



Research article

The boundedness of Calderón–Zygmund operators associated with admissible functions on spaces of homogeneous type in weighted Morrey-type spaces

Nan Zhao*

School of Mathematical Sciences, Xinjiang Normal University, Urumqi 830017, P.R. China

* **Correspondence:** Email: zhnmath@xjnu.edu.cn.

Abstract: In this paper, we introduce a class of weighted Morrey-type spaces associated with admissible functions on spaces of homogeneous type, and establish the boundedness of Calderón–Zygmund operators related to admissible functions and their commutators on these spaces. The results of this paper extend many known conclusions and remain novel even in the standard Euclidean setting.

Keywords: A_p^{ρ} weights; weighted Morrey-type spaces; Calderón–Zygmund operators; bounded mean oscillations; commutators

Mathematics Subject Classification: 42B20, 42B35

1. Introduction

In the past decades, many scholars have extensively studied operators in harmonic analysis derived from Schrödinger operators in Euclidean spaces $\mathbb{R}^n$, such as the maximal semigroup operator, Riesz transforms, Littlewood–Paley functions, and more. They obtain the boundedness of the above operators in weighted Lebesgue spaces, properly regular spaces and their weighted forms; see, for instance, [1–9] and the references therein.

The classical Morrey spaces were introduced by Morrey [10] to study the local behavior of solutions to second-order elliptic partial differential equations. Subsequently, various generalizations of Morrey spaces have been developed; see [11–13] and references therein. In particular, the generalized Morrey spaces $L^{p,\varphi}(\mathbb{R}^n)$ (also denoted by $M^{p,\varphi}(\mathbb{R}^n)$) were introduced independently by Guliyev [14], Mizuhara [15], and Nakai [16] (see, also [17–19]). Later, Komori and Shirai [20] defined the weighted Morrey spaces $L^{p,\kappa}(w)$ and established the boundedness of classical operators such as the Hardy–Littlewood maximal operator and Calderón–Zygmund operators on these spaces. Moreover, Guliyev [21] first introduced the generalized weighted Morrey spaces $L^{p,\varphi}(w)$ (also denoted by $M_w^{p,\varphi}(\mathbb{R}^n)$) and studied the boundedness of sublinear operators and their higher-order commutators on these spaces. The space $L^{p,\varphi}(w)$ can be viewed as an extension of both the generalized Morrey space $L^{p,\varphi}$ and the weighted Morrey space $L^{p,\kappa}(w)$.

In addition, lots of Morrey type spaces associated with a Schrödinger operator have been also studied (see [22–27]) to extend the well-known Morrey spaces.

Recall that (X, d, μ) is a space of homogeneous type in the sense of Coifman and Weiss if d is a quasi-metric, and μ is a nonzero measure satisfying the doubling condition. A quasi-metric $d : X \times X \rightarrow [0, \infty)$ satisfies

- (i) $d(x, y) = d(y, x) \geq 0$ for all $x, y \in X$;
- (ii) $d(x, y) = 0$ if and only if $x = y$;
- (iii) there is a constant $C \in [1, \infty)$ such that for all $x, y, z \in X$,

$$d(x, y) \leq C(d(x, z) + d(z, y)).$$

A nonzero measure μ satisfies the doubling condition if there is a constant C , depending on μ , such that for all $x \in X$ and $r > 0$,

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)) < \infty,$$

where $B(x, r)$ is the quasi-metric ball by $B(x, r) := \{y \in X : d(x, y) < r\}$ for $x \in X$ and $r > 0$.

We say that (X, d, μ) is a reverse doubling space if there exist constants $\kappa \in (0, n]$ and $C \geq 1$ such that for all $x \in X$, $r \in (0, \text{diam}(X)/2)$, and $\lambda \in [1, \text{diam}(X)/(2r))$,

$$C^{-1}\lambda^\kappa\mu(B(x, r)) \leq \mu(B(x, \lambda r)) \leq C\lambda^n\mu(B(x, r)),$$

where $\text{diam}(X) = \sup_{x, y \in X} d(x, y)$, and the parameter n is a measure of the dimension of the space. Clearly, any Ahlfors n -regular metric measure space (X, d, μ) , which means that there exists some $n > 0$ such that

$$\mu(B(x, r)) \sim r^n \text{ for all } x \in X \text{ and } r \in (0, \text{diam}(X)/2), \quad (1.1)$$

is a reverse doubling space. Throughout this paper, we assume that (X, d, μ) is an Ahlfors n -regular metric measure space and that $\mu(X) = \infty$. We use C to denote the positive constant, which is independent of main parameters and may be different at each occurrence. If $f \leq Cg$, we then write $f \lesssim g$ or $g \gtrsim f$, and if $f \lesssim g \lesssim f$, we then write $f \sim g$.

It turns out that numerous classical results in harmonic analysis within the Euclidean setting can be extended to both spaces of homogeneous type and reverse doubling spaces; see [28–31] and references therein. In their foundational work [32], Yang and Zhou first introduced the notion of admissible functions in spaces of homogeneous type as follows: A positive function $\rho : X \rightarrow (0, \infty)$ is said to be admissible if there exist positive constants C and l_0 satisfying

$$\rho(y) \leq C\rho(x)^{1/(1+l_0)}(\rho(x) + d(x, y))^{l_0/(1+l_0)} \quad (1.2)$$

for all $x, y \in X$. Then, they developed a real-variable theory for the localized Hardy spaces associated with admissible functions ρ , which has found broad applications, particularly in the analysis of the Schrödinger operator and the degenerate Schrödinger operator on $\mathbb{R}^n$. Further, Liu, Huang, and Dong [33] established the boundedness of the commutator

$$[b, T]f := bTf - T(bf)$$

on $L^p(X)$, $p \in (1, \infty)$, where T is a Calderón–Zygmund operator related to the admissible function ρ , and $b \in BMO_\rho(X)$. Recently, Qin and Fu [34] extended the results of [33] to the weighted case by employing a new class of weights that contains the classical Muckenhoupt weights (see Section 2). In particular, they investigated the boundedness properties of both the fractional integral operators associated with admissible functions ρ and their corresponding commutators in weighted $L^p(X)$ spaces. As a special class of spaces of homogeneous type, Wang [35] introduced Morrey spaces associated with Schrödinger operators on the Heisenberg group $\mathbb{H}^n$ and established the boundedness of both the Littlewood–Paley function and the Lusin area integral in these spaces. Further results concerning Morrey-type spaces on spaces of homogeneous type and the boundedness of certain operators on such spaces can be found in [36, 37].

Inspired by the previous work, the main aim of this paper is to obtain some weighted norm inequalities for Calderón–Zygmund operators related to admissible functions and their commutators on weighted Morrey-type spaces using the new class of weights (see Section 2). In what follows, we introduce a class of weighted Morrey-type space associated with admissible functions ρ on spaces of homogeneous type.

Definition 1.1. Let $(p, \kappa) \in (0, \infty) \times (0, 1)$ and w be a weight. Assume that $\Psi(\cdot)$ is a positive increasing function defined in $(0, \infty)$ and satisfies the following $\mathcal{D}_\kappa$ condition:

$$\frac{\Psi(x)}{x^\kappa} \lesssim \frac{\Psi(y)}{y^\kappa} \quad \text{for any } 0 < y < x < \infty. \quad (1.3)$$

For given $\theta \in [0, \infty)$, the space $L_{\rho, \theta}^{p, \Psi}(w)$ is defined as the set of all $L^p(X)$ locally integrable functions f satisfying

$$\|f\|_{L_{\rho, \theta}^{p, \Psi}(w)} := \sup_B \frac{1}{\varphi_\theta(B)} \left(\frac{1}{\Psi(w(B))} \int_B |f(x)|^p w(x) d\mu(x) \right)^{1/p} < \infty.$$

Here, the supremum is taken over all balls $B := B(x_0, r) \subset X$, and

$$\varphi_\theta(B) = \varphi(B)^\theta = \left(1 + \frac{r}{\rho(x_0)} \right)^\theta \quad (1.4)$$

with ρ being the admissible function from (1.2). Likewise, the weak space $WL_{\rho, \theta}^{p, \Psi}(w)$ is defined with

$$\|f\|_{WL_{\rho, \theta}^{p, \Psi}(w)} := \sup_B \frac{1}{\varphi_\theta(B)} \frac{1}{\Psi(w(B))^{1/p}} \sup_{\lambda > 0} \lambda w(\{x \in B : |f(x)| > \lambda\})^{1/p} < \infty.$$

Obviously, when $X = \mathbb{R}^n$, and $\theta = 0$, the new spaces $L_{\rho, \theta}^{p, \Psi}(w)$ and $WL_{\rho, \theta}^{p, \Psi}(w)$ are just the generalized weighted Morrey space $L^{p, \Psi}(w)$ and the generalized weighted weak Morrey space $WL^{p, \Psi}(w)$, which was first defined by Guliyev in [21].

If $X = \mathbb{R}^n$, $\theta = 0$, and $\Psi(x) = x^\kappa$ with $\kappa \in (0, 1)$, then the space $L_{\rho, \theta}^{p, \Psi}(w)$ is the weighted Morrey space $L^{p, \kappa}(w)$ introduced by Komori and Shirai [20], and $WL_{\rho, \theta}^{p, \Psi}(w)$ is the weighted weak Morrey space $WL^{p, \kappa}(w)$. Note that, Guliyev [21] gave the concept of generalized weighted Morrey space $L^{p, \Psi}(w)$, which could be viewed as an extension of both the generalized Morrey space $L^{p, \Psi}$ and weighted Morrey space $L^{p, \kappa}(w)$.

In the case $X = \mathbb{R}^n$, $\theta = 0$, and $w(x) = 1$, the space $L_{\rho, \theta}^{p, \Psi}(w)$ is the generalized Morrey space defined independently by Guliyev, Mizuhara, and Nakai in [14–16].

If $\mathcal{X} = \mathbb{R}^n$, $\theta \in (0, \infty)$, and $\Psi(x) = x^\kappa$ with $\kappa \in (0, 1)$, then the space $L_{\rho, \theta}^{p, \Psi}(w)$ is the weighted Morrey space $L_{\rho, \theta}^{p, \kappa}(w)$ related to a Schrödinger-type operator, and $WL_{\rho, \theta}^{p, \Psi}(w)$ is the weighted weak Morrey space $WL_{\rho, \theta}^{p, \kappa}(w)$; see [26].

