - 1)^{\kappa/2}}{(e^{2a_+t} - 1)^{s/2}} \right)^{\frac{1}{p}} \quad (1.8)$$

holds true for any $t > 0$, where $p' = \frac{p}{p-1}$.

Remark 1.4. In the conditions of Theorems 1.2 and 1.3, if $a_j = 0$ for some $j \in \{1, \dots, n-1\}$, then (1.7) and (1.8) become

$$\begin{aligned} \|v^\nu(\cdot, t) - v^E(\cdot, t)\|_{L^{p'}(\mathbb{R}^n)} &\leq C \nu^{\frac{\kappa-n+1}{\kappa+2p}} \|V_0\|_{W^{k,1}}^{\frac{1}{p}} \left(\frac{(e^{2|a'_-|t} - 1)^{\kappa/2}}{(e^{2a'_+t} - 1)^{s'/2}} \right)^{\frac{1}{p}}, \\ \|v^\nu(\cdot, t) - v^E(\cdot, t)\|_{L^{p'}(\mathbb{R}^n)} &\leq C \nu^{\min\{p, \frac{\kappa-n+1}{2}\}} \|V_0\|_{W^{k,1}}^{\frac{1}{p}} \left(\frac{(e^{2|a'_-|t} - 1)^{\kappa/2}}{(e^{2a'_+t} - 1)^{s'/2}} \right)^{\frac{1}{p}}, \end{aligned}$$

respectively. Here

$$s' = |\{j \in \{1, \dots, n\} | a_j > 0\}|, \quad a'_- = \min\{a_j | a_j < 0\}, \quad a'_+ = \min\{a_j | a_j > 0\},$$

and $|A|$ denotes the number of the elements of A .

The proof of Remark 1.4 will be given in the end of the paper.

Remark 1.5. In view of (1.7), (1.8), if $\kappa|a_-| \leq sa_+$, it follows that the solutions of the Navier-Stokes equations converge respectively to ones of the Euler equations uniformly on time $t \in [0, +\infty)$. That is,

$$\lim_{\nu \rightarrow 0} \|v^\nu(\cdot, t) - v^E(\cdot, t)\|_{L^{p'}(\mathbb{R}^n)} = 0, \quad \forall p \in [1, +\infty),$$

for any $t \in [0, +\infty)$. This is very different from (1.11) and (1.4) in which the convergence is for any $T > 0$ and $t \in (0, T)$. Moreover, if $\kappa|a_-| < sa_+$, then for given viscosity ν , it follows that

$$\lim_{t \rightarrow \infty} \|v^\nu(\cdot, t) - v^E(\cdot, t)\|_{L^{p'}(\mathbb{R}^n)} = 0, \quad \forall p \in [1, +\infty),$$

and the convergence rate is exponential one.

1.2. Sketch of proof

Now we introduce main ideas of our proof. First, substituting ν and p into (1.1), we obtain the initial problem of v^n :

$$\begin{cases} \partial_t v^n + \sum_{j=1}^{n-1} a_j x_j v_j^n = \nu \Delta v^n, \\ v^n|_{t=0} = v_0^n, \end{cases}$$

where $v_j^n = \partial_{x_j} v^n$. For convenience, we denote

$$V = v^n, \quad V_0 = v_0^n, \quad X = (x_1, \dots, x_{n-1}) \quad \text{and} \quad X_j = x_j, \quad j = 1, \dots, n-1.$$

Then it follows that

$$\begin{cases} \partial_t V + \sum_{j=1}^{n-1} a_j x_j V_j = \nu \Delta V, & \text{in } \mathbb{R}_+ \times \mathbb{R}^{n-1}, \\ V|_{t=0} = V_0, & \text{in } \mathbb{R}^{n-1}. \end{cases} \quad (1.9)$$

For the case $\nu = 0$ and $\nu > 0$, we denote the solutions of (1.1) by v^E and v^ν , and the solutions of (1.9) by V^E and V^ν , respectively.

