



Research article

Implicit Lagrangians, partial Dirac structures and extremals of the maximum principle in infinite dimensions

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Abstract: In a previous work, the notion of a Dirac structure was extended from finite-dimensional manifolds to the convenient setting of Frölicher, Kriegl, and Michor through the introduction of partial Dirac structures on convenient manifolds. This framework encompasses a wide range of infinite-dimensional settings, including Hilbert, Banach, and Fréchet manifolds, as well as direct limits of Banach manifolds. It is also particularly well-suited to variational methods, since the category of convenient vector spaces is Cartesian closed, unlike the category of locally convex vector spaces. In this paper, we develop a variational framework for the study of constrained Lagrangian systems on convenient manifolds and for optimal control problems in infinite dimensions. The variational tools introduced by Yoshimura and Marsden for constrained dynamics on Banach manifolds were extended to singular distributions arising in the context of the Pontryagin maximum principle. As a consequence, a characterization of normal extremals was obtained in this infinite-dimensional setting. The theory was then applied to conical Finsler geometry on Banach manifolds, yielding a characterization of normal geodesics.

Keywords: partial Dirac structures; convenient Lie algebroids; convenient manifolds; implicit Lagrangians; direct limits; extremals; geodesics; conical Finsler metrics

Mathematics Subject Classification: 46T05, 58B20, 70G45, 70H45

In loving memory of my friend Fernand Pelletier.

The key ideas and their development presented in this paper are his enduring contribution.

1. Introduction

An algebraic theory of Dirac structures first appeared in [1] within the Hamiltonian framework of integrable evolution equations. The geometric approach was later realized in [2] and [3] in the finite-dimensional setting, where Dirac structures were defined as subsets D of the Pontryagin bundle $TM \oplus T^*M$ satisfying specific conditions. This provides a unified treatment of weak symplectic and Poisson structures. The notion was further developed in finite dimensions, particularly in the study of nonholonomic mechanical systems. The Hamiltonian framework itself relies on the principle of least action and the Legendre transformation, expressed through Euler–Lagrange equations and a variational approach (cf. [4, 5]).

In order to extend this variational and geometric picture to infinite-dimensional systems arising in field theory and fluid mechanics, one is naturally led to a functional-analytic setting. However, the standard framework of locally convex topological vector spaces turns out to be poorly adapted to such problems. In particular, classical tools from the calculus of variations may fail: *le lemme de du Bois-Reymond* does not hold in this generality, which obstructs a direct extension of Euler–Lagrange-type arguments. By contrast, the category of convenient vector spaces provides a more suitable setting for infinite-dimensional differential geometry, in which a robust notion of smoothness is available and variational arguments can be recovered in a consistent way. This motivates working within the convenient framework when passing from finite-dimensional Dirac structures to infinite-dimensional contexts.

In this paper, the main contributions are the following:

- From Section 3 to Section 5 together with the applications developed in Section 6, and building on the framework introduced in [6] and [7] (see also the references therein), we extend to the convenient setting the finite-dimensional geometric framework of Dirac structures and its associated Lagrangian formalism (see, for instance, [4] and [5]). This provides a unified variational framework for constrained mechanical systems defined on subbundles of infinite-dimensional configuration manifolds. Such a framework is motivated by a variety of infinite-dimensional mechanical and control-theoretic problems arising in Hilbert, Banach, Fréchet, and more generally convenient manifolds. Examples include the Korteweg–de Vries hierarchy, the nonlinear Schrödinger equation, geodesic problems on diffeomorphism groups, and sub-Riemannian geometry (cf., for example [7], [8–15]).
- In Section 7, we extend the previous framework to mechanical systems subject to singular constraints, viewed as control systems. Within this setting, several characterizations of normal extremals are obtained. In particular, Theorem 7.18 provides a characterization of normal extremals in an infinite-dimensional context and constitutes the main result of the paper.
- In Section 8, the preceding theory is applied to conic Finsler metrics viewed as a generalization of sub-Riemannian problems in the Banach setting (cf. [14]).

Recall that, in [16], we introduced the notion of a partial Dirac structure on convenient manifolds and studied its geometric properties.

According to the framework of partial Poisson structures (cf. [17], Chap. 7), we introduce the notion on the partial bundle

$$T^{\mathfrak{p}}M = TM \oplus T^{\mathfrak{b}}M,$$

where $T^{\mathfrak{b}}M$ is a weak subbundle of the kinematic cotangent bundle $T'M$ of a convenient manifold¹.

As in finite dimension, this bundle can be equipped with a Courant bracket and an anchor map $\rho^D : T^{\mathfrak{p}}(M) \rightarrow TM$, and it defines a partial Dirac structure if the Courant bracket, restricted to local sections of $T^{\mathfrak{p}}(M)$, takes values in such a set of sections.

In finite dimensions, any Dirac structure is integrable, that is, the distribution defines a (singular) foliation equipped with a canonical pre-symplectic structure on each leaf. Unfortunately, even in the Hilbert setting, this result is no longer true. The interested reader will find sufficient conditions in the Banach setting under which a singular Banach foliation is associated to a Banach partial Dirac structure as well as conditions under which such a foliation can be equipped with a pre-symplectic structure. Throughout the text, the reader will find examples of partial Dirac structures.

In this paper, most of these geometric results will be applied to:

1. Non-degenerate and degenerate Lagrangians (generalizing [18] to our context);
2. Dynamical systems with non-degenerate Lagrangians subject to singular nonholonomic constraints, with application, in the Banach setting, to the conic sub-Finsler metric and, in particular, the (singular) sub-Riemannian metric, including a characterization of their geodesics, generalizing [14].

This paper is organized as follows:

- In Section 2, we review the fundamental elements of the convenient calculus, as developed by Frölicher, Kriegl, and Michor.
- In Section 3, we first recall the notion of the (weak) Pontryagin bundle $T^{\mathfrak{p}}M \rightarrow M$, where $T^{\mathfrak{p}}M = TM \oplus T^{\mathfrak{b}}M$, in the convenient setting. We then introduce key concepts such as the Courant bracket and partial almost-Dirac structures and study their properties.
- In Section 4, we extend the notion of implicit Lagrangians, as well as several finite-dimensional results by Yoshimura, Jacobs, and Marsden ([5]), to the convenient setting.
- In Section 5, we establish the equivalence between the variational formulation of implicit Lagrangian systems and their implicit Hamiltonian formulation in the convenient setting.
- Section 6 studies properties of direct limits of ascending sequences of almost-Dirac structures (including the Banach case) and provides sufficient conditions for such a direct limit of integrable Hilbert–Dirac structures to be a convenient integrable Dirac structure.
- In Section 7, we generalize, in the Banach setting, the partial almost-Dirac structure induced by a closed subbundle Δ_M of TM to singular distributions on M , interpreted as optimal control problems within a framework adapted to a suitable form of the maximum principle.
- The final section is devoted to the study of normal geodesics in a conic sub-Hilbert–Finsler structure on a Banach manifold.

¹That is, the inclusion $T^{\mathfrak{b}}M \rightarrow T'M$ is an injective convenient morphism.

2. Basic elements of the convenient setting

In this section, we recall the basic framework of convenient calculus developed by Frölicher, Kriegl, and Michor in [19] and [20].

This calculus is based on an infinite-dimensional extension of Boman's theorem ([21]), according to which smoothness can be characterized through smooth curves. A curve $c : \mathbb{R} \rightarrow \mathbb{E}$ is *smooth* if and only if, for every continuous linear functional $\lambda \in \mathbb{E}^*$, the composition $\lambda \circ c$ is smooth ([20], Theorem 2.14 (4)). We then get a second topology on $\mathbb{E}$, which is the final topology relative to the set of all smooth curves and is called the c^∞ -topology ([20], Definition 2.12). This last topology may be different from the locally convex topological vector space topology and, for this topology, $\mathbb{E}$ cannot be a topological vector space. However, for Fréchet (and so Banach) spaces, both topologies coincide. A map $f : \mathbb{E} \rightarrow \mathbb{R}$ is *smooth* if and only if $f \circ c : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth map for any smooth curve c in $\mathbb{E}$ ([20], Definition 3.11). Any composition of smooth maps is also smooth.

In Banach space theory, the Cauchy condition is a necessary condition for the convergence of a sequence, and completeness is defined by requiring that every Cauchy sequence converges. In the convenient setting, the weaker notion of Mackey–Cauchy sequence (cf. [20, Section 2]) is used. Let B be a bounded absolutely convex subset of a locally convex topological vector space $\mathbb{E}$. We define

$$\mathbb{E}_B := \bigcup_{t>0} tB = \text{span}(B).$$

The *Minkowski functional* associated with B is the map

$$\|\cdot\|_B : \mathbb{E}_B \rightarrow \mathbb{R}_+$$

defined by

$$\forall u \in \mathbb{E}_B, \|u\|_B = \inf\{t > 0 \mid u \in tB\}.$$

If B is closed and absolutely convex, then $\|\cdot\|_B$ is a norm on $\mathbb{E}_B$, and $(\mathbb{E}_B, \|\cdot\|_B)$ is a normed space. A sequence $(x_n)_{n \in \mathbb{N}}$ in $\mathbb{E}$ is called *Mackey–Cauchy* if there exists a bounded absolutely convex subset B of $\mathbb{E}$ such that (x_n) is a Cauchy sequence in the normed space $(\mathbb{E}_B, \|\cdot\|_B)$, that is,

$$\forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}, \forall n, m > n_\varepsilon, x_n - x_m \in \varepsilon B.$$

A locally convex topological vector space $\mathbb{E}$ is said to be *convenient*, or c^∞ -complete, if every Mackey–Cauchy sequence in $\mathbb{E}$ converges with respect to the original locally convex topology of $\mathbb{E}$. This property is called c^∞ -completeness.

The convenient calculus provides an appropriate extension of differential calculus to such spaces because, for any c^∞ -open set U of a convenient space $\mathbb{E}$ and any convenient space $\mathbb{F}$, the following properties hold:

- the space $C^\infty(U, \mathbb{F})$ of smooth maps may also be endowed with a structure of convenient space;

- the differential operator $d : C^\infty(U, \mathbb{F}) \rightarrow C^\infty(U, L(\mathbb{E}, \mathbb{F}))$ defined by:

$$df(x).v = \lim_{t \rightarrow 0} \frac{f(x + tv) - f(x)}{t}$$

exists for every $f \in C^\infty(U, \mathbb{F})$. Moreover, d is a linear and smooth map ([20, Section 3.18]). Here $L(\mathbb{E}, \mathbb{F})$ denotes the space of bounded (equivalently smooth) linear mappings² from $\mathbb{E}$ to $\mathbb{F}$;

- the chain rule holds ([20, Section 3.18]).

The notion of a *convenient manifold* modeled on a convenient vector space is then defined *via* an equivalent class of smooth atlases ([20, Section 27.1]).

On a convenient vector space $\mathbb{E}$, two kinds of tangent spaces at a point a of this space can be defined ([20, Section 28.1]): *the kinematic tangent space* and *the operational tangent space*. We are only interested in the first one³, $T_a\mathbb{E}$ consisting of all derivatives $c'(0)$ at 0 of smooth curves $c : \mathbb{R} \rightarrow \mathbb{E}$ with $c(0) = a$.

The *kinematic tangent bundle* TM of a convenient manifold M ([20, Section 29.9]) is the quotient space for the equivalence relation:

$$(\alpha, x, v) \sim (\beta, y, w) \iff \begin{cases} x = y \\ (d\phi_{\alpha\beta})_{\phi_\beta(x)} w = v \end{cases}$$

defined on the disjoint union $\bigcup_{\alpha \in A} \{\alpha\} \times U_\alpha \times \mathbb{M}$, where $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$ is a smooth atlas of M .

A *kinematic vector field* is a smooth section of the bundle TM ([20, Section 32.1]).

The *kinematic cotangent bundle* $T'M$ is defined as the union of the convenient vector spaces $T'_x M$ of all bounded linear functionals on the convenient vector space $T_x M$ endowed with an equivalence class of atlases ([20, Section 33.1]). A *kinematic 1-form* is a smooth section of the bundle $T'M$ ([20, Section 33.2]).

3. Partial almost-Dirac structures

3.1. The Pontryagin bundle

Throughout this section, M denotes a convenient manifold modeled on a convenient space $\mathbb{M}$. We write $p_{TM} : TM \rightarrow M$ for its kinematic bundle ([20, Section 28.1]) and $p_{T'M} : T'M \rightarrow M$ for its associated convenient dual bundle ([20, Section 33.1]).

A *weak subbundle* $p_{T^b M} : T^b M \rightarrow M$ of $T'M$ is a linear subbundle $T^b M$ of $T'M$ modeled on a convenient space $\mathbb{M}^b$ such that the inclusion $T^b M \hookrightarrow T'M$ is an injective convenient bundle morphism (cf. [17], Chap. 3). Without loss of generality, we shall assume that $\mathbb{M}^b$ is a subspace of $\mathbb{M}'$ and that the

²Let $\mathbb{E}$ and $\mathbb{F}$ be convenient vector spaces and let $T : \mathbb{E} \rightarrow \mathbb{F}$ be a linear map. Then the following conditions are equivalent:

- T is bounded, i.e., it maps bounded subsets of $\mathbb{E}$ to bounded subsets of $\mathbb{F}$;
- T is C^∞ -smooth;
- T is c^∞ -continuous, i.e., continuous with respect to the c^∞ -topologies on $\mathbb{E}$ and $\mathbb{F}$.

³The second one, $D_a\mathbb{E}$, corresponds to a family of bounded derivations.

inclusion $\mathbb{M}^b \hookrightarrow \mathbb{M}'$ is bounded. Thus $p_{T^b M}$ is simply the restriction of $p_{T' M}$ to $T^b M$.

Once a weak subbundle $T^b M$ has been fixed, the annihilator of a closed subbundle $\Delta \subset TM$ is defined fiberwise by:

$$\Delta_x^0 = \{\alpha \in T_x^b M : \forall u \in \Delta_x, \alpha(u) = 0\}$$

for every $x \in M$.

The bundle

$$\Delta^0 = \bigcup_{x \in M} \Delta_x^0$$

is called the *annihilator* of Δ in $T^b M$. It is a closed subbundle of $T^b M$.

The *Pontryagin bundle* associated with the weak subbundle $T^b M$ is the vector bundle

$$\pi_M^p : T^p M = TM \oplus T^b M \rightarrow M.$$

Let (U, ϕ) be a chart such that U is simply connected. As usual in convenient calculus, we may assume that U is a c^∞ -open set in M . The above bundles admit the following local trivialisations:

$$\begin{aligned} TM|_U &\cong U \times \mathbb{M}, & T' M|_U &\cong U \times \mathbb{M}', \\ T^b M|_U &\cong U \times \mathbb{M}^b, & T^p M &\cong U \times \mathbb{M} \times \mathbb{M}^b. \end{aligned}$$

The corresponding local coordinates are denoted by

$$\begin{aligned} (x, v) &\in U \times \mathbb{M}, & (x, p) &\in U \times \mathbb{M}', \\ (x, p) &\in U \times \mathbb{M}^b, & (x, v, p) &\in U \times \mathbb{M} \times \mathbb{M}^b. \end{aligned}$$

These trivialisations induce the following local expressions for the bundle projections:

$$\begin{aligned} p_{TM} &\cong (x, v) \mapsto x, & p_{T' M} &\cong (x, p) \mapsto x, \\ \text{pr} : T^p M \rightarrow TM &\cong (x, v, p) \mapsto (x, v), & \text{pr}^b : T^p M \rightarrow T^b M &\cong (x, v, p) \mapsto (x, p). \end{aligned}$$

3.2. Some canonical maps on double-vector bundles

We consider the two projections associated with the double-vector bundle structures of $TT' M$ and $T'TM$:

$$\pi_{T'TM} : T'TM \rightarrow TM \quad \text{and} \quad Tp_{T' M} : TT' M \rightarrow TM.$$

In local coordinates induced by a chart of M , they are given by

$$\pi_{T'TM} \cong (x, p, \dot{x}, \dot{p}) \mapsto (x, \dot{x}) \quad \text{and} \quad Tp_{T' M} \cong (x, \dot{x}, p, \dot{p}) \mapsto (x, \dot{x}).$$

As in finite dimensions, there exists a unique smooth diffeomorphism

$$\kappa_M : TT' M \rightarrow T'TM,$$

called the *canonical flip*, which identifies the two double-vector bundle structures. It is characterized by

$$\pi_{T'TM} \circ \kappa_M = Tp_{T' M}, \quad \text{and} \quad \pi_{TT' M} = p_{T' M} \circ \kappa_M^{-1}.$$

In local coordinates, the map κ_M is given by

$$\kappa_M \cong (x, p, \dot{x}, \dot{p}) \mapsto (x, \dot{x}, p, \dot{p}).$$

Thus, κ_M exchanges the two vector-bundle structures of the double-vector bundle while leaving the base variables unchanged.

Let Ω be the weak canonical symplectic form on $T'M$. Then $\Omega^b : TT'M \rightarrow T'T'M$ is an injective convenient morphism whose range is a weak subbundle $T^bT'M$ of $T'T'M$. Moreover, Ω^b induces an isomorphism of vector bundles between $TT'M$ and $T^bT'M$ (cf. [20, Section 48]). Since $T^bT'M$ is isomorphic to $TT'M$, over a chart domain U of M , we obtain the local trivialisation:

$$T^bT'M \cong (U \times \mathbb{M}') \times (\mathbb{M}' \times \mathbb{M}),$$

where $\mathbb{M}' \times \mathbb{M}$ is viewed as a subspace of $\mathbb{M}' \times \mathbb{M}''$.

Denote by (x, p, a, w) , the local coordinates on $T'T'M \cong (U \times \mathbb{M}') \times (\mathbb{M}' \times \mathbb{M}'')$. Thus, in local coordinates, the bundle morphism $\Omega^b \cong TT'M \rightarrow T'T'M$ is given by:

$$\Omega^b \cong (x, p, \dot{x}, \dot{p}) \mapsto (x, p, a = -\dot{p}, w = \dot{x}).$$

Here, in the target space, $\dot{x}$ is identified with its image under the canonical injection $\mathbb{M} \rightarrow \mathbb{M}''$, so that $w \in \mathbb{M}''$. Consequently, the composition

$$\gamma_M = \Omega^b \circ \kappa_M^{-1} : T'TM \rightarrow T^bT'M$$

is a smooth diffeomorphism.

In local coordinates, γ_M is given by:

$$\gamma_M \cong (x, \dot{x}, p, \dot{p}) \mapsto (x, p, -\dot{p}, \dot{x}).$$

In particular, the quadruple $(x, p, -\dot{p}, \dot{x})$ provides a system of local coordinates on the weak subbundle $T^bT'M$.

Remark 3.1. In the convenient setting, one obtains a canonical diffeomorphism that formally generalizes the classical Tulczyjew isomorphism in finite dimensions (cf. [22]). It should be emphasized that, in general, this is not a diffeomorphism $T'TM \rightarrow T'T'M$ but rather a diffeomorphism from $T'TM$ onto the weak subbundle $T^bT'M \subset T'T'M$. However, if the convenient space $\mathbb{M}$ is reflexive, then the canonical injection $\mathbb{M} \hookrightarrow \mathbb{M}''$ is an isomorphism and Ω^b is surjective. In this case,

$$T^bT'M = T'T'M,$$

and one recovers the classical Tulczyjew isomorphism exactly as in the finite-dimensional setting.

3.3. Partial almost-Dirac structures

Partial almost-Dirac structures provide a natural geometric framework for treating tangent and cotangent variables simultaneously. In the convenient setting, they are defined on a weak Pontryagin bundle

$$T^{\mathfrak{p}}M = TM \oplus T^{\flat}M,$$

where $T^{\flat}M$ is a weak subbundle of the kinematic cotangent bundle $T'M$. This construction extends the classical finite-dimensional theory by replacing the cotangent bundle with a suitable weak dual bundle.

Throughout this subsection, we fix a weak subbundle $T^{\flat}M$ of $T'M$ and consider the associated weak Pontryagin bundle $T^{\mathfrak{p}}M$.

Let $\langle \cdot, \cdot \rangle$ denote the canonical pairing between $T'M$ and TM . Its restriction to $T^{\flat}M \times TM$ induces a bilinear form, which is bounded since the injection $T^{\flat}M \hookrightarrow T'M$ is bounded. If there is no ambiguity, this restriction will be also denoted by $\langle \cdot, \cdot \rangle$.

The Pontryagin bundle $T^{\mathfrak{p}}M$ is endowed with the symmetric pairing $\langle\langle \cdot, \cdot \rangle\rangle$ defined as follows. For every $x \in M$, $X_x, Y_x \in T_xM$, and every $\alpha_x, \beta_x \in T_x^{\flat}M$, we set:

$$\langle\langle (X_x, \alpha_x), (Y_x, \beta_x) \rangle\rangle = \beta_x(X_x) + \alpha_x(Y_x). \quad (3.1)$$

Let D be a closed subbundle of $T^{\mathfrak{p}}M$. Its orthogonal complement $D^{\perp}$ is the subbundle whose fiber at $x \in M$ is defined by

$$D_x^{\perp} = \{(u, \alpha) \in T_x^{\mathfrak{p}}M : \forall (v, \beta) \in D_x, \langle\langle (u, \alpha), (v, \beta) \rangle\rangle = 0\}.$$

Definition 3.2. A partial almost-Dirac structure on M is a closed subbundle D of $T^{\mathfrak{p}}M$ satisfying

$$D^{\perp} = D.$$

For every open set U in M , the space of local sections $\Gamma(T^{\mathfrak{p}}M|_U)$ is endowed with a *Courant bracket* defined by

$$[(X, \alpha), (Y, \beta)]_C = \left([X, Y], L_X\beta - L_Y\alpha - \frac{1}{2}d(i_Y\alpha - i_X\beta) \right) \quad (3.2)$$

for all sections (X, α) and (Y, β) in $\Gamma(T^{\mathfrak{p}}M|_U)$. Here $[\cdot, \cdot]$ denotes the usual Lie bracket of vector fields, while L_X , i_X , and d denote, respectively, the Lie derivative, the contraction, and the exterior differential associated with the convenient differential calculus. The above expression defines a section of $T^{\mathfrak{p}}M|_U$ (cf. [16], 3.3). Note that $[(X, \alpha), (Y, \beta)]_C$ always belongs to $\Gamma(T^{\mathfrak{p}}M|_U)$ (cf. [17], 3.2).

3.4. Induced partial almost-Dirac structures

In this subsection, we define a partial almost-Dirac structure on the cotangent bundle that plays a key role for implicit Lagrangians mirroring the finite-dimensional case (cf. [5], for instance). We then adapt this framework to our current context.

Let Δ_M be a closed⁴ subbundle of TM called a *constraint distribution*. To lift these constraints to the cotangent bundle $T'M$, where the Hamiltonian formulation takes place, we consider the tangent directions of $T'M$ whose projection onto TM remains admissible. This leads naturally to the following induced distribution on $T'M$:

$$\Delta_{T'M} = (Tp_{T'M})^{-1}(\Delta_M) \subset TT'M,$$

where $Tp_{T'M} : TT'M \rightarrow TM$ is the tangent map of the projection $p_{T'M} : T'M \rightarrow M$.

Let Ω be the weak canonical symplectic form on $T'M$. We associate to $\Delta_{T'M}$ the subset D_{Δ_M} of $T^*T'M = TT'M \oplus T^*T'M$ defined, for each $(x, p) \in T'M$, by:

$$(D_{\Delta_M})_{(x,p)} = \left\{ (u, \alpha) \in T^*T'M : u \in (\Delta_{T'M})_{(x,p)}, \alpha - \Omega_{(x,p)}^b(u) \in (\Delta_{T'M}^0)_{(x,p)} \right\}.$$

We then set

$$D_{\Delta_M} := \bigcup_{(x,p) \in T'M} (D_{\Delta_M})_{(x,p)}.$$

As in [23, Theorem 2.3], we have the following result:

Proposition 3.3. *Let*

$$\widetilde{\Omega}^b : \Delta_{T'M} \rightarrow T^*T'M$$

denote the restriction of Ω^b to $\Delta_{T'M}$. Then D_{Δ_M} can be written as

$$D_{\Delta_M} = \text{Graph}(\widetilde{\Omega}^b) \oplus (\{0\} \times \Delta_{T'M}^0),$$

and it constitutes a partial almost-Dirac structure on $T'M$.

Proof. For each $(x, p) \in T'M$, by the definition of D_{Δ_M} , an element

$$(u, \alpha) \in (D_{\Delta_M})_{(x,p)}$$

satisfies

$$u \in (\Delta_{T'M})_{(x,p)} \quad \text{and} \quad \alpha - \Omega_{(x,p)}^b(u) \in (\Delta_{T'M}^0)_{(x,p)}.$$

Hence, there exists $\beta \in (\Delta_{T'M}^0)_{(x,p)}$ such that

$$\alpha = \widetilde{\Omega}_{(x,p)}^b(u) + \beta.$$

Therefore,

$$(D_{\Delta_M})_{(x,p)} = \text{Graph}(\widetilde{\Omega}_{(x,p)}^b) \oplus (\{0\} \times (\Delta_{T'M}^0)_{(x,p)}).$$

By Lemma 3.6 (2), this is a partial linear Dirac structure on $T_{(x,p)}T'M$.

Moreover, $\Delta_{T'M}$ is a closed subbundle of $TT'M$. Since $\Omega^b : TT'M \rightarrow T^*T'M$ is a vector bundle isomorphism, its restriction

$$\widetilde{\Omega}^b : \Delta_{T'M} \longrightarrow T^*T'M$$

has a closed image. Hence, $\widetilde{\Omega}^b(\Delta_{T'M})$ is a closed subbundle of $T^*T'M$. Consequently, by Definition 3.16 in [16], D_{Δ_M} is a partial almost-Dirac structure on $T'M$. $\square$

⁴The assumption that Δ_M is a closed subbundle is essential in the convenient setting. In contrast to the finite-dimensional case, a linear subspace of a convenient vector space need not be closed, and therefore need not define a convenient subspace.

Remark 3.4.