Inspired by a new class of weights A_p^ρ and the new bounded mean oscillation (BMO) function space BMO_ρ , we also define

$$L_\rho^{p, \Psi}(w) := \bigcup_{\theta \geq 0} L_{\rho, \theta}^{p, \Psi}(w) \quad \text{and} \quad WL_\rho^{p, \Psi}(w) := \bigcup_{\theta \geq 0} WL_{\rho, \theta}^{p, \Psi}(w).$$

According to the above definitions, it is easy to verify that

$$\begin{cases} L^{p, \Psi}(w) \subset L_{\rho, \theta_1}^{p, \Psi}(w) \subset L_{\rho, \theta_2}^{p, \Psi}(w); \\ WL^{p, \Psi}(w) \subset WL_{\rho, \theta_1}^{p, \Psi}(w) \subset WL_{\rho, \theta_2}^{p, \Psi}(w) \end{cases}$$

for $0 < \theta_1 < \theta_2 < \infty$. Hence,

$$L^{p, \Psi}(w) \subset L_\rho^{p, \Psi}(w), \quad \text{and} \quad WL^{p, \Psi}(w) \subset WL_\rho^{p, \Psi}(w)$$

for any $p \in (0, \infty)$ and Ψ satisfying $\mathcal{D}_\kappa$ in Condition (1.3).

We next recall some basic definitions and facts for Orlicz spaces. More details can be found in [38]. Let $\Phi : [0, \infty) \rightarrow [0, \infty)$ be a Young function, that is, a continuous, convex, increasing function with $\Phi(0) = 0$ such that $\Phi(t) \rightarrow \infty$ as $t \rightarrow \infty$. Given a Young function Φ and a weight function w , the $\Phi(w)$ -average of a function f over a ball B via the following Luxemburg norm is defined by

$$\|f\|_{\Phi(w), B} := \inf \left\{ \lambda > 0 : \frac{1}{w(B)} \int_B \Phi\left(\frac{|f(y)|}{\lambda}\right) w(y) d\mu(y) \leq 1 \right\}.$$

Associated with each Young function Φ , one can define its complementary function $\bar{\Phi}$ as follows:

$$\bar{\Phi}(s) := \sup_{t > 0} \{st - \Phi(t)\}.$$

It is easy to see that $\Phi(t) = t(1 + \log^+ t)$ is a Young function and its complementary function $\bar{\Phi}(t) = e^t$. In this case, we write $\|f\|_{L(\log L)(w), B} = \|f\|_{\Phi(w), B}$ and $\|f\|_{\exp L(w), B} = \|f\|_{\bar{\Phi}(w), B}$. For a weight w and for any ball B in $\mathcal{X}$, we can see the following generalized Hölder's inequality in the weighted setting:

$$\frac{1}{w(B)} \int_B |f(y)g(y)| w(y) d\mu(y) \leq C \|f\|_{L(\log L)(w), B} \|g\|_{\exp L(w), B} \quad (1.5)$$

holds (see [39]). Note that $t \leq t(1 + \log^+ t)$ for all $t > 0$; then, the inequality

$$\|f\|_{L^1(w), B} = \frac{1}{w(B)} \int_B |f(y)| w(y) d\mu(y) \leq \|f\|_{L(\log L)(w), B} \quad (1.6)$$

holds for any ball $B \subset \mathcal{X}$.

In order to deal with the boundedness of commutators at the endpoints, we introduce a class of weighted $L \log L$ type Morrey space associated with the admissible function ρ .

Definition 1.2. Let $\kappa \in (0, 1)$, and w be a weight. Assume that $\Psi(\cdot)$ satisfies $\mathcal{D}_\kappa$ Condition (1.3). For $\theta \in [0, \infty)$, the space $(L \log L)_{\rho, \theta}^{1, \Psi}(w)$ is defined as the set of all locally integrable functions f satisfying

$$\|f\|_{(L \log L)_{\rho, \theta}^{1, \Psi}(w)} := \sup_B \frac{w(B)}{\varphi_\theta(B) \Psi(w(B))} \|f\|_{L(\log L)(w), B} < \infty,$$

where the supremum is taken over all balls B in $\mathcal{X}$, and $\varphi_\theta(B)$ appears in (1.4).

From (1.6), it is easy to verify that $(L \log L)_{\rho, \theta}^{1, \Psi}(w) \subset L_{\rho, \theta}^{1, \Psi}(w)$ for any $\theta \in (0, \infty)$. Moreover, if $\mathcal{X} = \mathbb{R}^n$, and $\theta = 0$, then the space $(L \log L)_{\rho, \theta}^{1, \Psi}(w)$ reduces to the generalized weighted Morrey space of $L \log L$ type, which was defined by Wang in [40]. When $\mathcal{X} = \mathbb{R}^n$, $\theta \in (0, \infty)$, and $\Psi(x) = x^\kappa$ with $\kappa \in (0, 1)$, the space $(L \log L)_{\rho, \theta}^{1, \Psi}(w)$ is a weighted Morrey space $(L \log L)_{\rho, \theta}^{1, \kappa}(w)$ related to a Schrödinger type-operator; see [26].

We now state our main results as follows.

Theorem 1.3. Let $(p, \kappa) \in (0, \infty) \times (0, 1)$, and let w be a weight. Assume that $\Psi(\cdot)$ satisfies $\mathcal{D}_\kappa$ Condition (1.3). Then, the following statements hold:

- (i) If $w \in A_p^\rho$ with $p \in (1, \infty)$, then T is bounded on $L_\rho^{p, \Psi}(w)$.
- (ii) If $w \in A_1^\rho$, then T is bounded from $L_\rho^{1, \Psi}(w)$ into $WL_\rho^{1, \Psi}(w)$.

Theorem 1.4. Let $(p, \kappa) \in (0, \infty) \times (0, 1)$ and w be a weight. Assume that $\Psi(\cdot)$ satisfies $\mathcal{D}_\kappa$ Condition (1.3), and $b \in BMO_\rho$. Then, the following statements hold:

- (i) If $w \in A_p^\rho$ with $p \in (1, \infty)$, then $[b, T]$ is bounded on $L_\rho^{p, \Psi}(w)$.
- (ii) If $w \in A_1^\rho$, then for any $\lambda > 0$ and any ball $B = B(x, r)$ in $\mathcal{X}$, there exists a positive constant ν such that the inequalities

$$\Psi(w(B))^{-1} w(\{y \in B(x, r) : |[b, T]f(y)| > \lambda\}) \lesssim \left(1 + \frac{r}{\rho(x)}\right)^\nu \left\| \Phi\left(\frac{|f|}{\lambda}\right) \right\|_{(L \log L)_{\rho, \theta}^{1, \Psi}(w)}$$

hold for any f such that $\Phi(|f|) \in (L \log L)_{\rho, \theta}^{1, \Psi}(w)$ with some fixed $\theta > 0$.

Remark 1.5. In Theorem 1.4 (ii), the parameter θ on the right-hand side of the inequality is precisely the same parameter as in the hypothesis $\Phi(|f|) \in (L \log L)_{\rho, \theta}^{1, \Psi}(w)$, so it is fixed by f and independent of other parameters. In the proof, the parameter θ serves as a known constant when choosing later constants and the exponent ν , but this does not alter its a priori status in the theorem.

The paper is organized as follows. We introduce some preliminary results and notations in Section 2. Section 3 is devoted to proving the main theorems.

1.1. Notation

Throughout this paper, all notations will be defined as needed. For any subset E of $\mathcal{X}$, denote by χ_E its characteristic function. For $x, y \in \mathcal{X}$ and $r > 0$, we set $B(x, r) = \{y \in \mathcal{X} : d(x, y) < r\}$ to be the open ball with radius $r > 0$ and center $x \in \mathcal{X}$, and

$$V_r(x) := V(x, r) = \mu(B(x, r)), \quad V(x, y) := \mu(B(x, d(x, y))).$$

For a locally integrable function f , denoted by f_B , the average $f_B = \mu(B)^{-1} \int_B f(y) d\mu(y)$. For $p \in [1, \infty]$, let p' be the conjugate exponent of p .

Given a weight function w on X , for $p \in [1, \infty)$, the weighted Lebesgue space $L^p(w)$ is defined as the set of all functions f such that

$$\|f\|_{L^p(w)} := \left(\int_X |f(y)|^p w(y) d\mu(y) \right)^{1/p} < \infty.$$

We also denote by $WL^p(w)$, $p \in [1, \infty)$ the weighted weak Lebesgue space consisting of all measurable functions f such that

$$\|f\|_{WL^p(w)} := \sup_{\lambda > 0} \lambda(w(\{y \in X : |f(y)| > \lambda\}))^{1/p} < \infty.$$

2. Some preliminaries and notations

In this section, we review fundamental concepts and notation, while presenting several lemmas that will be essential for subsequent proofs.

The admissible function ρ considered in this paper was originally introduced by Yang and Zhou, and it satisfies the following properties (see [32, Lemma 2.1]):

- (i) For any $C' > 0$, there exists a positive constant C , depending on C' , such that

$$\text{if } d(x, y) \leq C'\rho(x), \text{ then } C^{-1}\rho(y) \leq \rho(x) \leq C\rho(y).$$

- (ii) There exists a positive constant C such that for all $x, y \in X$,

$$C^{-1}(\rho(x) + d(x, y)) \leq \rho(y) + d(x, y) \leq C(\rho(x) + d(x, y)).$$

- (iii) There exists a positive constant C such that for all $x, y \in X$,

$$\rho(y) \geq C(\rho(x))^{1+l_0}(\rho(x) + d(x, y))^{-l_0}.$$

As a consequence of properties (ii)-(iii) above, Liu, Huang, and Dong demonstrated that there exists $l_0 > 0$ such that

$$C^{-1}\rho(x)\left(1 + \frac{d(x, y)}{\rho(x)}\right)^{-l_0} \leq \rho(y) \leq C\rho(x)\left(1 + \frac{d(x, y)}{\rho(x)}\right)^{\frac{l_0}{l_0+1}} \quad (2.1)$$

for all $x, y \in X$; see [33, Corollary 2.3]. Following the proof scheme in [27, Lemma 2.6], one can readily verify that the admissible functions ρ possess the following crucial properties: For all $N \in \mathbb{N}$, $y \in B = B(x, r)$, and $z \in 2^{j+1}B \setminus 2^jB$, there exists $l_0 \in \mathbb{N}$ such that

$$\left(1 + \frac{d(y, z)}{\rho(y)}\right)^{-N} \lesssim \left(1 + \frac{2^{j+1}r}{\rho(x)}\right)^{-N/(l_0+1)}. \quad (2.2)$$

Moreover, we also have

$$\frac{(1 + \frac{r}{\rho(x)})^\alpha}{(1 + \frac{2^{j+1}r}{\rho(x)})^{N/(l_0+1)-D}} \lesssim \left(1 + \frac{2^{j+1}r}{\rho(x)}\right)^\alpha$$

for every $j \in \mathbb{N}$ and $N > (l_0 + 1)(|\alpha| - \alpha + D)$, where $\alpha \in \mathbb{R}$, $D > 0$ is a positive constant.

The following result will be useful in the sequel; see [32, Lemmas 2.3 and 2.4].