Obviously, if $a = 0$, then V solves

$$\begin{cases} \partial_t V = \nu \Delta V, & \text{in } \mathbb{R}_+ \times \mathbb{R}^{n-1}, \\ V|_{t=0} = V_0, & \text{in } \mathbb{R}^{n-1}. \end{cases} \quad (1.10)$$

When $n = 3$ and $V = V(t, X_1)$, it was shown in [1] that if $V_0 \in W^{1,\infty}(\mathbb{R})$, then

$$\|V^E(\cdot, t) - V^\nu(\cdot, t)\|_{L^\infty(\mathbb{R}^3)} \leq C(\nu t)^{\frac{1}{2}}. \quad (1.11)$$

If $a = (-\gamma, \gamma)$ with $\gamma > 0$, the vorticity $\omega = \omega(t, X_1) = \text{curl} V(t, X_1)$ satisfies

$$\begin{cases} \partial_t \omega - \gamma x \partial_x \omega = \gamma \omega + \nu \Delta \omega, \\ \omega|_{t=0} = \omega_0. \end{cases} \quad (1.12)$$

We denote ω^E and ω^ν the solutions of (1.12) with $\nu = 0$ and $\nu > 0$, respectively. It was shown in [1] that if $\omega_0 \in W^{1,\infty}(\mathbb{R})$, then

$$\|\omega^E(\cdot, t) - \omega^\nu(\cdot, t)\|_{L^\infty} \leq C(\|\omega_0\|_{W^{1,\infty}})e^{\nu t} \left(\frac{\nu}{2\gamma} (e^{2\gamma t} - 1) \right)^{\frac{1}{2}}.$$

To prove our main results, we will first present some new estimates on the integral

$$\int_{\mathbb{R}^n} |f(x)| (1 - e^{-\nu|x|^2})^p dx \lesssim \|(1 + |\cdot|^2)^{\frac{\alpha}{2}} f\|_{L^\infty} \nu^{\frac{(\alpha-n)p}{\alpha+2p}}$$

for any $p \in [1, +\infty)$. Then we solve the Navier-Stokes equations and the Euler equations explicitly by utilizing the Fourier transforms under the assumption (1.2). Based on the expression of the solution, some precise and delicate estimates are presented.

The paper is organized as follows. In Section 2, we prove key lemmas which will be needed later. Section 3 is the proof of the main results.

2. Preliminaries

In this section, we establish some key lemmas needed later.

Lemma 2.1. *Suppose that $(1 + |\cdot|^2)^{\frac{\alpha}{2}} f \in L^\infty(\mathbb{R}^n)$ for some $\alpha > n$. Then for any $p \in [1, +\infty)$, there exists a constant $C = C(n, p, \alpha)$ such that*

$$\int_{\mathbb{R}^n} |f(x)| (1 - e^{-\nu|x|^2})^p dx \leq C \|(1 + |\cdot|^2)^{\frac{\alpha}{2}} f\|_{L^\infty} \nu^{\frac{(\alpha-n)p}{\alpha+2p}}$$

holds for any $\nu > 0$.

Proof. For any $\nu > 0$, one has

$$\begin{aligned} & \int_{\mathbb{R}^n} |f(x)| (1 - e^{-\nu|x|^2})^p dx \\ &= \int_{\mathbb{R}^n \setminus B_R} |f(x)| (1 - e^{-\nu|x|^2})^p dx + \int_{B_R} |x|^{2p} |f(x)| \left(\int_0^\nu e^{-w|x|^2} dw \right)^p dx \\ &: -I_1 + I_2, \end{aligned} \tag{2.1}$$

where B_R is a ball centered at the origin with radius $R > 0$ is to be determined.

To estimate I_1 , we denote $M := \|(1 + |\cdot|^2)^{\frac{\alpha}{2}} f\|_{L^\infty}$. Then it is clear that

$$\|f\|_{L^\infty} \leq M, \quad \| |\cdot|^\alpha f \|_{L^\infty} \leq M.$$

It follows that

$$\begin{aligned} I_1 &\leq C(p)M \int_R^\infty \int_{\partial B_r} r^{-\alpha} dS dr \\ &= C(n, p)M \int_R^\infty r^{n-\alpha-1} dr \leq C(n, \alpha, p)MR^{n-\alpha}. \end{aligned}$$

To estimate I_2 , using Hölder's inequality and Fubini's theorem, one has

$$\begin{aligned} I_2 &\leq \nu^{p-1} M \int_{B_R} \int_0^\nu |x|^{2p} e^{-pw|x|^2} dw dx \\ &\leq \nu^{p-1} M \int_0^\nu \int_0^R \int_{\partial B_r} r^{2p} dS dr dw \\ &\leq C(n, p) M \nu^p R^{2p+n}. \end{aligned}$$