1. In local coordinates induced by the chart (U, ϕ) , and as in the finite-dimensional case (cf. [23]), the partial almost-Dirac structure D_{Δ_M} is locally described by

$$D_{\Delta_M} \cong \{((x, p, \dot{x}, \dot{p}), (x, p, a, w)) : \\ x \in U = \phi(U), \quad p \in \mathbb{M}', \quad \dot{x} \in \mathbb{E}, \quad w = \dot{x}, \quad a + \dot{p} \in \mathbb{E}^0\}, \quad (3.3)$$

where $\mathbb{E}$ denotes the typical fiber of Δ_M .

2. According to [16], Proposition 3.27, since Ω is closed, the almost-Dirac structure D_{Δ_M} is a Dirac structure if and only if the constraint distribution Δ_M is involutive.

4. Implicit Lagrangian and Hamiltonian systems

In this section, we extend to the convenient setting the theory of implicit Lagrangian and Hamiltonian systems developed by H. Yoshimura, H. Jacobs, J. E. Marsden, and A. J. van der Schaft (see, for instance, [5, 23–26]). We first formulate implicit Lagrangian systems associated with constrained Lagrangians and establish their fundamental properties. We then introduce the notion of an implicit Hamiltonian system associated with a partial almost-Dirac structure. Finally, we show that every regular constrained Lagrangian naturally induces an implicit Hamiltonian system through the restricted Legendre transformation. In particular, we prove the conservation of the generalized energy and establish the correspondence between the Lagrangian and Hamiltonian formulations.

Unless otherwise stated, throughout this section, M denotes a fixed convenient connected manifold and Δ_M a closed subbundle of TM . According to § 3.4, we consider the associated closed subbundle

$$\Delta_{T'M} = (Tp_{T'M})^{-1}(\Delta_M) \subset TT'M.$$

Let Ω be the weak canonical symplectic form on $T'M$ and let $T^b T'M$ denote the range of the bundle morphism

$$\Omega^b : TT'M \rightarrow T'T'M.$$

Then $T^b T'M$ is a weak subbundle of $T'T'M$. The weak Pontryagin bundle associated with $T'M$ is therefore

$$T^p T'M = TT'M \oplus T^b T'M.$$

The subbundle $\Delta_{T'M}$ induces a partial almost-Dirac structure D_{Δ_M} on $T'M$ as described in § 3.4. We shall use throughout the local coordinates introduced in § 3.2.

4.1. Legendre transformation

A smooth map $\mathbf{L} : TM \rightarrow \mathbb{R}$ is classically called a *Lagrangian*⁵.

The associated *Legendre transformation* is defined by

$$\mathbb{F}\mathbf{L}_x(v).w = \frac{d}{ds} (\mathbf{L}(v + sw))|_{s=0}, \quad (4.1)$$

⁵In infinite-dimensional settings, for certain physical or mechanical problems, the Lagrangian is not defined on the whole tangent bundle TM . Instead, there exists a weak immersed convenient submanifold M_0 modeled on $\mathbb{M}_0$, which is in general open or dense in M , such that the Lagrangian is only defined on the pull-back of TM to M_0 (cf. [27], p. 108). This situation is studied in [7]. For simplicity, we assume throughout that the Lagrangian L is defined on the entire tangent bundle TM .

where v and w belong to $T_x M$. Here $\mathbb{F}\mathbf{L}_x(v).w$ denotes the vertical derivative of $\mathbf{L}$ at (x, v) in the direction w . The construction of [28, Section 3.5] extends to the convenient setting. Since the map $\mathbf{L}$ is smooth, it follows that the induced map $\mathbb{F}\mathbf{L} : TM \rightarrow T'M$ is a smooth fiber-preserving map. In local coordinates on TM and $T'M$, this map is given by:

$$\mathbb{F}\mathbf{L} \cong (x, v) \mapsto \left(x, p = \frac{\partial \mathbf{L}}{\partial v} \right).$$

In finite-dimensional mechanics, the Legendre transformation is called *non-degenerate* if $\mathbb{F}\mathbf{L}$ is a local diffeomorphism. In the convenient setting, such a condition does not hold in general. Therefore, in our context, we say that $\mathbb{F}\mathbf{L}$ is *non-degenerate* if $\mathbb{F}\mathbf{L}$ is locally injective on some open subset $U \subset M$; otherwise, we say that $\mathbb{F}\mathbf{L}$ is *degenerate*. In view of the notion of an implicit Lagrangian, we shall work in this more general setting, allowing $\mathbb{F}\mathbf{L}$ to be degenerate.

4.2. Dirac differential operator associated with a Lagrangian

Let $\mathbf{L} : TM \rightarrow \mathbb{R}$ be a Lagrangian. Then its differential $d\mathbf{L}$ is a 1-form on TM . As introduced in [5, 23], the *Dirac differential operator* of $\mathbf{L}$ is defined by

$$\mathfrak{D}\mathbf{L} = \gamma_M \circ d\mathbf{L},$$

where γ_M is the diffeomorphism

$$\gamma_M = \Omega^b \circ \kappa_M^{-1} : T'TM \rightarrow T^bT'M$$

introduced in § 3.2. In local coordinates, the Dirac differential operator is given by

$$\mathfrak{D}\mathbf{L}(x, v) \cong \left(x, \frac{\partial \mathbf{L}}{\partial v}, -\frac{\partial \mathbf{L}}{\partial x}, v \right). \quad (4.2)$$

4.3. Implicit Lagrangian systems

In the setting of § 3.4, following [5], we introduce the following notion.

Definition 4.1. Consider a Lagrangian $\mathbf{L} : TM \rightarrow \mathbb{R}$ and a closed subbundle Δ_M of TM . Let $\Delta_{T'M} = (Tp_{T'M})^{-1}(\Delta_M)$ denote the associated closed subbundle of $TT'M$, let D_{Δ_M} be the induced partial almost-Dirac structure on $T'M$, and let

$$\mathfrak{D}\mathbf{L} : TM \rightarrow T^bT'M$$

be the Dirac differential operator of $\mathbf{L}$.

The triple $(M, \Delta_M, \mathfrak{D}\mathbf{L})$ is called an *implicit Lagrangian system*.

An *integral curve of an implicit Lagrangian system* $(M, \Delta_M, \mathfrak{D}\mathbf{L})$ is a curve

$$\begin{aligned} \gamma : I &\rightarrow TM \oplus T'M \\ t &\mapsto (x(t), v(t), p(t)) \end{aligned}$$

defined on an interval $I \subset \mathbb{R}$, satisfying

$$(x(t), p(t), \dot{x}(t), \dot{p}(t), \mathfrak{D}\mathbf{L}(x(t), v(t))) \in D_{\Delta_M}(x(t), p(t)) \quad (4.3)$$

for all $t \in I$.

We define

$$\mathbb{P} := \mathbb{FL}(\Delta_M) \subset T'M.$$

Remark 4.2. Let $t \mapsto (x(t), v(t), p(t))$ be an integral curve of the implicit Lagrangian system $(M, \Delta_M, \mathfrak{DL})$. Then

$$(x(t), v(t)) \in \Delta_M, \quad \mathbb{FL}(x(t), v(t)) = (x(t), p(t))$$

for every t . Consequently,

$$(x(t), v(t), p(t)) \in \Delta_M \oplus \mathbb{P}.$$

In the local coordinates introduced in § 3.4, if

$$(x(t), v(t), p(t)) \cong (\mathbf{x}(t), \mathbf{v}(t), \mathbf{p}(t)),$$

then, by (3.3) and (4.2), condition (4.3) is equivalent to

$$\mathbf{p} = \frac{\partial \mathbf{L}}{\partial \mathbf{v}}, \quad \dot{\mathbf{x}} \in \mathbb{E}, \quad \dot{\mathbf{x}} = \mathbf{v}, \quad \dot{\mathbf{p}} - \frac{\partial \mathbf{L}}{\partial \mathbf{x}} \in \mathbb{E}^0. \quad (4.4)$$

Equivalently, if

$$\mathbf{L}(x, v) \cong \mathbf{L}(\mathbf{x}, \mathbf{v}),$$

then an integral curve

$$t \mapsto (x(t), v(t), p(t)) \cong (\mathbf{x}(t), \mathbf{v}(t), \mathbf{p}(t))$$

satisfies the following implicit system of differential equations:

$$\begin{cases} \begin{pmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{p}} \end{pmatrix} = \begin{pmatrix} 0 & \text{Id}_{\mathbb{M}} \\ -\text{Id}_{\mathbb{M}'} & 0 \end{pmatrix} \begin{pmatrix} -\frac{\partial \mathbf{L}}{\partial \mathbf{x}} \\ \mathbf{v} \end{pmatrix} + \begin{pmatrix} 0 \\ \alpha(\mathbf{x}) \end{pmatrix}, & \text{for some } \alpha : U \rightarrow \mathbb{E}^0, \\ \mathbf{p} = \frac{\partial \mathbf{L}}{\partial \mathbf{v}}, \\ \langle \beta(\mathbf{x}), \mathbf{v} \rangle = 0, & \text{for any } \beta : U \rightarrow \mathbb{E}^0. \end{cases} \quad (4.5)$$

4.4. Non-degenerate constraint Lagrangians

In this subsection, we allow the Lagrangian $\mathbf{L}$ to be degenerate.

Nevertheless, the Legendre transformation $\mathbb{FL} : TM \rightarrow T'M$ is still well-defined. We set

$$\mathbb{P} := \mathbb{FL}(\Delta_M) \subset T'M.$$

We say that $\mathbf{L}$ is a *non-degenerate Δ_M -constraint Lagrangian* if, for every $x \in M$, there exists an open neighborhood U of x such that the restriction

$$\mathbb{FL}|_{(\Delta_M)|_U} : (\Delta_M)|_U \rightarrow T'M$$

is a weak immersion⁶. In local coordinates, this condition is equivalent to the requirement that the map

$$\mathbf{x} \mapsto \frac{\partial^2 \mathbf{L}}{\partial \mathbf{v}^2}(\mathbf{x})$$

⁶See [17, Section 3.7].

defines, on some open subset $\mathbb{U}$ of $\mathbb{M}$, a smooth field of weak non-degenerate bilinear forms on $\mathbb{E}$. Consequently, we have:

$$\mathbb{P} \cong \left\{ \left(x, \frac{\partial \mathbf{L}}{\partial \mathbf{v}}(x, v) \right), (x, v) \in \mathbb{U} \times \mathbb{E} \right\}. \quad (4.6)$$

Indeed, in local coordinates, the tangent map

$$T\mathbb{FL} : TM|_{\mathbb{U}} \cong \mathbb{U} \times \mathbb{M} \rightarrow T'M|_{\mathbb{U}} \cong \mathbb{U} \times \mathbb{M}'$$

is represented by the block matrix

$$\begin{pmatrix} \text{Id} & 0 \\ \frac{\partial^2 \mathbf{L}}{\partial x \partial v} & \frac{\partial^2 \mathbf{L}}{\partial v^2} \end{pmatrix}. \quad (4.7)$$

Therefore, the weak immersion property of the restricted Legendre transformation is determined by the vertical block

$$\frac{\partial^2 \mathbf{L}}{\partial v^2},$$

which defines a smooth field of weak non-degenerate bilinear forms on $\mathbb{E}$.

If $\mathbf{L}$ is a non-degenerate Δ_M -constraint Lagrangian, then $\mathbb{P}$ inherits the structure of a weak convenient submanifold of $T'M$, modeled on $\mathbb{M} \times \mathbb{E}^7$ via the projection onto the $\mathbb{M} \times \mathbb{E}$ coordinates. In finite dimensions, when $\mathbb{P}$ is a submanifold of $T'M$, it is called the *primary constraint* in [29]. Such situations occur in the following examples.

Example 4.3. Let M be a smooth paracompact Banach manifold and let Δ_M be a closed subbundle of TM of codimension k . Assume that there exists a weak Riemannian metric g on Δ_M . Since Δ_M is supplemented, let F be a k -dimensional supplementary subbundle of Δ_M in TM . Since M is smooth and paracompact, we can extend g to a smooth map $\mathbf{L} : TM \rightarrow \mathbb{R}$. For instance, let h be a Riemannian metric on F and extend it to a weak Riemannian metric $\tilde{g}$ on TM defined by

$$\tilde{g}|_F = h, \quad \tilde{g}|_{\Delta_M} = g, \quad \text{and} \quad \tilde{g}_x(u, v) = 0, \quad u \in (\Delta_M)_x, \quad v \in F_x.$$

Then, by the same arguments as in Example 4.3, one shows that

$$\mathbb{P} = \mathbb{FL}(\Delta_M)$$

is a weak immersed submanifold of $T'M$.

Therefore, over a coordinate domain $\mathbb{U} = \phi(U) \subset \mathbb{M}$, there exist smooth maps

$$\alpha_i : \mathbb{U} \rightarrow \mathbb{E}^0, \quad i \in \{1, \dots, k\},$$

such that

$$(\alpha_1(x), \dots, \alpha_k(x))$$

⁷Recall that $\mathbb{E}$ denotes the typical fiber of Δ_M .

forms a basis of $\mathbb{E}^0$. Hence, in the associated local coordinates, there exist Lagrange multipliers $\lambda_1, \dots, \lambda_k$ such that the associated implicit Lagrangian system can be written as (cf. [30]):

$$\begin{cases} \dot{x} = v, & \dot{p} = \frac{\partial L}{\partial x}(x, v) + \sum_{i=1}^k \lambda_i(x) \alpha_i(x), \\ \langle \alpha_i(x), v \rangle = 0, \quad i \in \{1, \dots, k\}, & p = \frac{\partial L}{\partial v}(x, v). \end{cases} \quad (4.8)$$

Note that these equations only depend on the restriction of L to Δ_M .

Example 4.4. The heavy rope.

Consider a non-elastic rope. We model such a rope by a map

$$x : [0, \ell] \rightarrow \mathbb{R}^2$$

belonging to the Sobolev space $H^1([0, \ell], \mathbb{R}^2)$ ⁸ and satisfying $\|x'\|_2 = 1$, a.e. (cf. [31]).

The configuration manifold is then the Hilbert space $M = H^1([0, \ell], \mathbb{R}^2)$. This space is isomorphic⁹ to $\mathbb{R}^2 \times L^2([0, \ell], \mathbb{R}^2)$. The constraint condition is the vector subbundle¹⁰

$$\Delta_M = \left\{ (x, v) \in H^1([0, \ell], \mathbb{R}^2) \times H^1([0, \ell], \mathbb{R}^2) : \int_0^\ell x'(\sigma) v(\sigma)^\top d\sigma = 0 \right\}$$

of TM of codimension 1. The rope has a mass density per unit length

$$\rho : [0, \ell] \rightarrow]0, +\infty[,$$

which we assume to be smooth. For every $\sigma \in [0, \ell]$, $\rho(\sigma) > 0$, by a compactness argument, it follows that we have some positive constant C such that $0 < \|\rho(\sigma)\| \leq C$. The rope is subjected to a uniform gravitational field $(0, g)$. The corresponding potential energy is given by:

$$E_{\text{pot}}(x) = \int_0^\ell \rho(\sigma) x_2(\sigma) g d\sigma,$$

where x_2 denotes the second component of $x \in H^1([0, \ell], \mathbb{R}^2)$.

The kinematic energy is:

$$E_{\text{kin}}(v) = \frac{1}{2} \int_0^\ell \rho(\sigma) \|v\|_2^2 d\sigma.$$

But, in this context, using the Riesz representation for the Hilbert space $H^1([0, \ell], \mathbb{R}^2)$ according to the inner product

$$\langle (x, v), (y, w) \rangle = \langle x, y \rangle_{\mathbb{R}^2} + \langle v, w \rangle_{L^2},$$

the bundle Δ_M is defined by the constraint $v \mapsto \langle \alpha(x), v \rangle := \langle x', v \rangle_{L^2}$. The Lagrangian associated to this physical problem is

$$L(x, v) = \langle \rho \cdot v, v \rangle_{L^2} - \langle \rho \cdot x', (0, g) \rangle_{L^2}.$$

⁸That is, x is continuous and its weak derivative x' belongs to $L^2([0, \ell], \mathbb{R}^2)$.

⁹This is the classical identification of $H^1([0, \ell], \mathbb{R}^2)$ with $\mathbb{R}^2 \times L^2([0, \ell], \mathbb{R}^2)$ via the map $x \mapsto (x(0), x')$, a consequence of the fundamental theorem of calculus for Sobolev functions.

¹⁰Here x' denotes the derivative of x relative to σ .

Thus, by assumption on ρ , the restriction of the Hessian $\frac{\partial^2 \mathbf{L}}{\partial \mathbf{v}^2}(x)$, in local coordinates, corresponding to the restriction of $\mathbf{L}$ to $(\Delta_M)_x$ is non-degenerate, since it is positive definite.

Therefore $\mathbb{P}$ is an immersed submanifold of

$$T'H^1([0, \ell], \mathbb{R}^2) \equiv TH^1([0, \ell], \mathbb{R}^2) = H^1([0, \ell], \mathbb{R}^2) \times H^1([0, \ell], \mathbb{R}^2),$$

which, via the Riesz representation theorem, can be identified with Δ_M .

The associated implicit Lagrangian problem is then

$$\begin{cases} \dot{x} = v, & \dot{p} = -\rho \begin{pmatrix} 0 \\ g \end{pmatrix} + \lambda(x)x', \\ \langle x', v \rangle = 0, & p = \rho.v. \end{cases} \quad (4.9)$$

4.5. Conservation of the generalized energy

Given an implicit Lagrangian system $(M, D_{\Delta_M}, \mathfrak{D}\mathbf{L})$, following [5] and [23] in finite dimensions, the generalized energy $\mathbf{E}$ is defined on the Whitney sum $TM \oplus T'M$ by

$$\mathbf{E}(x, v, p) := \langle p, v \rangle - \mathbf{L}(x, v). \quad (4.10)$$

Let $t \mapsto (x(t), v(t), p(t))$ be an integral curve of (4.3). To analyze the evolution of the generalized energy along such a curve, we compute the total time derivative of $\mathbf{E}$:

$$\begin{aligned} \frac{d}{dt} \mathbf{E}((x(t), v(t), p(t))) &= \langle \dot{p}, v \rangle + \langle p, \dot{v} \rangle - \frac{\partial \mathbf{L}}{\partial x} \dot{x} - \frac{\partial \mathbf{L}}{\partial v} \dot{v} \\ &= \langle \dot{p} - \frac{\partial \mathbf{L}}{\partial x}, v \rangle \\ &= 0 \end{aligned}$$

since $\dot{x} = v \in \Delta_M$ and $\dot{p} - \frac{\partial \mathbf{L}}{\partial x} \in \Delta_M^0$.

Thus, the generalized energy $\mathbf{E}$ is conserved along any integral curve of the implicit Lagrangian system. Note that given a generalized energy $\mathbf{E}$ on $TM \oplus T'M$, one can recover the Lagrangian from the relation (4.10). Moreover, starting from $\mathbf{E}$, the implicit Lagrangian system (4.3) can equivalently be reformulated as the implicit system:

$$(x(t), p(t), \dot{x}(t), \dot{p}(t), d\mathbf{E}(x(t), v(t), p(t))) \in D_{\Delta_M}(x(t), p(t)) \quad (4.11)$$

4.6. Implicit Hamiltonian systems

The concept of an implicit Hamiltonian system associated with a Dirac structure is classical in finite dimensions. In this section, we adapt this notion to the convenient setting of partial almost-Dirac structures, following the approach of A. J. van der Schaft [24–26].

Definition 4.5. Let D be a partial almost-Dirac structure on a convenient manifold M , relative to the Pontryagin bundle

$$T^{\mathfrak{p}}M = TM \oplus T^{\flat}M,$$

and let $H : M \rightarrow \mathbb{R}$ be a smooth Hamiltonian.

1. The implicit Hamiltonian system associated with (M, D, H) is the differential inclusion

$$(\dot{x}(t), d_{x(t)}H) \in D_{x(t)}.$$

2. A smooth curve $\gamma : I \rightarrow M$ satisfying

$$(\dot{\gamma}(t), d_{\gamma(t)}H) \in D_{\gamma(t)}$$

for every $t \in I$ is called an integral curve of the implicit Hamiltonian system.

3. Let S be a weak immersed submanifold of M such that $H|_S$ is smooth. The induced implicit Hamiltonian system on S is defined by

$$(M, D, H|_S),$$

whose integral curves are the smooth curves $\gamma : I \rightarrow S$ satisfying

$$(\dot{\gamma}(t), d_{\gamma(t)}H|_S) \in D_{\gamma(t)}.$$

Remark 4.6. If γ is an integral curve of an implicit Hamiltonian system (M, D, H) , then

$$\frac{d}{dt}H(\gamma(t)) = \langle dH, \dot{\gamma} \rangle = 0,$$

since $\langle \alpha, u \rangle = 0$ for every $(u, \alpha) \in D$. Thus the Hamiltonian is conserved along integral curves.

4.7. Implicit Hamiltonian system associated to a Lagrangian

Let $\mathbf{L}$ be a non-degenerate Δ_M -constraint Lagrangian on TM and set

$$\mathbb{P} = \mathbb{FL}(\Delta_M).$$

Then $\mathbb{P}$ is a weak immersed submanifold of $T'M$. We now show that $\mathbf{L}$ naturally induces an implicit Hamiltonian system on $\mathbb{P}$. More precisely, we adapt to the convenient setting the construction of the Hamiltonian associated with a constrained Lagrangian developed by H. Yoshimura and J. E. Marsden (see, for instance, [5, 23]).

Theorem 4.7. Let M be a convenient manifold and let Δ_M be a closed convenient subbundle of TM . Assume that $\mathbf{L}$ is a Δ_M -constraint regular Lagrangian and that the restricted Legendre transform

$$\mathbb{FL}|_{\Delta_M} : \Delta_M \longrightarrow \mathbb{P}$$

is injective.

Then the following assertions hold.

(1) The function $H_{\Delta_M} : \Delta_M \rightarrow \mathbb{R}$, given by

$$H_{\Delta_M}(x, v) = \mathbf{E}(x, v, \mathbb{FL}|_{\Delta_M}(x, v)),$$

is smooth. Moreover, it induces a smooth function

$$H_{\mathbb{P}} : \mathbb{P} \longrightarrow \mathbb{R},$$

defined by

$$H_{\mathbb{P}}(x, p) = H_{\Delta_M}((\mathbb{FL}|_{\Delta_M})^{-1}(x, p)).$$

(2) Let

$$H(x, v, p) = \langle p, v \rangle - \mathbf{L}(x, v).$$

The restriction of H to the graph of the restricted Legendre transform

$$\text{Graph}(\mathbb{F}\mathbf{L}|_{\Delta_M}) \subset \Delta_M \times T'M$$

coincides with $H_{\mathbb{P}}$. Consequently, the partial almost-Dirac structure D_{Δ_M} defines the implicit Hamiltonian system

$$(M, D_{\Delta_M}, H_{\mathbb{P}})$$

on $\mathbb{P}$.

In local coordinates, this system is characterized by

$$\dot{x} = \frac{\partial H}{\partial p}(x, v, p), \quad \dot{p} + \frac{\partial H}{\partial x}(x, v, p) \in \mathbb{E}^0,$$

together with the constraint

$$\left. \frac{\partial H}{\partial v}(x, v, p) \right|_{\mathbb{E}} = 0.$$

(3) A smooth curve $t \mapsto (x(t), p(t))$ in $\mathbb{P}$ is an integral curve of the induced implicit Hamiltonian system

$$(M, D_{\Delta_M}, H_{\mathbb{P}})$$

if and only if there exists a smooth curve $t \mapsto (x(t), v(t))$ in Δ_M such that

$$\mathbb{F}\mathbf{L}|_{\Delta_M}(x(t), v(t)) = (x(t), p(t)),$$

and such that $t \mapsto (x(t), v(t))$ is an integral curve of the implicit Lagrangian system

$$(M, \Delta_M, \mathfrak{D}\mathbf{L}).$$

This theorem motivates the following definition:

Definition 4.8. Let $\mathbf{L}$ be a Δ_M -constraint regular Lagrangian such that the restricted Legendre transform

$$\mathbb{F}\mathbf{L}|_{\Delta_M} : \Delta_M \rightarrow \mathbb{P}$$

is injective.

The implicit Hamiltonian system $(M, D_{\Delta_M}, H_{\mathbb{P}})$, obtained from the construction of the previous theorem, is called the induced implicit Hamiltonian system associated with $\mathbf{L}$.

The weak immersed submanifold $\mathbb{P} = \mathbb{F}\mathbf{L}(\Delta_M)$ is called the integrability domain of the induced implicit Hamiltonian system $(M, D_{\Delta_M}, H_{\mathbb{P}})$.

Remark 4.9.

1. The terminology *integrability domain* of $(M, D_{\Delta_M}, H_{\mathbb{P}})$ is justified by items (2) and (3) of Theorem 4.7, which establish the correspondence between integral curves of the induced implicit Hamiltonian system and those of the implicit Lagrangian system.

2. Unlike the finite-dimensional case, in the convenient infinite-dimensional setting, the manifold $\mathbb{P}$ is, in general, only a weak immersed submanifold of T^bM . Consequently, one cannot, in general, expect the existence of a smooth extension of

$$H_{\mathbb{P}} : \mathbb{P} \rightarrow \mathbb{R}$$

to a neighborhood of $\mathbb{P}$ in T^bM . The preceding theorem shows that such an extension is not required for the definition of the induced implicit Hamiltonian system.

3. In local coordinates, by item (2) of Theorem 4.7, and under the constraint

$$\frac{\partial H}{\partial v}(x, v, p)_{|\mathbb{E}} = 0,$$

the Hamiltonian part of the induced implicit Hamiltonian system is determined only by $H_{\mathbb{P}}$. Indeed, if $H_{\mathbb{P}} \cong H_{\mathbb{P}}$, then one has

$$\dot{x} = \frac{\partial H_{\mathbb{P}}}{\partial p}(x, v, p), \quad \dot{p} + \frac{\partial H_{\mathbb{P}}}{\partial x}(x, v, p) \in \mathbb{E}^0,$$

which involves only $H_{\mathbb{P}}$ and does not require any smooth extension of $H_{\mathbb{P}}$ to a neighborhood of $\mathbb{P}$ in T^bM . This differs from the finite-dimensional construction of H. Yoshimura and J. E. Marsden, where such extensions naturally appear.