Lemma 2.1. *Let ρ be an admissible function on $\mathcal{X}$. Then, there exists a sequence of points $\{x_j\}_{j \in I} \subset \mathcal{X}$ such that*

- (i) $\bigcup_{j \in I} B(x_j, \rho(x_j)) = \mathcal{X}$.
- (ii) *There exist constants C and N_1 such that for every $\lambda \geq 1$, we have*

$$\sum_{j \in I} \chi_{B(x_j, \lambda \rho(x_j))} \leq C \lambda^{N_1}.$$

A weight will always mean a positive function which is locally integrable. As in [34], we say that a weight w belongs to the class $A_p^{\rho, \theta}$ for $p \in [1, \infty)$, if there is a positive constant C such that for all balls $B = B(x, r)$ in $\mathcal{X}$,

$$\left(\frac{1}{\varphi_\theta(B)\mu(B)} \int_B w(y) d\mu(y) \right) \left(\frac{1}{\varphi_\theta(B)\mu(B)} \int_B w(y)^{-\frac{1}{p-1}} d\mu(y) \right)^{p-1} \leq C$$

for $p \in (1, \infty)$, and

$$\left(\frac{1}{\varphi_\theta(B)\mu(B)} \int_B w(y) d\mu(y) \right) \leq C \inf_{y \in B} w(y)$$

for $p = 1$. Because $\varphi_\theta(B) \geq 1$, it is obvious that $A_p \subset A_p^{\rho, \theta}$ for $p \in [1, \infty)$, where A_p denotes the classical Muckenhoupt weight; see [41].

Hereafter, fix $\theta \in (0, \infty)$. $\varphi_\theta(B)$ and $A_p^{\rho, \theta}$ are simply denoted by $\varphi(B)$ and A_p^ρ , respectively. The following properties for A_p^ρ can be seen from [34].

Lemma 2.2. *Let $p \in (1, \infty)$. Then, the following statements hold:*

- (i) *If $1 \leq p < q < \infty$, then $A_p^\rho \subset A_q^\rho$.*
- (ii) *$w \in A_p^\rho$ if and only if $w^{1-p'} \in A_{p'}^\rho$, where $1/p + 1/p' = 1$.*
- (iii) *If $w \in A_p^\rho$ for $p \in [1, \infty)$, then*

$$\frac{1}{\varphi(B)\mu(B)} \int_B |f(y)| d\mu(y) \lesssim \left(\frac{1}{w(5B)} \int_B |f(y)|^p w(y) d\mu(y) \right)^{1/p},$$

where

$$w(E) = \int_E w(y) d\mu(y).$$

As in the classical theory of weights, the following result reveals the reverse Hölder property of A_p^ρ weights, which generalizes [2, Lemma 5].

Proposition 2.3. *If $w \in A_p^\rho$ with $p \geq 1$, then there exist positive constants ε, η , and C so that for all balls B in $\mathcal{X}$,*

$$\left(\frac{1}{\mu(B)} \int_B w^{1+\varepsilon}(y) d\mu(y) \right)^{1/(1+\varepsilon)} \lesssim \left(\frac{1}{\mu(B)} \int_B w(y) d\mu(y) \right) \left(1 + \frac{r}{\rho(x)} \right)^\eta.$$

Proof. Let $B = B(x, r)$. We discuss the two cases $r \leq \rho(x)$ and $r > \rho(x)$ separately. If $r \leq \rho(x)$, it is clear that there is

$$\left(\frac{1}{\mu(B)} \int_B w^{1+\varepsilon}(y) d\mu(y) \right)^{1/(1+\varepsilon)} \lesssim \frac{w(B)}{\mu(B)} \quad (2.3)$$

following the classical theory for A_p .

For $r > \rho(x)$, let $\mathcal{F} = \{j : B_j \cap B \neq \emptyset\}$, where $B_j = B(x_j, \rho(x_j))$, and $\{x_j\}_{j \in \mathcal{I}}$ is the sequence of Lemma 2.1. For $j \in \mathcal{F}$, it follows from (2.1) that

$$\rho(x_j) \lesssim \left(1 + \frac{r}{\rho(x)}\right)^{l_0/(l_0+1)} r,$$

which implies

$$\bigcup_{j \in \mathcal{F}} B_j \subset \tilde{c}B, \quad (2.4)$$

where $\tilde{c} := C(1 + r\rho(x)^{-1})^{l_0/(l_0+1)}$. On the other hand, we deduce from (2.1) that

$$\rho(x_j) \gtrsim \rho(x) \left(1 + \frac{r}{\rho(x)}\right)^{-l_0} \quad (2.5)$$

for any $j \in \mathcal{F}$. Therefore, by using (2.3)–(2.5) and Lemma 2.1,

$$\begin{aligned} \left(\int_B w^{1+\varepsilon}(y) d\mu(y)\right)^{1/(1+\varepsilon)} &\leq \sum_{j \in \mathcal{F}} \left(\int_{B_j} w^{1+\varepsilon}(y) d\mu(y)\right)^{1/(1+\varepsilon)} \\ &\lesssim \sum_{j \in \mathcal{F}} w(B_j) \mu(B_j)^{-\varepsilon/(1+\varepsilon)} \\ &\lesssim w(\tilde{c}B) \rho(x)^{-n\varepsilon/(1+\varepsilon)} \left(1 + \frac{r}{\rho(x)}\right)^{n\varepsilon l_0/(1+\varepsilon)}. \end{aligned} \quad (2.6)$$

Furthermore, the A_p^ρ weight condition of w and (1.1) yield that

$$\begin{aligned} w(\tilde{c}B) &\lesssim \mu(\tilde{c}B)^p \left(\int_{\tilde{c}B} w^{-\frac{1}{p-1}}(y) d\mu(y)\right)^{1-p} \varphi_\theta(\tilde{c}B)^p \\ &\lesssim w(B) \left(1 + \frac{r}{\rho(x)}\right)^{\theta p + (\theta+n)pl_0/(l_0+1)}, \end{aligned} \quad (2.7)$$

where in the last inequality, we have used that

$$\mu(B)^p = \left(\int_X w(y)^{\frac{1}{p}} w(y)^{-\frac{1}{p}} \chi_B(y) d\mu(y)\right)^p \leq w(B) \left(\int_B w^{-\frac{1}{p-1}}(y) d\mu(y)\right)^{p-1}.$$

Combining (2.6) and (2.7), we have

$$\left(\int_B w^{1+\varepsilon}(y) d\mu(y)\right)^{1/(1+\varepsilon)} \lesssim w(B) \left(1 + \frac{r}{\rho(x)}\right)^\eta,$$

where $\eta = \theta p + (\theta+n)pl_0(l_0+1)^{-1} + (l_0+1)n\varepsilon(1+\varepsilon)^{-1}$. □

Applying Proposition 2.3 and Hölder's inequality, we can obtain the following Lemma 2.4.

Lemma 2.4. *Let $p \in [1, \infty)$. If $w \in A_p^\rho$, then there exist positive constants $\delta \in (0, 1)$, η , and C such that*

$$\frac{w(E)}{w(B)} \lesssim \left(1 + \frac{r}{\rho(x)}\right)^\eta \left(\frac{\mu(E)}{\mu(B)}\right)^\delta$$

for any measurable subset E of a ball B in $\mathcal{X}$.

Proof. Suppose that $E \subset B$. For any given ball B , by Hölder's inequality with exponent $1 + \varepsilon$ and Proposition 2.3, we obtain

$$\begin{aligned} w(E) &= \int_B \chi_E(y) w(y) d\mu(y) \\ &\leq \left(\int_B w(y)^{1+\varepsilon} d\mu(y) \right)^{\frac{1}{1+\varepsilon}} \left(\int_B \chi_E(y)^{\frac{1+\varepsilon}{\varepsilon}} d\mu(y) \right)^{\frac{\varepsilon}{1+\varepsilon}} \\ &\lesssim \mu(B)^{\frac{1}{1+\varepsilon}} \left(\frac{1}{\mu(B)} \int_B w(y) d\mu(y) \right) \left(1 + \frac{r}{\rho(x)} \right)^\eta \mu(E)^{\frac{\varepsilon}{1+\varepsilon}} \\ &\sim \left(\frac{\mu(E)}{\mu(B)} \right)^{\frac{\varepsilon}{1+\varepsilon}} \left(1 + \frac{r}{\rho(x)} \right)^\eta w(B), \end{aligned}$$

which proves Lemma 2.4 with $\delta = \varepsilon/(1 + \varepsilon)$. $\square$

Applying Hölder's inequality and the definition of A_p^ρ deduces that the following lemma holds.

Lemma 2.5. *Let $p \in [1, \infty)$. If $w \in A_p^\rho$, then there exists a positive constant C such that for any $\varrho > 1$,*

$$w(\varrho B) \lesssim \varphi(\varrho B)^p w(B)$$

holds for any ball B in $\mathcal{X}$.

Proof. For $p \in (1, \infty)$ and $\varrho > 1$, we deduce from Hölder's inequality and the definition of A_p^ρ that

$$\begin{aligned} &\frac{1}{\varphi(\varrho B)\mu(\varrho B)} \int_{\varrho B} |f(y)| d\mu(y) \\ &= \frac{1}{\varphi(\varrho B)\mu(\varrho B)} \int_{\varrho B} |f(y)| w(y)^{1/p} w(y)^{-1/p} d\mu(y) \\ &\leq \left(\frac{1}{\varphi(\varrho B)\mu(\varrho B)} \int_{\varrho B} |f(y)|^p w(y) d\mu(y) \right)^{1/p} \left(\frac{1}{\varphi(\varrho B)\mu(\varrho B)} \int_{\varrho B} w(y)^{-p'/p} d\mu(y) \right)^{1/p'} \\ &\lesssim \left(\frac{1}{\varphi(\varrho B)\mu(\varrho B)} \int_{\varrho B} |f(y)|^p w(y) d\mu(y) \right)^{1/p} \left(\frac{1}{\varphi(\varrho B)\mu(\varrho B)} \int_{\varrho B} w(y) d\mu(y) \right)^{-1/p} \\ &\lesssim \left(\int_{\varrho B} |f(y)|^p w(y) d\mu(y) \right)^{1/p} w(\varrho B)^{-1/p}, \end{aligned} \tag{2.8}$$

where $1/p + 1/p' = 1$, and $w(\varrho B) = \int_{\varrho B} w(y) d\mu(y)$. If we take $f(y) = \chi_B(y)$ in (2.8), noting that $B \subseteq \varrho B$ and (1.1), then we have

$$\frac{1}{\varphi(\varrho B)\varrho^n} \sim \frac{\mu(B)}{\varphi(\varrho B)\mu(\varrho B)} \lesssim w(\varrho B)^{-1/p} w(B)^{1/p},$$

which derives

$$w(\varrho B) \leq C' \varphi(\varrho B)^p w(B)$$

with $C' = \varrho^{np}$.