Choosing suitable $R > 0$ such that

$$\nu^p R^{n+2p} = R^{n-\alpha},$$

we obtain $R = \nu^{-\frac{p}{2p+\alpha}}$ and

$$\int_{\mathbb{R}^n} |f(x)| (1 - e^{-\nu|x|^2})^p dx \leq C M \nu^{\frac{(\alpha-n)p}{\alpha+2p}}.$$

The proof of the lemma is complete. $\square$

Lemma 2.2. Assume that the conditions of Lemma 2.1 hold. Then for $\alpha \in (n, n+2)$ and $p \in [1, +\infty)$, there exists a constant $C = C(n, p, \alpha, \|(1 + |\cdot|^2)^{\frac{\alpha}{2}} f\|_{L^\infty})$ such that

$$\int_{\mathbb{R}^n} |f(x)| (1 - e^{-\nu|x|^2})^p dx \leq C \nu^{\min\{p, \frac{\alpha-n}{2}\}}, \quad \forall \nu > 0$$

holds true for any $\nu > 0$.

Proof. Due to (2.1), it is direct to estimate

$$\begin{aligned} I_1 &\leq \nu^{p-1} M \int_0^\nu \int_{\mathbb{R}^n \setminus B_1} |x|^{-\alpha+2p} e^{-pw|x|^2} dx dw \\ &\leq C(n, M) \nu^{p-1} \int_0^\nu \int_0^\infty r^{-\alpha+2p+n-1} e^{-pwr^2} dr dw \\ &= C(n, p, M) \nu^{p-1} \int_0^\nu w^{\frac{\alpha-2p-n}{2}} dw \int_0^\infty r^{-\alpha+2p+n-1} e^{-r^2} dr \\ &= C(n, \alpha, p, M) \nu^{\frac{\alpha-n}{2}}, \end{aligned}$$

and

$$I_2 \leq M \nu^{p-1} \int_0^\nu \int_{B_1} |x|^{2p} e^{-wp|x|^2} dx dw \leq C(n, M) \nu^p,$$

where M is same as in Lemma 2.1. Combining the estimates of I_1, I_2 , we obtain

$$\int_{\mathbb{R}^n} |f(x)| (1 - e^{-\nu|x|^2})^p dx \leq C \nu^{\min\{p, \frac{\alpha-n}{2}\}}.$$

The proof of the lemma is finished. $\square$

The following is about the explicit solutions to (1.9). More precisely, we have

Lemma 2.3. Assume that $\hat{V}_0 \in L^\infty(\mathbb{R}^{n-1})$, then the explicit solutions to (1.9) are

- (i) Case $\nu = 0$: $V^E(t, X) = \mathcal{F}^{-1} \left(\tilde{V}_0(t, \cdot) \right) (X)$;
(ii) Case $\nu > 0$: $V^\nu(t, X) = \mathcal{F}^{-1} \left(\tilde{V}_0(t, \cdot) \tilde{E}(t, \cdot) \right) (X)$,

where $\mathcal{F}^{-1}$ is the inverse of the Fourier transform and

$$\begin{aligned} \tilde{V}_0(t, \xi) &= \hat{V}_0(\xi_1 e^{a_1 t}, \dots, \xi_{n-1} e^{a_{n-1} t}), \\ \tilde{E}(t, \xi) &= \exp \left(\nu \sum_{j=1}^{n-1} \frac{1}{2a_j} (1 - e^{2a_j t}) \xi_j^2 \right). \end{aligned}$$

Proof. Taking the Fourier transform on both sides of (1.9) with respect to the space variables, we obtain

$$\begin{cases} \hat{V}_t - \sum_{j=1}^{n-1} a_j \xi_j \hat{V}_j + \nu |\xi|^2 \hat{V} = 0, \\ \hat{V}|_{t=0} = \hat{V}_0, \end{cases} \quad (2.2)$$

where $\hat{V}(\xi) = \mathcal{F}(V)(\xi)$, $\hat{V}_j = \partial_{\xi_j} \hat{V}$.