Proof of Theorem 4.7. (1) Consider the subset

$$\text{Stab}(\mathbf{E}) = \left\{ (x, p, v) \in TM \oplus T^*M : \frac{\partial \mathbf{E}}{\partial v}(x, p, v) = 0 \right\}.$$

Since

$$\mathbf{E}(x, v, p) = \langle p, v \rangle - \mathbf{L}(x, v),$$

we have

$$\frac{\partial \mathbf{E}}{\partial v}(x, p, v) = p - \frac{\partial \mathbf{L}}{\partial v}(x, v).$$

Hence,

$$(x, p, v) \in \text{Stab}(\mathbf{E}) \iff p = \mathbb{F}\mathbf{L}(x, v).$$

Therefore,

$$H_{\Delta_M}(x, v) = \mathbf{E}(x, v, \mathbb{F}\mathbf{L}_{|\Delta_M}(x, v))$$

defines a smooth function on Δ_M .

Since the restricted Legendre transform

$$\mathbb{F}\mathbf{L}_{|\Delta_M} : \Delta_M \longrightarrow \mathbb{P}$$

is assumed to be injective, every $(x, p) \in \mathbb{P}$ admits a unique preimage $(x, v) \in \Delta_M$. Consequently,

$$H_{\mathbb{P}}(x, p) = H_{\Delta_M} \left((\mathbb{F}\mathbf{L}_{|\Delta_M})^{-1}(x, p) \right)$$

is well-defined. Since $\mathbb{F}\mathbf{L}_{|\Delta_M}$ is regular, its local inverse is smooth, and therefore $H_{\mathbb{P}}$ is smooth on $\mathbb{P}$.

(2) Let

$$H(x, v, p) = \langle p, v \rangle - \mathbf{L}(x, v).$$

Its restriction to the graph of the restricted Legendre transform coincides with $H_{\mathbb{P}}$. Hence the induced implicit Hamiltonian system

$$(M, D_{\Delta_M}, H_{\mathbb{P}})$$

is well-defined.

In local coordinates,

$$p = \frac{\partial L}{\partial v}(x, v),$$

and

$$\frac{\partial H}{\partial p} = v, \quad \frac{\partial H}{\partial x} = -\frac{\partial L}{\partial x}, \quad \frac{\partial H}{\partial v} = p - \frac{\partial L}{\partial v}.$$

Therefore the implicit Hamilton equations are

$$\dot{x} = \frac{\partial H}{\partial p}, \quad \dot{p} + \frac{\partial H}{\partial x} \in \Delta_M^0,$$

together with the constraint

$$\frac{\partial H}{\partial v}|_{\Delta_M} = 0,$$

which is precisely the Legendre relation

$$p = \frac{\partial L}{\partial v}(x, v).$$

(3) Let $t \mapsto (x(t), p(t))$ be an integral curve of

$$(M, D_{\Delta_M}, H_{\mathbb{P}}),$$

and let

$$(x(t), v(t)) = (\mathbb{F}\mathbf{L}|_{\Delta_M})^{-1}(x(t), p(t)).$$

Using the identities established in (2), the Hamilton equations become

$$\dot{x} = v \in \Delta_M$$

and

$$\dot{p} - \frac{\partial L}{\partial x}(x, v) \in \Delta_M^0,$$

while the constraint

$$\frac{\partial H}{\partial v} = 0$$

gives

$$p = \frac{\partial L}{\partial v}(x, v).$$

These are exactly the local equations of the implicit Lagrangian system $(M, \Delta_M, \mathfrak{D}\mathbf{L})$.

Conversely, if $t \mapsto (x(t), v(t))$ is an integral curve of the implicit Lagrangian system and

$$p(t) = \mathbb{F}\mathbf{L}(x(t), v(t)),$$

then the above identities show that $t \mapsto (x(t), p(t))$ satisfies the implicit Hamilton equations. This completes the proof. $\square$

Example 4.10. In the setting of Example 4.3, let g be a strong Riemannian metric on a convenient manifold M . Since g is strong, the model space is necessarily a Hilbert space $\mathbb{M}$. Given any closed Banach subbundle Δ_M of TM , the strong Riemannian metric induces a strong Riemannian metric on Δ_M . In fact, the typical fiber of Δ_M is a Hilbert space $\mathbb{E}$. Hence, $\mathbb{P}$ can be identified with the dual bundle Δ'_M of Δ_M . Since $\bar{g} := g|_{\Delta_M}$ defines a strong Riemannian metric on Δ_M , the associated musical map $\bar{g}^b$ is a bundle isomorphism. We obtain a strong Riemannian co-metric $\tilde{g}$ on $\mathbb{P}$:

$$(g_{\mathbb{P}})_x(p, p') = \bar{g}\left((\bar{g}_x^b)^{-1}(p), (\bar{g}_x^b)^{-1}(p')\right).$$

The Hamiltonian is given by $H_{\mathbb{P}} = \frac{1}{2}g_{\mathbb{P}}$. We have

$$H(x, v, p) = \langle p, v \rangle - \frac{1}{2}g_x(v, v) = \frac{1}{2}g_x\left((g_x^b)^{-1}(p), (g_x^b)^{-1}(p)\right). \quad (4.12)$$

In local coordinates, if $g_{\mathbb{P}} \cong \tilde{g}_{\mathbb{P}}$, then the associated induced implicit Hamiltonian $(M, D_{\Delta_M}, H_{|T'M})$ reads:

$$\dot{x} = \frac{\partial(g_{\mathbb{P}})_x}{\partial p}(p) \in \mathbb{E}, \quad \dot{p} + \frac{1}{2} \frac{\partial(g_{\mathbb{P}})_x}{\partial x}(p) \in \mathbb{E}^0. \quad (4.13)$$

Example 4.11. Consider the setting of the heavy rope described in Example 4.4. Taking into account the Riesz representation, the Hamiltonian on the integrability domain $\mathbb{P} = \mathbb{F}\mathbf{L}(\Delta_M)$ is given by

$$H_{\mathbb{P}}(x, p) = \left\langle \frac{1}{\rho}p, (1 - \langle x', p \rangle_{L^2})p \right\rangle_{L^2} - \langle \rho x', (0, g) \rangle.$$

It is induced by the function

$$H(x, p) = \left\langle \frac{1}{\rho}p, p \right\rangle - \langle \rho x', (0, g) \rangle,$$

defined on the ambient cotangent bundle.

The corresponding induced implicit Hamiltonian system $(M, D_{\Delta_M}, H_{\mathbb{P}})$ is therefore

$$\begin{cases} \dot{x} = \frac{1}{\rho}p, & \dot{p} = -\rho \begin{pmatrix} 0 \\ g \end{pmatrix} + \lambda(x)x', \\ \langle x', p \rangle = 0, & p = \rho v. \end{cases}$$

This system is equivalent to the implicit Lagrangian system (4.9).

5. Variational approach

Following [7], we establish a connection between the variational approaches to implicit Lagrangian and Hamiltonian systems within the framework of convenient calculus, analogous to the finite-dimensional case (cf. [18]).

As emphasised in the Introduction, the general framework of locally convex spaces is ill-suited to infinite-dimensional problems arising in physics and fluid mechanics. In contrast, convenient calculus provides a natural and robust framework in which these variational and geometric constructions can be consistently developed.

5.1. Hamiltonian principle

Assume that M is a convenient manifold. Let $C^\infty(I, M)$ denote the set of smooth curves from an open interval with compact closure $I \subset \mathbb{R}$ to M . Recall that, if M admits a local addition¹¹, then $C^\infty(I, M)$ carries a structure of a convenient manifold, modeled on the convenient spaces $\Gamma_0(c^!TM)$ of compactly supported sections of the pull-back bundle $c^!TM$, where c is any curve in $C^\infty(I, M)$. In this setting, the tangent space $T_c C^\infty(I, M)$ can be identified with $\Gamma_0(c^!TM)$. In this case, it follows that, for any $X \in \Gamma_0(c^!TM)$, there exists a smooth curve

$$\hat{c} :]-\varepsilon, \varepsilon[\rightarrow C^\infty(I, M)$$

such that

$$\frac{d}{ds}\bigg|_{s=0} \hat{c} = X.$$

We now return to the general setting where M is a convenient manifold. For any $c \in C^\infty(I, M)$, the velocity field $\dot{c}$ defines a lift of c to a curve in $C^\infty(I, TM)$. Given a smooth¹² Lagrangian $\mathbf{L} : TM \rightarrow \mathbb{R}$, we define the *functional action*

$$\mathfrak{F}_{\mathbf{L}} : C^\infty(I, M \rightarrow \mathbb{R})$$

by:

$$\mathfrak{F}_{\mathbf{L}}(c) = \int_I \mathbf{L}(c(t), \dot{c}(t)) dt. \quad (5.1)$$

It remains to justify that this integral is well-defined. Let c be a smooth curve in M . Then its tangent lift $(c, \dot{c})$ is a smooth curve in the convenient manifold TM . As the Lagrangian $\mathbf{L} : TM \rightarrow \mathbb{R}$ is smooth in the convenient sense¹³, the composition

$$\mathbf{L} \circ (c, \dot{c}) : I \rightarrow \mathbb{R}$$

is a conveniently smooth curve. As $\mathbb{R}$ is finite-dimensional, conveniently smooth curves coincide with classically smooth curves (cf. [20]). Therefore,

$$t \mapsto \mathbf{L}(c(t), \dot{c}(t))$$

¹¹See [20, Section 42.4].

¹²Smoothness is understood in the convenient sense.

¹³One of the main advantages of convenient calculus is that smoothness is characterized through the behavior of smooth curves rather than through the underlying locally convex topology.

is a classically smooth real-valued function and is consequently Riemann integrable on the compact closure of I . Thus, the action functional $\mathfrak{F}_L$ is well-defined.

A *variation* of $c \in C^\infty(I, M)$ is a smooth map

$$\hat{c} :] - \varepsilon, \varepsilon[\times I \rightarrow M$$

such that

- $\forall t \in I, \hat{c}(0, t) = c(t);$
- the associated *variational vector field*

$$t \mapsto \delta c(t) = \frac{\partial \hat{c}}{\partial s}(0, t)$$

is a compactly supported section of the pull-back bundle $c^!TM$ with support contained in a compact subset of I .

In these conditions, we have:

Remark 5.1.

1. For every $s \in] - \varepsilon, \varepsilon[$, the curve $c_s := \hat{c}(s, \cdot)$ belongs to $C^\infty(I, M)$. Moreover, $\delta c(t) := \frac{\partial \hat{c}}{\partial s}(0, t)$ defines a compactly supported section of the pull-back bundle $c^!TM$, that is,

$$\delta c \in \Gamma_0(c^!TM).$$

Conversely, if $c \in C^\infty(I, M)$ and $\delta c \in \Gamma_0(c^!TM)$, then there exists a germ of a smooth curve $s \mapsto c_s$ in $C^\infty(I, M)$, with $c_0 = c$, whose infinitesimal variation is given by

$$\delta c(t) = \frac{\partial \hat{c}}{\partial s}(0, t).$$

Such a section δc is called an *infinitesimal variation* of c . Consequently, unless M is known to admit a local addition, the space of infinitesimal variations of c may be a proper subset of $\Gamma_0(c^!TM)$.

2. Assume that $I = [t_1, t_2]$. Let $\hat{c} :] - \varepsilon, \varepsilon[\times I \rightarrow M$ be a smooth map such that

$$\hat{c}(0, t) = c(t), \quad \hat{c}(s, t_1) = c(t_1), \quad \hat{c}(s, t_2) = c(t_2)$$

for all $s \in] - \varepsilon, \varepsilon[$. Then $\delta c(t) := \frac{\partial \hat{c}}{\partial s}(0, t)$ is an infinitesimal variation of c satisfying the endpoints condition

$$\delta c(t_1) = \delta c(t_2) = 0.$$

Now, as in the classical finite-dimensional setting, we have the following result.

Lemma 5.2. Assume that $I = [t_1, t_2]$.

(1) Let

$$\hat{c} :]-\varepsilon, \varepsilon[\times I \rightarrow M$$

be a variation of c , and set $c_s := \hat{c}(s, \cdot)$. Then the map

$$s \mapsto \mathfrak{F}_L(c_s) = \int_{t_1}^{t_2} \mathbf{L} \left(\hat{c}(s, t), \frac{\partial \hat{c}}{\partial t}(s, t) \right) dt$$

is well-defined and smooth.

(2) If the associated infinitesimal variation is

$$\delta c(t) = \frac{\partial \hat{c}}{\partial s}(0, t),$$

then the differential of $\mathfrak{F}_L$ at c in the direction δc is given by

$$\begin{aligned} d_c \mathfrak{F}_L(\delta c) &:= \lim_{s \rightarrow 0} \frac{\mathfrak{F}_L(c_s) - \mathfrak{F}_L(c)}{s} \\ &= \int_{t_1}^{t_2} \left(\frac{\partial \mathbf{L}}{\partial x}(c(t), \dot{c}(t)) - \frac{d}{dt} \frac{\partial \mathbf{L}}{\partial v}(c(t), \dot{c}(t)) \right) (\delta c(t)) dt \\ &\quad + \left[\frac{\partial \mathbf{L}}{\partial v}(c(t), \dot{c}(t)) (\delta c(t)) \right]_{t_1}^{t_2}. \end{aligned} \quad (5.2)$$

Definition 5.3. A curve $c \in C^\infty(I, M)$ is called a critical point with fixed endpoints of the action functional $\mathfrak{F}_L$ if

$$d_c \mathfrak{F}_L(\delta c) = 0$$

for every infinitesimal variation $\delta c \in \Gamma_0(c^1 TM)$ arising from a variation with fixed endpoints.

The proof of the Euler–Lagrange equations in the convenient setting relies on the following extension of the classical *lemme de du Bois-Reymond*. The continuity of the canonical pairing between a convenient vector space and its bornological dual with respect to the c^∞ -topology makes it possible to adapt the classical finite-dimensional argument to the convenient setting.

Lemma 5.4. Let $f : [t_1, t_2] \rightarrow \mathbb{M}'$ be a smooth map, where $\mathbb{M}'$ denotes the bornological dual of the convenient vector space $\mathbb{M}$. Let $\mathbb{E}$ be a closed subspace of $\mathbb{M}$. Assume that

$$\int_{t_1}^{t_2} \langle f(t), h(t) \rangle dt = 0$$

for every smooth map $h : [t_1, t_2] \rightarrow \mathbb{E}$ satisfying $h(t_1) = h(t_2) = 0$. Then

$$\forall t \in [t_1, t_2], f(t) \in \mathbb{E}^0.$$

In particular, if $\mathbb{E} = \mathbb{M}$, then $f \equiv 0$ on $[t_1, t_2]$.

Proof. Suppose, by contradiction, that there exists $\tau \in [t_1, t_2]$ such that $f(\tau) \notin \mathbb{E}^0$. Since f is continuous, this property holds throughout a neighborhood of τ , and we may therefore assume that $t_1 < \tau < t_2$. As $f(\tau)$ does not annihilate $\mathbb{E}$, there exists $u \in \mathbb{E}$ such that:

$$\langle f(\tau), u \rangle = K > 0.$$

Since both f and the canonical pairing

$$\begin{aligned} \mathbb{M}' \times \mathbb{M} &\rightarrow \mathbb{R} \\ (\xi, v) &\mapsto \langle \xi, v \rangle \end{aligned}$$

are continuous with respect to the c^∞ -topology, there exists an open interval $J \subset (t_1, t_2)$ containing τ such that

$$\langle f(t), u \rangle \geq \frac{K}{2}$$

for every $t \in J$.

Choose a smooth function $\theta : [t_1, t_2] \rightarrow [0, +\infty[$ whose support is contained in J and such that $\theta \equiv 1$ on a closed subinterval $J_0 \subset J$ containing τ . Define

$$h(t) = \theta(t)u.$$

Then h is a smooth map from $[t_1, t_2]$ into $\mathbb{E}$ satisfying $h(t_1) = h(t_2) = 0$. Moreover,

$$\begin{aligned} \int_{t_1}^{t_2} \langle f(t), h(t) \rangle dt &= \int_J \theta(t) \langle f(t), u \rangle dt \\ &\geq \int_{J_0} \langle f(t), u \rangle dt \\ &\geq \text{length}(J_0) \frac{K}{2} > 0, \end{aligned}$$

contradicting the hypothesis. Hence $f(t) \in \mathbb{E}^0$ for every $t \in [t_1, t_2]$. $\square$

The following theorem provides the variational characterization of critical points of the action functional. It is the convenient infinite-dimensional analogue of the classical Euler–Lagrange theorem (cf. [7]).

Theorem 5.5. *Let $\mathbf{L} : TM \rightarrow \mathbb{R}$ be a Lagrangian. Then a curve $c \in C^\infty([t_1, t_2], M)$ is a critical point with fixed endpoints of the action functional $\mathfrak{F}_L$ if and only if c satisfies the local Euler–Lagrange equations:*

$$\left(\frac{\partial \mathbf{L}}{\partial \mathbf{x}} - \frac{d}{dt} \frac{\partial \mathbf{L}}{\partial \mathbf{v}} \right)(c(t), \dot{c}(t)) = 0 \quad (5.3)$$

in any system of local coordinates associated with a chart whose domain meets the image of c .

Proof. By Lemma 5.2, the first variation of the action functional is given by (5.2). The conclusion then follows exactly as in the finite-dimensional setting by applying Lemma 5.4. We refer to [7] for the remaining details. $\square$

5.2. Infinite dimensional problems for the Pontryagin maximum principle

The Pontryagin maximum principle arises from the deep interplay between constrained variational problems, Euler–Lagrange equations, and Hamiltonian dynamics. In finite dimensions, this framework culminates in the notion of a bi-extremal, a cornerstone of optimal control theory.

We first recall this classical framework before investigating its extension to the infinite-dimensional setting of convenient manifolds. Assuming that the space of admissible curves admits a convenient manifold structure, we discuss the new phenomena arising from the geometry of the endpoint map, in particular, the appearance of elusive curves.

We then show that, under suitable assumptions in the Banach setting, the fundamental equivalence between constrained critical points, Euler–Lagrange equations, and implicit Hamiltonian and Lagrangian dynamics continues to hold. Finally, we explain why these results do not, in general, extend to a Pontryagin maximum principle in the convenient setting.

5.2.1. Finite-dimensional context

We recall some results of control theory and Pontryagin maximum principle in finite dimensions. Given an anchored vector bundle (E, π, M, ρ) , a curve

$$(c, v) : [t_1, t_2] \rightarrow E$$

is called *admissible* if

$$\dot{c}(t) = \rho(c(t), v(t)),$$

for almost every $t \in [t_1, t_2]$, where v is of class L^2 . We denote by $\mathcal{A}(x, [t_1, t_2], E)$ the set of admissible curves whose base curve starts at x

For any $x \in M$, the *reachable set of x* is the set $\mathcal{R}(x)$ consisting of all points y in M such that there exists an admissible curve (c, v) such that

$$c(t_1) = x \quad \text{and} \quad c(t_2) = y.$$

We define *the endpoint map* :

$$\begin{array}{ccc} \text{End: } \mathcal{A}(x, [t_1, t_2], E) & \rightarrow & M \\ c & \mapsto & c(t_2) \end{array}.$$

The reachable set of x is therefore the image of the endpoint map associated with x .

The classical Stefan–Sussmann theorem asserts that $\mathcal{R}(x)$ is an immersed submanifold of M , although not necessarily an embedded one (cf. [32, 33]). By definition of the reachable set in this setting, $y \in \mathcal{R}(x)$ if there exists a smooth piecewise admissible curve $(c, v) : [t_1, t_2] \rightarrow E$ joining x to y . In fact, one may always assume that this curve is smooth and admissible; namely, without loss of generality, there exists a smooth admissible curve joining x to y . This result is well-known, but in the absence of a precise reference, we provide a proof for completeness.

Indeed, recall that a *cutoff function*¹⁴ is a function $\tau_\epsilon \in C^\infty(\mathbb{R})$ satisfying:

- (a) $\tau_\epsilon(t) = 1$ for $t \geq \epsilon$, $\tau_\epsilon(t) = 0$ for $t \leq 0$;
- (b) $\dot{\tau}_\epsilon(t) > 0$ for $t \in]0, \epsilon[$.

¹⁴See [34].

Assume that (c, v) is a piecewise smooth admissible curve which fails to be smooth only at a point $t_0 \in]t_1, t_2[$. Using a cutoff function, one can smooth the curve in a neighborhood of t_0 by a suitable reparametrization. More precisely, for ϵ sufficiently small, let

$$\phi_\epsilon(t) = t_0 + \tau_\epsilon(t - t_0)$$

in a neighborhood of t_0 , and extend it smoothly so that it coincides with the identity outside this neighborhood. The reparametrized curve

$$t \mapsto c_\epsilon(t) = c(\phi_\epsilon(t))$$

is then smooth at t_0 . Since the anchor ρ is fiberwise linear, the corresponding control is given by

$$v_\epsilon(t) = \dot{\phi}_\epsilon(t) v(\phi_\epsilon(t)).$$

Indeed,

$$\dot{c}_\epsilon(t) = \dot{\phi}_\epsilon(t) \dot{c}(\phi_\epsilon(t)) = \dot{\phi}_\epsilon(t) \rho(v(\phi_\epsilon(t))) = \rho(v_\epsilon(t)),$$

so that (c_ϵ, v_ϵ) remains admissible. Moreover, for ϵ sufficiently small, it has the same endpoints as (c, v) and therefore joins x to y . By iterating this construction at each nonsmooth point of the original piecewise admissible curve, we obtain a smooth admissible curve joining x to y .

We now recall the Hamiltonian formulation underlying the Pontryagin maximum principle.

Given a Lagrangian $\mathbf{L} : E \rightarrow \mathbb{R}$ and a parameter $v \in \mathbb{R}^+$, we associate the map called the *control Hamiltonian*:

$$\begin{aligned} H^v : E \times_M T^*M &\rightarrow \mathbb{R} \\ (x, v, p) &\mapsto \langle p, \rho(x, v) \rangle - v\mathbf{L}(x, v). \end{aligned}$$

Assume that $E = M \times U$, where U is an open subset of $\mathbb{R}^m$. If an admissible curve (c_0, v_0) maximizes the functional $\mathfrak{F}_\mathbf{L}$ among all absolutely continuous admissible curves $(c, v) \in \mathcal{A}([t_1, t_2], E)$ satisfying the boundary conditions

$$c(t_1) = c_0(t_1) \quad \text{and} \quad c(t_2) = c_0(t_2),$$

then, by the Pontryagin maximum principle¹⁵, there exist local coordinates on any chart domain intersecting $c([t_1, t_2])$ in which the following equations hold (cf. [35]):

$$\begin{cases} \dot{x}(t) = \frac{\partial H^v}{\partial p}(x(t), v(t), p(t)) \\ \dot{p}(t) = -\frac{\partial H^v}{\partial x}(x(t), v(t), p(t)) \\ \frac{\partial H^v}{\partial v}(x(t), v(t), p(t)) = 0 \quad \text{a.e.} \end{cases} \quad (5.4)$$

A curve $t \mapsto (x(t), v(t), p(t))$ satisfying the implicit differential system (5.4) in local coordinates on any chart domain intersecting $c([t_1, t_2])$ is called¹⁶ a *bi-extremal of the Pontryagin principle*. It is well-known that bi-extremals in the Pontryagin maximum principle are divided into two classes:

¹⁵The Pontryagin maximum principle states that an optimal state-control pair admits a non-trivial adjoint covector $p(t)$, arising as a Lagrange multiplier associated with the dynamical constraint, such that the dynamics of the state and adjoint variables are governed by an associated Hamiltonian system. Moreover, the optimal control maximizes the Hamiltonian almost everywhere along the trajectory.

¹⁶The terminology refers to the fact that a bi-extremal is an extremal both for the state equation and for the dual Hamiltonian system arising in the Pontryagin maximum principle.

1. *normal extremals* corresponding to the case $\nu > 0$, for which the triple $(x(t), \nu(t), p(t))$ is smooth and satisfies the condition $p(t) \neq 0$ (in particular, $p(t_2) \neq 0$ under appropriate non-triviality conditions);
2. *abnormal extremals* corresponding to the case where $\nu = 0$. These extremals do not depend on the Lagrangian $\mathbf{L}$; rather, they are determined solely by the constraint structure.

Most of the following results are well known in finite-dimensional sub-Riemannian geometry (see, for instance, [14], and the classical references therein).

Theorem 5.6. Fix $c \in \mathcal{A}_{\Delta_M}(x, [t_1, t_2], M)$. Under the previous notations, the following statements are equivalent:

(1) The following properties hold:

(a) There exists $q \in T_{c(t_2)}^*M \setminus \{0\}$ such that

$$d_c \mathfrak{F}_{\bar{\mathbf{L}}} = T_c^* \text{End}_x(q).$$

(b) c is a critical point with fixed endpoints of $\mathfrak{F}_{\bar{\mathbf{L}}}$.

(2) In any local coordinates on a chart domain intersecting $c([t_1, t_2])$, the Euler–Lagrange equation holds:

$$\left(\frac{\partial \bar{\mathbf{L}}}{\partial x} - \frac{d}{dt} \frac{\partial \bar{\mathbf{L}}}{\partial v} \right) (c(t), \dot{c}(t)) \equiv 0 \quad (5.5)$$

together with the transversality condition

$$p(t_2) \neq 0,$$

where p is the lift of c in T^*M satisfying

$$\mathbb{F}\bar{\mathbf{L}}(c(t), \dot{c}(t)) = (c(t), p(t)).$$

(3) If $q : t \mapsto q(t)$ is a section of $c^!(T^*M)$ such that $q(t_2) \neq 0$, and

$$\mathbb{F}\bar{\mathbf{L}}(c(t), \dot{c}(t)) = (c(t), q(t)),$$

then $t \mapsto (c(t), \dot{c}(t), q(t))$ is a bi-extremal of the Pontryagin maximum principle with $\nu = 1$, for $E = \Delta_M$ and with anchor ρ given by the inclusion $\Delta_M \hookrightarrow TM$.

(4) If

$$(c(t), p(t)) = \mathbb{F}\bar{\mathbf{L}}(c(t), \dot{c}(t)),$$

then the curve $t \mapsto (c(t), p(t))$ is an integral curve of the Hamiltonian vector field $X_{\mathbb{F}\bar{\mathbf{L}}}$. In particular, it is smooth and satisfies $p(t_2) \neq 0$.