For $p = 1$ and $\varrho > 1$, it follows from the definition of A_1^ρ that

$$\frac{1}{\varphi(\varrho B)\mu(\varrho B)} \int_{\varrho B} |f(y)| d\mu(y) \cdot w(\varrho B)$$

$$\begin{aligned}
&= \int_{\varrho B} |f(y)| d\mu(y) \cdot \frac{1}{\varphi(\varrho B)\mu(\varrho B)} \int_{\varrho B} w(y) d\mu(y) \\
&\lesssim \int_{\varrho B} |f(y)| d\mu(y) \cdot \inf_{y \in \varrho B} w(y) \\
&\lesssim \int_{\varrho B} |f(y)| w(y) d\mu(y).
\end{aligned} \tag{2.9}$$

If we take $f(y) = \chi_B(y)$ in (2.9), noting that $B \subseteq \varrho B$ and (1.1) again, then we have

$$\frac{1}{\varphi(\varrho B)\varrho^n} \cdot w(\varrho B) \sim \frac{\mu(B)}{\varphi(\varrho B)\mu(\varrho B)} \cdot w(\varrho B) \lesssim w(B),$$

which derives

$$w(\varrho B) \leq C' \varphi(\varrho B) w(B)$$

with $C' = \varrho^n$. □

In what follows, we will recall the definition and some basic properties of the new type BMO function spaces. According to [33], we say a locally integrable function b is in $BMO_\theta(\rho)$ if

$$\|b\|_{BMO_\theta(\rho)} := \sup_{B \subset \mathbb{R}^n} \frac{1}{\varphi_\theta(B)\mu(B)} \int_B |b(y) - b_B| d\mu(y) < \infty,$$

where

$$b_B = \frac{1}{\mu(B)} \int_B f(y) d\mu(y) \quad \text{and} \quad \varphi_\theta(B) = \left(1 + \frac{r}{\rho(x)}\right)^\theta$$

with $B = B(x, r)$ and $\theta > 0$. Obviously, the classical BMO is properly contained in $BMO_\theta(\rho)$. For convenience, we let $BMO_\rho = \bigcup_{\theta > 0} BMO_\theta(\rho)$. The following result can be considered to be a variant of John–Nirenberg inequality for the space BMO_ρ ; see [33].

Lemma 2.6. *Let $s \geq 1$. If $b \in BMO_\rho$, then, for all balls B in $\mathcal{X}$,*

- (i) $\left(\mu(B)^{-1} \int_B |b(y) - b_B|^s d\mu(y)\right)^{\frac{1}{s}} \lesssim \|b\|_{BMO_\rho} \varphi(B)^{l_0+1};$
- (ii) $\left(\mu(2^k B)^{-1} \int_{2^k B} |b(y) - b_B|^s d\mu(y)\right)^{\frac{1}{s}} \lesssim k \|b\|_{BMO_\rho} \varphi(2^k B)^{l_0+1}$ for all $k \in \mathbb{N}$.

Lemma 2.7. *If $f \in BMO_\rho$, then there exist positive constants C_1 and C_2 such that for all balls B in $\mathcal{X}$ and every $\lambda > 0$, we have*

$$\mu(\{y \in B : |f(y) - f_B| > \lambda\}) \leq C_1 \mu(B) \exp \left\{ -\frac{C_2 \lambda}{\|f\|_{BMO_\rho} \varphi(B)^{l_0+1}} \right\}.$$

Proof. The proof of Lemma 2.7 is accomplished immediately by applying Lemma 2.6 (i) and adopting the proof technique established in [6, Proposition 4.2]. □

Lemma 2.8. *If $f \in BMO_\rho$, and $w \in A_p^\rho$ with $p \in [1, \infty)$, then there exist positive constants C and τ such that for all balls B in $\mathcal{X}$, we have*

$$\left(\frac{1}{w(B)} \int_B |f(y) - f_B|^p w(y) d\mu(y) \right)^{1/p} \lesssim \varphi(B)^{\tau/p} \|f\|_{BMO_\rho}.$$

Proof. Applying Lemma 2.4 and Lemma 2.7, we have

$$\begin{aligned} & w(\{y \in B : |f(y) - f_B| > \lambda\}) \\ & \leq CC_1^\delta \varphi(B)^{\eta/\theta} \exp\left\{-\frac{C_2\delta\lambda}{\|f\|_{BMO_\rho} \varphi(B)^{l_0+1}}\right\} w(B). \end{aligned} \quad (2.10)$$

Let $\tau = \eta/\theta + (l_0 + 1)p$. Then, for any ball B ,

$$\begin{aligned} & \frac{1}{w(B)} \int_B |f(y) - f_B|^p w(y) d\mu(y) \\ & = \frac{p}{w(B)} \int_0^\infty \lambda^{p-1} w(\{y \in B : |f(y) - f_B| > \lambda\}) d\lambda \\ & \leq CC_1^\delta p \varphi(B)^{\eta/\theta} \int_0^\infty \lambda^{p-1} \exp\left\{-\frac{C_2\delta\lambda}{\|f\|_{BMO_\rho} \varphi(B)^{l_0+1}}\right\} d\lambda \\ & \leq C \varphi(B)^\tau \|f\|_{BMO_\rho}^p. \end{aligned}$$

This completes the proof of Lemma 2.8. $\square$

As a consequence of Lemma 2.7, we can then obtain the following result.

Lemma 2.9. *If $f \in BMO_\rho$, and $w \in A_p^\rho$ with $p \in [1, \infty)$, then there exist positive constants γ and C such that*

$$\sup_{B \subseteq X} \frac{1}{\varphi(B)^{\eta/\theta} w(B)} \int_B \exp\left\{\frac{\gamma|f(y) - f_B|}{\varphi(B)^{l_0+1} \|f\|_{BMO_\rho}}\right\} w(y) d\mu(y) \leq C.$$

Proof. It follows from (2.10) that

$$w(\{y \in B : |f(y) - f_B| > \gamma\}) \leq CC_1^\delta \varphi(B)^{\eta/\theta} \exp\left\{-\frac{C_2\delta\gamma}{\varphi(B)^{l_0+1} \|f\|_{BMO_\rho}}\right\} w(B).$$

Choosing $\gamma = C_2\delta/2$, we deduce from the last inequality above that

$$\begin{aligned} & \int_B \exp\left\{\frac{\gamma|f(y) - f_B|}{\varphi(B)^{l_0+1} \|f\|_{BMO_\rho}}\right\} w(y) d\mu(y) \\ & = \int_0^\infty w(\{y \in B : \exp\left\{\frac{\gamma|f(y) - f_B|}{\varphi(B)^{l_0+1} \|f\|_{BMO_\rho}}\right\} > t\}) dt \\ & \leq w(B) + \int_1^\infty w\left(\{y \in B : |f(y) - f_B| > \frac{\varphi(B)^{l_0+1} \|f\|_{BMO_\rho} \log t}{\gamma}\right) dt \\ & \leq w(B) + CC_1^\delta \varphi(B)^{\eta/\theta} w(B) \int_1^\infty \exp\left\{-\frac{C_2\delta \log t}{\gamma}\right\} dt \\ & \leq C \varphi(B)^{\eta/\theta} w(B). \end{aligned}$$

Thus, the proof is completed. $\square$

3. Proof of the main results

Let $L_b^\infty(\mathcal{X})$ denote the space of all bounded measurable functions with bounded support. In this paper, we always assume that T is a Calderón–Zygmund operator associated with the admissible function ρ , which is bounded on $L^2(\mathcal{X})$, defined by

$$Tf(x) := \int_{\mathcal{X}} K(x, y)f(y)d\mu(y), \quad \forall f \in L_b^\infty(\mathcal{X}), \quad \forall x \in \text{supp } f$$

with the kernel K satisfying the following:

(i) For any l , there exists a positive constant C_l such that

$$|K(x, y)| \leq \frac{C_l}{\left[1 + d(x, y)\left(\frac{1}{\rho(x)} + \frac{1}{\rho(y)}\right)\right]^l V(x, y)}. \quad (3.1)$$

(ii) For any $x, y, z \in \mathcal{X}$ with $x \neq z$,

$$\begin{aligned} & |K(x, z) - K(y, z)| + |K(z, x) - K(z, y)| \\ & \leq \frac{C_l}{\left[1 + d(x, z)\left(\frac{1}{\rho(x)} + \frac{1}{\rho(z)}\right)\right]^l} \frac{d(x, y)^\delta}{V(x, z)d(x, z)^\delta}, \end{aligned}$$

whenever $d(x, y) < d(x, z)/2$, and $\delta \in (0, 1]$.

Recall that the commutator $[b, T]$ of a linear operator T and a function $b \in L_{loc}^1(\mathcal{X})$ are defined by setting, for any suitable function f ,

$$[b, T]f(x) := T(bf)(x) - b(x)Tf(x), \quad \forall x \in \mathcal{X}.$$

Recently, Qin and Fu [34] proved the $L^p(w)$ boundedness for Calderón–Zygmund operator, which will be satisfied if $w \in A_p^\rho$. When $b \in BMO_\rho$, they also proved that the commutator $[b, T]$ is bounded on the weighted Lebesgue space $L^p(w)$ with $p \in (1, \infty)$ and $w \in A_p^\rho$. Meanwhile, the estimate for the commutator at the endpoints were also provided.

Theorem 3.1. ([34]) *Let T be a Calderón–Zygmund operator associated with the admissible function ρ . Then, the following statements hold:*

- (i) *If $w \in A_p^\rho$ with $p \in (1, \infty)$, then T is bounded on $L^p(w)$.*
- (ii) *If $w \in A_1^\rho$, then T is bounded from $L^1(w)$ into $WL^1(w)$.*

Theorem 3.2. ([34]) *Let T be a Calderón–Zygmund operator associated with the admissible function ρ and $b \in BMO_\rho$. Then, the following statements hold:*

- (i) *If $w \in A_p^\rho$ with $p \in (1, \infty)$, then $[b, T]$ is bounded on $L^p(w)$.*
- (ii) *If $w \in A_1^\rho$, then for any measurable function f and $t \in (0, \infty)$,*

$$w(\{x \in \mathcal{X} : |[b, T]f(x)| > t\}) \lesssim \int_{\mathcal{X}} \Phi\left(\frac{|f(x)|}{t}\right) w(x) d\mu(x),$$

where $\Phi(t) = t(1 + \log^+ t)$, and $\log^+ t = \max\{\log t, 0\}$.

We now present the proof of the main theorem in this paper.

Proof of Theorem 1.3. (i) Let $(p, \kappa) \in (1, \infty) \times (0, 1)$. For any given ball $B = B(x_0, r)$ in $\mathcal{X}$, it suffices to prove that there exists some $\nu > 0$ such that

$$\left(\frac{1}{\Psi(w(B))} \int_B |Tf(x)|^p w(x) d\mu(x) \right)^{1/p} \lesssim \left(1 + \frac{r}{\rho(x_0)} \right)^\nu \quad (3.2)$$

holds for any $f \in L_{\rho}^{p, \Psi}(w)$ with $\|f\|_{L_{\rho}^{p, \Psi}(w)} = 1$ and $w \in A_p^\rho$.