Case 1: $\nu = 0$. If $\nu = 0$, (2.2) reduces to

$$\begin{cases} \hat{V}_t - \sum_{j=1}^{n-1} a_j \xi_j \hat{V}_j = 0, \\ \hat{V}|_{t=0} = \hat{V}_0. \end{cases} \quad (2.3)$$

Denote

$$\eta_j = \xi_j e^{a_j t} \quad (j = 1, \dots, n-1), \quad t' = t.$$

Then

$$\begin{cases} \partial_{t'} \hat{V} = 0, \\ \hat{V}|_{t'=0} = \hat{V}_0, \end{cases}$$

and

$$\hat{V}^E(t, \xi) = \tilde{V}_0(t, \xi) := \hat{V}_0(\xi_1 e^{a_1 t}, \dots, \xi_{n-1} e^{a_{n-1} t}).$$

The L^∞ -boundedness of $\hat{V}_0$ implies that $\tilde{V}(t, \cdot) \in L^\infty(\mathbb{R}^{n-1}) \subset \mathcal{S}'(\mathbb{R}^{n-1})$ for any $t > 0$, where $\mathcal{S}'(\mathbb{R}^{n-1})$ is the tempered distributions space (see [12]). Since the Fourier transform is an automorphism from $\mathcal{S}'$ to itself (see [13]), it follows that

$$V^E(t, X) = \mathcal{F}^{-1} \left(\tilde{V}_0(t, \cdot) \right) (X). \quad (2.4)$$

Case 2: $\nu > 0$. In this case, we define

$$\eta_j = \xi_j e^{a_j t} \quad (j = 1, \dots, n-1), \quad t' = t$$

and (2.2) is transformed into

$$\begin{cases} \hat{V}_{t'} + \nu \sum_{j=1}^{n-1} e^{-2a_j t'} \eta_j^2 \hat{V} = 0, \\ \hat{V}|_{t'=0} = \hat{V}_0. \end{cases}$$

Let

$$\hat{W}(t', \eta) = \hat{V}(t', \eta) \exp \left(-\nu \sum_{j=1}^{n-1} \frac{1}{2a_j} e^{-2a_j t'} \eta_j^2 \right).$$

Then we have $\hat{W}_{t'} = 0$ and

$$\hat{V}^\nu(t, \xi) = \tilde{V}_0(t, \xi) \exp \left(\nu \sum_{j=1}^{n-1} \frac{1}{2a_j} (1 - e^{2a_j t}) \xi_j^2 \right).$$

Denote

$$\tilde{E}(t, \xi) = \exp \left(\nu \sum_{j=1}^{n-1} \frac{1}{2a_j} (1 - e^{2a_j t}) \xi_j^2 \right).$$

Then $\tilde{E}(t, \cdot) \in L^\infty(\mathbb{R}^{n-1})$ for any $t > 0$. It yields

$$V^\nu(t, X) = \mathcal{F}^{-1} \left(\tilde{V}_0(t, \cdot) \tilde{E}(t, \cdot) \right) (X). \quad (2.5)$$

The proof of the lemma is finished. $\square$

3. Proof of the main results

Proof of Theorem 1.1. Clearly V^E satisfies the equation $\partial_t V^E = 0$ and thus $V^E = V_0$. For the case $\nu > 0$, V^ν satisfies the equation (1.10) and can be solved out as

$$V^\nu(t, X) = \int_{\mathbb{R}^{n-1}} H(X - Y, \nu t) V_0(Y) dY,$$

where H is the heat kernel

$$H(t, X) = \frac{1}{(4\pi t)^{(n-1)/2}} e^{-|X|^2/4t}.$$

Since $\int_{\mathbb{R}^n} H(x, t) dx = 1$ for all $t > 0$, it deduces that

$$\begin{aligned} |v^\nu(t, x) - v^E(t, x)| &= |V^\nu(t, X) - V^E(t, X)| \\ &\leq \frac{1}{(4\pi \nu t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|X-Y|^2}{4\nu t}} |V_0(Y) - V_0(X)| dY \\ &\leq \|V_0\|_{D^{1,\infty}} \frac{1}{(4\pi \nu t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|X-Y|^2}{4\nu t}} |X - Y| dY \\ &= \|V_0\|_{D^{1,\infty}} \frac{\sqrt{\nu t}}{(4\pi)^{n/2}} \int_{\mathbb{R}^n} |Y| e^{-|Y|^2/4} dY \\ &\leq C(n) \|V_0\|_{D^{1,\infty}} \sqrt{\nu t}. \end{aligned}$$