Definition 5.7. A curve c in $\mathcal{A}(x, [t_1, t_2], \Delta_M)$ is called a *normal extremal* if it satisfies any of the equivalent conditions of Theorem 5.6.

Remark 5.8.

1. By Assertion (4), any normal bi-extremal $t \mapsto (c(t), p(t))$ is smooth. In particular, c is smooth.

2. The condition that the range of $T_c \text{End}_x$ is $T_{c(t_2)}\mathcal{R}(x)$ implies:
there exists $q \in T'_{c(t_2)}M \setminus \{0\}$ such that:

$$d_c \mathfrak{F}_{\bar{L}} = T_c^* \text{End}_x(q).$$

Note that, for $\nu = 0$, a bi-extremal $(c(t), \dot{c}(t), p(t))$ is abnormal if and only if c is a singular point of the endpoint map, that is:

there exists $q \in T'_{c(t_2)}\mathcal{R}(x) \setminus \{0\}$ such that:

$$T_c^* \text{End}_x(q) = 0.$$

In this case, c is called an *abnormal extremal*.

5.2.2. Convenient setting context

In infinite-dimensional settings, it is well known that the Pontryagin maximum principle (PMP) does not hold in full generality. In this section, we survey several reasons why the notion of a bi-extremal associated with the PMP in the finite-dimensional framework cannot be straightforwardly generalized to the convenient setting.

Let Δ_M be a closed subbundle of TM . Denote by $C_{\Delta_M}^\infty([t_1, t_2], M)$ the set of smooth curves $c : [t_1, t_2] \rightarrow M$ whose tangent vectors lie in Δ_M . In general, this space does not admit a convenient manifold structure.

Henceforth, we assume that $C_{\Delta_M}^k(x, [t_1, t_2], M)$ is a convenient manifold¹⁷.

Under this assumption, the endpoint map

$$\text{End}_x : C_{\Delta_M}^\infty(x, [t_1, t_2], M) \rightarrow M$$

is smooth. For $c \in C_{\Delta_M}^\infty(x, [t_1, t_2], M)$, the range of $T_c \text{End}_x$ may be

- equal to $T_{c(t_2)}M$; c is called a *normal curve*;
- dense in $T_{c(t_2)}M$ ¹⁸; c is called *elusive* (cf. [14])¹⁹;
- of codimension strictly positive; c is called an *abnormal curve* or *semi-rigid* (cf. [13]).

Let $\bar{L}$ be a Lagrangian on Δ_M . By the same argument as in the proof of Lemma 5.2, the reduced action functional $\mathfrak{F}_{\bar{L}}$ is differentiable. When the range of $T_c \text{End}_x$ coincides with $T_{c(t_2)}M$, there exists a non-zero covector $q \in T'_{c(t_2)}M$ such that $T_c^* \text{End}_x(q) = d_c \mathfrak{F}_{\bar{L}}$. Consequently, by Proposition 7.14 (with $\nu = 1$ and $E_0 = \Delta_M$), one obtains a characterization of c in terms of a Hamiltonian condition. This characterization remains valid under the assumptions previously introduced in the Banach setting. To recover the notion of a normal bi-extremal, and in view of Remark 5.8, we may consider the situation in which there exists a nonzero covector $q \in T'_{c(t_2)}M$ satisfying:

$$T_c^* \text{End}_x(q) = d_c \mathfrak{F}_{\bar{L}}.$$

¹⁷If M is a Banach manifold, see Theorem 7.13; if M is a convenient manifold, see Theorem 6.3.

¹⁸For the existence of such curves, see [14] or [15].

¹⁹Elusive curves cannot occur in finite-dimensional settings.

In this case, Proposition 7.14 (with $\nu = 1$ and $E_0 = \Delta_M$) allows us to recover the notion of a normal bi-extremal. When c is elusive, $T_c \text{End}_x$ is not a submersion. Therefore the set

$$C_{\Delta_M}^\infty(x, y, [t_1, t_2], M) := \text{End}_x^{-1}(\{y\})$$

is, in general, not a submanifold of $C^\infty \Delta_M(x, [t_1, t_2], M)$, even locally. In particular, there may exist a tangent vector $\delta c \in T_c C_{\Delta_M}^\infty(x, [t_1, t_2], M)$ such that $\delta c(t_2) = 0$, but for which there is no horizontal variation $\hat{c}$ with fixed endpoints satisfying $\frac{\partial \hat{c}}{\partial s}(0, t) = \delta c$. Note that, by duality, if $T_c \text{End}_x$ is a submersion or if its range is dense, then the adjoint map $T_c^* \text{End}_x$ is injective. Consequently, this dual approach does not distinguish between these two situations.

Finally, observe that a curve c is abnormal if and only if it is a singular point of the endpoint map.

Remark 5.9. The existence of elusive and abnormal curves is closely related to the issue of strong controllability of Δ_M , namely, the property that, for every pair of points $(x, y) \in M^2$, there exists a piecewise smooth curve $c : [0, 1] \rightarrow M$ tangent to Δ_M , such that $c(0) = x$ and $c(1) = y$. For instance, when Δ_M is of finite codimension, then normal curves always exist. On the other hand, if Δ_M is not strongly controllable, then normal curves need not exist. For further details, the reader is referred to [14].

Warning 5.10. From this point until the end of the section, we assume that M is a Banach manifold. Then, for every $x \in M$ and every integer $k \geq 2$, the space $C_{\Delta_M}^k(x, [t_1, t_2], M)$ consisting of all C^k curves $c : [t_1, t_2] \rightarrow M$ tangent to Δ_M with $c(t_1) = x$ carries a natural Banach manifold structure.

As in the finite-dimensional case, for any $\nu \geq 0$, we consider the associated control Hamiltonian $H^\nu : \mathbb{R}$ defined by:

$$\begin{aligned} H^\nu : \Delta_M \oplus T'M &\rightarrow \mathbb{R} \\ (x, v, p) &\mapsto \langle p, v \rangle - \nu \bar{\mathbf{L}}(x, v) \end{aligned}$$

Of course, there are several difficulties in recovering a situation comparable to the finite-dimensional case, as discussed in the previous section. The first difficulty is that, in the absence of a decomposition²⁰ $TM = \Delta_M \oplus T$, there is no natural way, for $\nu > 0$, to extend H^ν to a Hamiltonian $\tilde{H}^\nu$ on $TM \oplus T'M$ such that the restriction of $d\tilde{H}^\nu$ to the tangent space of TM vanishes identically. Note that even if a Lagrangian $\mathbf{L}$ is defined on TM , its restriction $\bar{\mathbf{L}}$ to Δ_M need not be smooth if Δ_M is not supplemented. In any case, we have a natural map

$$\tilde{H}^\nu(x, v, p) = \langle p, v \rangle - \nu \mathbf{L}(x, v)$$

defined on $TM \oplus T'M$. However, its restriction to $\Delta_M \oplus T'M$ defines H^ν , which in general is not smooth.

In connection with the setting of Theorem 4.7, we propose a framework in which one can define a normal extremal c for the functional $\mathfrak{F}_{\bar{\mathbf{L}}}$, in a way that is consistent with Definition 5.7.

Assume that Δ_M is a closed split Banach subbundle of TM . Let $\bar{\mathbf{L}}$ be a Lagrangian defined on Δ_M , and assume that there exists a Lagrangian $\mathbf{L} : TM \rightarrow \mathbb{R}$ such that $\mathbf{L}|_{\Delta_M} = \bar{\mathbf{L}}$ and $\mathbf{L}$ is Δ_M -constraint regular.

Then

$$\tilde{H}^\nu(x, v, p) = \langle p, v \rangle - \nu \mathbf{L}(x, v)$$

²⁰In the Banach setting, the obstruction is essentially the lack of a canonical supplement to a closed subbundle.

is an extension of

$$H^1(x, v, p) = \langle p, v \rangle - \bar{\mathbf{L}}(x, v)$$

whose restriction to $\mathbb{P} = \mathbb{F}\mathbf{L}(\Delta_M)$ is smooth. Therefore, according to Theorem 4.7 and Theorem 5.6, we introduce:

Definition 5.11. *Under the previous assumptions, a curve $c \in C^\infty([t_1, t_2], \Delta_M)$ is called a normal extremal if, in any local chart whose domain intersects $c([t_1, t_2])$, the curve*

$$t \mapsto (c(t), p(t)) := \mathbb{F}\mathbf{L}(c(t), \dot{c}(t))$$

is a solution of the implicit differential system:

$$\begin{cases} \dot{x}(t) = \frac{\partial H^1}{\partial p}(x(t), v(t), p(t)), \\ \dot{p}(t) = -\frac{\partial H^1}{\partial x}(x(t), v(t), p(t)), \\ \frac{\partial H^1}{\partial v}(x(t), v(t), p(t)) = 0, \end{cases} \quad (5.6)$$

with $p(t_2) \neq 0$.

Remark 5.12. In finite dimensions, this definition is coherent with the classical one (cf. Theorem 5.6). In the Banach setting (cf. [14]), if the anchored bundle $(\mathcal{H}, M, \xi)$ is (Δ_M, M, Id) , then a normal sub-Riemannian geodesic satisfies the above definition, according to Proposition 7.14 with $\nu = 1$ and $E_0 = \Delta_M$. More generally, Definition 7 in [13] of a normal sub-Riemannian geodesic satisfies the previous definition (cf. [14]).

5.3. Characterization of normal extremals

In this subsection, we introduce two variational Hamiltonian–Pontryagin principles associated with the restriction of a Δ_M -constraint Lagrangian to the constraint submanifold Δ_M . Unlike the finite-dimensional approach of [18], our formulation is adapted to the convenient setting and leads directly to a characterization of normal extremals.

Let $\mathbf{L} : TM \rightarrow \mathbb{R}$ be a Δ_M -constraint regular Lagrangian. We denote by

$$\bar{\mathbf{L}} := \mathbf{L}|_{\Delta_M}$$

its restriction to the constraint submanifold.

Let $C^\infty([t_1, t_2], \Delta_M)$ denote the space of smooth curves

$$(c, v) : [t_1, t_2] \rightarrow \Delta_M.$$

We define $C_{\Delta_M}^\infty([t_1, t_2], M)$ as the subset of curves satisfying the tangency condition

$$\dot{c}(t) = v(t).$$

Equivalently,

$$C_{\Delta_M}^\infty([t_1, t_2], M) = \{(c, v) \in C^\infty([t_1, t_2], \Delta_M) : \dot{c} = v\}. \quad (5.7)$$

Thus, $C_{\Delta_M}^\infty([t_1, t_2], M)$ is the space of smooth Δ_M -admissible curves. The variational functional associated with $\bar{\mathbf{L}}$ is

$$\mathfrak{F}_{\bar{\mathbf{L}}}(c) = \int_{t_1}^{t_2} \bar{\mathbf{L}}(c(t), \dot{c}(t)) dt.$$

A *horizontal variation* of $c \in C_{\Delta_M}^\infty([t_1, t_2], M)$ is a smooth map

$$\hat{c} :]-\varepsilon, \varepsilon[\times [t_1, t_2] \rightarrow M$$

such that

$$c_s := \hat{c}(s, \cdot) \in C_{\Delta_M}^\infty([t_1, t_2], M)$$

for every s , and

$$\hat{c}(0, t) = c(t)$$

for every $t \in [t_1, t_2]$.

The variation is said to have fixed endpoints if

$$c_s(t_1) = c(t_1) \quad \text{and} \quad c_s(t_2) = c(t_2),$$

for every s .

The *infinitesimal horizontal variation* is defined by

$$\delta c(t) = \frac{\partial \hat{c}}{\partial s}(0, t).$$

Since every c_s is an admissible curve, the infinitesimal variation satisfies the corresponding linearized constraint equations.

As in the proof of Lemma 5.2, for every horizontal variation $\hat{c}$, the map

$$c_s \mapsto \mathfrak{F}_{\bar{\mathbf{L}}}(c_s)$$

is differentiable at $s = 0$. Its derivative is denoted by

$$d_c \mathfrak{F}_{\bar{\mathbf{L}}}(\delta c).$$

Definition 5.13. A curve $c \in C_{\Delta_M}^\infty([t_1, t_2], M)$ is called a critical point with fixed endpoints of $\mathfrak{F}_{\bar{\mathbf{L}}}$ if

$$d_c \mathfrak{F}_{\bar{\mathbf{L}}}(\delta c) = 0$$

for every infinitesimal horizontal variation δc associated with a horizontal variation with fixed endpoints.

In view of (5.7), the same variational problem can be described by introducing the Hamilton–Pontryagin functional

$$\widehat{\mathfrak{F}}_{\bar{\mathbf{L}}}(c, v, p) = \int_{t_1}^{t_2} (\mathbf{L}(c(t), v(t)) + \langle p(t), \dot{c}(t) - v(t) \rangle) dt, \quad (5.8)$$

where $(c, v, p) : [t_1, t_2] \rightarrow TM \oplus T^*M$ is a smooth curve satisfying

$$(c(t), v(t)) \in \Delta_M.$$

The role of the variables is the following: the pair (c, v) is constrained to lie in Δ_M , whereas p is an arbitrary covector field along c . In particular, variations of (c, v, p) must preserve the constraint at the level of the varied curves.

Definition 5.14. Let $(c, v, p) : [t_1, t_2] \rightarrow TM \oplus T'M$ be a smooth curve such that

$$(c(t), v(t)) \in \Delta_M$$

for every t .

A variation of (c, v, p) is a smooth map

$$(\hat{c}, \hat{v}, \hat{p}) :]-\varepsilon, \varepsilon[\times [t_1, t_2] \rightarrow TM \oplus T'M$$

such that

$$(\hat{c}(0, t), \hat{v}(0, t), \hat{p}(0, t)) = (c(t), v(t), p(t))$$

and

$$(\hat{c}(s, t), \hat{v}(s, t)) \in \Delta_M$$

for every (s, t) .

The variation has fixed endpoints if

$$\hat{c}(s, t_1) = c(t_1) \quad \text{and} \quad \hat{c}(s, t_2) = c(t_2)$$

for every s .

The associated infinitesimal variation is

$$(\delta c, \delta v, \delta p) = \left(\frac{\partial \hat{c}}{\partial s}, \frac{\partial \hat{v}}{\partial s}, \frac{\partial \hat{p}}{\partial s} \right)_{|s=0}.$$

A curve (c, v, p) is called a critical point with fixed endpoints of $\widehat{\mathfrak{F}}_{\mathbf{L}}$ if

$$d\widehat{\mathfrak{F}}_{\mathbf{L}}(\delta c, \delta v, \delta p) = 0$$

for every infinitesimal variation associated with a variation satisfying the above conditions.

Theorem 5.15. Let $\mathbf{L} : TM \rightarrow \mathbb{R}$ be a Δ_M -constraint regular Lagrangian and let

$$\mathbb{P} = \mathbb{FL}(\Delta_M).$$

Assume that the restricted Legendre transform

$$\mathbb{FL}|_{\Delta_M} : \Delta_M \rightarrow \mathbb{P}$$

is injective.

Then the following assertions are equivalent:

(1) There exists a section

$$p \in \Gamma(c^!T'M)$$

such that

$$t \mapsto (c(t), \dot{c}(t), p(t))$$

is a critical point with fixed endpoints of the Hamilton–Pontryagin functional $\widehat{\mathfrak{F}}_{\mathbf{L}}$.

- (2) The curve $c \in C_{\Delta_M}^\infty([t_1, t_2], M)$ is a critical point with fixed endpoints of $\mathfrak{F}_L$.
 (3) The constrained Euler–Lagrange equation holds. In local coordinates,

$$\left(\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial v} \right)(c(t), \dot{c}(t)) \in \mathbb{E}^0.$$

- (4) Let $H_{\mathbb{P}}$ be the Hamiltonian induced on $\mathbb{P}$ by L . Setting

$$p(t) = \mathbb{F}L_{c(t)}(\dot{c}(t)),$$

the curve $t \mapsto (c(t), \dot{c}(t), p(t))$ is an integral curve of the implicit Hamiltonian system

$$(M, D_{\Delta_M}, H_{\mathbb{P}}).$$

Equivalently, in local coordinates,

$$\dot{x} = \frac{\partial H_{\mathbb{P}}}{\partial p}(x, v, p) \in \mathbb{E},$$

$$\dot{p} + \frac{\partial H_{\mathbb{P}}}{\partial x}(x, v, p) \in \mathbb{E}^0,$$

together with

$$\frac{\partial H_{\mathbb{P}}}{\partial v}(x, v, p)|_{\mathbb{E}} = 0.$$

- (5) With

$$p(t) = \mathbb{F}L_{c(t)}(\dot{c}(t)),$$

the curve $t \mapsto (c(t), \dot{c}(t), p(t))$ is an integral curve of the implicit Lagrangian system $(M, \Delta_M, \mathfrak{D}L)$.

Proof. We first prove (1) $\iff$ (3).

Let $(c, \dot{c}, p)$ be a critical point of the Hamilton–Pontryagin functional.

For a variation $(\hat{c}, \hat{v}, \hat{p})$ in the sense of Definition 7.17, the infinitesimal variation satisfies

$$(\delta c, \delta v) \in T_{(c,v)}\Delta_M.$$

In particular, δv is not an arbitrary vector in TM , but the linearization of an admissible variation.

A direct computation gives:

$$\begin{aligned} d\widehat{\mathfrak{F}}_L(\delta c, \delta v, \delta p) &= \int_{t_1}^{t_2} \langle \dot{c} - v, \delta p \rangle dt \\ &\quad + \int_{t_1}^{t_2} \left\langle -\dot{p} + \frac{\partial L}{\partial x}, \delta c \right\rangle dt \\ &\quad + \int_{t_1}^{t_2} \left\langle -p + \frac{\partial L}{\partial v}, \delta v \right\rangle dt. \end{aligned}$$

Since δp is arbitrary, we obtain:

$$\dot{c} = v.$$

Hence the critical curve is admissible.

The remaining variations satisfy

$$(\delta c, \delta v) \in T\Delta_M.$$

Therefore the vanishing of the remaining terms means precisely that

$$\left(-\dot{p} + \frac{\partial \mathbf{L}}{\partial x}, -p + \frac{\partial \mathbf{L}}{\partial v}\right)$$

annihilates $T\Delta_M$.

The second component gives the Legendre relation

$$p = \frac{\partial \mathbf{L}}{\partial v}$$

because the restricted Legendre map is injective on Δ_M .

Consequently,

$$\dot{p} - \frac{\partial \mathbf{L}}{\partial x} \in \Delta_M^0.$$

Since $p = \frac{\partial \mathbf{L}}{\partial v}$, we obtain

$$\frac{\partial \mathbf{L}}{\partial x} - \frac{d}{dt} \frac{\partial \mathbf{L}}{\partial v} \in \Delta_M^0,$$

which is exactly the constrained Euler–Lagrange equation.

Conversely, assume (3). Define

$$p(t) = \mathbb{F}\mathbf{L}_{c(t)}(\dot{c}(t)).$$

Then $p = \frac{\partial \mathbf{L}}{\partial v}$. The constrained Euler–Lagrange equation implies

$$\dot{p} - \frac{\partial \mathbf{L}}{\partial x} \in \Delta_M^0.$$

Hence the first variation of $\widehat{\mathfrak{F}}_{\mathbf{L}}$ vanishes for every admissible variation. Therefore (1) holds.

(2) $\iff$ (3). Let δc be an infinitesimal horizontal variation. Because the variation remains in $C_{\Delta_M}^\infty$, $\delta c(t) \in \Delta_{c(t)}$.

The first variation is

$$d\widehat{\mathfrak{F}}_{\mathbf{L}}(\delta c) = \int_{t_1}^{t_2} \left\langle \frac{\partial \mathbf{L}}{\partial x} - \frac{d}{dt} \frac{\partial \mathbf{L}}{\partial v}, \delta c \right\rangle dt.$$

Therefore this vanishes for all admissible variations if and only if

$$\frac{\partial \mathbf{L}}{\partial x} - \frac{d}{dt} \frac{\partial \mathbf{L}}{\partial v} \in \Delta_M^0,$$

which proves the equivalence.

(3) $\iff$ (4). Let $p = \mathbb{F}\mathbf{L}(\dot{c})$.

By the definition of the Hamiltonian on $\mathbb{P}$,

$$\begin{cases} \frac{\partial H_{\mathbb{P}}}{\partial p} = v, \\ \frac{\partial H_{\mathbb{P}}}{\partial x} = -\frac{\partial \mathbf{L}}{\partial x}. \end{cases}$$

Hence the Hamiltonian equations become

$$\begin{cases} \dot{x} = v, \\ \dot{p} - \frac{\partial \mathbf{L}}{\partial x} \in \Delta_M^0. \end{cases}$$

Using $p = \frac{\partial \mathbf{L}}{\partial v}$, the latter equation is exactly the constrained Euler–Lagrange equation.

(4) $\iff$ (5). The local implicit Lagrangian system is characterized by

$$\begin{cases} p = \frac{\partial \mathbf{L}}{\partial v}, \\ \dot{x} = v \in \mathbb{E}, \\ \dot{p} - \frac{\partial \mathbf{L}}{\partial x} \in \mathbb{E}^0. \end{cases}$$

The Legendre transformation identifies these equations with the implicit Hamiltonian equations obtained from $(D_{\Delta_M}, H_{\mathbb{P}})$. Hence the two systems are equivalent. $\square$

Corollary 5.16. *Assume that one of the equivalent assertions of Theorem 5.15 holds and that $p(t_2) \neq 0$. Then c is a normal extremal of $\mathfrak{F}_{\mathbf{L}}$.*

Proof. It is sufficient to assume that Assertion (4) of Theorem 5.15 holds. Hence, with

$$p(t) = \mathbb{F}\mathbf{L}_{c(t)}(\dot{c}(t)),$$

the curve $t \mapsto (c(t), \dot{c}(t), p(t))$ is an integral curve of the induced implicit Hamiltonian system

$$(M, D_{\Delta_M}, H_{\mathbb{P}}).$$

By Proposition 7.14, applied with

$$E = \Delta_M \quad \text{and} \quad \nu = 1,$$

the corresponding Hamiltonian integral curve defines an extremal of $\mathfrak{F}_{\mathbf{L}}$ in the sense of Definition 5.11.

Moreover, the condition

$$p(t_2) \neq 0$$

excludes the trivial extremal and ensures that the multiplier associated with the variational problem is non-zero. Therefore the extremal is a normal extremal.

Hence c is a normal extremal of $\mathfrak{F}_{\mathbf{L}}$. $\square$

6. Dynamics and implicit Lagrangian systems on direct limits of finite-dimensional manifolds

In this section, we extend the implicit Lagrangian formalism to direct limits of finite-dimensional manifolds. We first introduce direct limits of finite-dimensional Dirac structures and then define the dynamical system associated with an implicit Lagrangian on the resulting direct-limit manifold. Finally, we illustrate the theory through two classes of examples: direct limits of sub-Riemannian Lie groups and direct limits of electrical circuits. These examples show how the proposed framework naturally applies to infinite-dimensional systems arising as limits of finite-dimensional models.

6.1. Direct limits of finite-dimensional Dirac structures

Let $(M_n)_{n \in \mathbb{N}}$ be an ascending sequence of finite-dimensional manifolds such that M_n is a submanifold of M_{n+1} . Assume that each M_n is equipped with its Pontryagin bundle $T^p M_n = TM_n \oplus T' M_n$ and a Dirac structure D_n such that D_n is the pull-back of D_{n+1} along the inclusion $i_n^{n+1} : M_n \rightarrow M_{n+1}$. The sequence $(M_n, D_n)_{n \in \mathbb{N}}$ is called an *ascending sequence of finite-dimensional Dirac structures*.

The direct limit $M = \varinjlim M_n = \bigcup_{n \in \mathbb{N}} M_n$ carries a natural convenient manifold structure modeled on $\mathbb{R}^\infty$ (the space of finite real sequences). Likewise, the direct limit TM of the sequence of tangent bundles TM_n inherits a convenient vector bundle over M . On the other hand, the projective limit $T'M$ of the sequence of the cotangent bundles $T'M_n$ is naturally a Fréchet bundle over M . We denote by

$$p_n^b : T^b M_n \rightarrow TM_n \quad \text{and} \quad p_n' : T^b M_n \rightarrow T'M_n$$

the canonical projections. The following results are immediate consequences of [16].

Proposition 6.1. *Let $(M_n, D_n)_{n \in \mathbb{N}}$ be an ascending sequence of finite-dimensional Dirac structures.*

1. *There exist convenient vector bundle projections*

$$p : T^p M \rightarrow TM \quad \text{and} \quad p' : T^p M \rightarrow T'M$$

given, respectively, by $p = \varinjlim p_n$ and $p' = \varprojlim p_n'$.

2. *The limit $D = \varinjlim p_n(D_n) \times \varprojlim p_n'(D_n)$ defines an almost-Dirac structure on M .*
3. *A dual pairing $\langle \cdot, \cdot \rangle$ is defined as follows.*

Let $\alpha = \varprojlim \alpha_n \in T'_x M$ and $u = \varinjlim u_n \in T_x M$. If i is the smallest integer such that α is represented by an element of $T'_x M_i$, then

$$\langle \alpha, u \rangle = \varprojlim_{n \geq j} \langle \alpha_n, u_j \rangle = \varinjlim_{n \geq i} \langle \alpha_i, u_n \rangle.$$

4. *The Courant bracket*

$$[(X, \alpha), (Y, \beta)]_C = \left([X, Y], L_X \beta - L_Y \alpha - \frac{1}{2} d(i_Y \alpha - i_X \beta) \right) \quad (6.1)$$

is well-defined for all sections (X, α) and (Y, β) of $T^p M$ such that $X = \varinjlim X_n$ (resp., $Y = \varinjlim Y_n$) and $\alpha = \varprojlim \alpha_n$ (resp., $\beta = \varprojlim \beta_n$), where (X_n, α_n) (resp., (Y_n, β_n)) is a section of $T^p M_n$ ²¹. This implies that:

$$[(X, \alpha), (Y, \beta)]_C = \left(\varinjlim [X_n, Y_n], \varprojlim p_n^b([(X_n, \alpha_n), (Y_n, \beta_n)]_C) \right). \quad (6.2)$$

²¹In general, a section of $T^p M$ is not the direct limit of an ascending sequence of sections of $T^p M_n$; consequently, the Courant bracket is not defined on all sections of $T^p M$ (see Remark 5.11 in [16]).