Suppose that $f \in L_{\rho, \theta}^{p, \Psi}(w)$ with $\|f\|_{L_{\rho, \theta}^{p, \Psi}(w)} = 1$ for some $\theta > 0$ and that $\Psi(\cdot)$ satisfies $\mathcal{D}_\kappa$ Condition (1.3). We decompose f by

$$f = f\chi_{2B} + f\chi_{(2B)^c} =: f_1 + f_2.$$

Then we have

$$\begin{aligned} & \left(\frac{1}{\Psi(w(B))} \int_B |Tf(x)|^p w(x) d\mu(x) \right)^{1/p} \\ & \leq \left(\frac{1}{\Psi(w(B))} \int_B |Tf_1(x)|^p w(x) d\mu(x) \right)^{1/p} \\ & \quad + \left(\frac{1}{\Psi(w(B))} \int_B |Tf_2(x)|^p w(x) d\mu(x) \right)^{1/p} \\ & =: I + II. \end{aligned}$$

Let us first estimate the term I . By using Theorem 3.1 (i) and $\mathcal{D}_\kappa$ Condition (1.3),

$$\begin{aligned} I & \lesssim \frac{1}{\Psi(w(B))^{1/p}} \left(\int_{2B} |f(x)|^p w(x) dx \right)^{1/p} \\ & = \|f\|_{L_{\rho, \theta}^{p, \Psi}(w)} \left(\frac{\Psi(w(2B))}{\Psi(w(B))} \right)^{1/p} \varphi(2B) \\ & \lesssim \|f\|_{L_{\rho, \theta}^{p, \Psi}(w)} \left(\frac{w(2B)}{w(B)} \right)^{\kappa/p} \varphi(2B) \\ & \lesssim \varphi(B)^{1+\kappa}, \end{aligned}$$

where in the last inequality, we have used Lemma 2.5.

For the term II , notice that for any $x \in B$ and $y \in (2B)^c$, we have $d(x, y) \sim d(x_0, y)$. It follows from (3.1) that

$$\begin{aligned} |Tf_2(x)| & \lesssim \int_{(2B)^c} \frac{|f(y)|}{(1 + d(x, y)/\rho(x))^l V(x, y)} d\mu(y) \\ & \lesssim \int_{(2B)^c} \frac{|f(y)|}{(1 + d(x_0, y)/\rho(x))^l V(x_0, y)} d\mu(y) \\ & \lesssim \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^k B} \frac{|f(y)|}{(1 + d(x_0, y)/\rho(x))^l V(x_0, y)} d\mu(y) \\ & \lesssim \sum_{k=1}^{\infty} \int_{2^{k+1}B} \frac{|f(y)|}{(1 + 2^{k+1}r/\rho(x))^l V(x_0, 2^{k+1}r)} d\mu(y) \end{aligned}$$

$$\lesssim \sum_{k=1}^{\infty} \int_{2^{k+1}B} \frac{|f(y)|}{(1 + 2^{k+1}r/\rho(x_0))^{l/(l_0+1)} V(x_0, 2^{k+1}r)} d\mu(y), \quad (3.3)$$

which implies

$$II \lesssim \left(\frac{w(B)}{\Psi(w(B))} \right)^{1/p} \sum_{k=1}^{\infty} \int_{2^{k+1}B} \frac{|f(y)|}{(1 + 2^{k+1}r/\rho(x_0))^{l/(l_0+1)} V(x_0, 2^{k+1}r)} d\mu(y).$$

Because $w \in A_p^\rho$, we deduce from Hölder's inequality that

$$\begin{aligned} \int_{2^{k+1}B} |f(y)| d\mu(y) &\leq \left(\int_{2^{k+1}B} |f(y)|^p w(y) d\mu(y) \right)^{1/p} \left(\int_{2^{k+1}B} w(y)^{-p'/p} d\mu(y) \right)^{1/p'} \\ &\lesssim \left(\frac{\Psi(w(2^{k+1}B))}{w(2^{k+1}B)} \right)^{1/p} \varphi(2^{k+1}B)^2 \mu(2^{k+1}B) \|f\|_{L_{\rho,\theta}^{p,\Psi}(w)}. \end{aligned} \quad (3.4)$$

Therefore, the $\mathcal{D}_\kappa$ conditions (1.3) of $\Psi(\cdot)$, Lemma 2.4, and (1.1) lead to

$$\begin{aligned} II &\lesssim \sum_{k=1}^{\infty} \left(\frac{w(B)}{w(2^{k+1}B)} \right)^{(1-\kappa)/p} \varphi(2^{k+1}B)^{2-l/(l_0+1)\theta} \|f\|_{L_{\rho,\theta}^{p,\Psi}(w)} \\ &\lesssim \|f\|_{L_{\rho,\theta}^{p,\Psi}(w)} \sum_{k=1}^{\infty} 2^{-\frac{(k+1)n(1-\kappa)\delta}{p}} \varphi(2^{k+1}B)^{2-\frac{l}{(l_0+1)\theta} + \frac{(1-\kappa)\eta}{p\theta}} \\ &\lesssim \varphi(B)^0, \end{aligned}$$

where we choose l large enough such that $l \geq (l_0 + 1)(2p\theta + (1 - \kappa)\eta)/p$.

Finally, collecting the above estimates for I and II , and letting $\nu = \max\{0, 1 + \kappa\} = 1 + \kappa$, the desired inequality (3.2) is obtained.

(ii) The proof can be given by replacing (3.2) with the corresponding weak estimate. $\square$

Proof of Theorem 1.4. (i) Let $(p, \kappa) \in (1, \infty) \times (0, 1)$. For any given ball $B = B(x_0, r)$ in $\mathcal{X}$, it suffices to prove that there exists some $\nu > 0$ such that

$$\left(\frac{1}{\Psi(w(B))} \int_B |[b, T]f(x)|^p w(x) d\mu(x) \right)^{1/p} \lesssim \left(1 + \frac{r}{\rho(x_0)} \right)^\nu \quad (3.5)$$

holds for any $f \in L_{\rho}^{p,\Psi}(w)$ with $\|f\|_{L_{\rho}^{p,\Psi}(w)} = 1$ and $w \in A_p^\rho$, and $b \in BMO_\rho$ with $\|b\|_{BMO_\rho} = 1$.

Suppose that $f \in L_{\rho,\theta}^{p,\Psi}(w)$ with $\|f\|_{L_{\rho,\theta}^{p,\Psi}(w)} = 1$ for some $\theta > 0$. Using the representation $f = f_1 + f_2$ with the same f_1 and f_2 as in the proof of Theorem 1.3, we see that

$$\begin{aligned} &\left(\frac{1}{\Psi(w(B))} \int_B |[b, T]f(x)|^p w(x) d\mu(x) \right)^{1/p} \\ &\leq \left(\frac{1}{\Psi(w(B))} \int_B |[b, T]f_1(x)|^p w(x) d\mu(x) \right)^{1/p} \\ &\quad + \left(\frac{1}{\Psi(w(B))} \int_B |[b, T]f_2(x)|^p w(x) d\mu(x) \right)^{1/p} \\ &=: III + IV. \end{aligned}$$

From the weighted $L^p(w)$ boundedness of $[b, T]$ (see Theorem 3.2 (i)) and the $\mathcal{D}_\kappa$ condition (1.3) of $\Psi(\cdot)$, we thus obtain

$$\begin{aligned} III &\lesssim \frac{1}{\Psi(w(B))^{1/p}} \left(\int_{2B} |f(x)|^p w(x) d\mu(x) \right)^{1/p} \\ &\lesssim \|f\|_{L_{\rho, \theta}^{p, \Psi}(w)} \left(\frac{\Psi(w(2B))}{\Psi(w(B))} \right)^{1/p} \varphi(2B) \\ &\lesssim \|f\|_{L_{\rho, \theta}^{p, \Psi}(w)} \left(\frac{w(2B)}{w(B)} \right)^{\kappa/p} \varphi(2B) \\ &\lesssim \varphi(B)^{1+\kappa}, \end{aligned}$$

where in the last inequality, we have used Lemma 2.5.

Let us now turn to the estimate of IV . For any $x \in B(x_0, r)$, notice that

$$|[b, T]f_2(x)| \leq |b(x) - b_B| |Tf_2(x)| + |T((b_B - b)f_2(x))|. \quad (3.6)$$

Following the same arguments as in (3.3), we obtain

$$\begin{aligned} |T((b_B - b)f_2(x))| &\lesssim \int_X \frac{|b_B - b(y)| |f_2(y)|}{(1 + d(x, y)/\rho(x))^l V(x, y)} d\mu(y) \\ &\lesssim \int_{(2B)^c} \frac{|b_B - b(y)| |f(y)|}{(1 + d(x_0, y)/\rho(x))^l V(x_0, y)} d\mu(y) \\ &\lesssim \sum_{k=1}^{\infty} \int_{2^{k+1}B} \frac{|b_B - b(y)| |f(y)|}{(1 + 2^{k+1}r/\rho(x_0))^{l/(l_0+1)} V(x_0, 2^{k+1}r)} d\mu(y), \end{aligned} \quad (3.7)$$

which, together with (3.3), implies that

$$\begin{aligned} IV &\lesssim \left(\frac{1}{\Psi(w(B))} \int_B |b(x) - b_B|^p w(x) d\mu(x) \right)^{1/p} \\ &\quad \times \sum_{k=1}^{\infty} \int_{2^{k+1}B} \frac{|f(y)|}{(1 + 2^{k+1}r/\rho(x_0))^{l/(l_0+1)} V(x_0, 2^{k+1}r)} d\mu(y) \\ &\quad + \left(\frac{w(B)}{\Psi(w(B))} \right)^{1/p} \sum_{k=1}^{\infty} \int_{2^{k+1}B} \frac{|b_B - b(y)| |f(y)|}{(1 + 2^{k+1}r/\rho(x_0))^{l/(l_0+1)} V(x_0, 2^{k+1}r)} d\mu(y) \\ &\lesssim \left(\frac{1}{\Psi(w(B))} \int_B |b(x) - b_B|^p w(x) d\mu(x) \right)^{1/p} \\ &\quad \times \sum_{k=1}^{\infty} \int_{2^{k+1}B} \frac{|f(y)|}{(1 + 2^{k+1}r/\rho(x_0))^{l/(l_0+1)} V(x_0, 2^{k+1}r)} d\mu(y) \\ &\quad + \left(\frac{w(B)}{\Psi(w(B))} \right)^{1/p} \sum_{k=1}^{\infty} \int_{2^{k+1}B} \frac{|b_{2^{k+1}B} - b_B| |f(y)|}{(1 + 2^{k+1}r/\rho(x_0))^{l/(l_0+1)} V(x_0, 2^{k+1}r)} d\mu(y) \\ &\quad + \left(\frac{w(B)}{\Psi(w(B))} \right)^{1/p} \sum_{k=1}^{\infty} \int_{2^{k+1}B} \frac{|b(y) - b_{2^{k+1}B}| |f(y)|}{(1 + 2^{k+1}r/\rho(x_0))^{l/(l_0+1)} V(x_0, 2^{k+1}r)} d\mu(y) \\ &=: IV_1 + IV_2 + IV_3. \end{aligned}$$