The proof of Theorem 1.1 is complete. $\square$

Proof of Theorem 1.2. Applying for Lemma 2.3, we get

$$\begin{aligned} v^\nu(x, t) - v^E(x, t) &= (0, \dots, 0, V^\nu(t, X) - V^E(t, X)) \\ &= (0, \dots, 0, \mathcal{F}^{-1} (\tilde{V}(t, \cdot) (\tilde{E}(t, \cdot) - 1))) . \end{aligned}$$

Since $\mathcal{F} : L^{p'} \rightarrow L^p$ is a linear and bounded operator for $p \in (1, +\infty)$, we have

$$\|v^\nu(\cdot, t) - v^E(\cdot, t)\|_{L^{p'}(\mathbb{R}^n)} \leq C(p) \|\tilde{V}(t, \cdot) (\tilde{E}(t, \cdot) - 1)\|_{L^p(\mathbb{R}^{n-1})}.$$

Denote

$$\begin{cases} \zeta_{j_k} = \left(\frac{e^{2a_{j_k}t} - 1}{2|a_{j_k}|} \right)^{1/2} \xi_{j_k}, & k = 1, \dots, s, \\ \zeta_j = \left(\frac{1 - e^{2a_jt}}{2|a_j|} \right)^{1/2} \xi_j, & j \in \Sigma. \end{cases} \quad (3.1)$$

and

$$\begin{cases} A_{j_k}(t) = \left(\frac{e^{2a_{j_k}t} - 1}{2|a_{j_k}|} \right)^{-1/2}, & k = 1, \dots, s, \\ A_j(t) = \left(\frac{1 - e^{2a_jt}}{2|a_j|} \right)^{-1/2}, & j \in \Sigma, \end{cases} \quad (3.2)$$

Then it follows that

$$\begin{aligned} \|\tilde{V}(t, \cdot)(\tilde{E}(t, \cdot) - 1)\|_{L^p(\mathbb{R}^{n-1})}^p &\leq C(n, a) \prod_{k=1}^s (e^{2a_{j_k}t} - 1)^{-1/2} \prod_{j \in \Sigma} (1 - e^{2a_jt})^{-1/2} \\ &\quad \int_{\mathbb{R}^{n-1}} f^p(t, \zeta) (1 - e^{-\nu|\zeta|^2})^p d\zeta, \end{aligned}$$

where

$$f(t, \zeta) := |\hat{V}_0(A_1(t)e^{a_1t}\zeta_1, \dots, A_{n-1}(t)e^{a_{n-1}t}\zeta_{n-1})|.$$

Note that the assumption (1.6) implies that

$$\|(1 + |\cdot|^2)^{\frac{\kappa}{2}} \hat{V}_0\|_{L^\infty} \leq \|V_0\|_{W^{k,1}}. \quad (3.3)$$

Consequently, one has

$$\begin{aligned} |f(t, \zeta)| &\leq \|V_0\|_{W^{k,1}} \left(\sum_{i=1}^{n-1} A_i(t)^2 e^{2a_it} \zeta_i^2 \right)^{-\frac{\kappa}{2}} \\ &\leq C(n) \|V_0\|_{W^{k,1}} \max \{A_i(t)^{-\kappa} e^{-\kappa a_it}\} |\zeta|^{-\kappa}, \end{aligned}$$

for $\forall \zeta \in \mathbb{R}^{n-1}$.

Thanks to Lemma 2.1 and (3.3), it yields

$$\begin{aligned} &\nu^{\frac{-(\kappa-n+1)p}{\kappa+2p}} \|\tilde{V}(t, \cdot)(\tilde{E}(t, \cdot) - 1)\|_{L^p(\mathbb{R}^{n-1})}^p \\ &\leq C(n, a, p, \kappa) \|V_0\|_{W^{k,1}} \max \{A_i(t)^{-\kappa} e^{-\kappa a_it}\} \\ &\quad \prod_{k=1}^s (e^{2a_{j_k}t} - 1)^{-1/2} \prod_{j \in \Sigma} (1 - e^{2a_jt})^{-1/2} \\ &\leq C(n, a, p, \kappa) \|V_0\|_{W^{k,1}} \max \left\{ (1 - e^{-2|a_{j_k}|t})^{\kappa/2}, (e^{2|a_j|t} - 1)^{\kappa/2} \right\} \\ &\quad \prod_{k=1}^s (e^{2a_{j_k}t} - 1)^{-1/2} \prod_{j \in \Sigma} (1 - e^{2a_jt})^{-1/2}. \end{aligned} \quad (3.4)$$