The next proposition provides a partial converse to Proposition 6.1. Its proof follows directly from the constructions developed in [16] and is therefore omitted.

Proposition 6.2. *Let $(M_n, \varepsilon_n^{n+1})_{n \in \mathbb{N}}$ be an ascending sequence of finite-dimensional manifolds satisfying the direct limit chart property, and let $M = \varinjlim M_n$.*

For each n , let

$$\iota_n : T^*M_n \rightarrow T^*M$$

denote the canonical injection into the convenient kinematic cotangent bundle

$$T^*M = \varprojlim T^*M_n.$$

Then the following assertions hold.

(1) *The weak canonical symplectic form Ω on T^*M induces the partial Dirac structure*

$$D = \text{graph}(\Omega^b) \subset T^p(T^*M) = T(T^*M) \oplus T^b(T^*M).$$

Moreover, for every n , the pull-back

$$D_n = \iota_n^! D$$

coincides with the Dirac structure on T^*M_n associated with the canonical symplectic form Ω_n . Consequently,

$$D = \varinjlim p_n(D_n) \times \varprojlim p'_n(D_n),$$

so that D is completely determined by the compatible sequence $(D_n)_{n \in \mathbb{N}}$.

(2) *Let U be an open subset of M . The space of local sections $\Gamma(T^p M|_U)$ admits the Courant bracket defined by (3.2) whenever the sections arise as limits of compatible local sections on the finite-dimensional approximations. Under this identification, the Courant bracket is the direct limit of the finite-dimensional Courant brackets, namely,*

$$[(X, \alpha), (Y, \beta)]_C = (\varinjlim [X_n, Y_n], \varprojlim p'_n([(X_n, \alpha_n), (Y_n, \beta_n)]_C)),$$

and therefore satisfies the compatibility property stated in Proposition 6.1, (4).

6.2. Dynamical system associated to an implicit Lagrangian on the direct limit of finite-dimensional manifolds

First, we state the following results on direct limits of implicit non-degenerate constraint Lagrangians.

Theorem 6.3. *We consider the following data:*

- *an increasing sequence of integers $(m_n)_{n \in \mathbb{N}}$ such that $0 < m_n \leq n$;*
- *an ascending sequence $(M_n)_{n \in \mathbb{N}}$ of manifolds with $\dim M_n = n$;*
- *an ascending sequence $(\Delta_{M_n})_{n \in \mathbb{N}}$ of subbundles of TM_n of rank m_n ;*
- *for each n , a Lagrangian $\mathbf{L}_n : TM_n \rightarrow \mathbb{R}$, which is non-degenerate on Δ_{M_n} .*

Then the following assertions hold.

(1) On $M = \varinjlim M_n$, the direct limit

$$\mathbf{L} = \varinjlim \mathbf{L}_n$$

defines a Lagrangian on TM , which is non-degenerate on

$$\Delta_{TM} = \varinjlim \Delta_{TM_n}.$$

(2) The Legendre transformation satisfies

$$\mathbb{F}\mathbf{L} = \varinjlim \mathbb{F}\mathbf{L}_n,$$

and $\mathbb{P} := \mathbb{F}\mathbf{L}$ is the direct limit of

$$\mathbb{P}_n := \mathbb{F}\mathbf{L}_n.$$

(3) If

$$H_n(x_n, v_n, p_n) = \langle p_n, v_n \rangle - \mathbf{L}_n(x_n, v_n)$$

is the Hamiltonian associated with $\mathbf{L}_n$, then

$$H = \varinjlim H_n$$

is the Hamiltonian associated with $\mathbf{L}$.

(4) If $t \mapsto (c_n(t), v_n(t), p_n(t))$ is an integral curve of the implicit Lagrangian system $(M_n, \Delta_{M_n}, \mathfrak{D}\mathbf{L}_n)$, then

$$t \mapsto (c(t), v(t), p(t)) = \left(\varinjlim c_n(t), \varinjlim v_n(t), \varinjlim p_n(t) \right)$$

is an integral curve of $(M, \Delta_M, \mathfrak{D}\mathbf{L})$.

(5) For $x_n \in M_n$ and $k \geq 2$, consider the Banach manifold

$$C_{\Delta_{M_n}}^k(x_n, [t_1, t_2], M_n)$$

of C^k curves tangent to Δ_{M_n} satisfying $c(t_1) = x_n$. Let $x = \varinjlim x_n$. For sufficiently small $[t_1, t_2]$,

$$C_{\Delta_M}^k(x, [t_1, t_2], M) = \varinjlim C_{\Delta_{M_n}}^k(x_n, [t_1, t_2], M_n)$$

is a convenient manifold, and the endpoint map

$$\text{End}_x : C_{\Delta_M}^k(x, [t_1, t_2], M) \rightarrow M$$

is the direct limit of the maps

$$\text{End}_{x_n} : C_{\Delta_{M_n}}^k(x_n, [t_1, t_2], M_n) \rightarrow M_n.$$

In particular, the range of $T_c \text{End}_x$ is the direct limit of the ranges of $T_{c_n} \text{End}_{x_n}$, and is closed in $T_{c(t_2)}M$.

(6) There are no elusive curves in $C_{\Delta_M}^k(x, [t_1, t_2], M)$. Moreover, the curve

$$t \mapsto (c(t), v(t), p(t)) = \left(\varinjlim c_n(t), \varinjlim v_n(t), \varinjlim p_n(t) \right)$$

is normal (respectively, abnormal) if and only if there exists an integer n_0 such that

$$t \mapsto (c_n(t), v_n(t), p_n(t))$$

is normal (respectively, abnormal) for every $n \geq n_0$.

(7) For $x = \varinjlim x_n$, there exists an ascending sequence of immersed submanifolds

$$\mathcal{R}_n(x_n) \subset M_n$$

such that:

- $\mathcal{A}_{\Delta_{M_n}}(x_n, [t_1, t_2], M_n)$ is a submanifold of $\mathcal{A}(x_n, [t_1, t_2], \mathcal{R}_n(x_n))$;
- End_{x_n} takes values in $\mathcal{R}_n(x_n)$ and is a submersion;
- $\mathcal{R}(x) := \bigcup_{n \in \mathbb{N}} \mathcal{R}_n(x_n)$ is an immersed convenient submanifold of M .

Moreover, every point $y \in \mathcal{R}(x)$ can be joined to x by a horizontal C^k curve, and End_x takes values in $\mathcal{R}(x)$ and is a submersion.

Proof. Assertions (1), (2), and (3) follow directly from the definition of the direct limit of maps. To prove that $\mathbf{L}$ is non-degenerate, let $x \in M$. Then there exists n such that $x \in M_n$. Since the Hessian of $\mathbf{L}$ restricted to Δ_{M_n} coincides with the Hessian of $\mathbf{L}_n$, and the latter is non-degenerate by assumption, the result follows.

For Assertion (4), let $\mathfrak{D}\mathbf{L}_n$ denote the Dirac differential associated with $\mathbf{L}_n$. By the definition of the Dirac differential, Assertion (1), and the definition of the direct limit of maps, we have

$$\mathfrak{D}\mathbf{L} = \varinjlim \mathfrak{D}\mathbf{L}_n.$$

The conclusion therefore follows immediately from the definition of an integral curve of an implicit Lagrangian system together with Proposition 6.2 (1).

To prove Assertion (5), observe first that the sequence

$$\left(C_{\Delta_{M_n}}^k(x_n, [t_1, t_2], M_n) \right)_{n \in \mathbb{N}}$$

is an ascending sequence of Banach manifolds. Hence, its direct limit is a convenient manifold (cf. [17], Chap. 5) naturally contained in $C_{\Delta_{M_n}}^k(x, [t_1, t_2], M)$.

On the other hand, if $c : [t_1, t_2] \rightarrow M$ is tangent to Δ_M , and if $c(t_1) = x = \varinjlim x_n$, let n_0 be the smallest integer such that $\dot{c}(t_1)$ belongs to $\Delta_{M_{n_0}}$. There exists $\epsilon > 0$ such that $c([t_1, \epsilon]) \subset M_{n_0}$. Since $\Delta_{M_{n_0}}$ is closed, $\dot{c}(\epsilon)$ must belong to $\Delta_{M_{n_0}}$. By a connectedness argument, it follows that $c \in C_{\Delta_{M_n}}^k(x_n, [t_1, t_2], M_{n_0})$. Therefore, we have

$$C_{\Delta_{M_n}}^k(x; [t_1, t_2], M) = \varinjlim C_{\Delta_{M_n}}^k(x_n; [t_1, t_2], M_n).$$

The last part of the assertion follows from the properties of direct limits of ascending sequences of Banach spaces (cf. [17], Chap. 5).

For Assertion (6), since the range of $T_c \text{End}_x$ is closed in $T_{c(t_1)}M$, it is either equal to $T_{c(t_1)}M$, or there exists a nonzero covector $q \in T'_{c(t_1)}M$ such that q annihilates the range of $T_c \text{End}_x$:

$$q \in (\text{Im } T_c \text{End}_x)^0.$$

In the first case, the curve is normal, whereas in the second, it is abnormal. In the normal case, this is equivalent to the fact that End_{x_n} is a submersion for all $n \geq n_0$ for some n_0 , that is, the curves $t \mapsto (c_n(t), v_n(t), p_n(t))$ are normal for all $n \geq n_0$. In the abnormal case, one similarly obtains that $t \mapsto (c_n(t), v_n(t), p_n(t))$ are abnormal for all sufficiently large n .

Finally, Assertion (7) follows from the results established at the beginning of § 5.2.1 together with the properties of direct limits of ascending sequences of finite-dimensional manifolds (cf. [36]). The corresponding property of the endpoint map End_x is then an immediate consequence of the functoriality of direct limits and Remark 5.8. $\square$

Conversely, let $\mathbf{L} : TM \rightarrow \mathbb{R}$ be an implicit Lagrangian defined on the direct limit $M = \varinjlim M_n$, and let Δ_M be a subbundle of TM . We denote by

$$\Delta_{T'M} := (Tp_{T'M})^{-1}(\Delta_M)$$

the associated closed subbundle of $TT'M$.

According to Definition 4.1, we obtain an induced partial almost-Dirac structure D_{Δ_M} on $T'M$, where $\mathbb{F}\mathbf{L}$ denotes the Legendre transformation of $\mathbf{L}$ and $\mathcal{D}\mathbf{L}$ its Dirac operator. We then set

$$\mathbb{P} := \mathbb{F}\mathbf{L}(\Delta_M) \subset T'M.$$

Proposition 6.4. *Under the previous context, we have:*

1. Let $\mathbf{L}_n := \iota'_n \mathbf{L}$. Then

$$\mathbf{L} = \varinjlim \mathbf{L}_n.$$

2. For each n , the intersection $\Delta_M \cap TM_n$ defines a subbundle Δ_{M_n} of TM_n . This yields an ascending sequence of vector bundles $(\Delta_{M_n})_{n \in \mathbb{N}}$ over the ascending sequence of manifolds $(M_n)_{n \in \mathbb{N}}$ such that

$$\Delta_M = \varinjlim \Delta_{M_n}.$$

Moreover,

$$\Delta_{T'M} = \varinjlim \Delta_{T'M_n}.$$

3. The sequence $(M_n, \Delta_{M_n}, \mathbf{L}_n)_{n \in \mathbb{N}}$ satisfies the assumptions of Theorem 6.3. Consequently, the implicit Lagrangian system $(M, \Delta_M, \mathcal{D}\mathbf{L})$ is obtained as the direct limit of the sequence of the implicit Lagrangian systems $(M_n, \Delta_{M_n}, \mathcal{D}\mathbf{L}_n)_{n \in \mathbb{N}}$ as described in the conclusion of that theorem.

Proof. Assertion 1 is immediate.

For Assertion 2, for any $x = \varinjlim x_n \in M$, according to [36], there exists an ascending sequence of charts $(U_n, \phi_n)_{n \in \mathbb{N}}$ around x_n such that the pair $(U = \varinjlim U_n, \phi = \varinjlim \phi_n)$ is a chart around x in M . Without loss of generality, we may assume that $\phi_n(U_n)$ is an open set $\bar{U}_n$ of $\mathbb{M}_n \equiv \mathbb{R}^n$ around 0 so that $U = \varinjlim U_n$ is an open set in $\mathbb{R}^\infty$ around 0. After restricting each U_n , if necessary, we may assume that

$$(\Delta_M)|_U = U \times \mathbb{E} \subset TM|_U = U \times \mathbb{R}^\infty.$$

Since $\mathbb{R}^\infty = \bigcup_{n \in \mathbb{N}} \mathbb{R}^n$, it follows that $E_n := \mathbb{E} \cap \mathbb{R}^n$ is a vector subspace of $\mathbb{R}^n$. Hence, we have

$$(\Delta_M \cap TM_n)|_{\mathbb{U}_n} = \mathbb{U}_n \times \mathbb{E}_n.$$

Consequently, $\Delta_M \cap TM_n$ is a subbundle Δ_{M_n} of TM_n and we also have $\Delta_M = \varinjlim \Delta_{M_n}$. Finally, we obtain $\Delta_{T'M} = \varinjlim \Delta_{T'M_n}$.

Assertion 3 follows directly from Theorem 6.3. □

6.3. Direct limit of sub-Riemannian Lie groups

Consider an ascending sequence of Lie groups $(G_n)_{n \in \mathbb{N}}$, each equipped with an ascending sequence of left invariant distributions $(\Delta_{G_n})_{n \in \mathbb{N}}$ and a left invariant sub-Riemannian metric²² $\langle \cdot, \cdot \rangle^n$.

We denote such a structure by $(G_n, \Delta_{G_n}, \langle \cdot, \cdot \rangle^n)$ and refer to it as an *ascending sequence of sub-Riemannian Lie groups*.

From properties of direct limits (cf. [17]), it follows that $G = \varinjlim G_n$ defines a convenient regular Lie group, that $\Delta_G = \varinjlim \Delta_{G_n}$ defines a left invariant distribution on G , and that $\langle \cdot, \cdot \rangle_g := \varinjlim \langle \cdot, \cdot \rangle_{g_n}^n$ is a left invariant sub-Riemannian metric on G . Moreover, Δ_G is bracket generating.

By an application of Theorem 6.3, normal curves associated with this direct-limit structure always exist and arise as direct limits of normal curves associated with the sub-Riemannian structures on G_n . Likewise, abnormal curves are direct limits of abnormal curves associated with the sub-Riemannian structures on G_n . In particular, elusive curves do not exist.

As an application of this setting, we consider $G_n = \mathrm{GL}(n, \mathbb{R})$. Such an ascending sequence of sub-Riemannian structures induces a sub-Riemannian structure on the direct limit²³ $\mathrm{GL}(\infty, \mathbb{R})$. The same holds for any ascending sequence of Lie subgroups of $\mathrm{GL}(n, \mathbb{R})$.

Conversely, by Proposition 6.4, any sub-Riemannian structure on $\mathrm{GL}(\infty, \mathbb{R})$ induces compatible sub-Riemannian structures on each $\mathrm{GL}(n, \mathbb{R})$. In this sense, $\mathrm{GL}(\infty, \mathbb{R})$ can be regarded as a classifying object for sub-Riemannian structures on the sequence $(\mathrm{GL}(n, \mathbb{R}))_{n \in \mathbb{N}}$.

6.4. Direct limits of electric circuits

This subsection develops a geometric and variational framework for electric circuits viewed as inductive limits of finite networks of inductors and capacitors (LC networks). Starting from the Dirac formulation of constrained Lagrangian systems encoding Kirchhoff's laws, we construct compatible sequences of finite-dimensional circuit manifolds with constraint distributions and Lagrangians. The resulting direct limit provides a setting for large-scale circuits in which the Dirac structure and energy formulation are preserved, and Kirchhoff's laws persist as distributional constraints inherited from the finite levels.

²²For the study of sub-Riemannian finite Lie groups, see [37, Chap. 5] or [38].

²³See [20, Section 47.7], for basic properties of $\mathrm{GL}(\infty, \mathbb{R})$.

6.4.1. Electric circuits

We first recall the formalism of the constraint Lagrangian on LC circuits. For this section, we refer to [39] and [23].

For a given electric circuit graph, Kirchhoff's current law (KCL) can be formulated on an n -dimensional manifold Q , called the *set of charges*, together with a finite family of 1-forms $(\omega^a)_{1 \leq a \leq n-m}$, where $m < n$, such that:

$$\langle \omega^a, f \rangle = 0,$$

where $f = (f^1, \dots, f^n) \in T_q Q$ represents the *set of currents* associated with the branches of the circuit. The 1-forms ω^a are given, using Einstein notation, by:

$$\omega^a = \omega_k^a dq^k, \quad a \in \{1, \dots, n-m\}, \quad k \in \{1, \dots, n\},$$

where the coefficients $\omega_k^a \in \{-1, 0, 1\}$ are determined by the topology of the circuit. The set of all branch currents $f = (f^1, \dots, f^n)$ satisfying the KCL constraints forms an m -dimensional subspace $\Delta_Q(q) \subset T_q Q$ defined by:

$$\Delta_Q(q) = \{f \in T_q Q : \forall a \in \{1, \dots, n-m\}, \langle \omega^a, f \rangle = 0\}.$$

This subspace is called the *Kirchhoff current law distribution*, or the *KCL distribution* for short.

Let $\Delta_Q^0(q) \subset T_q^* Q$ denote the annihilator of $\Delta_Q(q)$. We assume that $\Delta_Q^0(q)$ is spanned by m linearly independent 1-forms $\omega^1, \dots, \omega^m$, which represent the *Kirchhoff voltage law* (KVL).

Let $T\pi_Q : TT^*Q \rightarrow TQ$ denote the tangent map of the cotangent bundle projection $\pi_Q : T^*Q \rightarrow Q$.

We define the distribution Δ_{T^*Q} on T^*Q as the pull-back of the constrained distribution $\Delta_Q \subset TQ$ via $T\pi_Q$, namely,

$$\Delta_{T^*Q} := (T\pi_Q)^{-1}(\Delta_Q) \subset TT^*Q.$$

The annihilator $\Delta_{T^*Q}^0$ of Δ_{T^*Q} is given for each $z = (q, p) \in T^*Q$ by

$$\Delta_{T^*Q}^0(z) = \{\alpha_z \in T_z^* T^*Q : \forall w_z \in \Delta_{T^*Q}, \langle \alpha_z, w_z \rangle = 0\}.$$

Consider the bundle morphism $\Omega^b : TT^*Q \rightarrow T^*T^*Q$ associated with the Liouville 2-form Ω (canonical symplectic structure on T^*Q). The Dirac structure D_{Δ_Q} on T^*Q , which is induced from the Kirchhoff distribution $\Delta_Q \subset TQ$ is defined, at each point $z = (q, p) \in T^*Q$ by:

$$D_{\Delta_Q}(z) = \{(v_z, \alpha_z) \in T_z T^*Q \oplus T_z^* T^*Q : v_z \in \Delta_{T^*Q}(z), \alpha_z - \Omega^b(v_z) \in \Delta_{T^*Q}^0(z)\}.$$

For simplicity, we henceforth assume that Q is an open submanifold of $\mathbb{R}^n$ and work in the standard coordinates on $\mathbb{R}^n$. Let Δ_Q be a subbundle of TQ and let Δ_Q^0 denote its annihilator.

We consider the Lagrangian given by

$$\mathbf{L}(q, v) = \frac{1}{2} (v^\top \mathbf{L}v - q^\top \mathbf{C}q)^{24},$$

²⁴More general Lagrangians can be considered as in [39].

where $\mathbf{L} = \text{diag}(L_1, \dots, L_n)$ and $\mathbf{C} = \text{diag}\left(\frac{1}{C_1}, \dots, \frac{1}{C_n}\right)$ are constant diagonal matrices.

Therefore, the Legendre transform is given by:

$$\mathbb{F}\mathbf{L}(q, v) = \left(q, \frac{\partial \mathbf{L}}{\partial v}(q, v) \right) = (q, \mathbf{L}v) = (q, p).$$

Again, let $\mathbb{P} = \mathbb{F}\mathbf{L}(\Delta_Q)$. If $\mathbb{E}$ denotes the typical fiber of Δ_Q , then the equations of motion of the associated nonholonomic Kirchhoff circuit are locally given by:

$$\begin{cases} \dot{q} = v, \\ \dot{p} + \mathbf{C}q \in \mathbb{E}^0, \\ p = \mathbf{L}v, \\ v \in \mathbb{E}. \end{cases} \quad (6.3)$$

We assume that each $L_i > 0$ and $C_i > 0$ for $i \in \{1, \dots, n\}$. Hence the Legendre transformation is non-degenerate and consequently $\mathbb{P}$ is a submanifold of T^*Q (cf. [39]).

Remark 6.5. Every integral curve of this system with initial condition

$$f^0 := (f_1^0, \dots, f_n^0) \in Q$$

remains in the reachable set $\mathcal{R}(f^0)$. Moreover, every integral curve of the previous dynamical system restricted to the immersed submanifold $\mathcal{R}(f^0)$ is a normal bi-extremal. It is an abnormal bi-extremal if and only if $\mathcal{R}(f^0) \neq Q$.

6.4.2. Direct limits of LC electric circuits

We begin by fixing the following data:

- an increasing sequence of integers $(m_n)_{n \in \mathbb{N}}$ such that $0 < m_n \leq n$;
- infinite sequences $(L_n)_{n \in \mathbb{N}}$ and $(C_n)_{n \in \mathbb{N}}$ of strictly positive real numbers;
- an ascending sequence (Q_n) where Q_n is an open submanifold of $\mathbb{R}^n$;
- an ascending sequence Δ_{Q_n} where Q_n is a subbundle of TQ_n of rank m_n .

For each n , we set

$$\mathbf{L}_n = \text{diag}(L_1, \dots, L_n) \quad \text{and} \quad \mathbf{C} = \text{diag}\left(\frac{1}{C_1}, \dots, \frac{1}{C_n}\right).$$

Given these data, for each fixed sequence m_n , ascending sequence Q_n of open submanifolds of $\mathbb{R}^n$, and ascending sequence Δ_{Q_n} of rank m_n , we define the direct limits: $Q := \varinjlim Q_n$, $\Delta_Q = \varinjlim \Delta_{Q_n}$, $\mathbf{L} = \varinjlim \mathbf{L}_n$. By application of Theorem 6.3, we obtain an implicit Lagrangian system $(Q, \Delta_Q, \mathfrak{D}\mathbf{L})$, which satisfies the conclusions of that theorem. Conversely, any such implicit Lagrangian system $(Q, \Delta_Q, \mathfrak{D}\mathbf{L})$ induces on each Q_n a finite-dimensional Lagrangian system $(Q_n, \Delta_{Q_n}, \mathfrak{D}_n \mathbf{L}_n)$ as stated in Proposition 6.4.

7. Banach Dirac structures and implicit Lagrangian systems with singular constraints

In this section, we work entirely in the Banach setting.

Our first objective is to extend the construction of induced partial almost-Dirac structures associated with closed subbundles of TM to the case of singular constraints. To this end, we consider an anchored Banach bundle (E, M, ρ) , where $\pi : E \rightarrow M$ is a vector bundle with reflexive typical fiber $\mathbb{E}$, together with an open subset $E_0 \subset E$ such that the restriction $\pi_0 : E_0 \rightarrow M$ is a locally trivial bundle whose typical fiber $\mathbb{E}_0$ is an open subset of $\mathbb{E}$. Let ρ_0 denote the restriction of the anchor ρ to E_0 , and define

$$\Sigma_M := \rho_0(E_0) \subset TM.$$

We call Σ_M a singular nonholonomic constraint.

We then introduce the corresponding notion of singular partial almost-Dirac structure and show that it provides a natural geometric framework for implicit Lagrangian systems associated with non-degenerate Lagrangians on E_0 . This extends the formalism developed in Sections 3, 4, and 5.

7.1. Singular Induced Partial almost-Dirac Structures

In this subsection, we extend the notion of a partial almost-Dirac structure associated with a regular distribution, introduced in § 3.4, to the singular setting. More precisely, let (E, M, ρ) be an anchored Banach bundle whose typical fiber $\mathbb{E}$ is reflexive, and let $E_0 \subset E$ be a locally trivial subbundle with conic²⁵ typical fiber $\mathbb{E}_0$.

Throughout this subsection, we assume that, for every $x \in M$, the kernel of ρ_x is supplemented in the fiber $E_x = \pi^{-1}(x)$, and that its range is a closed subspace of $T_x M$ ²⁶. These assumptions are always satisfied if ρ_x is a Fredholm operator for all $x \in M$ (cf. [16], Examples A.12 (4)).

We now introduce the singular induced partial almost-Dirac structure on $T'M$ associated with Δ_M . To this end, we first fix some notation.

Notations 7.1. Let (U, ϕ) be a chart on M such that $E|_U$ is trivializable, and set $\mathbb{U} = \phi(U)$. We have a trivialization $\Psi : E|_U \rightarrow \mathbb{U} \times \mathbb{E}$. The restriction Ψ_0 to $(E_0)|_U$ is a trivialization onto $\mathbb{U} \times \mathbb{E}_0$. In these local coordinates, for $(x, v) \in E|_U$, the representation $\Psi(x, v)$ will be again denoted as $(x, v) \cong (\mathbf{x}, \mathbf{v})$. The tangent map $T\Psi$ induces a trivialization of $T(E|_U)$ onto $\mathbb{U} \times \mathbb{E} \times \mathbb{M} \times \mathbb{E}$, whose associated local coordinates are denoted as $(\mathbf{x}, \mathbf{v}, \dot{\mathbf{x}}, \dot{\mathbf{v}})$. We also have a trivialization $T\phi : TM|_U \rightarrow \mathbb{U} \times \mathbb{M}$ under which the anchor map is locally expressed as $\rho(x, v) := \rho_x(v) \cong r_{\mathbf{x}}(\mathbf{v})$.