For IV_1 , by using Lemma 2.8, (3.4), and the $\mathcal{D}_\kappa$ condition (1.3),

$$\begin{aligned} IV_1 &\lesssim \left(\frac{w(B)}{\Psi(w(B))}\right)^{1/p} \varphi(B)^{\tau/p} \|b\|_{BMO_\rho} \sum_{k=1}^{\infty} \int_{2^{k+1}B} \frac{|f(y)|}{\varphi(2^{k+1}B)^{l/(l_0+1)\theta} V(x_0, 2^{k+1}r)} d\mu(y) \\ &\lesssim \|b\|_{BMO_\rho} \|f\|_{L_{\rho,\theta}^{p,\Psi}(w)} \sum_{k=1}^{\infty} \left(\frac{w(B)}{w(2^{k+1}B)}\right)^{(1-\kappa)/p} \frac{\varphi(B)^{\tau/p}}{\varphi(2^{k+1}B)^{l/(l_0+1)\theta-2}} \\ &\lesssim \|b\|_{BMO_\rho} \|f\|_{L_{\rho,\theta}^{p,\Psi}(w)} \sum_{k=1}^{\infty} 2^{-\frac{(k+1)n\delta(1-\kappa)}{p}} \frac{\varphi(2^{k+1}B)^{\frac{\tau}{p} + \frac{(1-\kappa)\eta}{p\theta}}}{\varphi(2^{k+1}B)^{l/(l_0+1)\theta-2}} \\ &\lesssim \varphi(B)^0, \end{aligned}$$

where we have used Lemma 2.4 and let $l \geq (2p\theta + 2\theta + (1 - \kappa)\eta)(l_0 + 1)/p$, observe that for any $b \in BMO_\rho$, there is

$$|b_{2^{k+1}B} - b_B| \lesssim (k+1)\varphi(2^{k+1}B)\|b\|_{BMO_\rho}. \quad (3.8)$$

Thus, we obtain

$$\begin{aligned} IV_2 &\lesssim \left(\frac{w(B)}{\Psi(w(B))}\right)^{1/p} \|b\|_{BMO_\rho} \sum_{k=1}^{\infty} \int_{2^{k+1}B} \frac{(k+1)\varphi(2^{k+1}B)|f(y)|}{\varphi(2^{k+1}B)^{l/(l_0+1)\theta} V(x_0, 2^{k+1}r)} d\mu(y) \\ &\lesssim \|b\|_{BMO_\rho} \|f\|_{L_{\rho,\theta}^{p,\Psi}(w)} \sum_{k=1}^{\infty} \left(\frac{w(B)}{w(2^{k+1}B)}\right)^{(1-\kappa)/p} \frac{(k+1)\varphi(2^{k+1}B)}{\varphi(2^{k+1}B)^{l/(l_0+1)\theta-2}} \\ &\lesssim \|b\|_{BMO_\rho} \|f\|_{L_{\rho,\theta}^{p,\Psi}(w)} \sum_{k=1}^{\infty} (k+1) 2^{-\frac{(k+1)n\delta(1-\kappa)}{p}} \frac{\varphi(2^{k+1}B)^{1 + \frac{(1-\kappa)\eta}{p\theta}}}{\varphi(2^{k+1}B)^{l/(l_0+1)\theta-2}}, \end{aligned}$$

where we let $l \geq (3p\theta + (1 - \kappa)\eta)(l_0 + 1)/p$.

We now turn to deal with the term IV_3 . Because $w \in A_\rho^p$, we know from Lemma 2.2 (ii) that $w^{-p'/p} \in A_{\rho'}^p$. Thus, by Hölder's inequality and Lemma 2.8,

$$\begin{aligned} &\int_{2^{k+1}B} |b(y) - b_{2^{k+1}B}| |f(y)| d\mu(y) \\ &\lesssim \left(\int_{2^{k+1}B} |f(y)|^p w(y) d\mu(y)\right)^{1/p} \left(\int_{2^{k+1}B} |b(y) - b_{2^{k+1}B}|^{p'} w(y)^{-p'/p} d\mu(y)\right)^{1/p'} \\ &\lesssim \|b\|_{BMO_\rho} \left(\int_{2^{k+1}B} |f(y)|^p w(y) d\mu(y)\right)^{1/p} \varphi(2^{k+1}B)^{\tau/p'} \left(\int_{2^{k+1}B} w(y)^{-p'/p} d\mu(y)\right)^{1/p'} \\ &\lesssim \|b\|_{BMO_\rho} \left(\int_{2^{k+1}B} |f(y)|^p w(y) d\mu(y)\right)^{1/p} \frac{\varphi(2^{k+1}B)^{\tau/p'+1} \mu(2^{k+1}B)}{w(2^{k+1}B)^{1/p}} \\ &= \|b\|_{BMO_\rho} \|f\|_{L_{\rho,\theta}^{p,\Psi}(w)} \left(\frac{\Psi(w(2^{k+1}B))}{w(2^{k+1}B)}\right)^{1/p} \varphi(2^{k+1}B)^{\tau/p'+2} \mu(2^{k+1}B), \end{aligned}$$

which, combined with the $\mathcal{D}_\kappa$ condition (1.3) and Lemma 2.4, further implies that

$$IV_3 \lesssim \left(\frac{w(B)}{\Psi(w(B))}\right)^{1/p} \sum_{k=1}^{\infty} \int_{2^{k+1}B} \frac{|b(y) - b_{2^{k+1}B}| |f(y)|}{\varphi(2^{k+1}B)^{l/(l_0+1)\theta} V(x_0, 2^{k+1}r)} d\mu(y)$$

$$\begin{aligned}
&\lesssim \|b\|_{BMO_\rho} \|f\|_{L_{\rho,\theta}^{p,\Psi}(w)} \sum_{k=1}^{\infty} \left(\frac{w(B)}{w(2^{k+1}B)} \right)^{(1-\kappa)/p} \frac{\varphi(2^{k+1}B)^{\tau/p'+2}}{\varphi(2^{k+1}B)^{l/(l_0+1)\theta}} \\
&\lesssim \|b\|_{BMO_\rho} \|f\|_{L_{\rho,\theta}^{p,\Psi}(w)} \sum_{k=1}^{\infty} 2^{-\frac{(k+1)n\delta(1-\kappa)}{p}} \frac{\varphi(2^{k+1}B)^{\frac{\tau}{p'}+2+\frac{(1-\kappa)\eta}{p\theta}}}{\varphi(2^{k+1}B)^{l/(l_0+1)\theta}} \\
&\lesssim \varphi(B)^0,
\end{aligned}$$

where we let $l \geq (\tau(p-1)\theta + 2p\theta + (1-\kappa)\eta)(l_0+1)/p$.

Combining the estimates of the terms *III* and *IV*, and letting $\nu = \max\{1+\kappa, 0\} = 1+\kappa$, we obtain the desired result (3.5).

(ii) Let $p = 1$, and $\kappa \in (0, 1)$. For any $\lambda > 0$ and any given ball $B = B(x_0, r)$ in $\mathcal{X}$, it suffices to prove that there exists some $\nu > 0$ such that

$$\frac{1}{\Psi(w(B))} w(\{x \in B : |[b, T]f(x)| > \lambda\}) \lesssim \left(1 + \frac{r}{\rho(x_0)}\right)^\nu \left\| \Phi\left(\frac{|f|}{\lambda}\right) \right\|_{(L \log L)_{\rho,\theta}^{1,\Psi}(w)} \quad (3.9)$$

holds for any f such that $\Phi(|f|) \in (L \log L)_{\rho,\theta}^{1,\Psi}(w)$ with $w \in A_\rho^p$ and $b \in BMO_\rho$.

Without loss of generality, suppose that $\|b\|_{BMO_\rho} = 1$. Using the representation $f = f_1 + f_2$, where $f_1 = f\chi_{2B}$, $f_2 = f - f_1$, we have

$$\begin{aligned}
&\frac{1}{\Psi(w(B))} w(\{x \in B : |[b, T]f(x)| > \lambda\}) \\
&\leq \frac{1}{\Psi(w(B))} w(\{x \in B : |[b, T]f_1(x)| > \lambda/2\}) \\
&\quad + \frac{1}{\Psi(w(B))} w(\{x \in B : |[b, T]f_2(x)| > \lambda/2\}) \\
&=: III' + IV'.
\end{aligned}$$

By using Theorem 3.2 (ii) and (1.6),

$$\begin{aligned}
III' &\lesssim \frac{1}{\Psi(w(B))} \int_{2B} \Phi\left(\frac{|f(x)|}{\lambda}\right) w(x) d\mu(x) \\
&\lesssim \frac{w(2B)}{\Psi(w(B))} \left\| \Phi\left(\frac{|f|}{\lambda}\right) \right\|_{L(\log L)(w), 2B} \\
&= \frac{\varphi(2B)\Psi(w(2B))}{\Psi(w(B))} \left\| \Phi\left(\frac{|f|}{\lambda}\right) \right\|_{(L \log L)_{\rho,\theta}^{1,\Psi}(w)} \\
&\lesssim \varphi(B)^{1+\kappa} \left\| \Phi\left(\frac{|f|}{\lambda}\right) \right\|_{(L \log L)_{\rho,\theta}^{1,\Psi}(w)},
\end{aligned}$$

where in the last inequality, we have used the $\mathcal{D}_\kappa$ condition (1.3) and Lemma 2.5.