Moreover, it is clear that the functions

$$\prod_{j \in \Sigma} (1 - e^{2a_j t})^{-1/2}, \quad (1 - e^{-2|a_{j_k}|t})^{\kappa/2}$$

are bounded with respect to t for any $k = 1, \dots, s$ and $j \in \Sigma$. Therefore, for large enough t , we have

$$\begin{aligned} & \nu^{\frac{-(\kappa-n+1)p}{\kappa+2p}} \|\tilde{V}(t, \cdot)(\tilde{E}(t, \cdot) - 1)\|_{L^p(\mathbb{R}^{n-1})}^p \\ & \leq C(n, a, p, \kappa) \|V_0\|_{W^{k,1}} \prod_{k=1}^s (e^{2a_{j_k} t} - 1)^{-1/2} \max_{j \in \Sigma} (e^{2|a_j|t} - 1)^{\kappa/2} \\ & \leq C(n, a, p, \kappa) \|V_0\|_{W^{k,1}} (e^{2a_+ t} - 1)^{-s/2} (e^{2|a_-|t} - 1)^{\kappa/2}. \end{aligned}$$

Thus, we deduce

$$\|v^v(\cdot, t) - v^E(\cdot, t)\|_{L^{p'}(\mathbb{R}^n)} \leq C \nu^{\frac{\kappa-n+1}{\kappa+2p}} \|V_0\|_{W^{k,1}}^{\frac{1}{p}} \left(\frac{(e^{2|a_-|t} - 1)^{\kappa/2}}{(e^{2a_+ t} - 1)^{s/2}} \right)^{\frac{1}{p}}.$$

where C is a constant that depends only on n, a, p, κ . The proof of Theorem 1.2 is completed. $\square$

Proof of Theorem 1.3. The proof is similar as that of Theorem 1.2 except for estimate (3.4). Utilizing Lemma 2.2 gives

$$\|v^v(\cdot, t) - v^E(\cdot, t)\|_{L^{p'}(\mathbb{R}^n)} \leq C \nu^{\min\{p, \frac{\kappa-n+1}{2}\}} \|V_0\|_{W^{k,1}}^{\frac{1}{p}} \left(\frac{(e^{2|a_-|t} - 1)^{\kappa/2}}{(e^{2a_+ t} - 1)^{s/2}} \right)^{\frac{1}{p}},$$

which is used to accomplish the proof of Theorem 1.2. $\square$

Proof of Remark 1.4. If there exists a positive integer $j_0 \in \{1, \dots, n-1\}$ such that $a_{j_0} = 0$, we rewrite the notations (3.1), (3.2) as

$$\zeta_{j_0} := \xi_{j_0}, \quad A_{j_0}(t) = 1.$$

Then the estimate (3.4) can be written as

$$\begin{aligned} & \nu^{\frac{-(\kappa-n+1)p}{\kappa+2p}} \|\tilde{V}(t, \cdot)(\tilde{E}(t, \cdot) - 1)\|_{L^p(\mathbb{R}^{n-1})}^p \\ & \leq C(n, a, p, \kappa) \|V_0\|_{W^{k,1}} \max \{A_i(t)^{-\kappa} e^{-\kappa a_i t} | a_i \neq 0\} \\ & \quad \prod_{a_{j_k} > 0} (e^{2a_{j_k} t} - 1)^{-1/2} \prod_{j \in \Sigma'} (1 - e^{2a_j t})^{-1/2} \\ & \leq C \|V_0\|_{W^{k,1}} \frac{(e^{2|a'_-|t} - 1)^{\kappa/2}}{(e^{2a'_+ t} - 1)^{\frac{s'}{2}}}, \end{aligned}$$

which leads to the proof of Remark 1.4. $\square$

Author contributions

The two authors contributed equally to this work.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This research was supported by National Natural Science Foundation of China (No. 12671278) and the Interdisciplinary Project of Capital Normal University(No. 2026JCYY04).

Conflict of interest

The authors declare no conflict of interest.

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