On the other hand, let $\pi' : E' \rightarrow M$ denote the dual bundle of $\pi : E \rightarrow M$. As before, let (U, ϕ) be a chart in M such that $E|_U$ is trivializable. Then the dual bundle E' is also trivializable and $(\Psi^*)^{-1} : E'|_U \rightarrow \mathbb{U} \times \mathbb{E}'$ is a trivialization²⁷. The representation $(\Psi^*)^{-1}(x, \mu)$ is denoted as $(\mathbf{x}, \mathbf{m})$.

²⁵A subset $\mathbb{E}_0 \subset \mathbb{E}$ is *conic* if $\lambda \mathbb{E}_0 \subset \mathbb{E}_0$, for every $\lambda > 0$.

²⁶For further details on this geometric framework, see [17, Section 8.1].

²⁷Following the convention in [20], the adjoint of a morphism Ψ between two Banach bundles is denoted as Ψ^* .

Moreover, $T(\Psi^*)^{-1}$ defines a trivialization $TE' \rightarrow \mathbb{U} \times \mathbb{E}' \times \mathbb{M} \times E'$ and the associated local coordinates are denoted $(x, m, \dot{x}, \dot{m})$.

Given any $x \in M$, we have $(\Delta_M)_x \cong \{(x, r_x(v), v \in \mathbb{E})\}$. Now for every $p \in T'_x M$, we set:

$$(\Delta_{T'M})_{(x,p)} := (Tp_{T'M})^{-1}((\Delta_M)_x).$$

Then we obtain:

$$(\Delta_{T'M})_{(x,p)} \cong \{(x, p, \dot{x} = r_x(v), \dot{p}, u \in \mathbb{E}, \dot{p} \in \mathbb{M}'\}.$$

Considering the weak canonical symplectic form Ω on $T'M$, as in § 3.4, to any $(x, p) \in T'M$, we associate the following vector space $(D_{\Delta_M})_{(x,p)}$ of $T_{(x,p)}^p T'M$:

$$(D_{\Delta_M})_{(x,p)} = \left\{ (\dot{x}, \dot{p}, \alpha) \in T_{(x,p)}(T'M) : \begin{array}{l} \dot{x} = \rho_x(v), \quad v \in E_x, \\ \alpha - \Omega_{(x,p)}^b(v) \in (\Delta_{T'M}^0)_{(x,p)} \end{array} \right\}.$$

From the proof of Proposition 3.3, we obtain the following result:

Proposition 7.2. *One has*

$$(D_{\Delta_M})_{(x,p)} = \text{Graph}(\widetilde{\Omega}_{(x,p)}^b) \oplus (\{0\} \times (\Delta_{T'M}^0)_{(x,p)}),$$

where $\widetilde{\Omega}^b$ denotes the restriction of Ω^b to $\Delta_{T'M}$. In particular, $(D_{\Delta_M})_{(x,p)}$ defines a partial linear Dirac structure on $T_{(x,p)}(T'M)$.

As in Remark 3.4 (1), we have:

$$(D_{\Delta_M})_{(x,p)} \cong \{((x, p, r_x(u), \dot{p}), (x, p, -\dot{p}, r_x(u))) : u \in \mathbb{E}, \dot{p} \in \mathbb{M}'\}. \quad (7.1)$$

Definition 7.3. *The map*

$$D_{\Delta_M} : (x, p) \mapsto (D_{\Delta_M})_{(x,p)} \subset T_{(x,p)}^p T'M$$

is called the singular Dirac structure associated to Δ_M .

Note that if E is a closed subbundle of TM , with the anchor given by the inclusion $E \rightarrow TM$, then $E = \Delta_M$ and the above construction of D_{Δ_M} reduces to the induced partial almost-Dirac structure associated with Δ_M .

Proposition 7.4. *For any $x \in M$ and any $(x, p) \in T'_x M$ fixed, there exist an open neighborhood U of $x \in M$ and a smooth map*

$$\Theta^p : T'M|_U \times (D_{\Delta_M})_{(x,p)} \rightarrow T^p M$$

such that:

- (1) $\forall (x', p') \in T'M|_U, \Theta^p(\{(x', p')\} \times (D_{\Delta_M})_{(x,p)}) \subset (D_{\Delta_M})_{(x',p')}$;
- (2) for each $(x', p') \in T'M|_U$,

$$\Theta_{(x',p')}^p := \Theta^p((x', p'), \cdot) : (D_{\Delta_M})_{(x,p)} \rightarrow T_{(x',p')}^p M$$

is a continuous operator and $\Theta_{(x,p)}^p : (D_{\Delta_M})_{(x,p)} \rightarrow T_{(x,p)}^p M$ is the natural inclusion;

(3) $\Theta^p_{(x',p')} : (D_{\Delta_M})_{(x,p)} \rightarrow (D_{\Delta_M})_{(x',p')}$ is an isomorphism onto its range.

Definition 7.5. By abuse of terminology, we say that the induced partial almost-Dirac D_{Δ_M} is a smooth distribution in $T^p M$.

Proof. Assume that there exists a smooth map $\Theta : U \times (\Delta_M)_x \rightarrow TM$ satisfying Properties (1) and (2). Then by (1), for any $x' \in U$ and $(x', p') \in T'M|_U$, we have:

$$[(Tp_{T'M})^{-1}(\Theta_{x'}((\Delta_M)_x))](x', p') \subset (\Delta_{T'M})_{(x',p')}. \quad (7.2)$$

On the other hand, since the problem is local, we may assume without loss of generality that $M = U \subset \mathbb{M}$. In this setting, we shall use the notations introduced in § 3.2 and in Notations 7.1.

For a fixed $(x, p) \in T'M$, we define $\widehat{\Theta} : T'M|_U \times (\Delta_{T'M})_{(x,p)} \rightarrow T_{(x',p')}(T'M)$ by

$$\widehat{\Theta}((x', p'), (\dot{x}, \dot{p})) = ((x', p'), (\Theta(x', \dot{x}), \dot{p})).$$

Then, for any $(x', p') \in T'M$, the map

$$\widehat{\Theta}_{(x',p')} := \widehat{\Theta}((x', p'), \cdot)$$

defines a continuous operator and, by (7.2), we have:

$$\widehat{\Theta}_{(x',p')}((\Delta_{T'M})_{(x,p)}) \subset (\Delta_{T'M})_{(x',p')}, \quad (7.3)$$

$$\widehat{\Theta}_{(x,p)} = \text{Id}_{(\Delta_{T'M})_{(x,p)}}, \quad (7.4)$$

$$\widehat{\Theta}_{(x',p')} \text{ is an isomorphism } (\Delta_{T'M})_{(x,p)} \rightarrow \widehat{\Theta}_{(x',p')}((\Delta_{T'M})_{(x,p)}). \quad (7.5)$$

Now, from Proposition 7.2, since

$$(D_{\Delta_M})_{(x,p)} = (\Delta_{T'M})_{(x,p)} \oplus \Omega^b((\Delta_{T'M})_{(x,p)}),$$

an using (7.2), we obtain:

$$\begin{aligned} \widehat{\Theta}_{(x,p)}((\Delta_{T'M})_{(x,p)}) \oplus \Omega^b(\widehat{\Theta}_{(x,p)}((\Delta_{T'M})_{(x,p)})) \\ \subset (\Delta_{T'M})_{(x',p')} \oplus \Omega^b((\Delta_{T'M})_{(x',p')}). \end{aligned} \quad (7.6)$$

Thus, we can consider the map

$$\Theta^p : T'M|_U \times (D_{\Delta_M})_{(x,p)} \rightarrow T^p M$$

defined by

$$\Theta^p((x, p), (\dot{x}, \dot{p}), \Omega^b_{(x,p)}(\dot{x}, \dot{p})) = (\widehat{\Theta}_{(x,p)}(\dot{x}, \dot{p}), \Omega^b_{(x,p)} \circ \widehat{\Theta}_{(x,p)}(\dot{x}, \dot{p})).$$

According to this definition, using the relations (7.3), (7.4), and (7.5), we obtain the announced results. $\square$

7.2. Lagrangians and associated integrability domains with singular constraints

As announced in the introduction to this section, let E_0 be an open subset of E such that the restriction $\pi_0 : E_0 \rightarrow M$ of π is a locally trivial bundle with typical fiber $\mathbb{E}_0$, where $\mathbb{E}_0$ is a conic open subset of the typical fiber $\mathbb{E}$ of E . We denote by ρ_0 the restriction of ρ to E_0 . Then $\Sigma_M := \rho(E_0)$ is a singular nonholonomic constraint.

Consider a smooth map $\mathbf{L} : E_0 \rightarrow \mathbb{R}$. As in § 4, for $(x, v) \in E_0$, since $\pi_0^{-1}(x)$ is conic, we can define:

$$(\mathbb{F}\mathbf{L})_x(v).w = \frac{d}{ds} (\mathbf{L}(x, v + sw))|_{s=0}, \quad (7.7)$$

where v and w belong to $\pi_0^{-1}(x)$. Thus $\mathbb{F}\mathbf{L}_x(v).w$ is the derivative of $\mathbf{L}$ along the vertical fiber of $TE_0 \rightarrow E_0$ at point (x, v) , in the vertical direction determined by $w \in E_x$. Since E_0 is an open conic subbundle of E , for every $x \in M$, the map

$$(\mathbb{F}\mathbf{L})_x : \pi_0^{-1}(x) \rightarrow E'_x$$

is well-defined. Moreover, for each $v \in \pi_0^{-1}(x)$, the element $(\mathbb{F}\mathbf{L})_x(v) \in E'_x$ is a continuous linear functional on E_x ²⁸. Since $\mathbf{L}$ is smooth on the open conic subbundle E_0 of E , it follows that the associated Legendre map $\mathbb{F}\mathbf{L} : E_0 \rightarrow E'$ is a smooth fiber-preserving map.

In local coordinates on E and E' , the Legendre map is given by:

$$\mathbb{F}\mathbf{L} \cong (x, v) \mapsto \left(x, p = \frac{\partial \mathbf{L}}{\partial v} \right).$$

Since $\mathbb{E}$ is reflexive, there is no *a priori* obstruction to the Legendre map

$$\mathbb{F}\mathbf{L} : E_0 \rightarrow E'$$

being a local diffeomorphism. If this holds at every point $(x, v) \in E_0$, we shall say that $\mathbf{L}$ is *regular*.

On the other hand, consider the pull-back $E \times_M T'M$ of E along the projection $p_{T'M} : T'M \rightarrow M$. Then we have a bundle morphism $\tilde{p}_{T'M} : E \times_M T'M \rightarrow E$, which is a fiberwise isomorphism and a projection $\tilde{\pi}_{T'M} : E \times_M T'M \rightarrow T'M$. These maps fit into the following commutative diagram:

$$\begin{array}{ccc} E \times_M T'M & \xrightarrow{\tilde{p}_{T'M}} & E \\ \downarrow \tilde{\pi} & & \downarrow \pi \\ T'M & \xrightarrow{p_{T'M}} & M \end{array} \quad (7.8)$$

Since E_0 is an open subbundle of E , the same diagram holds upon restriction to E_0 :

$$\begin{array}{ccc} E_0 \times_M T'M & \xrightarrow{\tilde{p}_{T'M}} & E_0 \\ \downarrow \tilde{\pi}_0 & & \downarrow \pi \\ T'M & \xrightarrow{p_{T'M}} & M \end{array} \quad (7.9)$$

²⁸Here, E'_x denotes the fiber over x of the dual bundle E' .

Lemma 7.6. *If the Lagrangian $\mathbf{L} : E_0 \rightarrow \mathbb{R}$ is regular, then for every $x \in M$, there exist a chart (U, ϕ) around x and a trivialization $\Phi : E|_U \rightarrow \phi(U) \times \mathbb{E}$ such that, in the associated local coordinates, the map*

$$(z, v) \mapsto \frac{\partial^2 \mathbf{L}}{\partial v^2}(z, v)$$

defines a smooth field of bilinear forms on $\mathbb{E}$, which is strongly nondegenerate at (x, u) .

Proof. Let $(x, u) \in E_0$. We work in a local trivialization of E over a chart (U, ϕ) around x , identifying a neighborhood of (x, u) with an open subset of $\mathbb{U} \times \mathbb{E}$, where $\mathbb{U} = \phi(U)$. In these coordinates, the Legendre map $\mathbb{F}\mathbf{L}$ is given by

$$(z, v) \rightarrow \left(z, \frac{\partial \mathbf{L}}{\partial v}(z, v) \right).$$

Its differential at (x, u) has the block form:

$$T_{(x,u)}\mathbb{F}\mathbf{L} = \begin{pmatrix} \text{Id} & 0 \\ \frac{\partial^2 \mathbf{L}}{\partial x \partial v} & \frac{\partial^2 \mathbf{L}}{\partial v^2} \end{pmatrix}. \quad (7.10)$$

Since the upper-left block is the identity, the map $T_{(x,u)}\mathbb{F}\mathbf{L}$ is an isomorphism if and only if the vertical block

$$\frac{\partial^2 \mathbf{L}}{\partial v^2}(x, u)$$

is a strongly non-degenerate bilinear form on $\mathbb{E}$. By the inverse function theorem, $\mathbb{F}\mathbf{L}$ is a local diffeomorphism around (x, u) if and only if its differential at (x, u) is an isomorphism. Therefore, regularity of $\mathbf{L}$ is equivalent to strong non-degeneracy of the fiberwise Hessian, as claimed. $\square$

Proposition 7.7. *Assume that the Lagrangian $\mathbf{L}$ is regular.*

(1) *The set*

$$\mathbb{P} := \{(z, v, p) \in E_0 \times T'M : \rho_z^*(p) = \mathbb{F}\mathbf{L}(z, v)\}$$

is a split Banach immersed submanifold of $E_0 \times_M T'M$ modeled on the space $\mathbb{M} \times \mathbb{M}'$. Moreover, the restriction $\Theta_{\mathbf{L}}$ of $\tilde{\pi}_0$ to $\mathbb{P}$ is a surjective local diffeomorphism from $\mathbb{P}$ into $T'M$.

(2) *Let $\tilde{\Omega}$ be the pull-back $\tilde{\pi}_0^*\Omega$. Then $\tilde{\Omega}$ is a strong symplectic 2-form on $E_0 \times_M T'M$ ²⁹ whose kernel is $\ker T\tilde{\pi}_0$ and whose supplementary distribution is $\ker T\tilde{p}_{T'M}$. Consequently, the graph of $\tilde{\Omega}$ defines a partial Dirac structure $D_{\tilde{\Omega}}$ on $E_0 \times_M T'M$. In particular, the induced Dirac structure $D_{\mathbb{P}}$ on $\mathbb{P}$ is the graph of the (weak) symplectic 2-form $\Omega_{\mathbb{P}}$ induced by $\tilde{\Omega}$ on $\mathbb{P}$. Equivalently, $\Theta_{\mathbf{L}}^*(\Omega)$ is a weak symplectic form on $\mathbb{P}$ whose associated Dirac structure $D_{\Omega_{\mathbb{P}}}$ is also the pull-back $\Theta_{\mathbf{L}}^!(D_{\Delta_M})$ on $\mathbb{P}$.*

Proof. (1) Let (U, ϕ) be a chart on M such that $E|_U$ is trivial, and let $\Phi : E|_U \rightarrow \phi(U) \times \mathbb{E}$ be a local trivialization. Since E_0 is an open subbundle of E , the restriction $\Phi : E_0|_U \rightarrow \phi(U) \times \mathbb{E}_0$ is also a trivialization. Likewise, the cotangent bundle $T'M|_U$ is trivialized by the adjoint map $T^*\phi$. In the same way, $T'M|_U$ is trivialized by the adjoint map $T^*\phi$. Since the statement is local, we may identify U with its image $\mathbb{U} = \phi(U)$ and write

$$E_{0|U} \cong \mathbb{U} \times \mathbb{E}_0, \quad T'M|_U \cong \mathbb{U} \times \mathbb{M}',$$

²⁹See [16, Section 4.1].

so that

$$E_{0|U} \times_M T'M|_U \cong U \times \mathbb{E}_0 \times \mathbb{M}'.$$

In this situation, we have

$$\mathbb{P} = \left\{ (z, v, p) : \rho_z^*(p) - \frac{\partial \mathbf{L}}{\partial u}(z, v) = 0 \right\}. \quad (7.11)$$

Let $(x, u, p_0) \in \mathbb{P}$ and write

$$\mathbb{FL}(x, u) = (x, \xi).$$

By Lemma 7.6, the differential of $\mathbb{FL}$ at (x, u) is an isomorphism. Hence, by the inverse function theorem for Banach manifolds, after possibly shrinking U , there exist open neighborhoods

$$U \times V \subset U \times \mathbb{E}_0 \quad \text{and} \quad U \times V' \subset U \times \mathbb{E}'$$

of (x, u) and (x, ξ) , respectively, such that

$$\mathbb{FL} : U \times V \rightarrow U \times V'.$$

Define

$$W = \left\{ (z, p) \in U \times \mathbb{M}' : (z, \rho_z^*(p)) \in U \times V' \right\}. \quad (7.12)$$

Since ρ^* is smooth, W is an open subset of $U \times \mathbb{M}'$. Moreover, on W , the local inverse of the Legendre map is well-defined, and $\mathbb{P}$ is locally the graph of the smooth map

$$(z, p) \mapsto (\mathbb{FL}_{|U \times V})^{-1}(z, \rho_z^*(p)).$$

More precisely,

$$\mathbb{P} \cap (U \times V \times \mathbb{M}') = \left\{ (z, (\mathbb{FL}_{|U \times V})^{-1}(z, \rho_z^*(p)), p) : (z, p) \in W \right\}. \quad (7.13)$$

Therefore, $\mathbb{P}$ is a closed immersed submanifold in $E_0 \times_M T'M$ modeled on $\mathbb{M} \times \mathbb{M}'$ and whose tangent bundle is transverse to $\ker T\tilde{\pi}_0$, and so is a split immersed submanifold of $E_0 \times_M T'M$. The last property is clear.

(2) Consider the closed 2-form $\tilde{\pi}_0^*(\Omega)$ on $\mathbb{P}$. Using the local description established in the proof (1), we see that

$$\ker \tilde{\pi}_0^*(\Omega) = \ker T\tilde{\pi}_0.$$

It follows that the closed 2-form $\tilde{\Omega}$ is a strong pre-symplectic form.

On the other hand, $\tilde{\Omega}^b(T(E_0 \times_M T'M))$ is the pull-back of $T^b(T(T'M))$ over $T\tilde{\pi}_0$. Therefore the graph of $\tilde{\Omega}$ defines a partial Dirac structure $D_{\tilde{\Omega}}$ (cf. [16], Example 4.17). Moreover, by Assertion (1), the tangent bundle of $\mathbb{P}$ is transverse to the kernel of $\tilde{\pi}_0^*(\Omega)$. Hence the induced form $\Omega_{\mathbb{P}}$ on $\mathbb{P}$ is a weak symplectic form and its graph is a partial almost-Dirac structure by the same argument as for $\tilde{\Omega}$. Now, from [16, Lemma-Definition 5.1], the pull-back $\Theta_L^!(D_{\Delta_M})$ on $\mathbb{P}$ is the graph of $\tilde{\Omega}$. $\square$

In the context of Proposition 7.7, we set:

$$\tilde{M} = E_0 \times_M T'M \quad \text{and} \quad \tilde{z} = (z, v, p) \in \tilde{M}.$$

Definition 7.8. Let $H : \tilde{M} \rightarrow \mathbb{R}$ be a Hamiltonian function.

- (1) The implicit Hamiltonian system associated with $(\tilde{M}, D_{\tilde{\Omega}}, H)$ is the dynamical system defined by the differential inclusion

$$(\dot{\tilde{z}}(t), d_{\tilde{z}(t)}H) \in (D_{\tilde{\Omega}})_{\tilde{z}(t)}. \quad (7.14)$$

- (2) A smooth curve $\tilde{c} :]-\varepsilon, \varepsilon[\rightarrow \tilde{M}$ is called an integral curve of the system if it satisfies:

$$(\dot{\tilde{c}}(t), d_{\tilde{c}(t)}H) \in (D_{\tilde{\Omega}})_{\tilde{c}(t)} \quad \text{for all } t \in (-\varepsilon, \varepsilon).$$

Such a curve is said to pass through $\tilde{z} = \tilde{c}(0)$.

Remark 7.9.

1. Using the local context of the proof of Proposition 7.7, in local coordinates, we have

$$\tilde{\Omega}_{\tilde{z}}((\dot{x}, \dot{v}, \dot{p}), (\dot{z}', \dot{v}', \dot{p}')) \cong \dot{p}'(\dot{z}) - \dot{p}(\dot{z}'). \quad (7.15)$$

2. Let $\tilde{c} :]-\varepsilon, \varepsilon[\rightarrow \tilde{M}$ be an integral curve of the implicit Hamiltonian system (7.14) through $\tilde{z}_0 = \tilde{c}(0)$. This means that we have

$$\tilde{\Omega}(\dot{\tilde{c}}(t), \cdot) = -d_{\tilde{c}(t)}H.$$

If we write $\tilde{c} := (z(t), v(t), p(t))$, this implies $d_{\tilde{z}(t)}((\dot{z}(t), \dot{v}(t), \dot{p}(t))) = 0$ for all $t \in]-\varepsilon, \varepsilon[$ and so this implies $\frac{\partial H}{\partial v}(\tilde{c}(t)) = 0$ for all $t \in]-\varepsilon, \varepsilon[$. If we write $\tilde{c} := (z(t), v(t), p(t))$, its projection $(z(t), p(t))$ on $T'M$ is an integral curve of the singular partial almost-Dirac $(D_{\Delta_M})_{(z(t), p(t))}$; in particular, if we have $\dot{z}(t) = \rho_{z(t)}(v(t))$ on $]-\varepsilon, \varepsilon[$, then z is an admissible curve.

3. Conversely, consider a curve $t \mapsto (z(t), p(t))$ on $]-\varepsilon, \varepsilon[$ which is an integral curve of $(D_{\Delta_M})_{(z(t), p(t))}$. Then $\dot{z}(t)$ belongs to $(\Delta_M)_{z(t)}$ and we assume that $\dot{z}(t) = \rho(z(t), v(t))$. Then $t \mapsto (z(t), v(t), p(t))$ is an integral curve of the implicit Hamiltonian system (7.14) if and only if $\frac{\partial H}{\partial v}(\tilde{c}(t)) = 0$, for all $t \in]-\varepsilon, \varepsilon[$.
4. More generally, assume that H is a Hamiltonian; since $\tilde{\Omega}$ is a strong pre-symplectic form on $\tilde{M}$, then the equation

$$\tilde{\Omega}(X, \cdot) = -dH \quad (7.16)$$

has a solution if and only if $\frac{\partial H}{\partial v}(z, v, p) = 0$. Such a solution is determined only up to the addition of elements in $\ker \tilde{\Omega}$. For further details on this degeneracy, see [16], Proposition 4.6.

5. In local coordinates, the implicit Hamiltonian system $(\tilde{M}, D_{\tilde{\Omega}}, H)$ can be written as follows:

$$\begin{cases} \dot{z}(t) = \frac{\partial H}{\partial p}(z(t), v(t), p(t)), \\ \dot{p}(t) = -\frac{\partial H}{\partial x}(z(t), v(t), p(t)), \\ \frac{\partial H}{\partial v}(z(t), v(t), p(t)) = 0. \end{cases} \quad (7.17)$$

For simplicity, we set $\tilde{z}(t) = (z(t), v(t), p(t))$. Since $\frac{\partial H}{\partial v}(\tilde{z}(t)) = 0$, we obtain that

$$\begin{aligned} d_{\tilde{z}(t)}H(\tilde{z}, \dot{v}(t), \dot{p}(t)) &= \frac{\partial H}{\partial x}(\tilde{z}(t))\dot{z}(t) + \frac{\partial H}{\partial p}(\tilde{z}(t))\dot{p}(t) \\ &= \frac{\partial H}{\partial x}(\tilde{z}(t))\frac{\partial H}{\partial p}(\tilde{z}(t)) - \frac{\partial H}{\partial p}(\tilde{z}(t))\frac{\partial H}{\partial x}(\tilde{z}(t)) \\ &= 0. \end{aligned}$$

Hence, H is constant along any solution of (7.17).

6. An integral curve $t \mapsto \tilde{z}(t) = (z(t), v(t), p(t))$ of $(\tilde{M}, D_{\tilde{\Omega}}, H)$ satisfies that the curve $t \mapsto (z(t), v(t))$ is tangent to Δ_M and that

$$\mathbb{F}\mathbf{L}(z(t), v(t)) = (z(t), p(t)).$$

Hence, $\tilde{z}(t)$ takes values in $\mathbb{P}$. In particular, $\mathbb{P}$ is the integrability domain of $(\tilde{M}, D_{\tilde{\Omega}}, H)$. Since the restriction of $\tilde{\Omega}$ to $\mathbb{P}$ is weak symplectic, let H be a Hamiltonian function on $\tilde{M}$. If the differential of the restriction $H_{\mathbb{P}} := H|_{\mathbb{P}}$ satisfies the hypotheses of [16], Proposition 4.6, on some open subset $U \subset \mathbb{P}$, then there exists a unique Hamiltonian vector field X_H on $\mathbb{P}$. Moreover, $t \mapsto (z(t), v(t), p(t))$ is an integral curve of the implicit Hamiltonian system (7.14) on U if and only if it is an integral curve of X_H . Thus, by Assertion (5), there exists an open set $W \subset T^*M$ on which h_W is defined (cf. relation (8.11)). Moreover, h_W is constant along each integral curve of X_H contained in U .

To the Lagrangian $\mathbf{L}$, we associate the Hamiltonian on $\tilde{M}$ defined by:

$$H(z, v, p) = \langle p, \rho_z(v) \rangle - \mathbf{L}(z, v). \quad (7.18)$$

From Remark 7.9, we have:

Proposition 7.10. *Let $H_{\mathbf{L}}^{\mathbb{P}}$ denote the restriction of $H_{\mathbf{L}}$ to $\mathbb{P}$. Then the equation*

$$i_X \Omega_{\mathbb{P}} = -dH_{\mathbf{L}}^{\mathbb{P}}$$

admits a unique solution $X_{\mathbf{L}}$ on $\mathbb{P}$. Moreover, any integral curve of the implicit Hamiltonian system $(\tilde{M}, D_{\tilde{\Omega}}, H_{\mathbf{L}})$ through any point of $\mathbb{P}$ is an integral curve of $X_{\mathbf{L}}$. In particular, such curves are unique and tangent to $\mathbb{P}$.