For the term *IV'*, by (3.6), we can further decompose *IV'* as

$$\begin{aligned}
IV' &\lesssim \frac{1}{\Psi(w(B))} w(\{x \in B : |b(x) - b_B| |Tf_2(x)| > \lambda/4\}) \\
&\quad + \frac{1}{\Psi(w(B))} w(\{x \in B : |T((b_B - b)f_2)(x)| > \lambda/4\})
\end{aligned}$$

$$=: IV'_1 + IV'_2.$$

Using Chebyshev's inequality (in the measure-theoretic form: For any non-negative measurable function g and any $t > 0$, $\mu(\{x : g(x) > t\}) \leq t^{-1} \int_{\mathcal{X}} g(x) d\mu$), the pointwise estimate (3.3) and Lemma 2.8, we obtain

$$\begin{aligned} IV'_1 &\lesssim \frac{1}{\Psi(w(B))\lambda} \int_B |b(x) - b_B| T f_2(x) |w(x)| d\mu(x) \\ &\lesssim \sum_{k=1}^{\infty} \int_{2^{k+1}B} \frac{|f(y)|}{\varphi(2^{k+1}B)^{l/(l_0+1)\theta} V(x_0, 2^{k+1}r)} d\mu(y) \\ &\quad \times \frac{1}{\Psi(w(B))\lambda} \int_B |b(x) - b_B| w(x) d\mu(x) \\ &\lesssim \|b\|_{BMO_\rho} \sum_{k=1}^{\infty} \frac{w(B)}{\Psi(w(B))\lambda} \int_{2^{k+1}B} \frac{\varphi(B)^\tau |f(y)|}{\varphi(2^{k+1}B)^{l/(l_0+1)\theta} V(x_0, 2^{k+1}r)} d\mu(y). \end{aligned}$$

Moreover, we deduce from $w \in A_1^\rho$ that

$$\begin{aligned} \frac{1}{V(x_0, 2^{k+1}r)} \int_{2^{k+1}B} \frac{|f(y)|}{\lambda} d\mu(y) &\lesssim \frac{\inf_{y \in 2^{k+1}B} w(y)}{w(2^{k+1}B)} \varphi(2^{k+1}B) \int_{2^{k+1}B} \frac{|f(y)|}{\lambda} d\mu(y) \\ &\lesssim \frac{\varphi(2^{k+1}B)}{w(2^{k+1}B)} \int_{2^{k+1}B} \frac{|f(y)|}{\lambda} w(y) d\mu(y). \end{aligned}$$

Note that for any $t > 0$, $t \leq t(1 + \log^+ t) = \Phi(t)$. It then follows from (1.6) that

$$\begin{aligned} \frac{1}{V(x_0, 2^{k+1}r)} \int_{2^{k+1}B} \frac{|f(y)|}{\lambda} d\mu(y) &\lesssim \frac{\varphi(2^{k+1}B)}{w(2^{k+1}B)} \int_{2^{k+1}B} \Phi\left(\frac{|f(y)|}{\lambda}\right) w(y) d\mu(y) \\ &\lesssim \frac{\varphi(2^{k+1}B)^2 \Psi(w(2^{k+1}B))}{w(2^{k+1}B)} \left\| \Phi\left(\frac{|f|}{\lambda}\right) \right\|_{(L \log L)_{\rho, \theta}^{1, \Psi}(w)}, \end{aligned} \quad (3.10)$$

which, together with the $\mathcal{D}_\kappa$ condition (1.3) and Lemma 2.4, implies that

$$\begin{aligned} IV'_1 &\lesssim \|b\|_{BMO_\rho} \left\| \Phi\left(\frac{|f|}{\lambda}\right) \right\|_{(L \log L)_{\rho, \theta}^{1, \Psi}(w)} \sum_{k=1}^{\infty} \frac{w(B)}{w(2^{k+1}B)} \frac{\Psi(w(2^{k+1}B))}{\Psi(w(B))} \frac{\varphi(2^{k+1}B)^{\tau+2}}{\varphi(2^{k+1}B)^{l/(l_0+1)\theta}} \\ &\lesssim \|b\|_{BMO_\rho} \left\| \Phi\left(\frac{|f|}{\lambda}\right) \right\|_{(L \log L)_{\rho, \theta}^{1, \Psi}(w)} \sum_{k=1}^{\infty} \left(\frac{w(B)}{w(2^{k+1}B)} \right)^{1-\kappa} \frac{\varphi(2^{k+1}B)^{\tau+2}}{\varphi(2^{k+1}B)^{l/(l_0+1)\theta}} \\ &\lesssim \|b\|_{BMO_\rho} \left\| \Phi\left(\frac{|f|}{\lambda}\right) \right\|_{(L \log L)_{\rho, \theta}^{1, \Psi}(w)} \sum_{k=1}^{\infty} 2^{-(k+1)n\delta(1-\kappa)} \frac{\varphi(2^{k+1}B)^{\tau+2+\frac{(1-\kappa)\eta}{\theta}}}{\varphi(2^{k+1}B)^{l/(l_0+1)\theta}} \\ &\lesssim \varphi(B)^0 \left\| \Phi\left(\frac{|f|}{\lambda}\right) \right\|_{(L \log L)_{\rho, \theta}^{1, \Psi}(w)}, \end{aligned}$$

where we let $l \geq ((\tau + 2)\theta + (1 - \kappa)\eta)(l_0 + 1)$.

Next, let us deal with the term IV'_2 . By applying Chebyshev's inequality and the pointwise estimate (3.7), we obtain

$$IV'_2 \lesssim \frac{1}{\Psi(w(B))\lambda} \int_B |T((b_B - b)f_2)(x)| w(x) d\mu(x)$$

$$\begin{aligned}
&\lesssim \frac{w(B)}{\Psi(w(B))\lambda} \sum_{k=1}^{\infty} \int_{2^{k+1}B} \frac{|b_B - b(y)||f(y)|}{(1 + 2^{k+1}r/\rho(x_0))^{l/(l_0+1)} V(x_0, 2^{k+1}r)} d\mu(y) \\
&\lesssim \frac{w(B)}{\Psi(w(B))\lambda} \sum_{k=1}^{\infty} \int_{2^{k+1}B} \frac{|b_{2^{k+1}B} - b_B||f(y)|}{(1 + 2^{k+1}r/\rho(x_0))^{l/(l_0+1)} V(x_0, 2^{k+1}r)} d\mu(y) \\
&\quad + \frac{w(B)}{\Psi(w(B))\lambda} \sum_{k=1}^{\infty} \int_{2^{k+1}B} \frac{|b(y) - b_{2^{k+1}B}||f(y)|}{(1 + 2^{k+1}r/\rho(x_0))^{l/(l_0+1)} V(x_0, 2^{k+1}r)} d\mu(y) \\
&=: IV'_{21} + IV'_{22}.
\end{aligned}$$

It follows from (3.8) and the pointwise estimate (3.10) that

$$IV'_{21} \lesssim \|b\|_{BMO_\rho} \|\Phi\left(\frac{|f|}{\lambda}\right)\|_{(L \log L)_{\rho, \theta}^{1, \Psi}(w)} \frac{w(B)}{\Psi(w(B))} \sum_{k=1}^{\infty} (k+1) \frac{\varphi(2^{k+1}B)^3 \Psi(w(2^{k+1}B))}{\varphi(2^{k+1}B)^{l/(l_0+1)\theta} w(2^{k+1}B)}.$$

We further deduce from the $\mathcal{D}_\kappa$ condition (1.3) and Lemma 2.4 that

$$\begin{aligned}
IV'_{21} &\lesssim \|b\|_{BMO_\rho} \|\Phi\left(\frac{|f|}{\lambda}\right)\|_{(L \log L)_{\rho, \theta}^{1, \Psi}(w)} \sum_{k=1}^{\infty} (k+1) \left(\frac{w(B)}{w(2^{k+1}B)}\right)^{1-\kappa} \frac{\varphi(2^{k+1}B)^3}{\varphi(2^{k+1}B)^{l/(l_0+1)\theta}} \\
&\lesssim \|b\|_{BMO_\rho} \|\Phi\left(\frac{|f|}{\lambda}\right)\|_{(L \log L)_{\rho, \theta}^{1, \Psi}(w)} \sum_{k=1}^{\infty} \frac{(k+1)}{2^{(k+1)n\delta(1-\kappa)}} \frac{\varphi(2^{k+1}B)^{3+\frac{(1-\kappa)\eta}{\theta}}}{\varphi(2^{k+1}B)^{l/(l_0+1)\theta}} \\
&\lesssim \varphi(B)^0 \|\Phi\left(\frac{|f|}{\lambda}\right)\|_{(L \log L)_{\rho, \theta}^{1, \Psi}(w)},
\end{aligned}$$

where we let $l \geq (3\theta + (1 - \kappa)\eta)(l_0 + 1)$.

For the last term IV'_{22} , we proceed as follows. By using the A_1^ρ condition and the fact that $t \leq \Phi(t)$, there is

$$\begin{aligned}
IV'_{22} &\lesssim \frac{w(B)}{\Psi(w(B))} \sum_{k=1}^{\infty} \frac{\varphi(2^{k+1}B)}{w(2^{k+1}B)} \int_{2^{k+1}B} |b(y) - b_{2^{k+1}B}| \frac{|f(y)|}{\lambda} w(y) d\mu(y) \varphi(2^{k+1}B)^{-\frac{l}{(l_0+1)\theta}} \\
&\lesssim \frac{w(B)}{\Psi(w(B))} \sum_{k=1}^{\infty} \frac{\varphi(2^{k+1}B)}{w(2^{k+1}B)} \int_{2^{k+1}B} |b(y) - b_{2^{k+1}B}| \Phi\left(\frac{|f(y)|}{\lambda}\right) w(y) d\mu(y) \varphi(2^{k+1}B)^{-\frac{l}{(l_0+1)\theta}}.
\end{aligned}$$

From Lemma 2.9, it is easy to verify that

$$\|b - b_B\|_{\exp L(w), B} \lesssim \varphi(B)^{\frac{\eta}{\theta} + l_0 + 1} \|b\|_{BMO_\rho}.$$

Combining this and the generalized Hölder inequality (1.5) gives

$$\begin{aligned}
IV'_{22} &\lesssim \frac{w(B)}{\Psi(w(B))} \sum_{k=1}^{\infty} \frac{\varphi(2^{k+1}B)}{\varphi(2^{k+1}B)^{l/(l_0+1)\theta}} \|b - b_{2^{k+1}B}\|_{\exp L(w), 2^{k+1}B} \|\Phi\left(\frac{|f|}{\lambda}\right)\|_{L(\log L)(w), 2^{k+1}B} \\
&\lesssim \|b\|_{BMO_\rho} \frac{w(B)}{\Psi(w(B))} \sum_{k=1}^{\infty} \frac{\varphi(2^{k+1}B)^{l_0+2+\eta/\theta}}{\varphi(2^{k+1}B)^{l/(l_0+1)\theta}} \|\Phi\left(\frac{|f|}{\lambda}\right)\|_{L(\log L)(w), 2^{k+1}B} \\
&\lesssim \|b\|_{BMO_\rho} \|\Phi\left(\frac{|f|}{\lambda}\right)\|_{(L \log L)_{\rho, \theta}^{1, \Psi}(w)} \sum_{k=1}^{\infty} \frac{w(B)}{\Psi(w(B))} \frac{\Psi(w(2^{k+1}B))}{w(2^{k+1}B)} \frac{\varphi(2^{k+1}B)^{l_0+3+\eta/\theta}}{\varphi(2^{k+1}B)^{l/(l_0+1)\theta}}.
\end{aligned}$$

Furthermore, using the $\mathcal{D}_\kappa$ condition (1.3) of $\Psi(\cdot)$ and Lemma 2.4, we obtain

$$\begin{aligned} IV'_{22} &\lesssim \|b\|_{BMO_\rho} \left\| \Phi\left(\frac{|f|}{\lambda}\right) \right\|_{(L \log L)_{\rho, \theta}^{1, \Psi}(w)} \sum_{k=1}^{\infty} \left(\frac{w(B)}{w(2^{k+1}B)} \right)^{1-\kappa} \frac{\varphi(2^{k+1}B)^{l_0+3+\eta/\theta}}{\varphi(2^{k+1}B)^{l/(l_0+1)\theta}} \\ &\lesssim \|b\|_{BMO_\rho} \left\| \Phi\left(\frac{|f|}{\lambda}\right) \right\|_{(L \log L)_{\rho, \theta}^{1, \Psi}(w)} \sum_{k=1}^{\infty} 2^{-(k+1)n\delta(1-\kappa)} \frac{\varphi(2^{k+1}B)^{l_0+3+(2-\kappa)\eta/\theta}}{\varphi(2^{k+1}B)^{l/(l_0+1)\theta}} \\ &\lesssim \varphi(B)^0 \left\| \Phi\left(\frac{|f|}{\lambda}\right) \right\|_{(L \log L)_{\rho, \theta}^{1, \Psi}(w)}, \end{aligned}$$

where l can be chosen sufficiently large such that $l \geq [(l_0 + 3)\theta + (2 - \kappa)\eta](l_0 + 1)$.