Proof. Recall that $\Omega_{\mathbb{P}}$ is a weak symplectic form on $\mathbb{P}$. Now we have

$$\frac{\partial H_{\mathbf{L}}}{\partial v}(z, v, p) = \rho_z^* p - \mathbb{F}\mathbf{L}.$$

It follows that $\frac{\partial H_{\mathbf{L}}^{\mathbb{P}}}{\partial u} \equiv 0$. Thus $H_{\mathbf{L}}^{\mathbb{P}}$ trivially satisfies the hypothesis of [16], Proposition 4.6, and hence the Hamiltonian vector field $X_{\mathbf{L}}$ is well-defined. The final statement of the proposition follows immediately. $\square$

This proposition justifies the following definition:

Definition 7.11. *The immersed submanifold $\mathbb{P}$ of $\tilde{M}$ is called the integrability domain of the implicit Hamiltonian system $(\tilde{M}, D_{\tilde{\Omega}}, H_{\mathbf{L}})$ associated with $\mathbf{L}$.*

Remark 7.12. Taking into account Remark 7.9 (1), the Hamiltonian vector field X_L can be written as

$$X_L(z, v, p) = \left(\rho(z, v), 0, -\langle p, \frac{\partial}{\partial x}(\rho(z, v)) \rangle + \frac{\partial^2}{\partial x \partial v}(\mathbf{L}(z, v)) \right). \quad (7.19)$$

Moreover, according to (7.13), since $\mathbb{F}\mathbf{L}$ is invertible on an open set $U \times V$, there exists an open set $W \subset T'M$ on which the smooth map

$$v_W(z, p) := (\mathbb{F}\mathbf{L})^{-1}(\rho_z^*(p))$$

is well-defined. For $(z, p) \in W$, we have:

$$H_L^{\mathbb{P}}(z, v, p) = \langle p, \rho(z, v_W(z, p)) - \mathbf{L}(z, v_W(z, p)) \rangle := h_W(z, p). \quad (7.20)$$

Therefore, for $(z, p) \in W$, we have:

$$\begin{aligned} X_L(z, v, p) &= \left(\rho(z, v), 0, -\langle p, \frac{\partial(u_W)}{\partial x}(z, p) \rangle + \frac{\partial^2}{\partial x \partial u}(\mathbf{L}(z, u_W(z, v))) \right) \\ &= \left(\frac{\partial h_W}{\partial p}(z, p), 0, -\frac{\partial h_W}{\partial x}(z, p) \right). \end{aligned} \quad (7.21)$$

Note that h_W can be regarded as a Hamiltonian function on $W \subset T'M$, and hence X_L as a Hamiltonian vector field with respect to the canonical symplectic form Ω on $T'M$. Moreover, by Remark 7.9 (6), the function h_W is constant along any integral curve of X_L .

7.3. Normal extremals of a Lagrangian with singular constraints

In this subsection, we study the normal extremals of the dynamical system associated with a regular Lagrangian defined on E_0 in the presence of singular constraints.

Consider a regular Lagrangian $\mathbf{L} : E_0 \rightarrow \mathbb{R}$. For $2 \leq k \leq \infty$, let $C^k([t_1, t_2], E_0)$ denote the set of curves $(z, v) : [t_1, t_2] \rightarrow E_0$ of class C^k and let $C_{E_0}^k([t_1, t_2], M)$ denote the set of E_0 -admissible curves $c : [t_1, t_2] \rightarrow M$, that is,

$$C_{E_0}^k([t_1, t_2], M) = \left\{ (c, v) \in C^k([t_1, t_2], E_0) : \dot{c}(t) = \rho_{c(t)}(v(t)) \right\}. \quad (7.22)$$

Associated with the Lagrangian $\mathbf{L}$ is the action functional

$$\mathfrak{F}_L(c) = \int_{t_1}^{t_2} \mathbf{L}(c(t), v(t)) dt$$

defined on the space $C_{E_0}^k([t_1, t_2], M)$.

A horizontal variation of $(c, v) \in C_{E_0}^k([t_1, t_2], M)$ is a C^k map

$$\hat{c} :]-\varepsilon, \varepsilon[\times [t_1, t_2] \rightarrow E_0$$

such that

$$c_s := \hat{c}(s, \cdot) \in C_{E_0}^k([t_1, t_2], M) \quad \text{and} \quad \hat{c}(0, t) = c(t).$$

We say that $\hat{c}$ is a *horizontal variation with fixed origin (resp., ends)* if for all s , $c_s(t_1) = c(t_1)$ (resp., $c_s(t_1) = c(t_1)$ and $c_s(t_2) = c(t_2)$ for all s). $\delta c(t) := \frac{\partial \hat{c}}{\partial s}(0, t)$ is called an *infinitesimal horizontal variation* of c .

For any horizontal variation $\hat{c}$ of c , the map $s \mapsto \mathfrak{F}_L(c_s)$ is differentiable at $s = 0$. Its derivative at $s = 0$ is denoted by $d_c \mathfrak{F}_L(\delta c)$.

More precisely, for $k \geq 2$, in this Banach context (cf. Warning 5.10), we have (cf. [40]) and [41]):

Theorem 7.13.

- (1) The space $C^k([t_1, t_2], M)$ of C^k curves in M , as well as the space $C_E^k([t_1, t_2], M)$ of E -admissible curves $\gamma = (c, v) : [t_1, t_2] \rightarrow E$ of class C^k , carries a Banach manifold structure. Moreover, the natural projection $C_E^k([t_1, t_2], M) \rightarrow C^k([t_1, t_2], M)$ is a Banach bundle.
- (2) For every fixed $x \in M$, the subset

$$C_E^k(x, [t_1, t_2], M) = \{(c, v) \in C_E^k([t_1, t_2], M) : c(t_1) = x\}$$

is a split submanifold of $C_E^k([t_1, t_2], M)$ and the endpoint map $\text{End}_x : C_E^k(x, [t_1, t_2], M) \rightarrow M$ is smooth.

Since E_0 is an open subbundle of E , Theorem 7.13 remains valid after replacing E with E_0 .

We now present a global generalization of Proposition 2 in [14], extending it beyond the framework of sub-Riemannian geometry. Although the proof follows the same general strategy as in [14], several arguments require careful adaptation to the present setting. For the reader's convenience, we provide a complete proof.

Proposition 7.14. Consider a curve $c \in C_{E_0}^k(x, [t_1, t_2], M)$. Then there exists a nonzero covector $q \in T'_{c(t_2)} M$ such that

$$vd_c \mathfrak{F}_L - T_c^* \text{End}_{c(t_1)}(q) = 0$$

if and only if there exists a C^k section p of $c^! T' M$ such that $p(t_2) = q$ satisfying:

$$\begin{cases} \dot{c}(t) = \frac{\partial H^v}{\partial p}(c(t), v(t), p(t)) \\ \dot{p}(t) = -\frac{\partial H^v}{\partial x}(c(t), v(t), p(t)) \\ \frac{\partial H^v}{\partial v}(c(t), v(t), p(t)) = 0 \end{cases} \quad (7.23)$$

In this case, either $\gamma = (c, v)$ is a critical point of $\mathfrak{F}_L$ with fixed endpoints and $v = 1$, or it is a critical point of the endpoint map End_x with $v = 0$.

Remark 7.15. According to Proposition 7.14, Remark 7.9 (3), and Proposition 7.10, we have:

- if (γ, p) is a solution of the differential system (7.23), then $p = \mathbb{F}L(\gamma)$ with $p(t_2) \neq 0$ for $v = 1$ and so (γ, p) is an integral curve of the Hamiltonian vector field X_L ; therefore it is a smooth curve.
- Conversely, if $\gamma := (c, v)$ is an E_0 -lift of c such that $(\gamma, \mathbb{F}L(\gamma))$ is an integral curve of X_L with $p(t_2) \neq 0$, then γ is a critical point of $\mathfrak{F}_L$ with fixed endpoints with $v = 1$.

Proof of Proposition 7.14. Let $c \in C_{E_0}^k(x, [t_1, t_2], M)$, and let

$$\gamma : t \rightarrow \gamma(t) = (c(t), v(t))$$

be an E_0 -lift of c . We first prove that the existence of a covector $q \in T'_{c(t_2)}M$ satisfying

$$\nu d_c \mathfrak{F}_L - T_c^* \text{End}_{c(t_1)}(q) = 0$$

is equivalent to the existence of a section

$$p \in \Gamma^k(c^! T' M)$$

with $p(t_2) = q$ satisfying the Hamiltonian equations.

Let p be the solution of the adjoint equation

$$\dot{p}(t) = -\frac{\partial H^v}{\partial x}(c(t), v(t), p(t)), \quad p(t_2) = q.$$

In local coordinates, this equation becomes

$$\dot{p}(t) = -\left\langle p, \frac{\partial \rho}{\partial x}(c(t), v(t)) \right\rangle - v \frac{\partial \mathbf{L}}{\partial x}(c(t), v(t)). \quad (7.24)$$

Since the coefficients of this equation are of class C^k , the section p is also of class C^k . Moreover, since $\dot{c} = \rho(\gamma)$, we have, for every variation,

$$\delta \dot{c} = \frac{\partial \rho}{\partial x}(\gamma) \delta c + \rho(c, \delta v).$$

Using the definition of the differential of the action functional, we obtain

$$\begin{aligned} & \nu d_c \mathfrak{F}_L(\delta c) - T_c^* \text{End}_{c(t_1)}(q)(\delta c) \\ &= v \int_{t_1}^{t_2} \frac{\partial \mathbf{L}}{\partial x}(\gamma)(\delta c) dt + \int_{t_1}^{t_2} \left\langle p, \frac{\partial \rho}{\partial v}(\gamma)(\delta v) \right\rangle dt - p(t_2)(\delta c(t_2)). \end{aligned} \quad (7.25)$$

From (7.24), we obtain

$$v \frac{\partial \mathbf{L}}{\partial x}(\gamma)(\delta c) = -\left(\dot{p} + \left\langle p, \frac{\partial \rho}{\partial x}(\gamma) \right\rangle \right)(\delta c). \quad (7.26)$$

Substituting this expression into (7.25), we get

$$\begin{aligned} & \nu d_c \mathfrak{F}_L(\delta c) - T_c^* \text{End}_{c(t_1)}(q)(\delta c) \\ &= -\int_{t_1}^{t_2} \langle \dot{p}, \delta c \rangle dt - \int_{t_1}^{t_2} \left\langle p, \frac{\partial \rho}{\partial x}(\gamma)(\delta c) \right\rangle dt \\ & \quad + \int_{t_1}^{t_2} \left\langle p, \frac{\partial \rho}{\partial v}(\gamma)(\delta v) \right\rangle dt - p(t_2)(\delta c(t_2)). \end{aligned}$$

Integrating by parts, and using the fixed endpoint conditions

$$\delta c(t_1) = \delta c(t_2) = 0,$$

we have

$$\int_{t_1}^{t_2} \langle \dot{p}, \delta c \rangle dt = - \int_{t_1}^{t_2} \langle p, \delta \dot{c} \rangle dt.$$

Therefore,

$$\begin{aligned} & \nu d_c \mathfrak{F}_L(\delta c) - T_c^* \text{End}_{c(t_1)}(q)(\delta c) \\ &= \int_{t_1}^{t_2} \left\langle p, \delta \dot{c} - \frac{\partial \rho}{\partial x}(\gamma)(\delta c) \right\rangle dt \\ & \quad + \int_{t_1}^{t_2} \left\langle p, \frac{\partial \rho}{\partial v}(\gamma)(\delta v) \right\rangle dt. \end{aligned}$$

Using

$$\delta \dot{c} = \frac{\partial \rho}{\partial x}(\gamma)(\delta c) + \rho(c, \delta v),$$

the terms involving δc cancel and we obtain

$$\nu d_c \mathfrak{F}_L(\delta c) - T_c^* \text{End}_{c(t_1)}(q)(\delta c) = \int_{t_1}^{t_2} \langle p, \rho(c, \delta v) \rangle dt + \int_{t_1}^{t_2} \left\langle p, \frac{\partial \rho}{\partial v}(\gamma)(\delta v) \right\rangle dt.$$

By the definition of the Hamiltonian

$$H^\nu(x, v, p) = \langle p, \rho(x, v) \rangle - \nu L(x, v),$$

the last expression is exactly

$$- \int_{t_1}^{t_2} \left\langle \frac{\partial H^\nu}{\partial v}(\gamma(t), p(t)), \delta v(t) \right\rangle dt.$$

Hence

$$\nu d_c \mathfrak{F}_L - T_c^* \text{End}_{c(t_1)}(q) = - \int_{t_1}^{t_2} \left\langle \frac{\partial H^\nu}{\partial v}, \delta v \right\rangle dt. \quad (7.27)$$

Since the equality holds for every admissible variation δv , the fundamental lemma of the calculus of variations implies that

$$\frac{\partial H^\nu}{\partial v}(c(t), v(t), p(t)) = 0, \quad t \in [t_1, t_2].$$

Together with the equations defining p and $\dot{c}$, this gives the system

$$\begin{cases} \dot{c} = \frac{\partial H^\nu}{\partial p}, \\ \dot{p} = -\frac{\partial H^\nu}{\partial x}, \\ \frac{\partial H^\nu}{\partial v} = 0. \end{cases}$$

Conversely, if a section p satisfies these Hamiltonian equations and $p(t_2) = q$, reversing the above computation gives

$$\nu d_c \mathfrak{F}_L - T_c^* \text{End}_{c(t_1)}(q) = 0.$$

Finally, when $\nu = 1$, this condition is precisely the critical point condition for $\mathfrak{F}_L$ with fixed endpoints, whereas when $\nu = 0$, it is the critical point condition for the endpoint map. This completes the proof. $\square$

Definition 7.16. A curve $c \in C_{E_0}^\infty([t_1, t_2], M)$ is called a critical point of $\mathfrak{F}_L$ with fixed endpoints if

$$d\mathfrak{F}_L(\delta c) = 0 \quad (7.28)$$

for every infinitesimal horizontal variation δc with vanishing endpoints.

Once again, we observe that

$$C_{E_0}^\infty([t_1, t_2], M) = \{(c, v) \in C^\infty([t_1, t_2], E_0) : \dot{c}(t) - \rho_{c(t)}(v(t)) = 0\}. \quad (7.29)$$

Accordingly, on $C^\infty([t_1, t_2], \widetilde{M})$, we define the functional:

$$\widehat{\mathfrak{F}}_L = \int_{t_1}^{t_2} (\bar{L}(x(t), v(t)) + \langle p(t), (\dot{x}(t) - \rho_{x(t)}(v(t))) \rangle) dt. \quad (7.30)$$

Let $(\hat{c}, \hat{v}, \hat{p})$ be a variation defined on $] - \varepsilon, \varepsilon[\times [t_1, t_2]$ of (c, v, p) in $C^\infty([t_1, t_2], \widetilde{M})$. Then the map $s \mapsto \widehat{\mathfrak{F}}_L(c_s, v_s, p_s)$ is differentiable at $s = 0$. Its differential at (c, v, p) in the direction $(\delta v, \delta v, \delta p)$ is denoted by $d_c \widehat{\mathfrak{F}}_L(\delta c, \delta v, \delta p)$.

Definition 7.17. A curve $t \mapsto (c(t), v(t), p(t))$ in $\widetilde{M}$, defined on $[t_1, t_2]$, is said to be a critical point with fixed endpoints of $\widehat{\mathfrak{F}}_L$ if

$$d\widehat{\mathfrak{F}}_L(\delta c, \delta v, \delta p) = 0 \quad (7.31)$$

for every infinitesimal variation $(\delta c, \delta v, \delta p)$ associated with a variations $(\hat{c}, \hat{v}, \hat{p})$ satisfying

$$\hat{c}(s, t_1) = c(t_1) \quad \text{and} \quad \hat{c}(s, t_2) = c(t_2)$$

for all s .

The previous characterization admits a simplified form when the configuration curve is fixed. In this case, the variational principle is defined only with respect to vertical variations along the prescribed curve c . Consequently, no integration by parts with respect to the base curve is involved, and the variational equations are obtained directly from the vertical first variation. The proof follows the same strategy as that of Theorem 5.15, taking into account that the admissible variations are now vertical along the fixed curve.

Theorem 7.18. Let

$$c \in C_{E_0}^\infty([t_1, t_2], M)$$

be fixed.

Under the preceding assumptions, the following assertions are equivalent.

(1) There exists a section

$$p \in \Gamma(c^! T' M)$$

such that

$$t \mapsto (c(t), v(t), p(t))$$

is a critical point with fixed endpoints of the Hamilton–Pontryagin functional $\widehat{\mathfrak{F}}_L$.

(2) The prescribed curve c is a critical point with fixed endpoints of $\mathfrak{F}_L$.

(3) In every local coordinate system associated with a chart whose domain intersects $c([t_1, t_2])$, the Euler–Lagrange equation is

$$\frac{\partial \mathbf{L}}{\partial \mathbf{x}}(\mathbf{c}(t), \mathbf{v}(t)) - \frac{d}{dt} \frac{\partial \mathbf{L}}{\partial \mathbf{v}}(\mathbf{c}(t), \mathbf{v}(t)) = 0.$$

(4) Let $H_{\mathbf{L}}$ denote the Hamiltonian induced on $\mathbb{P}$ by $\mathbf{L}$. Then there exists a section

$$p \in \Gamma(c^!T'M)$$

such that

$$t \mapsto (c(t), v(t), p(t))$$

is an integral curve of the implicit Hamiltonian system

$$(\widetilde{M}, D_{\widetilde{\Omega}}, H_{\mathbf{L}}).$$

Equivalently, in local coordinates, the curve satisfies the system (7.17).

In reference to Definition 5.11, we introduce the following notion:

Definition 7.19. Under the previous assumptions, a curve $c \in C_{E_0}^\infty([t_1, t_2], M)$ is called a normal extremal if, in any local coordinate system associated with a chart domain intersecting $c([t_1, t_2])$, there exists a section $p : t \mapsto p(t)$ of $c^!T'M$ such that the curve

$$t \mapsto (c(t), p(t)) := \mathbb{F}\mathbf{L}(c(t), v(t))$$

is a solution of the implicit differential system (7.17) with $p(t_2) \neq 0$.

We then have the following corollary:

Corollary 7.20. Let $\mathbf{L}$ be a regular Lagrangian on E_0 and let $c \in C_{E_0}^\infty([t_1, t_2], M)$ be a curve. Assume that c satisfies any of the equivalent conditions of Theorem 7.18 and that there exists a section p of $c^!T'M$ such that $p(t_2) \neq 0$. Then c is a normal extremal of $\mathfrak{F}_{\mathbf{L}}$ and, moreover,

$$\mathbb{F}\mathbf{L}(c(t), v(t)) = (c(t), p(t)).$$

Proof. We may assume condition (4) in Theorem 7.18 together with $p(t_2) \neq 0$. It follows that $(c(t), v(t), p(t))$ satisfies the differential system (7.23) of Theorem 7.18. Hence c is a normal extremal since $p(t_2) \neq 0$. $\square$

Remark 7.21. According to Corollary 7.20 and Theorem 7.18, $c \in C_{E_0}^\infty([t_1, t_2], M)$ is a normal extremal if and only if c is a critical point of $\mathfrak{F}_{\mathbf{L}}$ and

$$\mathbb{F}\mathbf{L}(c(t_2), v(t_2)) \neq (c(t_2), 0).$$

8. Normal geodesics of a conic sub-Hilbert–Finsler structure on a Banach manifold

As in the previous section, we restrict ourselves in this section to the Banach setting.

We present an application of the results of Section 7, and we also characterize the normal geodesics associated with this metric.

8.1. Conic Minkowski norm on a Banach space

Let $\mathbb{E}$ be a Banach space. Recall that a *conic domain* $\mathbb{E}_0$ in $\mathbb{E}$ is an open subset of $\mathbb{E}$ such that if $u \in \mathbb{E}_0$, then λu belongs to $\mathbb{E}_0$ for any $\lambda > 0$ and any $u \in \mathbb{E}_0$.

A *weak conic Minkowski norm* on $\mathbb{E}$ is a map F from a conic domain $\mathbb{E}_0$ in $\mathbb{E}$ into $[0, \infty[$ with the following properties:

1. $\forall u \in \mathbb{E}_0 \setminus \{0\}, F(u) > 0$;
2. $\forall \lambda > 0, \forall u \in \mathbb{E}_0, F(\lambda u) = \lambda F(u)$;
3. F is smooth on $\mathbb{E}_0 \setminus \{0\}$, so that the Hessian of $\frac{1}{2}F^2$ is well-defined.

Note that if $0 \in \mathbb{E}_0$, then the conic property implies $\mathbb{E}_0 = \mathbb{E}$.

If $\mathbb{S}$ is a hypersurface of $\mathbb{E}$ not containing 0, then

$$\mathbb{E}_0 = \{\lambda u, u \in \mathbb{S}, \lambda > 0\}$$

is a conic domain in $\mathbb{E}$.

The Hessian g of F^2 is defined, for any $u \in \mathbb{E}_0$, any v and w in $\mathbb{E}$, by:

$$g_u(v, w) = \frac{\partial^2}{\partial s \partial t} \left(\frac{1}{2} F^2(u + sv + tw) \right) \Big|_{s=t=0}. \quad (8.1)$$

Therefore, g_u is positive definite on $\mathbb{E}$. In particular, g is a (weak) Riemannian metric on the Banach manifold $\mathbb{E}_0$ and each g_u defines an inner product on $\mathbb{E}$. From the definition of F , we obtain the following proposition (see [42], Proposition 2.2).

Proposition 8.1. *The following properties hold.*

1. $\forall \lambda > 0, g_{\lambda u} = g_u$.
2. $g_u(u, u) = \|u\|^2 := \mathcal{F}^2(u)$.
3. $g_u(u, w) = \frac{\partial}{\partial s} (F^2(u + sw)) \Big|_{s=0}$.

Definition 8.2. A conic Minkowski norm on $\mathbb{E}$ is a weak conic Minkowski norm F whose Hessian g defines a strong Riemannian metric on the domain $\mathbb{E}_0$.

When the domain $\mathbb{E}_0$ contains 0, a conic Minkowski norm will simply be called a *Minkowski norm*, even though it is not a norm on $\mathbb{E}$ in the classical sense.

If F is a conic Minkowski norm on $\mathbb{E}$, then each inner product g_u endows $\mathbb{E}$ with a Hilbert space structure. Conversely, if $\mathbb{E}$ is Hilbertizable, we consider an inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{E}$ and denote by F the associated norm. For any conic domain in $\mathbb{E}_0$, the restriction F to $\mathbb{E}_0$ is a conic Minkowski norm. More generally, given a non-zero vector ξ in $\mathbb{E}$ such that $F(\xi) < 1$, then the function $\bar{F} : u \mapsto F(u) + \langle \xi, u \rangle$ restricted to $\mathbb{E}_0$ is also a conic Minkowski norm. We will give further original examples in subsection 8.2.

Remark 8.3. The term "norm" is not strictly appropriate in the classical sense, since even when $E_0 = \mathbb{E}$, the definition of a conic Minkowski norm F does not imply that F is a norm on $\mathbb{E}$ in the usual sense. Indeed, in general, one has $F(u) \neq F(-u)$ (see the previous example).

Remark 8.4. Let $v, w \in \mathbb{E}_0$ be such that the segment $\{tv + (1-t)w, t \in]0, 1[\}$ is contained in $\mathbb{E}_0$. Then the strict triangle inequality

$$F(v+w) \leq F(v) + F(w)$$

holds, with equality if and only if $v = \lambda w$ for some $\lambda > 0$. In particular, if $\mathbb{E}_0$ is convex, then F is a convex function. This situation occurs, in particular, for any Minkowski norm.

8.2. Conic sub-Hilbert–Finsler structures

Let (E, M, ρ) be a Banach anchored bundle and let E_0 be a conic (open) subbundle of E whose typical fiber $\mathbb{E}_0$ is a convex conic domain in $\mathbb{E}$. A standard example is obtained as follows. Let $S \subset E$ be a hypersurface that does not intersect the zero section 0_E and assume that the restriction $\pi|_S : S \rightarrow M$ is a surjective fibration with a typical fiber a hypersurface $\mathbb{S} \subset \mathbb{E}$.

The positive cone generated by S ,

$$E_0 = \{(x, \lambda u), (x, u) \in S, \lambda > 0\},$$

is then an open conic subbundle of E .

Definition 8.5. A conic Hilbert–Finsler metric on a bundle $\pi : E \rightarrow M$ is a map $\mathcal{F} : E_0 \rightarrow [0, +\infty[$ defined on a conic subbundle E_0 of E , which is continuous on E_0 and smooth on $E_0 \setminus \{0_E\}$, and whose restriction $\mathcal{F}_x : (E_0)_x \rightarrow [0, +\infty[$ is a conic Minkowski norm for every $x \in M$.

When E is endowed with an anchor $\rho : E \rightarrow TM$, the pair $(E, \mathcal{F})$ is called a conic sub-Hilbert–Finsler structure on M . If $E_0 = E$, then $\mathcal{F}$ is simply called a Hilbert–Finsler metric on E , and the corresponding anchored bundle is called a sub-Hilbert–Finsler structure on M .

Example 8.6. Let g be a strong Riemannian metric on E . Then $\mathcal{F}(x, u) = \sqrt{g_x(u, u)}$ defines a conic Hilbert–Finsler metric on E (that is, a conic Hilbert–Finsler metric with the domain $E_0 = E$). More generally, suppose that there exists a global section X of E satisfying $g(X, X) < 1$. Then the function $\mathcal{F}$ defined by:

$$\mathcal{F}(x, u) = \sqrt{g_x(u, u)} - g_x(X(x), u)$$

also defines a Hilbert–Finsler metric on E . This is the infinite-dimensional analogue of a Randers metric. In general, its restriction to each fiber is not induced by a norm.

Example 8.7. A Hilbert–Finsler metric on E is a conic Hilbert–Finsler on $E_0 = E$ such that $\mathcal{F}_x$ is a Minkowski norm on E_x , for each $x \in M$. Thus it is a particular case of conic Hilbert–Finsler metric.

Example 8.8. Let $\mathbb{H}$ be a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$. On the trivial bundle $E = M \times \mathbb{H}$, define the strong Riemannian metric

$$g_x(u, v) = \langle u, v \rangle.$$

Fix a nonzero vector $\xi \in \mathbb{H}$ and let $0 < \alpha < \beta < 1$. Define the conic subset

$$E_0 = \left\{ (x, u) \in M \times (\mathbb{H} \setminus \{0\}) : \alpha < \frac{|\langle \xi, u \rangle|}{\|\xi\| \|u\|} < \beta \right\}.$$

Then E_0 is a conic subbundle of E , and the restriction

$$\mathcal{F}(x, u) = \sqrt{g_x(u, u)} = \|u\| \quad \text{on } E_0$$

defines a conic Hilbert–Finsler metric.

Example 8.9. As in finite dimensions (see [42]), let $\mathcal{F}_1, \dots, \mathcal{F}_k$ be conic Hilbert-Finsler metrics on E , defined on conic domains $\mathcal{K}_1, \dots, \mathcal{K}_k$, respectively. Set $E_0 := \bigcap_{i=1}^k \mathcal{K}_i$. Since each $\mathcal{K}_i$ is a conic domain, E_0 is again a conic domain, and all the functions $\mathcal{F}_i$ are well-defined on E_0 .

Then the sum $\mathcal{F} := \sum_{i=1}^k \mathcal{F}_i$ defines a conic Hilbert-Finsler metric on E with the domain E_0 .

More generally, any convex combination

$$\mathcal{F} := \sum_{i=1}^k \lambda_i \mathcal{F}_i, \quad 0 \leq \lambda_i \leq 1, \quad \sum_{i=1}^k \lambda_i = 1,$$

is also a conic Hilbert-Finsler metric on E with domain E_0 . Finally, for $q \geq 2$, the function $\mathcal{F}$ defined by:

$$\mathcal{F}(x, u) = \sqrt{\left(\sum_{i=1}^k \mathcal{F}_i(x, u)^{\frac{1}{q}} \right)^q}$$

defines a conic Hilbert-Finsler metric on E with domain E_0 .

Definition 8.10. A conic sub-Hilbert-Finsler structure on M is the data of an anchored bundle (E, M, ρ) together with a conic Hilbert-Finsler metric $\mathcal{F}$ defined on a conic domain $E_0 \subset E$. When $E_0 = E$, the structure is called a sub-Hilbert-Finsler structure.

Remark 8.11. A (singular) sub-Riemannian structure on M consists of an anchored bundle (E, M, ρ) together with a Riemannian metric g on E . The function

$$\mathcal{F}(x, u) = \sqrt{g_x(u, u)}$$

then defines a Hilbert-Finsler metric on E , and hence a particular case of a sub-Hilbert-Finsler structure on M (see [14]). Conversely, let $\mathcal{F}$ be a Hilbert-Finsler metric on E . If $\frac{1}{2}\mathcal{F}^2$ is smooth on E and its Hessian depends only on the base point x (and not on the fiber variable u), then $\mathcal{F}$ arises from a (singular) sub-Riemannian structure on M .

As in finite-dimensional Finsler geometry, one associates with $\mathcal{F}$ its Cartan tensor $C_{\mathcal{F}}$, defined by:

$$(C_{\mathcal{F}})(x, u)(u_1, u_2, u_3) = \frac{\partial^3}{\partial s_1 \partial s_2 \partial s_3} \mathcal{F}^2 \left(x, u + \sum_{i=1}^3 s_i u_i \right) \Big|_{s_1=s_2=s_3=0}.$$

When $\mathcal{F}^2$ is smooth on E , $\mathcal{F}^2$ gives rise to a sub-Riemannian structure on M if and only if the Cartan tensor $C_{\mathcal{F}}$ vanishes identically.

Let $(E, M, \rho, \mathcal{F})$ be a conic sub-Hilbert-Finsler structure with domain E_0 , and let $\Delta = \rho(E)$ be the associated distribution on M . For each $x \in M$, define

$$\Sigma_x := \rho_x((E_0)_x) \subset \Delta_x.$$

Then Σ_x is a conic domain in the vector space Δ_x . On Σ_x , define

$$F_x(v) := \inf\{\mathcal{F}_x(u), u \in (E_0)_x, \rho_x(u) = v\}.$$

For every $v \in \Sigma_x$, the following properties hold:

1. $F_x(v) > 0$ whenever $v \neq 0$;
2. $F_x(\lambda v) = \lambda F_x(v)$ for every $\lambda > 0$.

Unfortunately, F_x is not in general a conic Minkowski norm on $(\Delta_M)_x$ with the domain Σ_x , even when the typical fiber $\mathbb{E}_0$ is convex.

8.3. Extremals and geodesics of admissible curves in a sub-Hilbert–Finsler structure

8.3.1. Energy, length and geodesics

Consider a conic sub-Hilbert–Finsler metric $\mathcal{F}$ on an anchored bundle (E, M, ρ) defined on a conic domain $\mathbb{E}_0$.

Definition 8.12.

1. Let $\gamma := (c, v) : [t_1, t_2] \rightarrow E_0$ be a smooth curve.

(a) The energy of γ is defined by:

$$\mathcal{E}(\gamma) = \int_{t_1}^{t_2} \frac{1}{2} \mathcal{F}^2(c(t), v(t)) dt.$$

(b) The length of γ is defined by:

$$L(\gamma) = \int_{t_1}^{t_2} \mathcal{F}(c(t), v(t)) dt.$$

2. Let $c \in C_{E_0}^\infty([t_1, t_2], M)$.

A smooth curve $\gamma = (c, v) : [t_1, t_2] \rightarrow E_0$ is called a lift of c if

$$\pi \circ \gamma = c \quad \text{and} \quad \dot{c} = \rho(\gamma).$$

(a) The energy of c is

$$\mathcal{E}(c) = \inf\{\mathcal{E}(\gamma) \mid \gamma \text{ is a lift of } c\}.$$

(b) The length of c is

$$L(c) = \inf\{L(\gamma) \mid \gamma \text{ is a lift of } c\}.$$

Definition 8.13.

1. An E_0 -admissible curve $c : [a, b] \rightarrow M$ is said to be locally trivial if there exist $t_0 \in (a, b)$ and a subinterval $[t_1, t_2] \subset [a, b]$ containing t_0 such that $c|_{[t_1, t_2]}$ is constant.
2. An E_0 -admissible curve $c : [a, b] \rightarrow M$ that is not locally trivial is said to be locally length-minimizing if, for every $t_0 \in [a, b]$, there exist a subinterval $[t_1, t_2] \subset [a, b]$ containing t_0 and an open neighborhood U of $c([t_1, t_2])$ such that, for every $\bar{c} \in C_{E_0}^\infty([t_1, t_2], U)$ satisfying $\bar{c}(t_i) = c(t_i)$, $i \in \{1, 2\}$, one has

$$L(c|_{[t_1, t_2]}) \leq L(\bar{c}).$$

Such a curve is called a geodesic.

As in classical Finsler geometry, the following result relates the energy and the length of locally length-minimizing curves.

Proposition 8.14. *Let $c : [a, b] \rightarrow M$ be an E_0 -admissible curve admitting a lift $\gamma : [a, b] \rightarrow E_0 \setminus \{0_E\}$. Then the following properties hold.*

- (1) *For every lift $\gamma = (c, v) : [a, b] \rightarrow E_0$, the length $L(\gamma)$ is independent of the parametrization of c .*
- (2) *Let $\gamma = (c, v) : [a, b] \rightarrow E_0 \setminus \{0_E\}$ be a lift of c . The map τ defined, for every $t \in [a, b]$, by*

$$\tau(t) = a + \int_a^t \mathcal{F}(c(s), v(s)) ds$$

is a well-defined, smooth, strictly increasing diffeomorphism satisfying

$$\frac{d\tau}{dt} = \mathcal{F}(c(t), v(t)).$$

Consequently, the reparametrized lift has a constant speed equal to 1.

- (3) *The curve c is locally length-minimizing if and only if it admits an E_0 -lift that is locally energy-minimizing.*

Proof. Assertions (1) and (2) follow immediately from the change-of-variables formula.

For Assertion (3), let $\gamma = (c, v) : [t_1, t_2] \rightarrow E_0$ be a lift of c . By the Cauchy–Schwarz inequality,

$$\left(\int_{t_1}^{t_2} \mathcal{F}(\gamma(t)) dt \right)^2 \leq (t_2 - t_1) \int_{t_1}^{t_2} \mathcal{F}^2(\gamma(t)) dt,$$

or equivalently,

$$L(\gamma)^2 \leq 2(t_2 - t_1) \mathcal{E}(\gamma).$$

Moreover, equality holds if and only if the speed $t \mapsto \mathcal{F}(\gamma(t))$ is constant.

Assume first that γ is a locally energy-minimizing lift. By Assertion (2), γ admits a reparametrization with constant speed. For this parametrization, equality holds in the above inequality, and therefore minimizing the energy is equivalent to minimizing the length. Consequently, the underlying curve c is locally length-minimizing.

Conversely, suppose that c is locally length-minimizing. By Assertion (2), c admits a constant-speed lift γ . For such a lift, the previous inequality becomes

$$L(\gamma)^2 = 2(t_2 - t_1) \mathcal{E}(\gamma).$$

Since the factor $2(t_2 - t_1)$ is fixed, minimizing the length is equivalent to minimizing the energy among constant-speed lifts. Hence γ is a locally energy-minimizing lift.

Therefore, an E_0 -admissible curve is locally length-minimizing if and only if it admits an E_0 -lift that is locally energy-minimizing. $\square$

Remark 8.15. As follows from the proof of Proposition 8.14, for every $t_0 \in [a, b]$, there exists a subinterval $[t_1, t_2] \subset [a, b]$ containing t_0 such that the restriction $c|_{[t_1, t_2]}$ admits an E_0 -lift γ , which can be reparametrized with constant speed. For this parametrization,

$$L(c|_{[t_1, t_2]}) = L(\gamma)$$

and

$$L(\gamma)^2 = 2(t_2 - t_1) \mathcal{E}(\gamma).$$

Thus, for constant-speed lifts, minimizing the length is equivalent to minimizing the energy.

1. The *energy functional*

$$\mathcal{E} : C^\infty([t_1, t_2], E_0) \rightarrow \mathbb{R}$$

is defined by

$$\mathcal{E}(\gamma) = \int_{t_1}^{t_2} \frac{1}{2} \mathcal{F}^2(\gamma(t)) dt. \quad (8.2)$$

2. The *augmented energy functional*

$$\widehat{\mathcal{E}} : C^\infty([t_1, t_2], \widetilde{M}) \rightarrow \mathbb{R}$$

is defined by

$$\widehat{\mathcal{E}}(\gamma, p) = \int_{t_1}^{t_2} \left(\frac{1}{2} \mathcal{F}^2(\gamma(t)) + \langle p(t), \dot{c}(t) - \rho_{c(t)}(v(t)) \rangle \right) dt, \quad (8.3)$$

where $\gamma(t) = (c(t), v(t))$ and $(\gamma, p) \in C^\infty([t_1, t_2], \widetilde{M})$.

Then we have the following proposition.

Proposition 8.16. *Let $c : [a, b] \rightarrow M$ be a locally length-minimizing E_0 -admissible curve admitting a lift into E_0 . For every $t_0 \in [a, b]$, let $[t_1, t_2] \subset [a, b]$ and U be an open neighborhood of $c([t_1, t_2])$ satisfying the assumptions of Definition 8.13. Then there exists a lift*

$$\gamma : [t_1, t_2] \rightarrow E_0 \setminus \{0_E\}$$

of $c|_{[t_1, t_2]}$ such that γ is a critical point of the energy functional

$$\mathcal{E} : C^\infty([t_1, t_2], E_0) \rightarrow \mathbb{R}.$$

Proof. By Proposition 8.14, a locally length-minimizing curve admits a parametrization with constant speed, and the corresponding lift γ is locally minimizing for the energy functional $\mathcal{E}$. More precisely, after restricting to a sufficiently small interval $[t_1, t_2]$ containing t_0 , the lift can be chosen so that

$$\mathcal{E}(\gamma) \leq \mathcal{E}(\gamma_s)$$

for every sufficiently small variation γ_s with fixed endpoints.

Let us prove that this lift is a critical point of $\mathcal{E}$. Suppose, by contradiction, that

$$d_\gamma \mathcal{E} \neq 0.$$

Then there exists an admissible variation $\gamma_s = (c_s, v_s)$ with fixed endpoints such that the derivative of

$$s \mapsto \mathcal{E}(\gamma_s)$$

at $s = 0$ is non-zero. Changing the sign of the parameter s if necessary, one obtains, for sufficiently small s ,

$$\mathcal{E}(\gamma_s) < \mathcal{E}(\gamma).$$

This contradicts the local energy-minimizing property of γ . Consequently,

$$d_\gamma \mathcal{E} = 0,$$

and γ is a critical point of the energy functional. Finally, by Theorem 7.18, the corresponding curve is also a critical point of $\widehat{\mathcal{E}}$ on $C^\infty([t_1, t_2], \widetilde{M})$. $\square$

According to Remark 7.21, this proposition justifies the following definition.

Definition 8.17. A locally length-minimizing curve $c \in C_{E_0}^\infty([t_1, t_2], M)$ is called a normal geodesic if, for every $t_0 \in [t_1, t_2]$, there exist a subinterval $[t_1, t_2] \subset [a, b]$ containing t_0 , an open neighborhood U of $c([t_1, t_2])$ satisfying the assumptions of Definition 8.13, and a section p of c^*T^*M over $[t_1, t_2]$ such that $t \rightarrow (c(t), v(t), p(t))$ is a critical point of $\widehat{\mathcal{E}}$ on $C^\infty([t_1, t_2], U)$, with $p(t_2) \neq 0$.

8.3.2. Characterization of normal geodesics

On $\widetilde{M}$, the variational problem associated with the augmented energy functional $\widehat{\mathcal{E}}$ is governed by the Hamiltonian:

$$H(x, v, p) = \langle p, \rho(x, v) \rangle - \frac{1}{2} \mathcal{F}^2(x, v) = \langle p, \rho(x, v) \rangle - \frac{1}{2} g_{(x,v)}(v, v), \quad (8.4)$$

where $g_{(x,v)}$ denotes the fundamental tensor of the Hilbert–Finsler metric $\mathcal{F}$. By the elementary properties of the fundamental tensor on each fiber $(E_0)_x$ (cf. (8.1)), we have:

$$g_{(x,v)}(v, \cdot) = \frac{\partial \mathcal{F}_x^2}{\partial v}(v, \cdot).$$

Thus, the covector $g_{(x,v)}(v, \cdot)$ is the Legendre transform associated with the Lagrangian

$$L(x, v) = \frac{1}{2} \mathcal{F}^2(x, v).$$

Moreover, Proposition 8.1 gives

$$\mathcal{F}^2(x, v) = g_{(x,v)}(v, v).$$

Consequently,

1. the Hamiltonian can be written as

$$H(x, v, p) = \langle p, \rho(x, v) \rangle - \frac{1}{2} g_{(x,v)}(v, v); \quad (8.5)$$

2. its partial derivative with respect to the fiber variable v is:

$$\frac{\partial H}{\partial v}(x, v, p) = \rho_x^* p - g_{(x,v)}(v, \cdot); \quad (8.6)$$

3. the submanifold $\mathbb{P} \subset \widetilde{M}$ is characterized by:

$$\mathbb{P} = \{(x, v, p) \in \widetilde{M} ; \rho_x^* p = g_{(x,v)}(v, \cdot)\}, \quad (8.7)$$

and therefore

$$\frac{\partial H}{\partial v}(x, v, p) = 0 \quad \text{for every } (x, v, p) \in \mathbb{P}.$$

Now, as in [14], we obtain the following characterization of normal geodesics.

Theorem 8.18. *Let $c : [a, b] \rightarrow M$ be an E_0 -admissible curve. Then c is a normal geodesic if and only if there exists a lift*

$$\gamma = (c, v) : [a, b] \rightarrow E_0 \setminus \{0_E\}$$

such that the covector field $p : [a, b] \rightarrow E^*$, defined by:

$$p(t) := g_{\gamma(t)}(v(t), \cdot) \in E_{c(t)}^*,$$

satisfying

$$t \mapsto (c(t), p(t)),$$

is an integral curve of the Hamiltonian vector field $X_{\frac{1}{2}\mathcal{F}^2}$ ³⁰.

Proof. The proof of Theorem 8.18 will proceed in two steps:

- The first step for the necessary condition, which is very easy.
- The second step for the sufficient condition is much more technical and follows, step by step, the proof of the property "local minimizing geodesic" in the proof of Theorem 7 of [14].

1. Necessary condition.

Fix a normal geodesic $c : [a, b] \rightarrow M$. The necessary condition is local, so we work on a sufficiently small interval $[t_1, t_2] \subset [a, b]$. By (8.5) and (8.7), the restriction of the Hamiltonian to $\mathbb{P}$ is

$$H(x, v, p) = \frac{1}{2} g_{(x,v)}(v, v). \quad (8.8)$$

Since c is a geodesic, Proposition 8.16 ensures the existence of an E_0 -lift $\gamma = (c, v)$, which is a critical point of $\mathcal{E}$. By Theorem 7.18 and Remark 7.9, it follows from (8.8) that

$$g_{\gamma(t)}(v(t), v(t)) = \text{const} \quad (8.9)$$

for all $t \in [t_1, t_2]$. Since c is not locally trivial, this constant is non-zero, hence $p(t) = g_{\gamma(t)}(v(t), \cdot) \neq 0$ for all t , in particular at t_2 . As $g_{(x,v)}$ is non-degenerate, it follows that $v(t) \neq 0$ for all t . Therefore, by Corollary 7.20, c is a normal geodesic whose lift lies in $E_0 \setminus \{0_E\}$.

2. Sufficient condition.

We follow the proof of Theorem 7 in [14]. It suffices to construct a closed 1-form θ on a neighborhood of $c([t_1, t_2])$, called a *calibration*, satisfying the following properties:

There exists $\delta > 0$ such that

³⁰See Proposition 7.10 for the notation.

- for t close to t_0 ,

$$\theta_{\gamma(t)}(\dot{\gamma}(t)) = \delta \sqrt{H(\gamma(t), p(t))} \leq \delta \sqrt{g_{\gamma(t)}(v(t), v(t))};$$

- for all (x, v) near $\gamma(t_0)$,

$$|\theta_x(\rho_x(v))| \leq \delta \sqrt{g_{\gamma(t_0)}(v, v)}.$$

The remainder of the proof consists in constructing such a calibration, following the same strategy as in [14], and verifying that all arguments extend to the present setting.

We apply the setting of Remark 7.12 to the case of a sub-Hilbert–Finsler metric. Fix an E_0 -admissible curve $c : [a, b] \rightarrow M$ with lift $\gamma = (c, v) \in E_0 \setminus \{0_E\}$ and define

$$p(t) = g_{\gamma(t)}(v(t), \cdot).$$

Fix $t_0 \in [a, b]$, a subinterval $[t_1, t_2]$ containing t_0 , and an open set $U \subset M$ containing $c([t_1, t_2])$. Let W be a neighborhood of $(c(t_1, t_2), p(t_1, t_2))$ whose projection contains U . Since $g_{(x,v)}(v, \cdot)$ is the Legendre transform of $\frac{1}{2}\mathcal{F}^2$ and since, by assumption, $g_{(x,v)}(\cdot, \cdot)$ is a strong Riemannian metric on the manifold E_0 , it follows that the Lagrangian $\mathcal{F}^2$ is regular.

We denote by $g_{(x,v)}^b : E_x \rightarrow E_x^*$ the induced isomorphism³¹. From Remark 7.12, we may assume, after possibly shrinking $[t_1, t_2]$, U , and W , that:

$$v_W(x, p) = (g_{(x,v)}^b)^{-1}(\rho_x^* p). \quad (8.10)$$

On W , we have

$$\rho_x^* p = g_{(x, v_W(x, p))}(v_W(x, p), \cdot),$$

and by (8.8), we obtain:

$$\begin{aligned} h_W(x, p) &= \frac{1}{2} g_{(x, v_W(x, p))}(v_W(x, p), v_W(x, p)) \\ &= \frac{1}{2} \mathcal{F}^2(x, v_W(x, p)). \end{aligned} \quad (8.11)$$

Thus, by relations (8.10) and (8.11), h_W is a Hamiltonian on $W \subset T^*M$, homogeneous of degree 2 with respect to p .

Assume $t \mapsto (c(t), p(t))$ is an integral curve of the Hamiltonian vector field $X_{\frac{1}{2}\mathcal{F}^2}$. Since $v(t) \neq 0$ on $[a, b]$, Proposition 8.14 implies $\mathcal{F}(c(t), v(t)) = 1$. Hence, up to reparametrization, we may assume $[a, b] = [0, 1]$.

For simplicity, we write $v_W = v$, $h_W = h$, and denote X_h its Hamiltonian vector field.

Without loss of generality, we identify U with an open subset of $\mathbb{M}$ and W with an open subset of $T^*U \equiv U \times \mathbb{M}^*$. Moreover, W can be chosen of the form $U \times U'$, where U' is an open subset of $\mathbb{M}^*$. Thus, the Hamiltonian vector field X_h is defined on $U \times U'$. Moreover, we may assume that $x_0 = c(t_0) = 0 \in U$, and we set $p_0 = p(t_0) \in U'$, so that $\gamma(t_0) = (0, p_0) \in U \times U'$.

Construction of θ

After shrinking U , the flow of X_h is defined and written as

$$(t, x, p) \mapsto (\text{Fl}_1(t, x, p), \text{Fl}_2(t, x, p)).$$

³¹Recall that $g_{(x,v)}$ is a strong non-degenerate bilinear form on each fiber $E_{(x,v)}$.

Note that since $\gamma([0, 1]) \subset E_0 \setminus \{0_E\}$, we have $v(t) \neq 0$ for all $t \in [0, 1]$. Hence

$$p_0 = g_{\gamma(t_0)}(v(t_0), v(t_0)) > 0,$$

and in particular, $p_0 \neq 0$. Since the curve (c, p) is an integral curve of X_h passing through $(0, p_0)$, we have

$$\text{Fl}_1(t, 0, p_0) = (c(t), p(t))$$

for t sufficiently small, and $p(t_0) = p_0$. Let $\mathbb{K} = \ker p_0 \subset \mathbb{M}$ be a hyperplane, and set $U_0 = \mathbb{K} \cap U$. As in the proof of Theorem 7 in [14], for $\varepsilon > 0$ sufficiently small, we define a map

$$\mathfrak{f} :]t_0 - \varepsilon, t_0 + \varepsilon[\times U_0 \rightarrow U,$$

possibly after shrinking U_0 , by:

$$\mathfrak{f}(t, x) = \text{Fl}_1(t, x, v(x)p_0), \quad (8.12)$$

where

$$v(x) = \sqrt{\frac{h(0, p_0)}{h(x, p_0)}}.$$

Since h is 2-homogeneous in the variable p , we obtain:

$$\forall x \in U_0, \quad h(x, v(x)p_0) = h(0, p_0). \quad (8.13)$$

By the same arguments as in the proof of Lemma 5 in [14], possibly after further reducing ε , U_0 , and U , we obtain that $\mathfrak{f}$ is a diffeomorphism onto its image. Note that the map $t \mapsto \mathfrak{f}(t, x)$ coincides with the projection onto U of the integral curve of X_h with the initial condition $(x, v(x)p_0)$ at time $t = t_0$. For each $x \in U$, we set $(t(x), z(x)) = \mathfrak{f}^{-1}(x)$. The map $x \mapsto (t(x), z(x))$ is smooth.

Now consider the 1-form θ on U defined by:

$$\theta(x) = \text{Fl}_2(t(x), z(x), v(z(x))p_0).$$

By construction, we have $\theta(\mathfrak{f}(t, z)) = p_z(t)$, with the initial condition $p_z(t_0) = v(z)p_0$. Hence, with these notations, we obtain $\theta(x) = p_{z(x)}(t(x))$. We set $c_z(t) = \mathfrak{f}(t, z)$, so that $(c_z(t), p_z(t))$ is an integral curve of X_h with the initial conditions $c_z(t_0) = z$ and $p_z(t_0) = v(z)p_0$. As in the sub-Riemannian context (cf. [14]), we also have

$$\begin{aligned} |\theta(x)(\rho_x(u))| &= |\langle p_{z(x)}(t(x)), \rho_x(u) \rangle| \\ &= |g_{(x, v(x, p_{z(x)}(t(x))))}(v(x, p_{z(x)}(t(x))), u)| \\ &\leq \sqrt{g_{(x, v(x, p_{z(x)}(t(x))))}(v(x, p_{z(x)}(t(x))), v(x, p_{z(x)}(t(x))))} \\ &\quad \times \sqrt{g_{(x, v(x, p_{z(x)}(t(x))))}(u, u)} \\ &\leq \sqrt{2h(x, p_{z(x)}(t(x)))} \sqrt{g_{(x, v(x, p_{z(x)}(t(x))))}(u, u)}, \end{aligned}$$

since

$$g_{(x, v)}(v, v) = 2h(x, p) \quad \text{for } v = v(x, p_{z(x)}(t(x))).$$

Now, by (8.13), we obtain:

$$h(x, \theta(x)) = h(x, p_{z(x)}(t(x))) = h(z(x), v(z(x))p_0) = h(0, p_0).$$

Hence, setting $\delta = \sqrt{2h(0, p_0)}$, the previous estimate yields:

$$|\theta(x)(\rho_x(u))| \leq \delta \sqrt{g_{(x, v(x, p_{z(x)}(t(x))))}(u, u)}.$$

It remains to prove that θ is closed. Since $\mathfrak{f}$ is a diffeomorphism, it is sufficient to verify that the pull-back of θ by $\mathfrak{f}$ satisfies:

$$\mathfrak{f}^*\theta = \delta^2 dt.$$

This identity is precisely the content of Lemma 6 in [14], and its proof adapts verbatim to the present setting, as it relies only on the Hamiltonian flow structure and the homogeneity properties already established. We therefore omit the details. It follows that $\mathfrak{f}^*\theta$ is closed, and hence θ is closed as well. This completes the proof of Theorem 8.18. $\square$

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Use of Generative-AI tools declaration

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Conflict of interest

The authors declare no conflict of interest.

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