Finally, combining the above estimates for III' and IV' , and letting $\nu = \max\{1 + \kappa, 0\} = 1 + \kappa$, Ineq (3.9) is proved, and then the proof of Theorem 1.4 is completed. $\square$

Use of Generative-AI tools declaration

The author declare he has not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

Nan. Zhao was supported by the National Natural Science Foundation of China (No. 12526505) and the Doctoral Research Foundation of Xinjiang Normal University (No. XJNUZBS2514).

Conflict of interest

The author declare no conflict of interest.

References

1. B. Bongioanni, E. Haboure, O. Salinas, Commutators of Riesz transforms related to Schrödinger operators, *J. Fourier Anal. Appl.*, **17** (2011), 115–134. <https://doi.org/10.1007/s00041-010-9133-6>
2. B. Bongioanni, E. Haboure, O. Salinas, Classes of weights related to Schrödinger operators, *J. Math. Anal. Appl.*, **373** (2011), 563–579. <https://doi.org/10.1016/j.jmaa.2010.08.008>
3. S. Chang, J. Wilson, T. Wolff, Some weighted norm inequalities concerning the Schrödinger operators, *Comment. Math. Helv.*, **60** (1985), 217–246. <https://doi.org/10.1007/BF02567411>
4. F. K. Ly, Classes of weights and second order Riesz transforms associated to Schrödinger operators, *J. Math. Soc. Japan*, **68** (2016), 489–533. <https://doi.org/10.2969/jmsj/06820489>
5. Z. Shen, L^p estimates for Schrödinger operators with certain potentials, *Ann. Inst. Fourier (Grenoble)*, **45** (1995), 513–546. <https://doi.org/10.5802/aif.1463>
6. L. Tang, Weighted norm inequalities for Schrödinger type operators, *Forum Math.*, **27** (2015), 2491–2532. <https://doi.org/10.1515/forum-2013-0070>
7. L. Tang, Weighted norm inequalities for commutators of Littlewood-Paley functions related to Schrödinger operators, preprint, arXiv: 1109.0100. <https://doi.org/10.48550/arXiv.1109.0100>

8. L. Tang, J. Wang, H. Zhu, Weighted norm inequalities for area functions related to Schrödinger operators, *J. Contemp. Math. Anal.*, **54** (2019), 38–50. <https://doi.org/10.3103/S1068362319010059>
9. J. Q. Zhang, D. C. Yang, Quantitative boundedness of Littlewood-Paley functions on weighted Lebesgue spaces in the Schrödinger setting, *J. Math. Anal. Appl.*, **484** (2020), 123731. <https://doi.org/10.1016/j.jmaa.2019.123731>
10. C. Morrey, On the solutions of quasi-linear elliptic partial differential equations, *Trans. Amer. Math. Soc.*, **43** (1938), 126–166. <https://doi.org/10.1090/S0002-9947-1938-1501936-8>
11. C. Aykol, J. J. Hasanov, A. Serbetci, A characterization of Riesz potential and its commutator in local complementary generalized Orlicz-Morrey spaces, *J. Funct. Spaces*, **2025** (2025), 2528537. <https://doi.org/10.1155/jofs/2528537>
12. S. Gala, M. A. Ragusa, A logarithmically improved regularity criterion for the 3D MHD equations in Morrey-Campanato space, *AIMS Math.*, **2** (2017), 16–23. <https://doi.org/10.3934/math.2017.1.16>
13. M. M. Wang, S. P. Tao, Marcinkiewicz integral operators associated with the Dunkl setting and their commutators on some spaces, *Filomat*, **39** (2025), 11433–11450.
14. V. S. Guliyev, *Integral operators on function spaces on the homogeneous groups and on domains in $\mathbb{R}^n$* , Doctor degree dissertation, Mat. Inst. Steklov, Moscow 1994.
15. T. Mizuhara, Boundedness of some classical operators on generalized Morrey spaces, In: *ICM-90 Satellite Conference Proceedings*, Springer, Tokyo, 1991. https://doi.org/10.1007/978-4-431-68168-7_16
16. E. Nakai, Hardy-Littlewood maximal operator, singular integral operators and Riesz potentials on generalized Morrey spaces, *Math. Nachr.*, **166** (1994), 95–103. <https://doi.org/10.1002/mana.19941660108>
17. V. S. Guliyev, *Function spaces, integral operators and two weighted inequalities on homogeneous groups, Some applications*, Baku, 1999.
18. V. S. Guliyev, Boundedness of the maximal, potential and singular operators in the generalized Morrey spaces, *J. Inequal. Appl.*, **2009** (2009), 503948.
19. Y. Sawano, A thought on generalized Morrey spaces, *J. Indones. Math. Soc.*, **25** (2019), 210–281. <https://doi.org/10.22342/jims.25.3.819.210-281>
20. Y. Komori, S. Shirai, Weighted Morrey spaces and a singular integral operator, *Math. Nachr.*, **282** (2009), 219–231. <https://doi.org/10.1002/mana.200610733>
21. V. S. Guliyev, Generalized weighted Morrey spaces and higher order commutators of sublinear operators, *Eurasian Math. J.*, **3** (2012), 33–61.
22. J. F. Dong, L. Tang, Boundedness for some Schrödinger type operators on Morrey spaces related to certain nonnegative potentials, *J. Math. Anal. Appl.*, **355** (2009), 101–109. <https://doi.org/10.1016/j.jmaa.2009.01.043>
23. V. S. Guliyev, R. V. Guliyev, M. N. Omarova, M. A. Ragusa, Schrödinger type operators on local generalized Morrey spaces related to certain nonnegative potentials, *Discrete Contin. Dyn. Syst. Ser. B*, **25** (2020), 671–690. <https://doi.org/10.3934/dcdsb.2019260>
24. G. X. Pan, L. Tang, Boundedness for some Schrödinger type operators on weighted Morrey spaces, *J. Funct. Spaces*, **2014** (2014), 878629. <https://doi.org/10.1155/2014/878629>

25. N. N. Trong, L. X. Truong, Generalized Morrey spaces associated to Schrödinger operators and applications, *Czech. Math. J.*, **68** (2018), 953–986. <https://doi.org/10.21136/CMJ.2018.0039-17>
26. H. Wang, Weighted Morrey spaces related to certain nonnegative potentials and Riesz transforms, *J. Funct. Spaces*, **2019** (2019), 7057512. <https://doi.org/10.1155/2019/7057512>
27. T. L. Xuan, T. N. Nguyen, N. T. Nguyen, On boundedness property of singular integral operators associated to a Schrödinger operator in a generalized Morrey space and applications, *Acta Math. Sci.*, **40** (2020), 1171–1184. <https://doi.org/10.1007/s10473-020-0501-2>
28. T. A. Bui, T. Q. Bui, X. T. Duong, Quantitative estimates for square functions with new class of weights, *Potential Anal.*, **57** (2022), 545–569. <https://doi.org/10.1007/s11118-021-09927-y>
29. T. A. Bui, T. Q. Bui, X. T. Duong, Quantitative weighted estimates for some singular integrals related to critical functions, *J. Geom. Anal.*, **31** (2021), 10215–10245. <https://doi.org/10.1007/s12220-021-00641-0>
30. Y. S. Han, D. Müller, D. C. Yang, A theory of Besov and Triebel-Lizorkin spaces on metric measure spaces modeled on Carnot-Carathéodory spaces, *Abstr. Appl. Anal.*, **2008** (2008), 893409. <https://doi.org/10.1155/2008/893409>
31. N. N. Trong, N. X. V. Trung, L. X. Truong, T. D. D, A new class of weights associated with Schrödinger operator on Heisenberg groups and applications, *Anal. Math. Phys.*, **14** (2024), 25. <https://doi.org/10.1007/s13324-024-00886-2>
32. D. C. Yang, Y. Zhou, Localized Hardy spaces H^1 related to admissible functions on RD -spaces and applications to Schrödinger operators, *Trans. Amer. Math. Soc.*, **363** (2011), 1197–1239.
33. Y. Liu, J. Z. Huang, J. F. Dong, Commutators of Calderón-Zygmund operators related to admissible functions on spaces of homogeneous type and applications to Schrödinger operators, *Sci. China Math.*, **56** (2013), 1895–1913. <https://doi.org/10.1007/s11425-012-4551-3>
34. G. G. Qin, X. Fu, Weighted boundedness of fractional integrals associated with admissible functions on spaces of homogeneous type, *Fract. Calc. Appl. Anal.*, **27** (2024), 2718–2753. <https://doi.org/10.1007/s13540-024-00300-5>
35. H. Wang, Morrey spaces for Schrödinger operators with certain nonnegative potentials, Littlewood-Paley and Lusin functions on the Heisenberg groups, *Banach J. Math. Anal.*, **14** (2020), 1532–1557. <https://doi.org/10.1007/s43037-020-00076-9>
36. R. M. Gong, J. Li, E. Pozzi, M. Vempati, Commutators on weighted Morrey spaces on spaces of homogeneous type, *Anal. Geom. Metr. Spaces*, **8** (2020), 305–334. <https://doi.org/10.1515/agms-2020-0116>
37. V. Kokilashvili, A. Meskhi, The boundedness of sublinear operators in weighted Morrey spaces defined on spaces of homogeneous type, In: *Function Spaces and Inequalities*, Springer, Singapore, 2017. https://doi.org/10.1007/978-981-10-6119-6_9
38. M. M. Rao, Z. D. Ren, Theory of Orlicz spaces, In: *Monographs and Textbooks in Pure and Applied Mathematics*, New York, 1991.
39. P. Zhang, Weighted endpoint estimates for commutators of Marcinkiewicz integrals, *Acta Math. Sin.*, **26** (2010), 1709–1722. <https://doi.org/10.1007/s10114-010-8562-0>

40. H. Wang, Boundedness of θ -type Calderón-Zygmund operators and commutators in the generalized weighted Morrey spaces, *J. Funct. Spaces*, **2016** (2016), 1309348. <https://doi.org/10.1155/2016/1309348>
41. J. O. Strömberg, A. Torchinsky, Weighted hardy spaces, In: *Lecture Notes in Mathematics*, Springer-Verlag, Berlin, 1989. <https://doi.org/10.1007/BFb0091154>



AIMS Press

©2026 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (<http://creativecommons.org/licenses/by/4.0>)